Fourier Transform and Digital Image Processing

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Outline

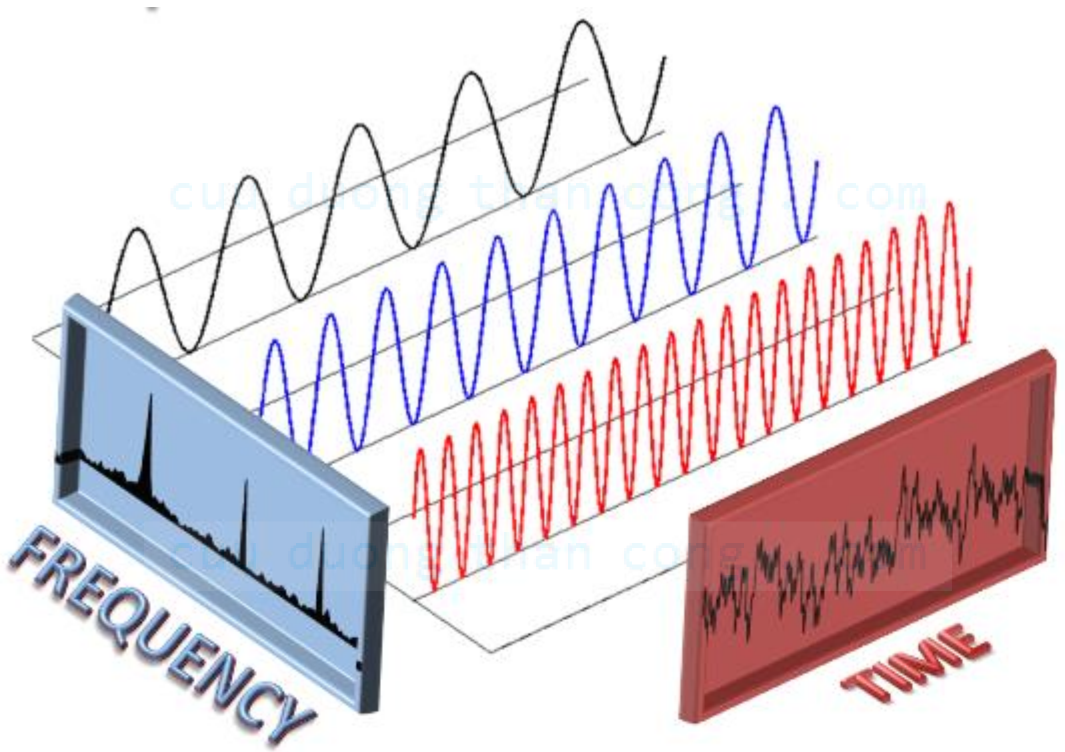
- Introduction to Fourier Transform
- Fourier Transform and Digital Image Processing

Section B.1

INTRODUCTION TO FOURIER TRANSFORM

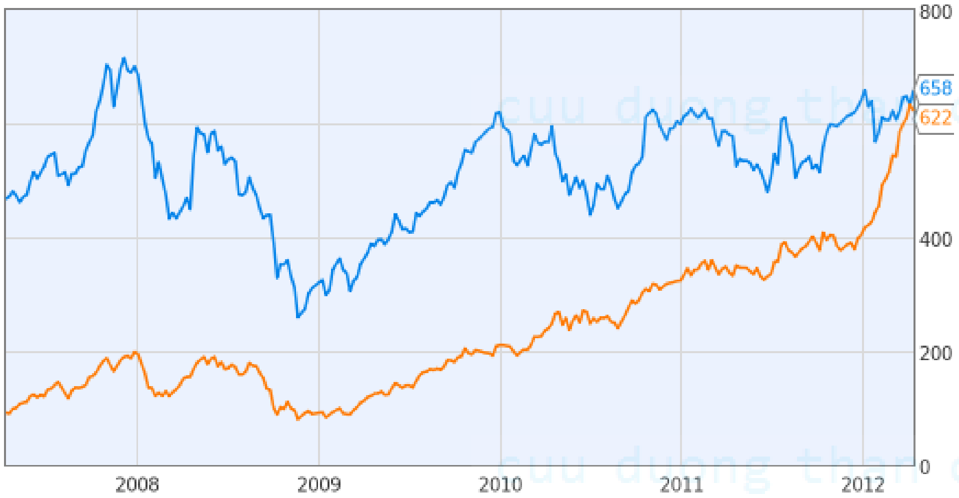
The Fourier Transform (FT)

- A tool that breaks a waveform (function or signal) into an alternate representation characterized by sines and cosines



Waveform representation

- Everything in the world can be virtually described via a waveform – a function of time, space or some other variable
  - E.g.. sound waves, electromagnetic fields, the elevation of a hill versus location, stock price versus time



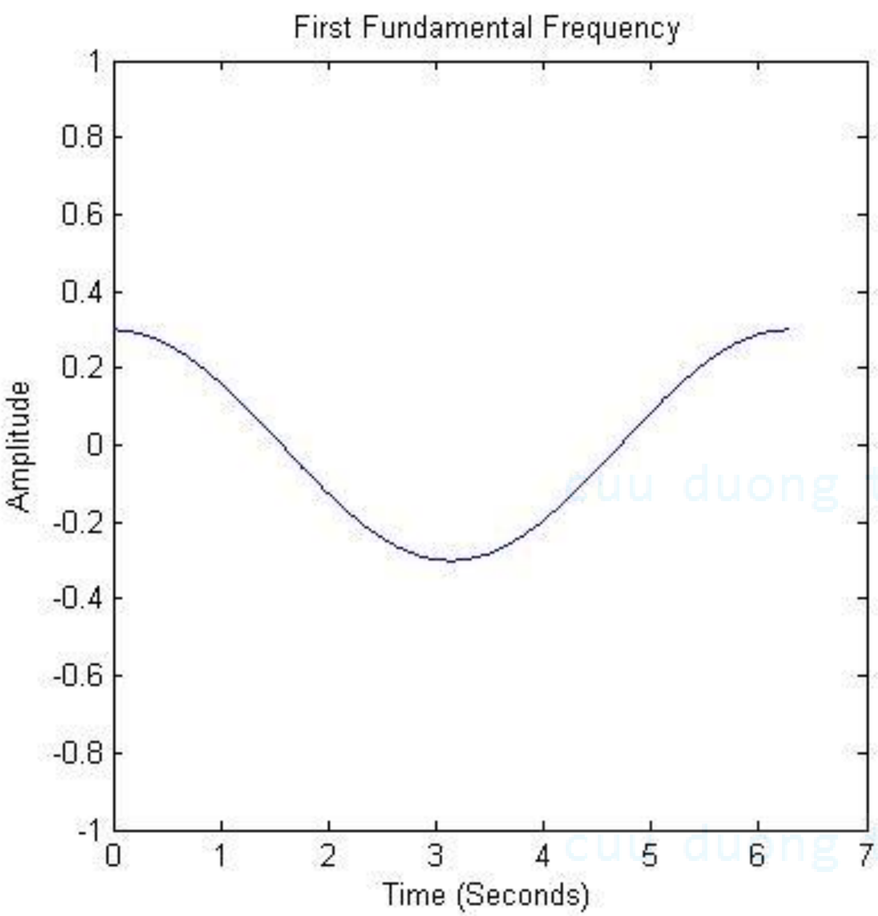
- Any waveform can be re-written as the sum of sinusoidal functions.

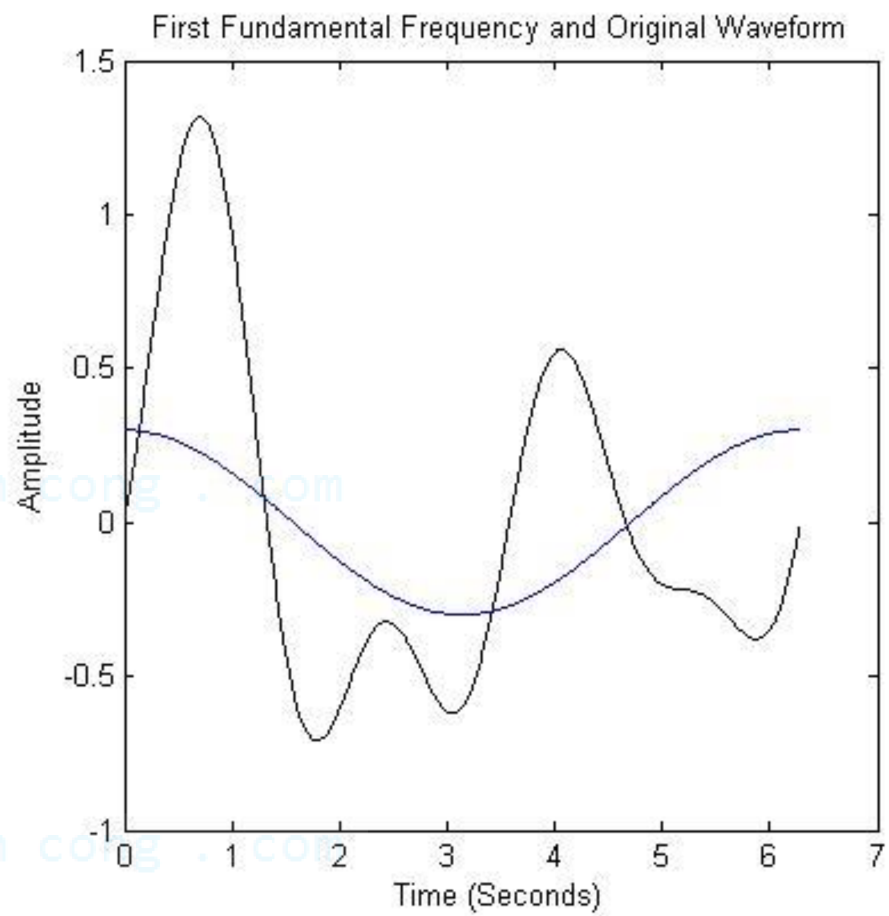
The origin of Fourier Transform

- Jean-Baptiste Joseph Fourier
  - 1768 – 1830, French mathematician and physicist
  - Investigation of Fourier series and their applications to heat transfer and vibrations
  - Also generally credited with the discovery of the greenhouse effect

- The ideas of Fourier Transform were first appeared in *La Théorie Analytique de la Chaleur (The Analytic Theory of Heat), 1822*

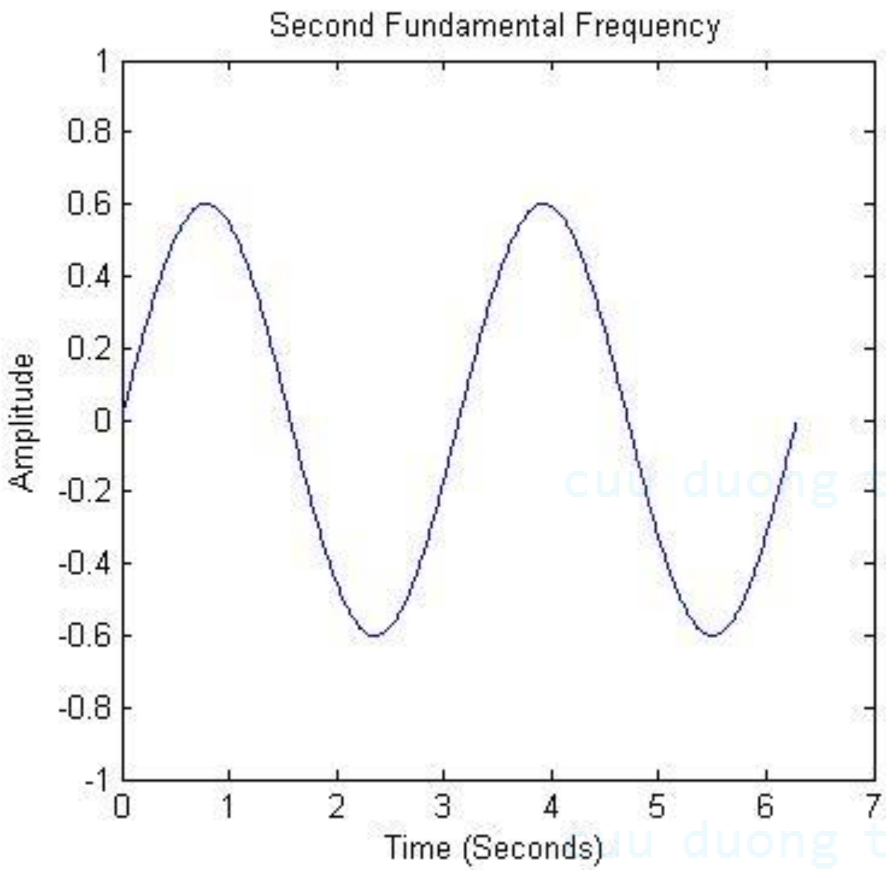
Waveform decomposition: An example

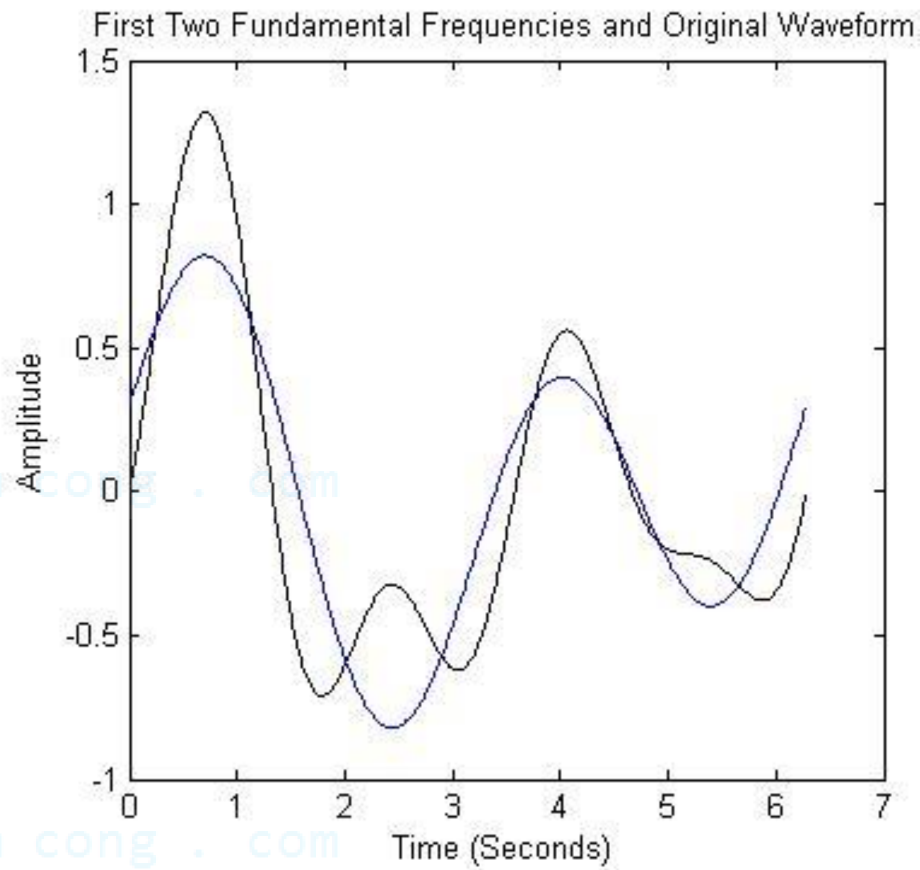




- The first component is a sinusoidal wave with period $T_1 = 2\pi$ and amplitude 0.3

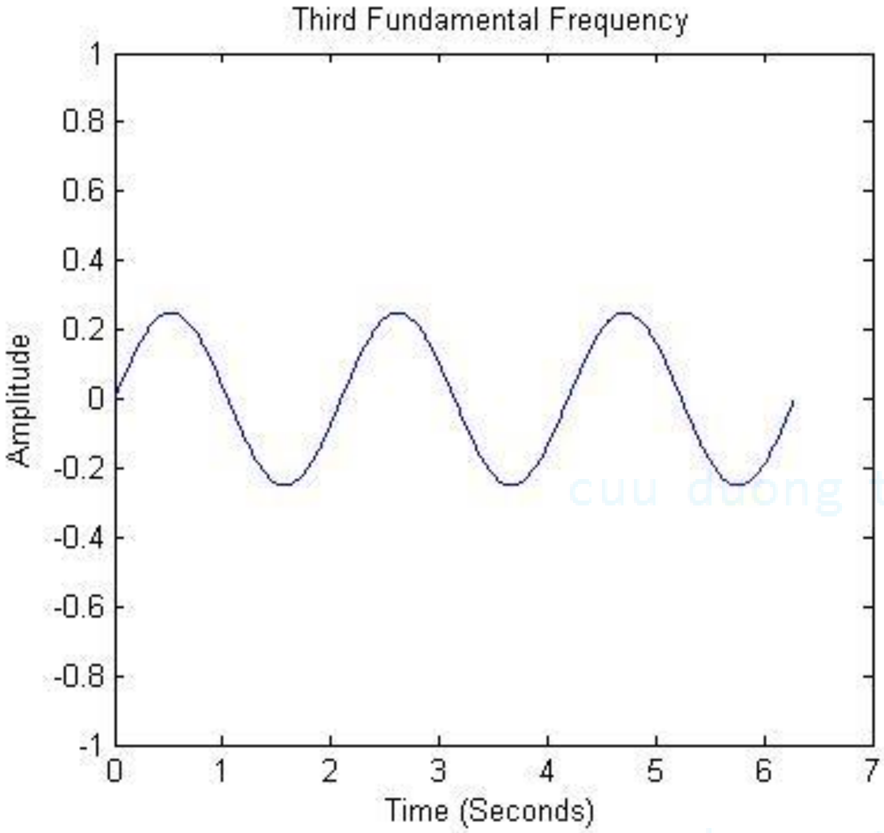
Waveform decomposition: Example

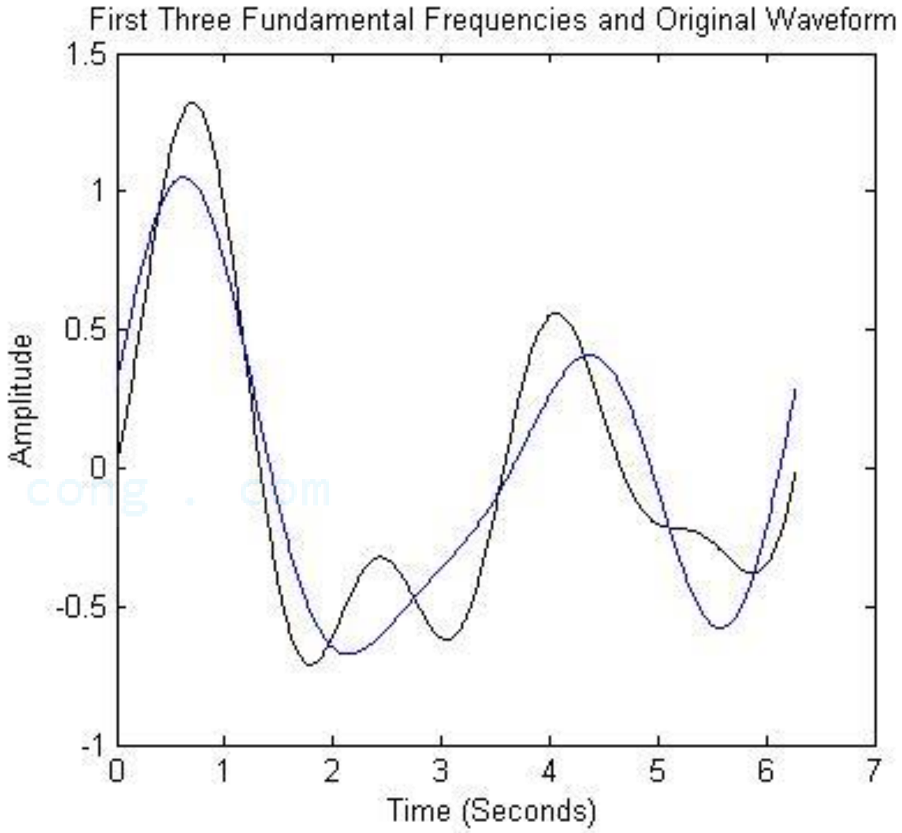




- The second frequency will have a period half as long as the first (twice the frequency), $T_2 = T_1/2$

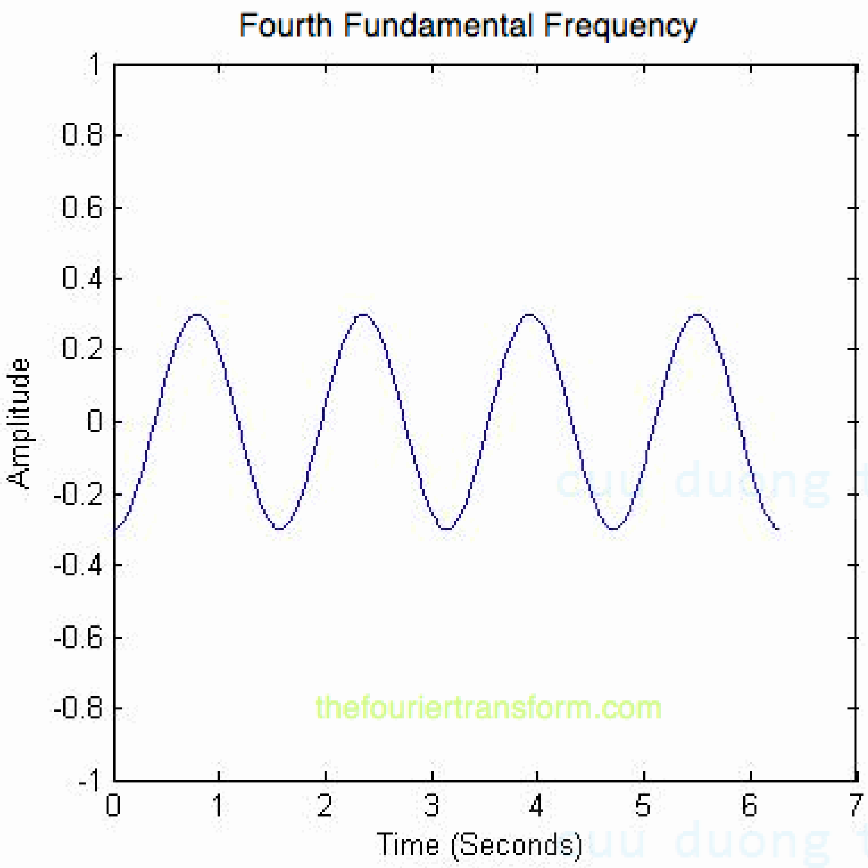
Waveform decomposition: Example

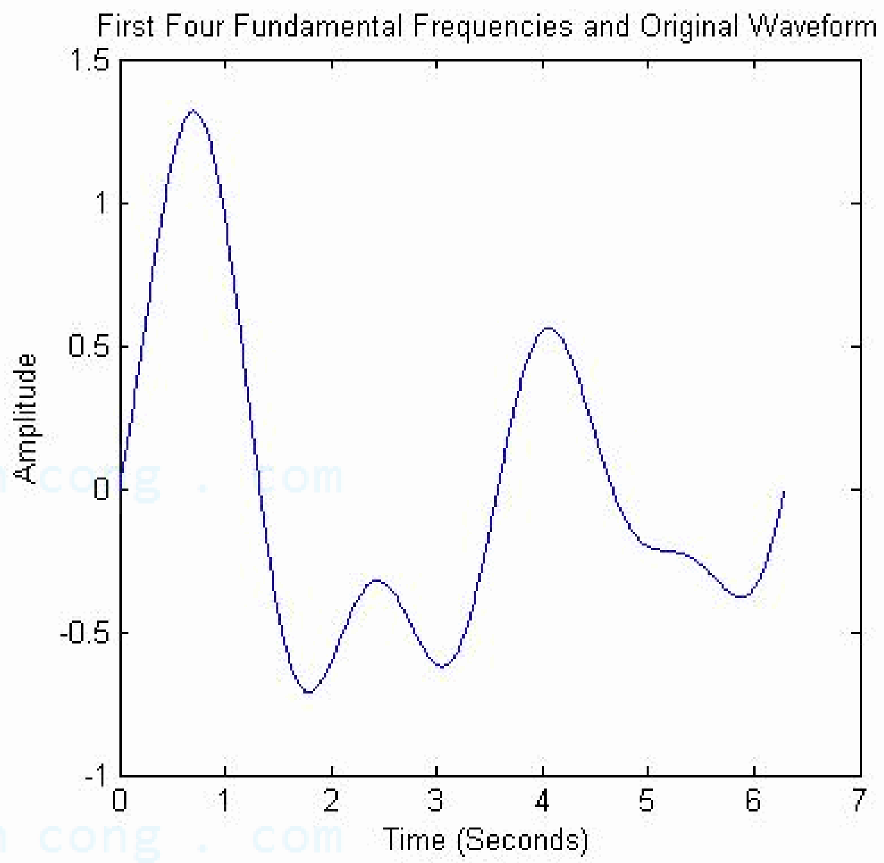




- The third frequency component is 3 times the frequency as the first, $T_3 = T_1/3$

Waveform decomposition: Example





- Finally, adding in the fourth frequency component, whose $T_4 = T_1/4$, the original waveform was revealed

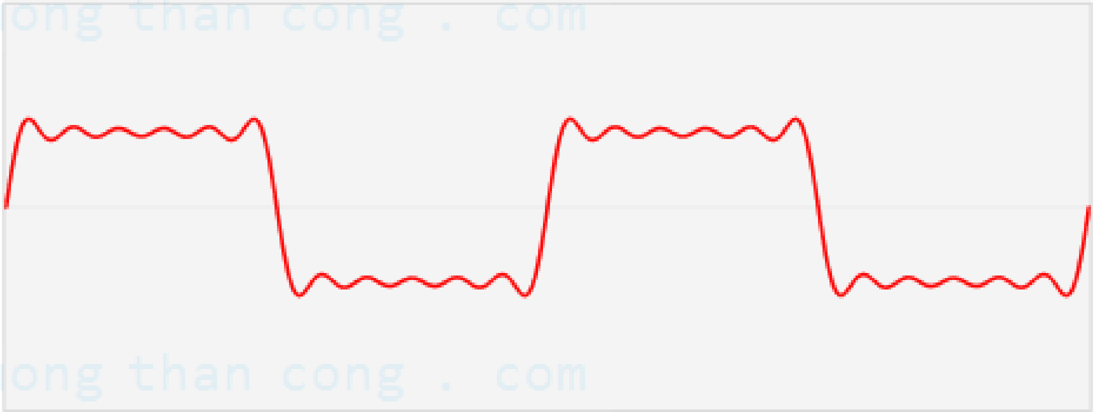
Section B.1.1

Fourier Series

The Fourier Series

- Break down a periodic function into the sum of sinusoidal functions
- It is the Fourier Transform for periodic functions

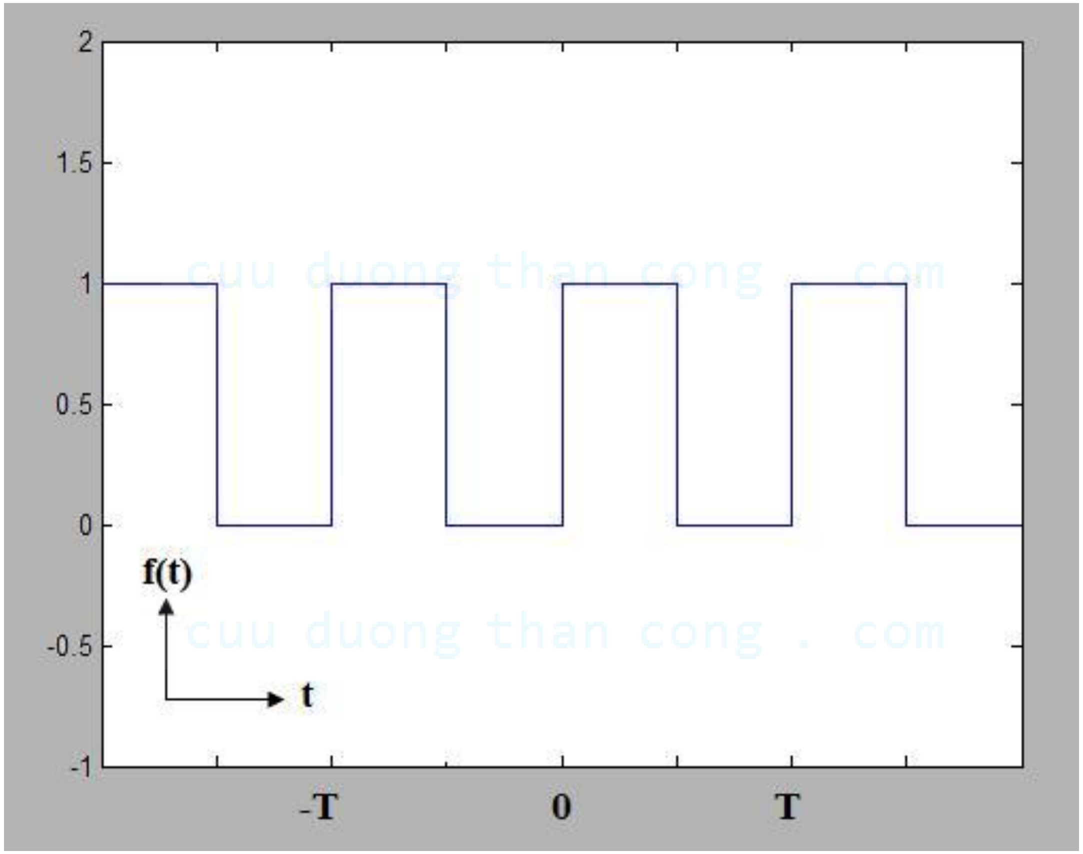
A function f is resolved into Fourier series. The component frequencies of these sines and cosines are represented as peaks in the frequency domain



Periodic function

- A function f is periodic with fundamental period T if

$$\boldsymbol{f(t + T) = f(t), \quad \forall t}$$



A periodic square waveform with period T

The Fourier Series

- A Fourier Series with period T is an infinite sum of sinusoidal functions (cosines and sines)

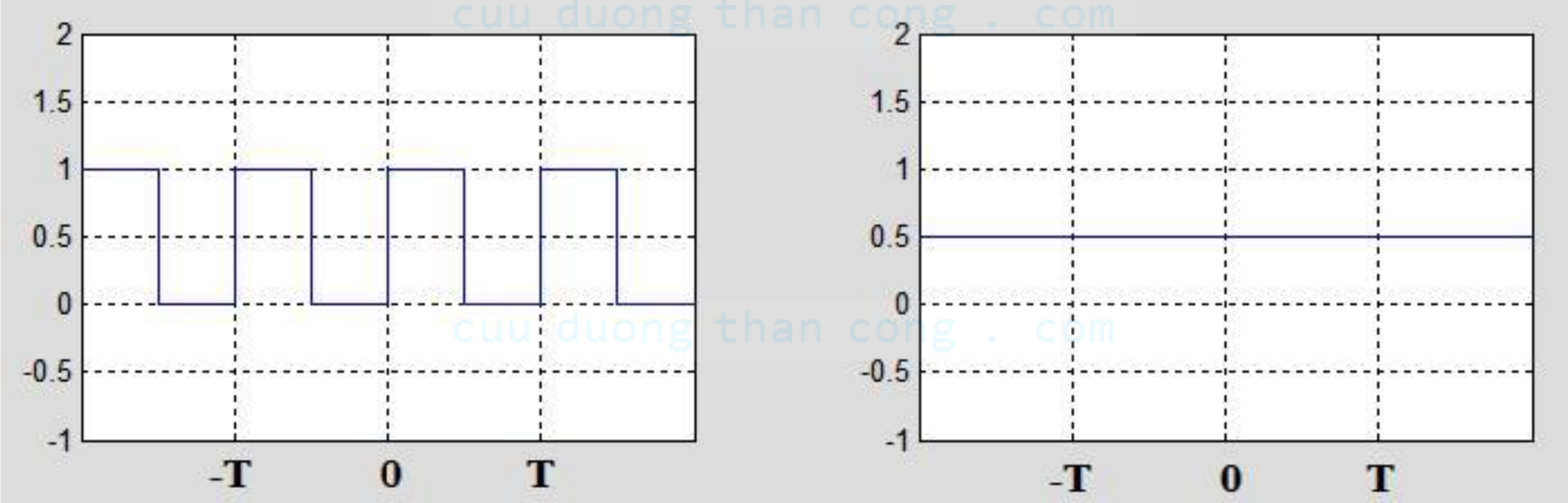
$$g(t) = a_0 + \sum_{m=1}^{\infty} a_m cos\left(\frac{2\pi mt}{T}\right) + \sum_{n=1}^{\infty} b_n sin\left(\frac{2\pi nt}{T}\right)$$

$$= \sum_{m=0}^{\infty} a_m cos\left(\frac{2\pi mt}{T}\right) + \sum_{n=1}^{\infty} b_n sin\left(\frac{2\pi nt}{T}\right)$$

  - where a_m and b_n are the coefficients of the Fourier Series, which determine the relative weights for each of the sinusoids

Fourier Series: Real coefficients

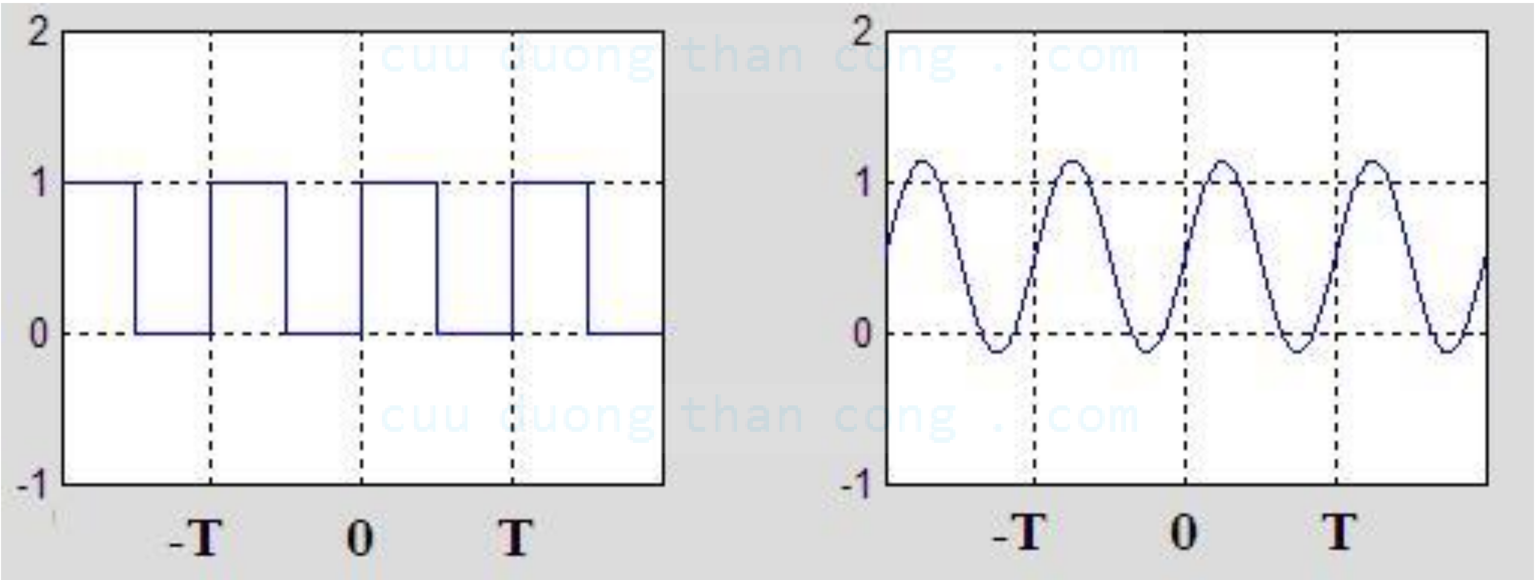
- We want to approximate a periodic function $f(t)$, with fundamental period T, with the Fourier Series
- One term approximation of the function: $g(t) = a_0$
  - $a_0 = \frac{1}{T}\int_0^T f(t)dt$: the average value of f



The square waveform and the one term (constant) expansion a_0

Fourier Series: Real coefficients

- Two term expansion of the function: $g(t) = a_0 + b_1 sin\left(\frac{2\pi t}{T}\right)$
  - $b_1 = \frac{2}{T}\int_0^T f(t)sin(\frac{2\pi t}{T})dt \quad \left(b_1 = \frac{2}{\pi}\right)$



The square waveform and the two term expansion (a_0, b_1)

Fourier Series: Real coefficients

- The optimal coefficients for all a_m and b_n are generally formulated as

$$a_0 = \frac{1}{T}\int_0^T f(t)dt$$

$$a_m = \frac{2}{T}\int_0^T f(t)cos(2\pi m/T)dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)sin(2\pi n/T)dt$$

- All of these represent the correlation of the function $f(t)$ with the basis sine and cosine functions

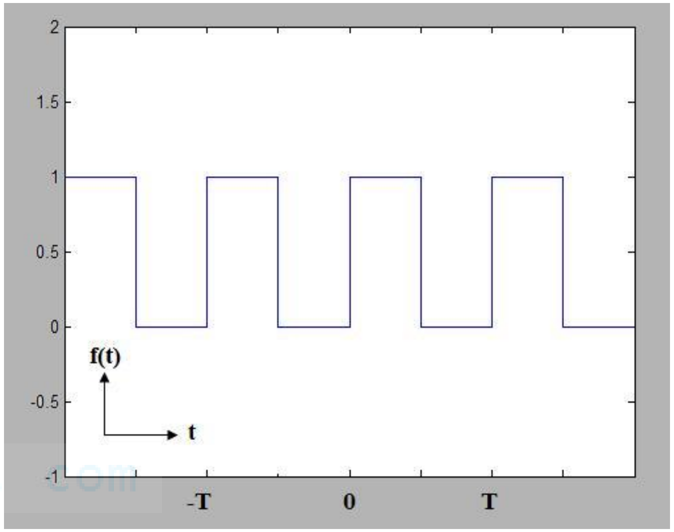
Fourier Series: Real coefficients

- For the specific square-wave $f(t)$, the optimal values come out to be

$$a_0 = \frac{1}{2}$$

$$a_m = 0, \qquad m = 1, 2, \ldots$$

$$b_n = \begin{cases} \frac{2}{\pi n}, & n\ odd \\ 0, & n\ even \end{cases}$$



- Note that all of the a_m values (except a_0) turn out to be zero
  - The function $g(t) = f(t) - a_0$ is an odd function, and the cosines are all even functions. Hence, all the values $a_1, a_2, \ldots$ do not contribute to $g(t)$ [or $f(t)$], so must be zero

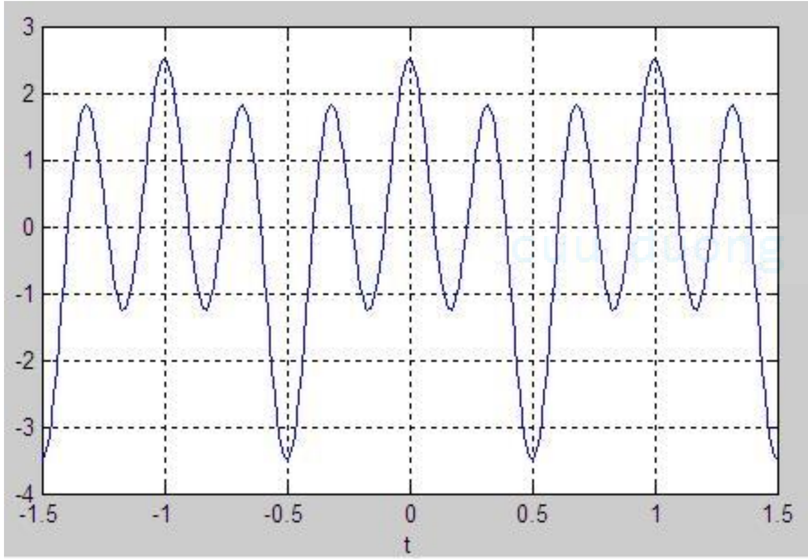
Odd functions vs. Even functions

- A function f is even if

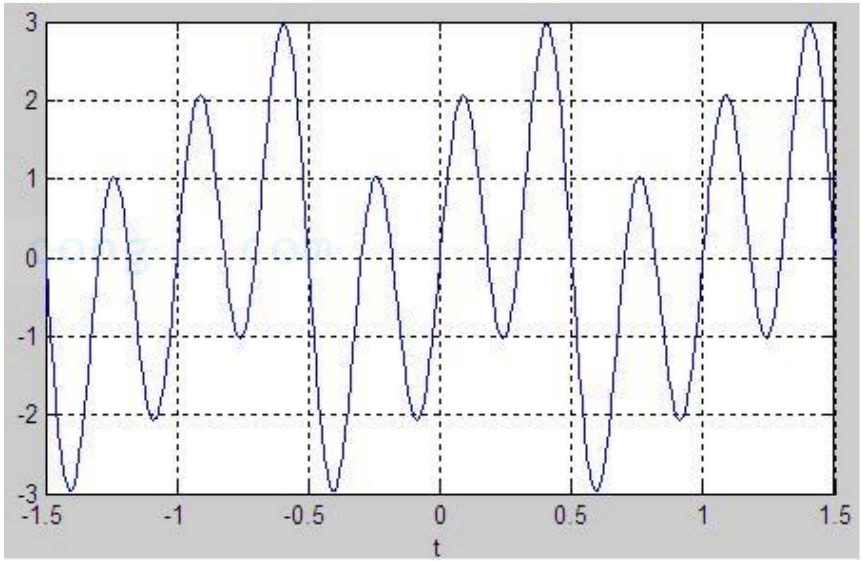
$$f(t) = f(-t), \forall t$$

- A function f is odd if

$$f(t) = -f(-t), \forall t$$



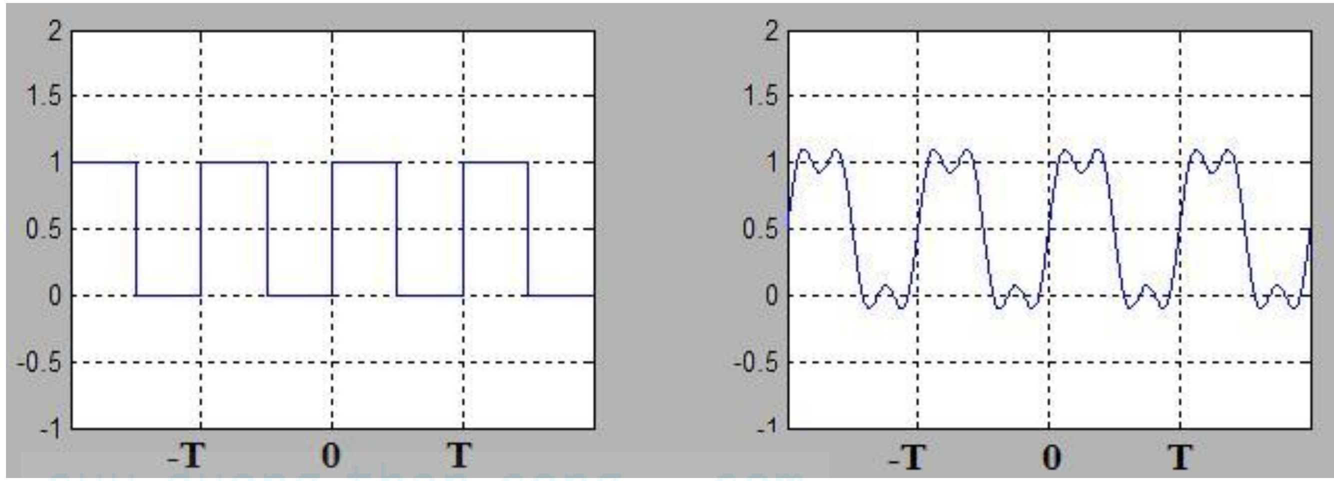
An even function



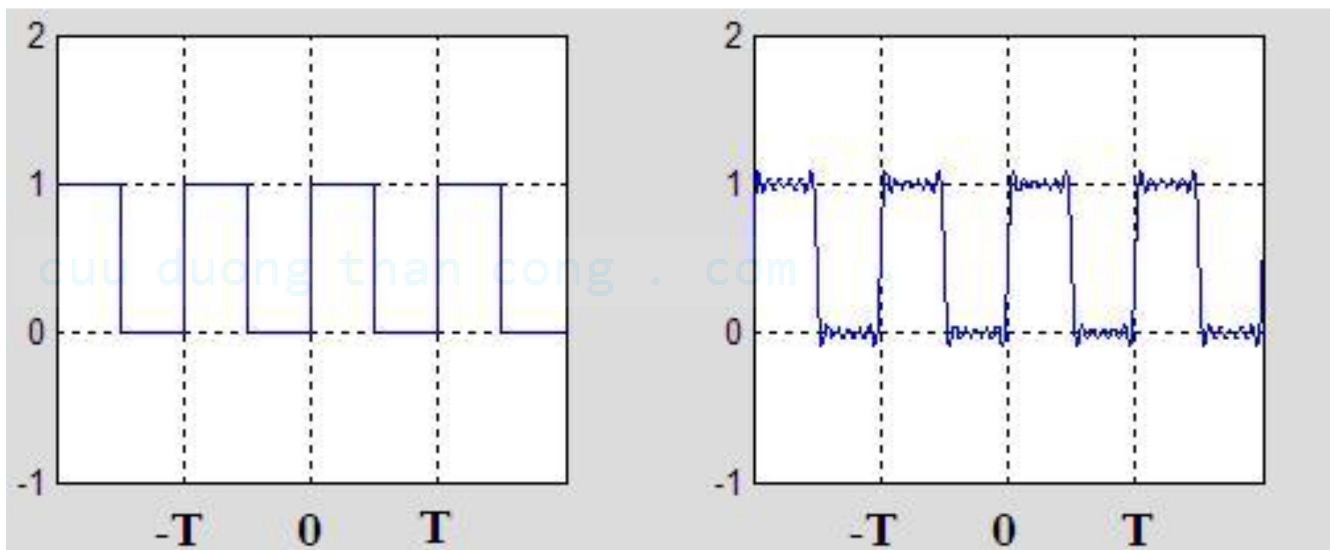
An odd function

Fourier Series: Real coefficients

Three (non-zero) term expansion (a_0, b_1, b_3)



Seven term expansion (a_0, b_1, b_3, b_5, b_7, b_9, b_{11})



Fourier Series: Complex coefficients

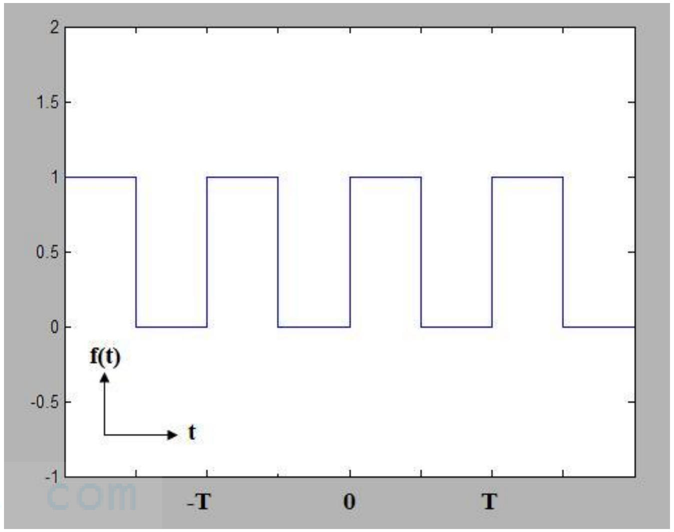
- The Fourier Series, with period T, is rewritten as

$$\boldsymbol{g(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{j\frac{2\pi n}{T}t}}$$

  - Euler formula: $e^{j\theta} = cos\theta + jsin\theta, \;\; j = \sqrt{-1}$
  - $c_n = \frac{1}{T}\int_0^T f(t)e^{-j\frac{2\pi n}{T}t}\,dt$ where $n = 0, \pm 1, \pm 2, \ldots$

Fourier Series: Complex coefficients

- Recall the square function $f(t)$



$$c_0 = \frac{1}{2}$$

$$c_n = \frac{1}{j\pi n}, \quad n = \pm 1, \pm 3, \pm 5, \ldots$$

$$c_n = 0, \qquad n = \pm 2, \pm 4, \pm 6, \ldots$$

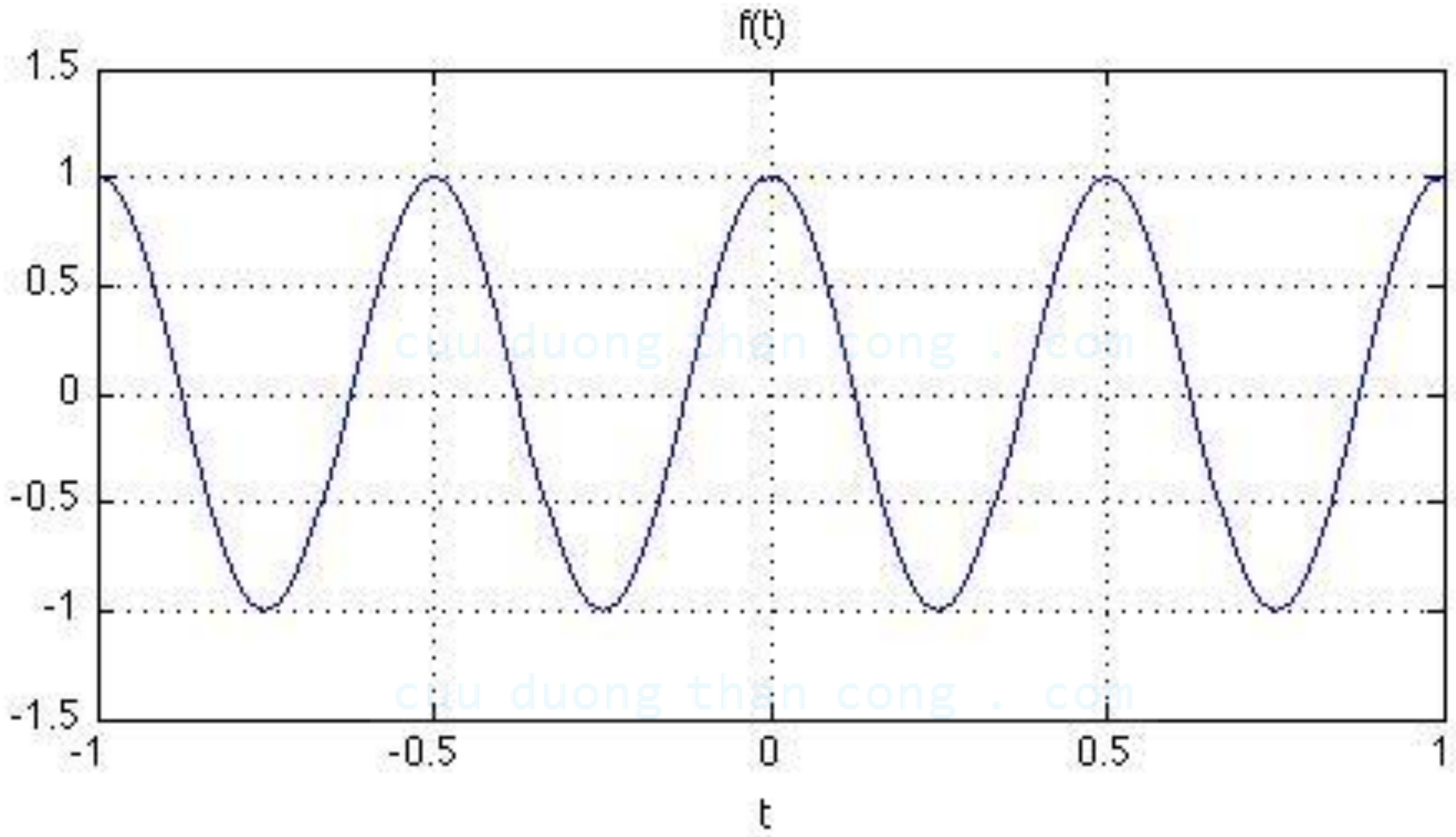
- Note that the resultant function $g(t)$ will be entirely real, though the Fourier coefficients are complex numbers, if

$$c_n^* = c_{-n}$$

  - where the * represents the complex conjugate

Fourier Series for the cosine function

- The cosine function $f(t) = \cos(4\pi t)$ with $T = 0.5$



$$c_n = \frac{1}{T}\int_0^T f(t)e^{-j\frac{2\pi nt}{T}}dt = \frac{1}{0.5}\int_0^{0.5} \cos(4\pi t)e^{-j\frac{2\pi nt}{0.5}}dt$$

The cosine function is rewritten (via Euler's identity) as: $cos(\theta) = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$

Therefore, $$c_n = 2\int_0^{0.5} \left(\frac{e^{j4\pi t} + e^{-j4\pi t}}{2}\right)e^{-j\frac{2\pi nt}{0.5}}dt$$

$$= \int_0^{0.5} e^{j4\pi t(1-n)}dt + \int_0^{0.5} e^{-j4\pi t(n+1)}dt$$

For the integral on the left

$$\int_0^{0.5} e^{j4\pi t(1-n)}dt = \left.\frac{e^{j4\pi t(1-n)}}{j4\pi(1-n)}\right|_0^{0.5} = \frac{e^{j2\pi(1-n)} - 1}{j4\pi(1-n)} = 0, \quad n \neq 1$$

$$c_1 = \int_0^{0.5} e^{j4\pi t(1-n)}dt = \int_0^{0.5} 1dt = 0.5$$

For the integral on the right $\int_{0}^{0.5} e^{-j4\pi t(n+1)} dt = 0,$ $\quad n \neq -1$

$$c_{-1} = 0.5$$

Hence, all the coefficients are zero except for the $n = 1$ and $n = -1$ terms. Finally, the Fourier Series $g(t)$ is given as

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi nt}{T}} = c_1 e^{j\frac{2\pi t}{T}} + c_{-1} e^{-j\frac{2\pi t}{T}}$$

$$= 0.5(e^{j4\pi t} + e^{-j4\pi t})$$

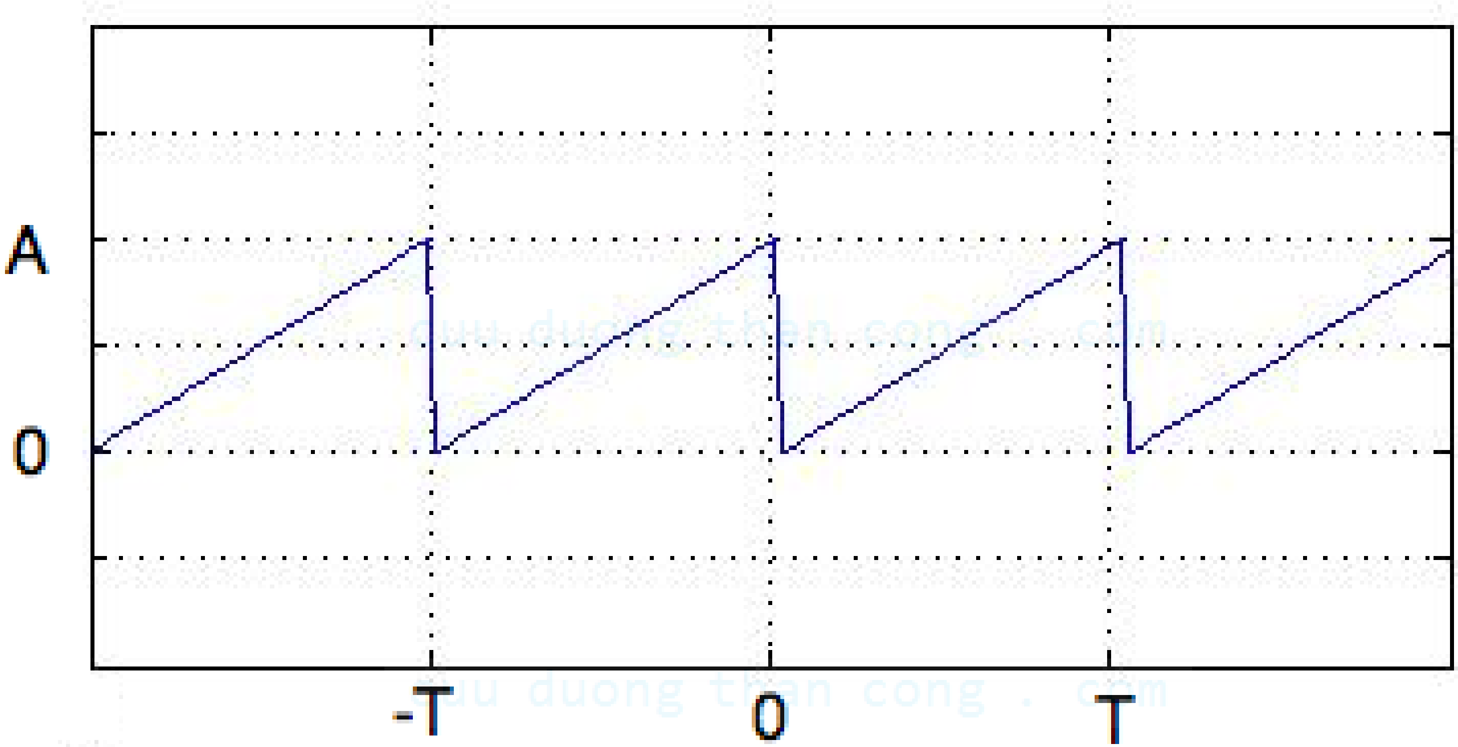
$$= \cos(4\pi t)$$

$$= f(t)$$

The Fourier Representation $g(t)$ yields exactly the cosine function $f(t)$

Fourier Series for the saw function

- The saw function $f(t) = \mathrm{A}\frac{t}{T},\ 0 \leq t \leq T$



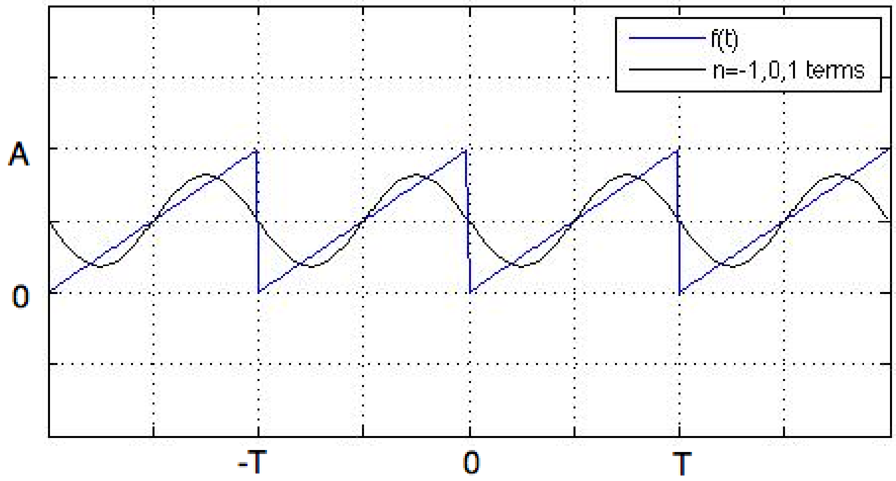
$$c_n = \frac{1}{T}\int_0^T f(t)e^{-j\frac{2\pi nt}{T}}dt = \frac{A}{T^2}\int_0^T te^{-j\frac{2\pi nt}{T}}dt$$

$$= \frac{A}{T^2}\left[\left\{\frac{tT}{-j2\pi n}e^{-j\frac{2\pi nt}{T}} + \frac{T^2}{(2\pi n)^2}e^{-j\frac{2\pi nt}{T}}\right\}\Bigg|_0^T\right]$$

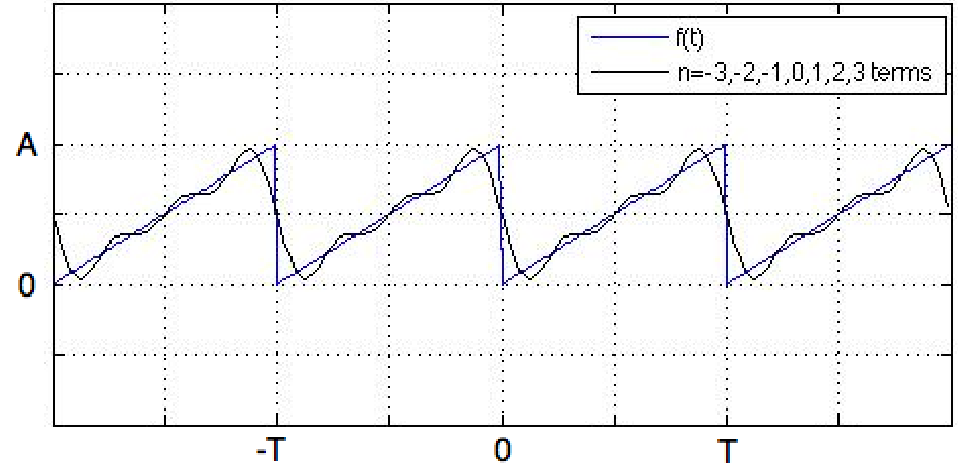
(Because $e^{-j2\pi n} = 1$) $$= \frac{A}{T^2}\left[\frac{T^2}{-j2\pi n}\right] = \frac{jA}{2\pi n}, \qquad n \neq 0$$

$$c_0 = \frac{A}{2}$$

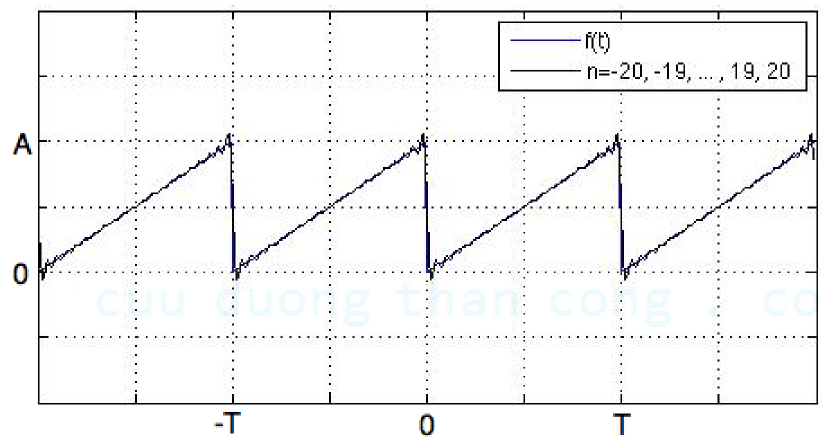
Finally, compute the Fourier series $g(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi nt}{T}}$ with c_n



The saw function with 3 Fourier coefficients
$(n = -1, 0, 1)$



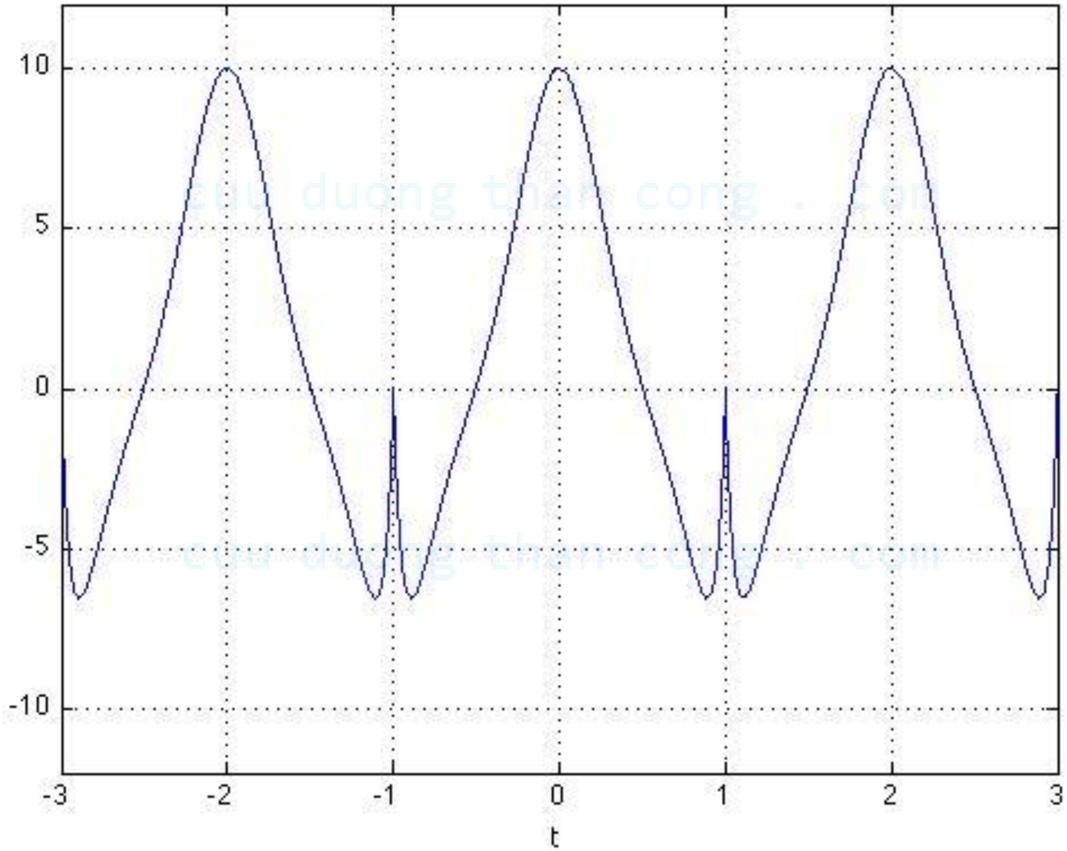
The saw function with 5 Fourier coefficients
$(n = -2, -1, 0, 1, 2)$



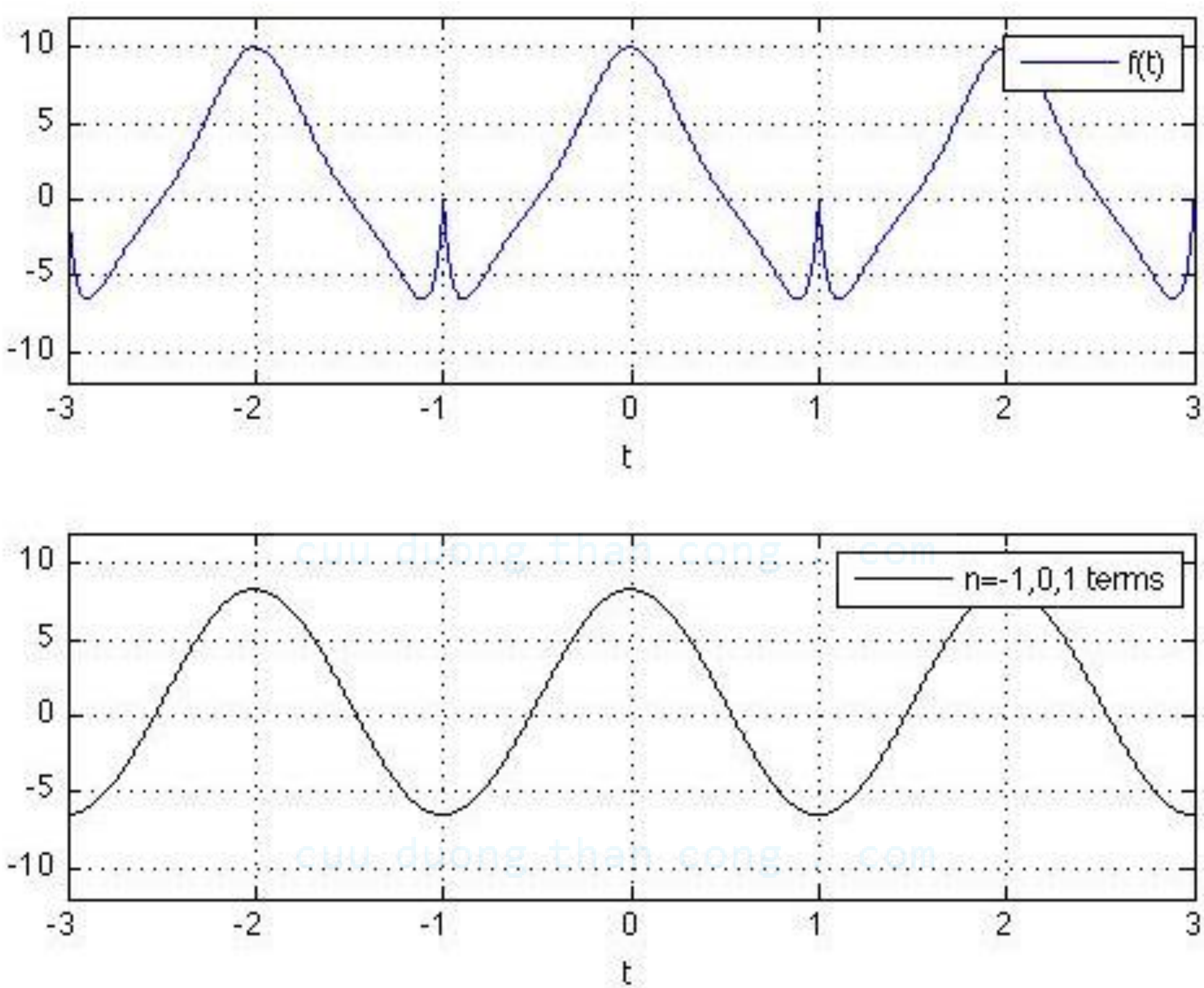
The saw function with 21 Fourier coefficients
$(n = -20, -19, \ldots, 19, 20)$

Fourier Series for a complicated function

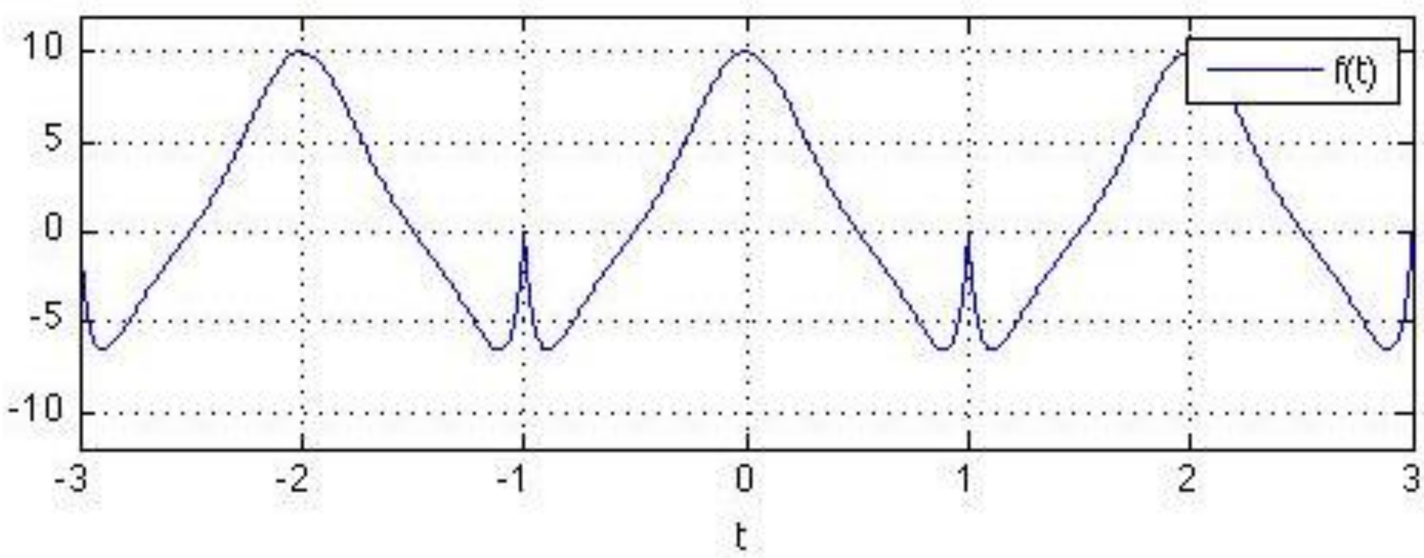
- The complicated function $f(t) = \frac{(t-1)(t+1)(2t)^4}{\cos(1.4t)} + 10e^{-9t^2}$ $(-1 \leq t \leq 1)$ with $T = 2$

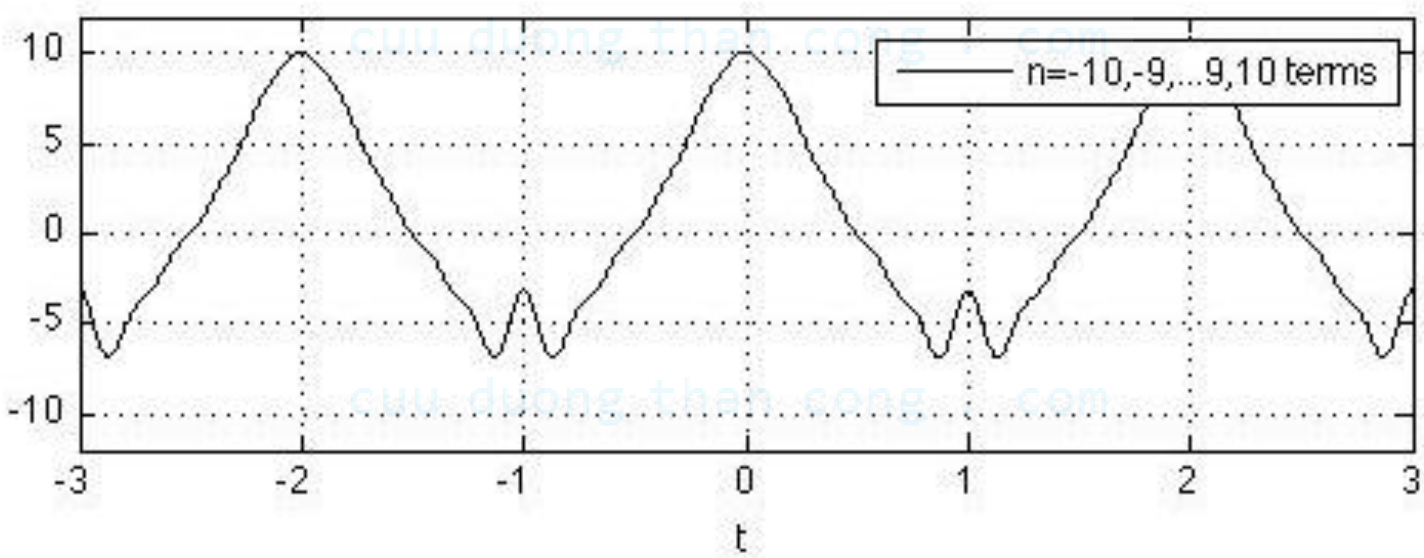


```
clear; close all;

% fundamental period
T = 2;

% this is the step size for the t-axis (how finely we sample the function)
delT = 0.001;

% declare the range of t over the fundamental period, -1<t<1
t = -1: delT : (1-delT);

% here is the function f(t)
gg = (t-1).*(t+1).*(2*t).^4.*1./cos(t*1.4) + 10*exp(-9*t.*t);

% evaluation of c0:  this is simply the integral of the f(t)*exp(0) = f(t)
% [the integral is numerically found as the discrete sum]
c0 = 1/T*sum(    gg * delT )

% numN is the number of coefficients to determine (-numN, -numN+1,...numN)
numN = 30;
for n=1:numN

    % this is the discrete integral for finding c_n
     c(n) = delT/T*sum( gg.*exp(-i*2*pi*n*t/T) );

     % this is the discrete integral for finding c_(-n)
     cn(n) = delT/T*sum( gg.*exp(i*2*pi*n*t/T) );
end
```

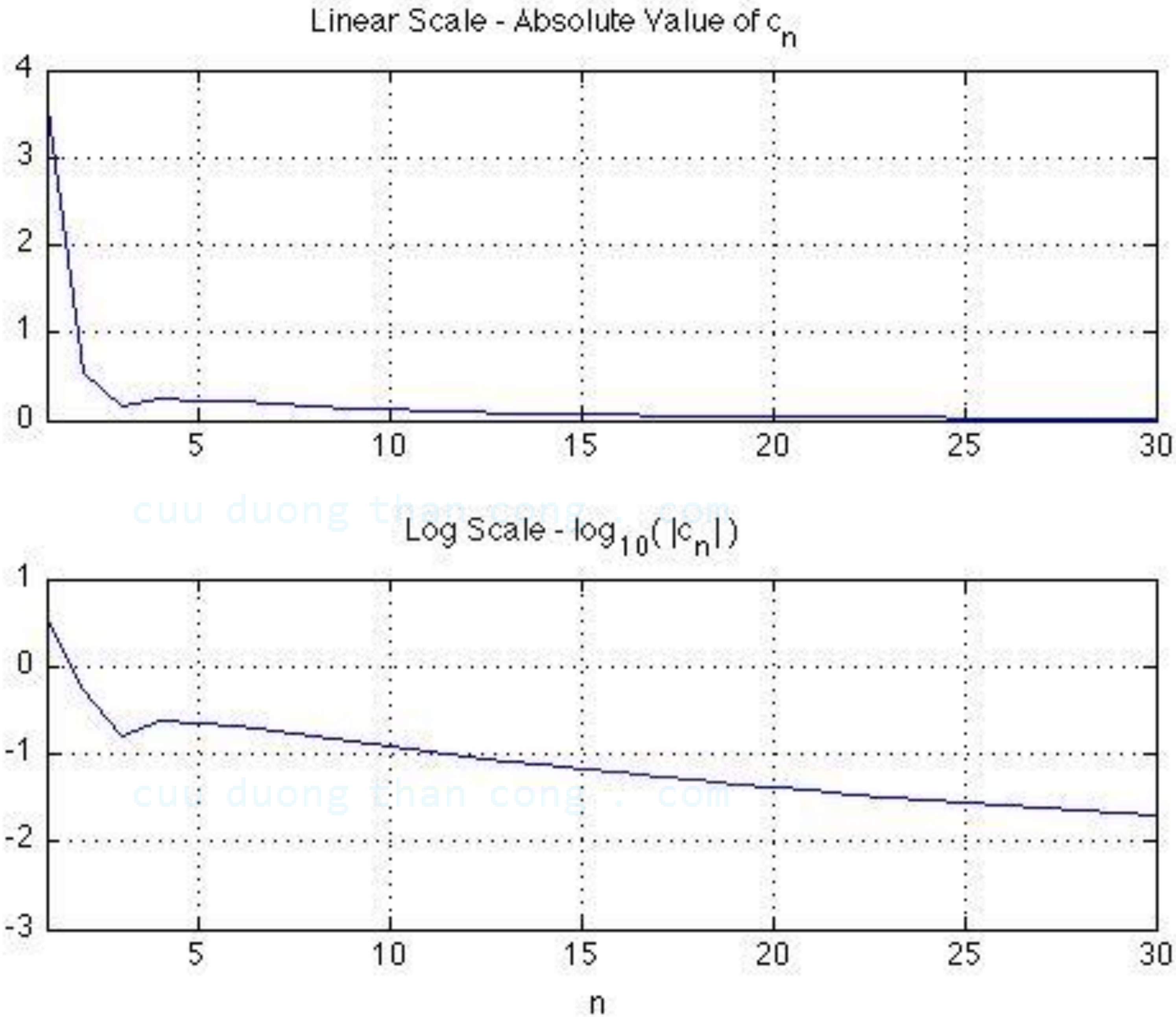


Original f(t) (top) and the first 3 terms of the Fourier Expansion (bottom)





Original f(t) (top) and the first 21 terms of the Fourier Expansion (bottom)



Magnitude of Fourier Series coefficients versus n using a linear scale (top) and a log scale of the same function (bottom)

Mean Squared Error (MSE)

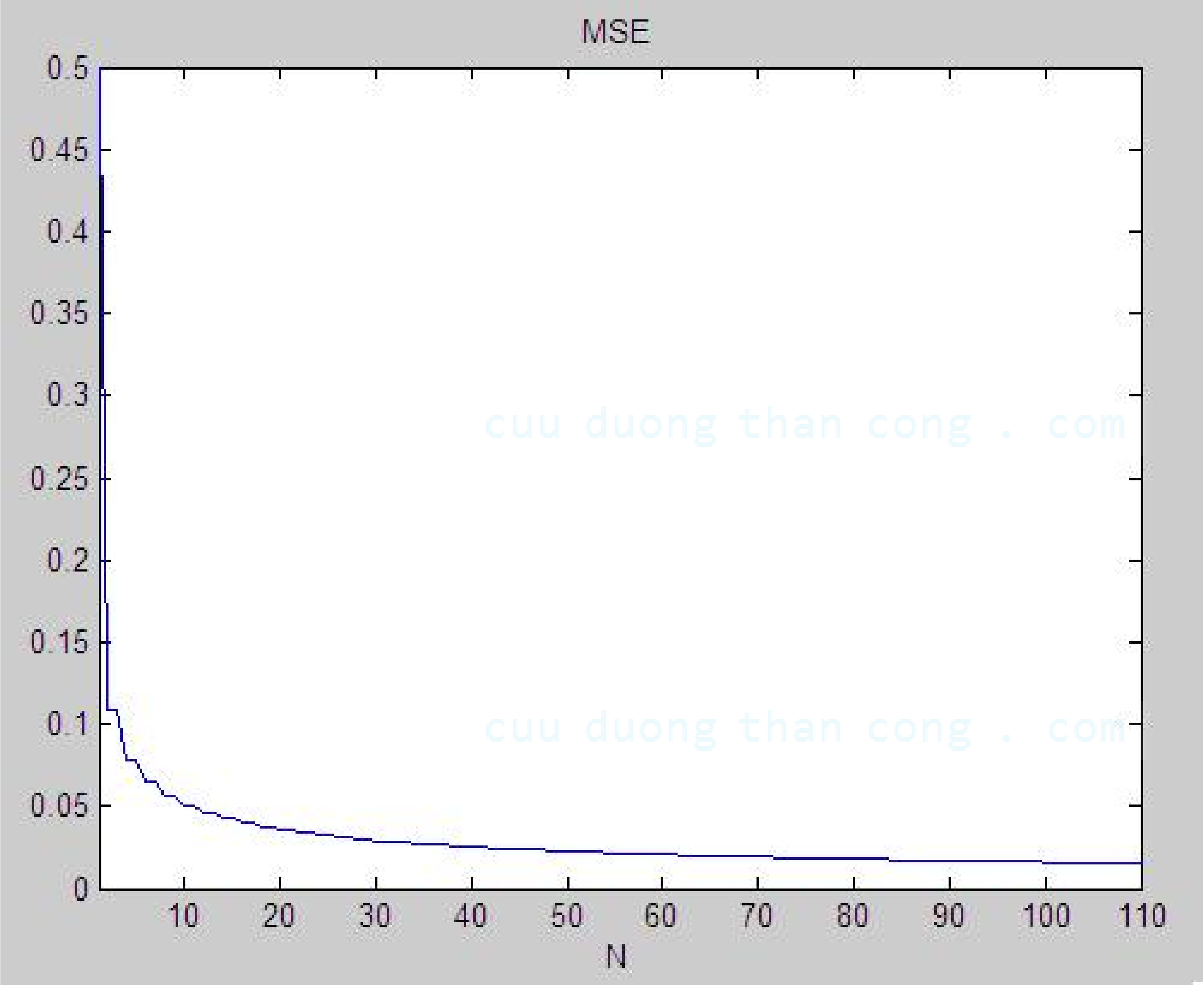
- Let $g_N(t) = \sum_{n=-N}^{N} c_n e^{j\frac{2\pi nt}{T}}$ be the Fourier Series with $2N + 1$ coefficients
  - The higher N gets, the more terms are in the finite Fourier Series $g_N(t)$, and the closer $g_N(t)$ will be to $f(t)$
- The distance (MSE) between $g_N(t)$ and $f(t)$ is given as

$$\boldsymbol{mse(N) = \|f(t) - g_N(t)\|}$$

$$\boldsymbol{= \sqrt{\int_0^T |f(t) - g_N(t)|^2\, dt}}$$

$$\boldsymbol{= \sqrt{\int_0^T \left| f(t) - \sum_{n=-N}^{N} c_n e^{j\frac{2\pi nt}{T}} \right|^2 dt}}$$

Mean Squared Error (MSE)



The MSE between $g_N(t)$ and the square wave $f(t)$

Section B.1.2

Fourier Transform

The Fourier Transform

- The **Fourier Transform** of a function $g(t)$ is defined by

$$\boldsymbol{\mathcal{F}\{g(t)\} = G(\mu) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi\mu t}dt}$$

  - $G(\mu)$ gives how much power $g(t)$ contains at the frequency μ. It is often called the spectrum of g

- Extend the Fourier Series **to non-periodic functions**

The Fourier Transform

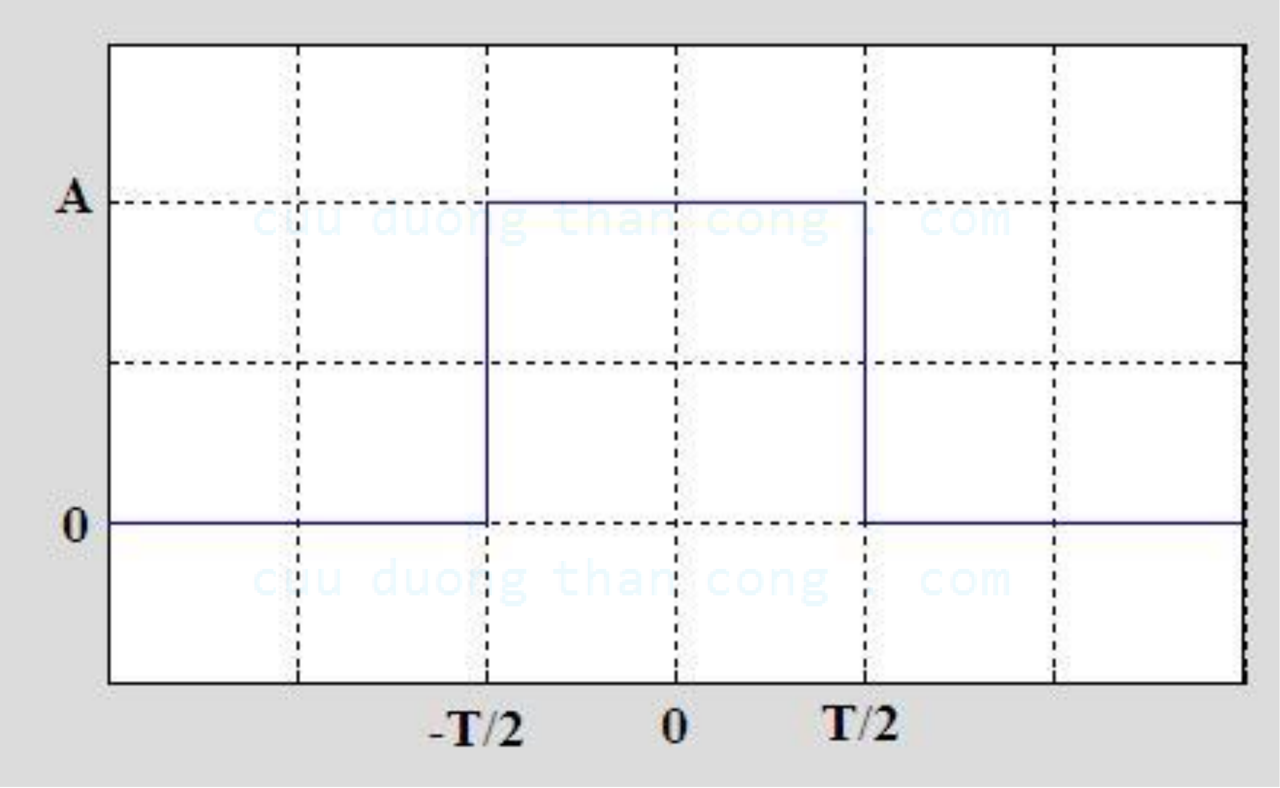
- In addition, $g(t)$ can be obtained from $G(\mu)$ via the **inverse Fourier Transform**:

$$\mathcal{F}^{-1}\{G(\mu)\} = \int_{-\infty}^{\infty} G(\mu)e^{j2\pi\mu t}d\mu = g(t)$$

- As a result, $g(t)$ and $G(\mu)$ form a **Fourier pair:** $g \overset{\mathcal{F}}{\Leftrightarrow} G$

Fourier Transform: The box function

- The box function $g(t) = \begin{cases} A & -T/2 \le t \le T/2 \\ 0 & otherwise \end{cases}$



$$\mathcal{F}\{g(t)\} = G(\mu) = \int_{-\infty}^{-\infty} g(t)e^{-j2\pi\mu t}dt$$

$$= \int_{-T/2}^{T/2} Ae^{-j2\pi\mu t}dt = \frac{A}{-j2\pi\mu}\left[e^{-j2\pi\mu t}\Big|_{-T/2}^{T/2}\right]$$

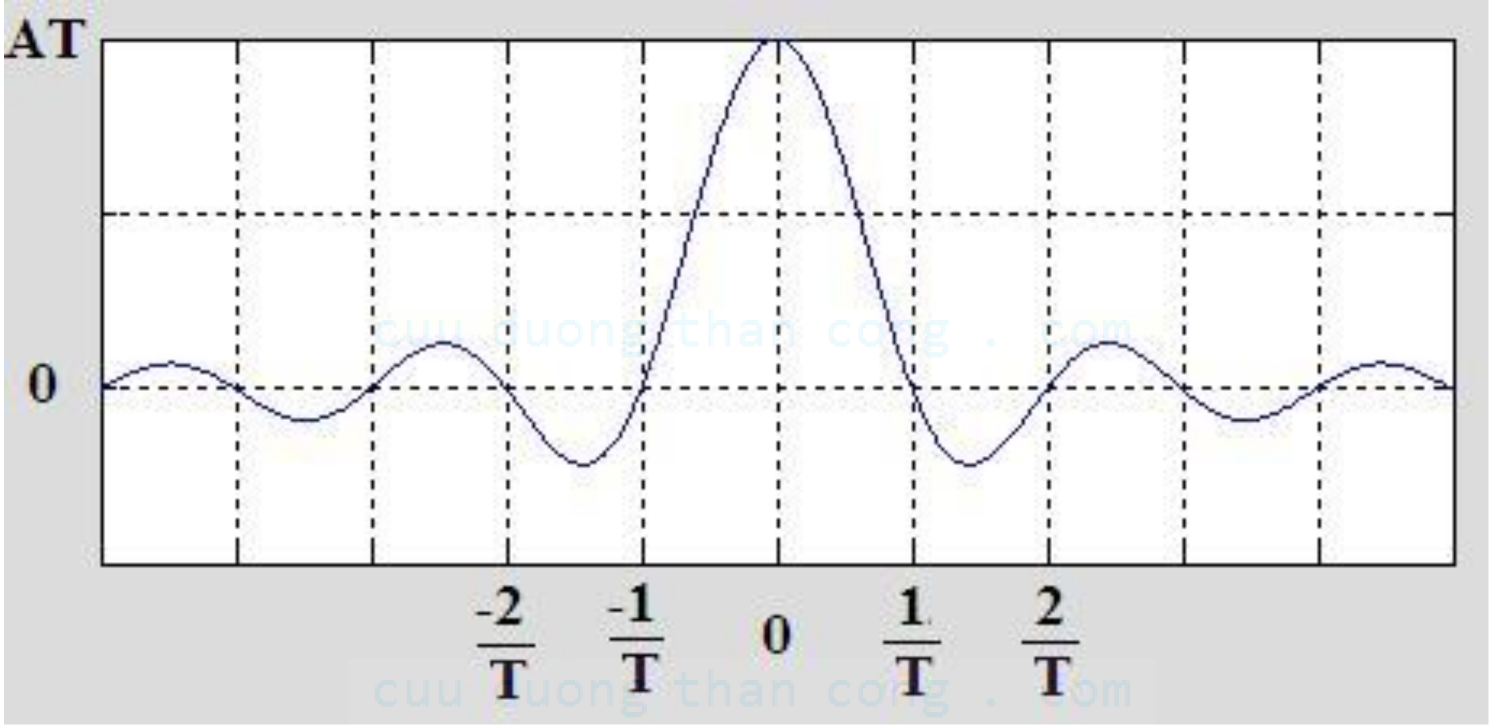
$$= \frac{A}{-j2\pi\mu}\left[e^{-j\pi\mu T} - e^{j\pi\mu T}\right] = \frac{A}{\pi\mu T}\left[\frac{e^{-j\pi\mu T} - e^{j\pi\mu T}}{2i}\right]$$

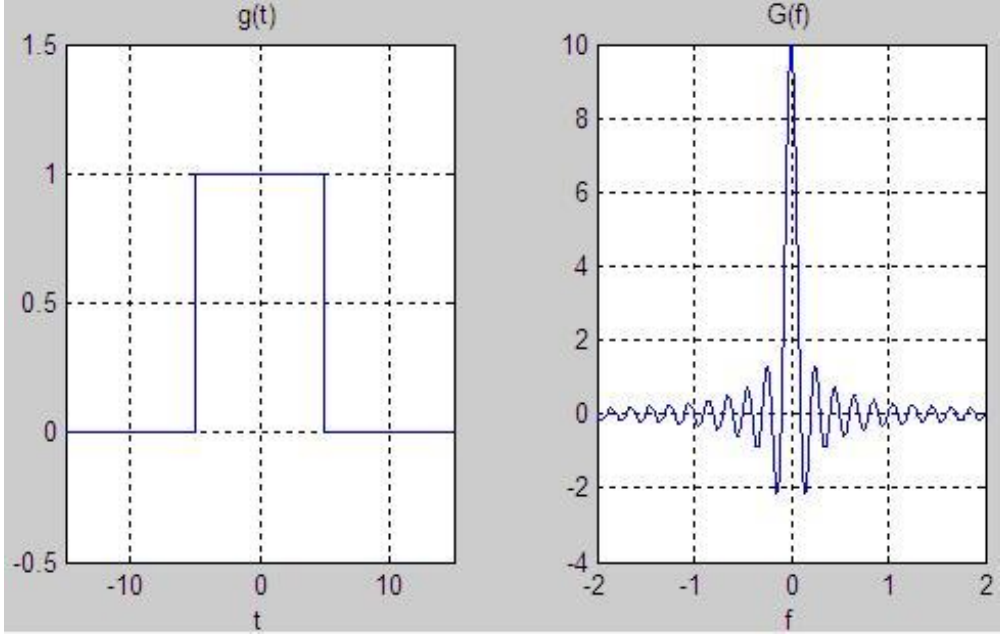
$$= \frac{A}{\pi\mu T}\sin(\pi\mu T) = AT[sinc(\mu T)]$$

where, $sinc(t) = \frac{\sin(\pi t)}{\pi t}$ and $sinc(0) = 1$ (using L'Hopitals rule)

Fourier Transform: The box function

- The Fourier Transform of the box function is a sinc function

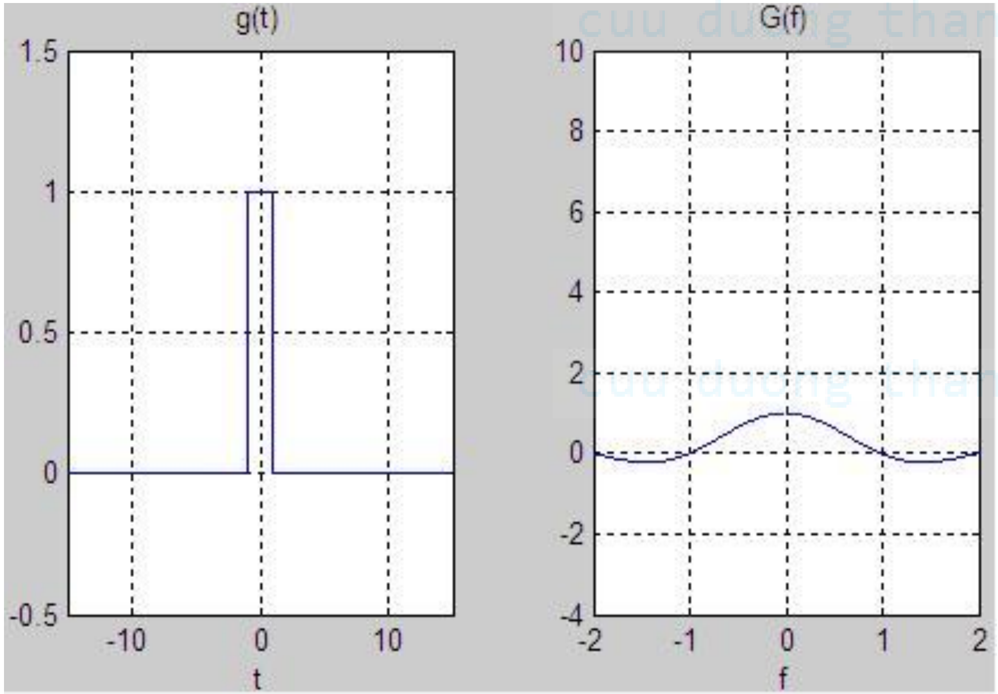




The box function with $T = 10$, and its Fourier Transform

Rapidly changing functions require more high frequency content (bottom). Functions that are moving more slowly in time will have less high frequency energy (top)

When the box function is shorter in time (bottom), so that it has less energy, there appears to be less energy in its Fourier Transform



The box function with $T = 1$, and its Fourier Transform

Linearity of the Fourier Transform

- The Fourier Transform is a linear transform

$$\boldsymbol{\mathcal{F}\{c_1 g(t) + c_2 h(t)\} = c_1 G(\mu) + c_2 H(\mu)}$$

  - where $G(\mu)$ and $H(\mu)$ are FTs of functions $g(t)$ and $h(t)$, respectively

- Proof: $$\mathcal{F}\{c_1 g(t) + c_2 h(t)\} = \int_{-\infty}^{\infty} c_1 g(t) e^{-i2\pi\mu t} dt + \int_{-\infty}^{\infty} c_2 h(t) e^{-i2\pi\mu t} dt$$
$$= c_1 \int_{-\infty}^{\infty} g(t) e^{-i2\pi\mu t} dt + c_2 \int_{-\infty}^{\infty} h(t) e^{-i2\pi\mu t} dt$$
$$= c_1 G(\mu) + c_2 H(\mu)$$

Shift Property of the Fourier Transform

- If $g(t)$ is shifted in time by a constant amount, it should have the same magnitude of the spectrum, $G(\mu)$

$$\boldsymbol{\mathcal{F}\{g(t-a)\} = e^{-j2\pi\mu a}G(\mu)}$$

- Proof: $\mathcal{F}\{g(t-a)\} = \int_{-\infty}^{\infty} g(t-a)e^{-i2\pi\mu t}dt = \int_{-\infty}^{\infty} g(u)e^{-i2\pi\mu(u+a)}du$

$$= e^{-i2\pi\mu a}\int_{-\infty}^{\infty} g(u)e^{-i2\pi\mu u}du = e^{-i2\pi\mu a}G(\mu)$$

  - The complex exponential always has a magnitude of 1, the time delay alters the phase of $G(\mu)$ but not its magnitude

Scaling property of the Fourier Transform

- If $g(t)$ is scaled in time by a non-zero constant c, it is written as $g(ct)$
- The resultant Fourier Transform will be given by

$$\boldsymbol{\mathcal{F}\{g(ct)\} = \frac{G\left(\frac{\mu}{c}\right)}{|c|}}$$

- Proof:
  - $\mathcal{F}\{g(ct)\} = \int_{-\infty}^{\infty} g(ct)e^{-i2\pi\mu t}$
  - Substitute $u = ct,\, du = cdt,\, F\{g(ct)\} = \int_{-\infty}^{\infty} \frac{g(u)}{c} e^{-i2\pi\mu\frac{u}{c}} du$
  - If $c > 0$: $F\{g(ct)\} = \int_{-\infty}^{\infty} \frac{g(u)}{c} e^{-i2\pi\mu\frac{u}{c}} du = \frac{G\left(\frac{\mu}{c}\right)}{c}$
  - If $c < 0$: $F\{g(ct)\} = \int_{\infty}^{-\infty} \frac{g(u)}{c} e^{-i2\pi\mu\frac{u}{c}} du = -\int_{-\infty}^{\infty} \frac{g(u)}{c} e^{-i2\pi\mu\frac{u}{c}} du = \frac{G\left(\frac{\mu}{c}\right)}{-c}$

Parseval's Theorem

- Let $g(t)$ have Fourier Transform $G(\mu)$.
- The energy of $g(t)$ is the same as the energy in $G(\mu)$

$$\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(\mu)|^2 d\mu$$

  - The integral of the squared magnitude of a function: the energy of the function

Other properties of the Fourier Transform

- **Derivative property (Differentiation):** The FT of the derivative of $g(t)$ is given by

$$\mathcal{F}\left\{\frac{dg(t)}{dt}\right\} = j2\pi\mu \cdot G(\mu)$$

- **Convolution property:** The FT of the convolution of $g(t)$ and $h(t)$ [with corresponding FTs $G(\mu)$ and $H(\mu)$] is

$$\mathcal{F}\{g(t)\bigstar h(t)\} = G(\mu)H(\mu)$$

  - where the convolution of two functions in time is defined by $g(t)\bigstar h(t) = \int_{-\infty}^{\infty} g(\tau)h(t-\tau)d\tau$

Other properties of the Fourier Transform

- **Modulation property:** A function is "modulated" by another function if they are multiplied in time.

$$\mathcal{F}\{g(t)\,h(t)\} = G(\mu)\star H(\mu)$$

- **Duality:** Suppose $g(t)$ has the Fourier Transform $G(\mu)$. The Fourier Transform of the function $G(t)$ is

$$\mathcal{F}\{G(t)\} = g(-\mu)$$

Section B.1.3

Discrete Fourier Transform

Discrete Fourier Transform (DFT)

- Equivalent of the continuous Fourier Transform for signals known only at N instants separated by sample times T
- Let $g(t)$ be the continuous signal which is the source of the data and N samples be $g[0]$, $g[1]$, $g[2]$,…, $g[N-1]$
- The Fourier Transform of the original signal, $g(t)$, would be

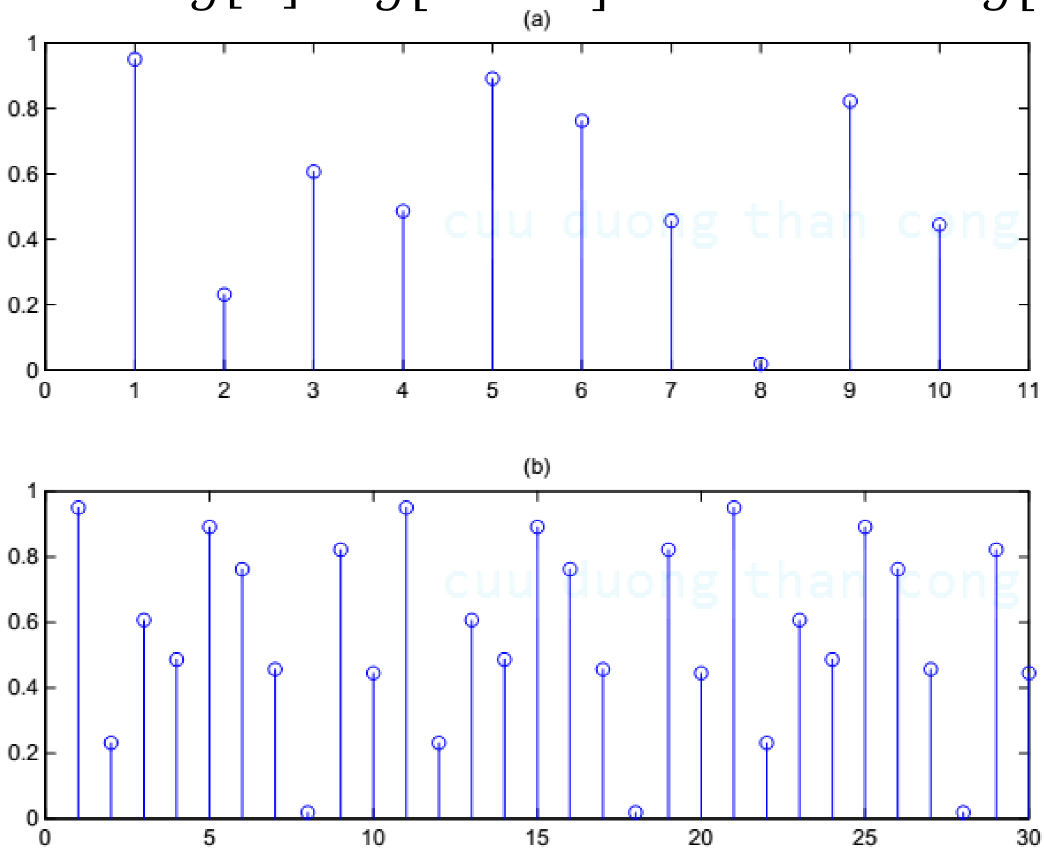
$$\mathcal{F}\{g(t)\} = G(\mu) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt, \text{ where } \omega = 2\pi\mu$$

$$= g_0 e^{-j0} + g_1 e^{-j\omega T} + \cdots + g_k e^{-j\omega kT} + \cdots + g_{N-1} e^{-j\omega(N-1)T}$$

$$= \sum_{k=0}^{N-1} g_k e^{-j\omega kT}$$

Discrete Fourier Transform (DFT)

- There are only a finite number of input data points $\rightarrow$ DFT treats the data as if it were periodic
  - I.e. $g[N]$ to $g[2N-1]$ is the same as $g[0]$ to $g[N-1]$



Sequence of $N = 10$ samples (top) and implicit periodicity in DFT (bottom)

Discrete Fourier Transform (DFT)

- The operation treats the data as if it were periodic
- Tthe DFT equation is evaluated for the fundamental frequency, $\frac{1}{NT}$, and its harmonics
  - I.e. set $\omega = 0, \frac{2\pi}{NT}, \frac{2\pi}{NT} \times 2, \ldots, \frac{2\pi}{NT} \times n, \ldots, \frac{2\pi}{NT} \times (N-1)$
- Hence, the DFT $\mathcal{F}_n$ of the sequence g_k is given as

$$\boldsymbol{\mathcal{F}_n\{g_k(t)\} = \sum_{k=0}^{N-1} g_k e^{-j\frac{2\pi}{N}nk}} \quad \text{where } n = 0, \ldots, N-1$$

- And its inverse counterpart is $\boldsymbol{g_k = \frac{1}{N} \sum_{n=0}^{N-1} \mathcal{F}_n e^{j\frac{2\pi}{N}nk}}$

Discrete Fourier Transform (DFT)

- The DFT equation can be represented in matrix form as

$$\begin{pmatrix} \mathcal{F}_0 \\ \mathcal{F}_1 \\ \mathcal{F}_2 \\ \vdots \\ \mathcal{F}_{N-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & W & W^2 & W^3 & \cdots & W^{N-1} \\ 1 & W^2 & W^4 & W^6 & \cdots & W^{N-2} \\ 1 & W^3 & W^6 & W^9 & \cdots & W^{N-3} \\ \vdots & & & & & \\ 1 & W^{N-1} & W^{N-2} & W^{N-3} & \cdots & W \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_{N-1} \end{pmatrix}$$

  - where $W = e^{-j2\pi/N}$ and $W = W^{2N} etc. = 1$

- The inverse matrix is $\frac{1}{N}$ times the complex conjugate of the original (symmetric) matrix

Discrete Fourier Transform (DFT)

- The evaluation of DFT on a digital computer requires $O(N^2)$ operations on complex numbers
  - N^2 multiplications and $N(N-1)$ additions
  - where N is the length of the transform
- For most problems, N is chosen to be at least 256 in order to get a reasonable approximation for the spectrum of the sequence under consideration

Computational speed becomes a major.

Fast Fourier Transform (FFT)

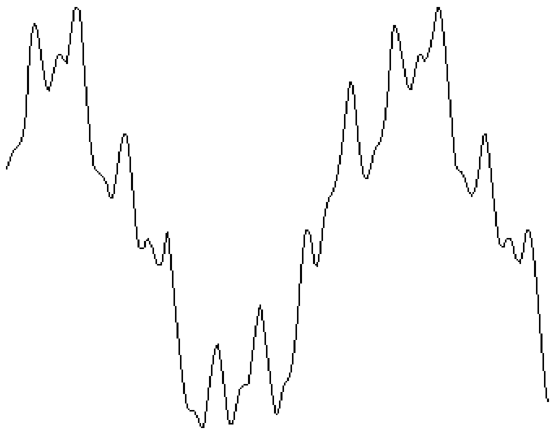
- Highly efficient computer algorithms for estimating DFT
  - Modern algorithm invented by Cooley and Turkey in 1965
- All known FFT algorithms require $\Theta(NlogN)$ operations
- The FFT computes the DFT and **produces exactly the same result** as evaluating the DFT definition directly, yet **it is much faster**

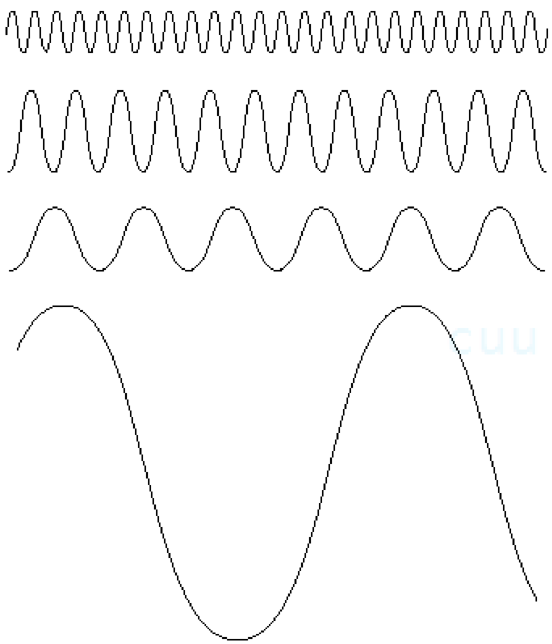
	Evaluating DFT directly	FFT from Cooley – Tukey (1965)
Multiplications	N^2	$N/2log2(N)$
Additions	$N(N-1)$	$Nlog2(N)$

Section B.2

FOURIER TRANSFORM AND DIGITAL IMAGE PROCESSING

Fourier Series



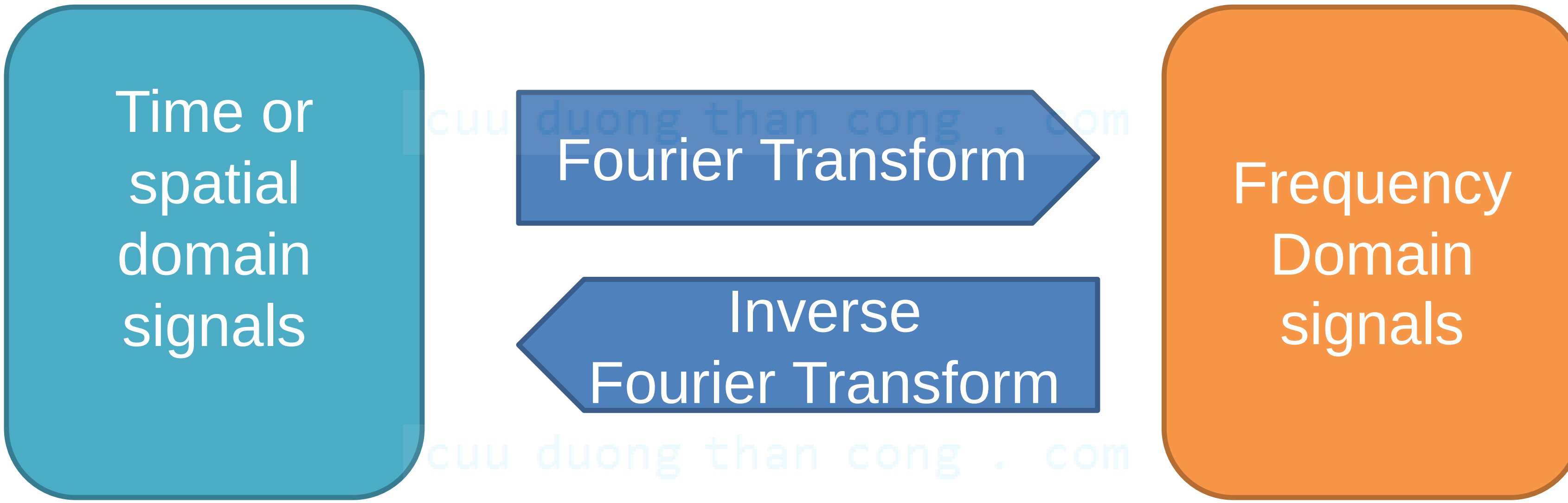


Any periodic signals can be viewed as weighted sum of sinusoidal signals with different frequencies

Frequency Domain: view frequency as an independent variable

Fourier Transform

- The signals in the spatial domain are transformed into signals of different frequencies in the frequency domain, and vice versa



1-D Fourier Transform

- 1-D FT for the continuous case

$$\mathcal{F}(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt \qquad f(t) = \int_{-\infty}^{\infty} \mathcal{F}(\mu) e^{j2\pi\mu x} d\mu$$

- 1-D FT for the discrete case

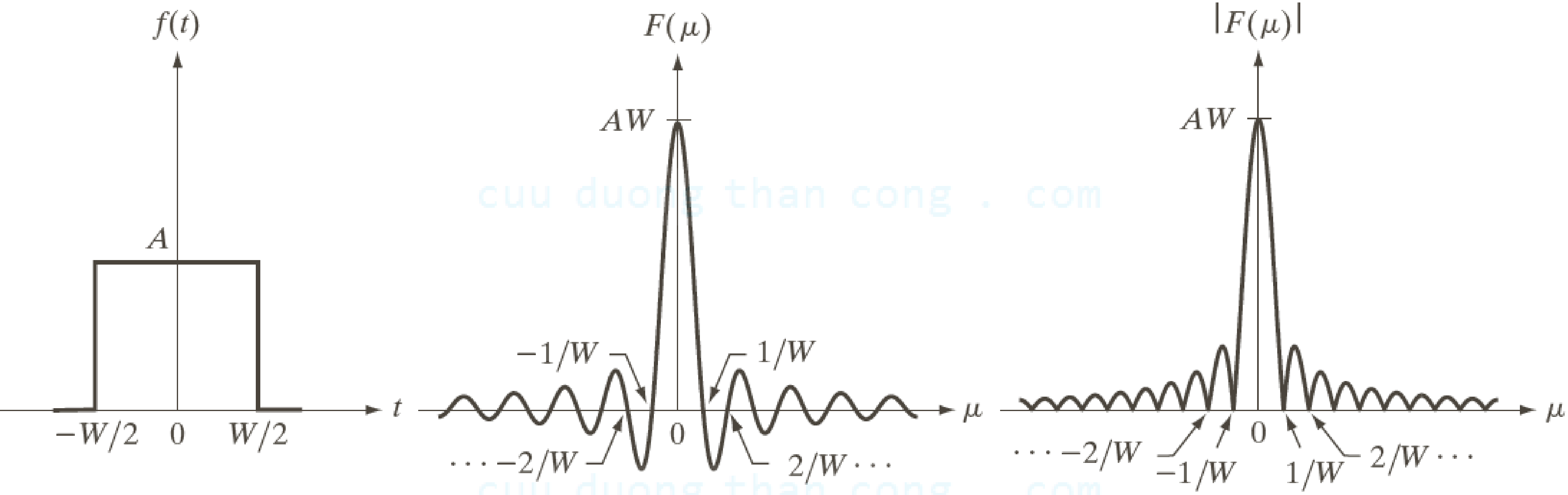
$$\mathcal{F}_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M}, \qquad m = 0, 1, 2, \dots, M-1$$

$$f_n = \frac{1}{M} \sum_{m=0}^{M-1} \mathcal{F}_m e^{j2\pi mn/M}, \qquad n = 0, 1, 2, \dots, M-1$$

- $\mathcal{F}(\mu)$ can be written as $\mathcal{F}(\mu) = R(\mu) + jI(\mu) = |\mathcal{F}(\mu)| e^{-j\phi(\mu)}$
  - Magnitude (Fourier spectrum) $|\mathcal{F}(\mu)| = \sqrt{R(\mu)^2 + I(\mu)^2}$
  - Phase angle $\phi(\mu) = \arctan\left(\frac{I(\mu)}{R(\mu)}\right)$

a b c

(a) A simple function. (b) Fourier transform of (a). (c) The spectrum. All functions extend to infinity in both directions.

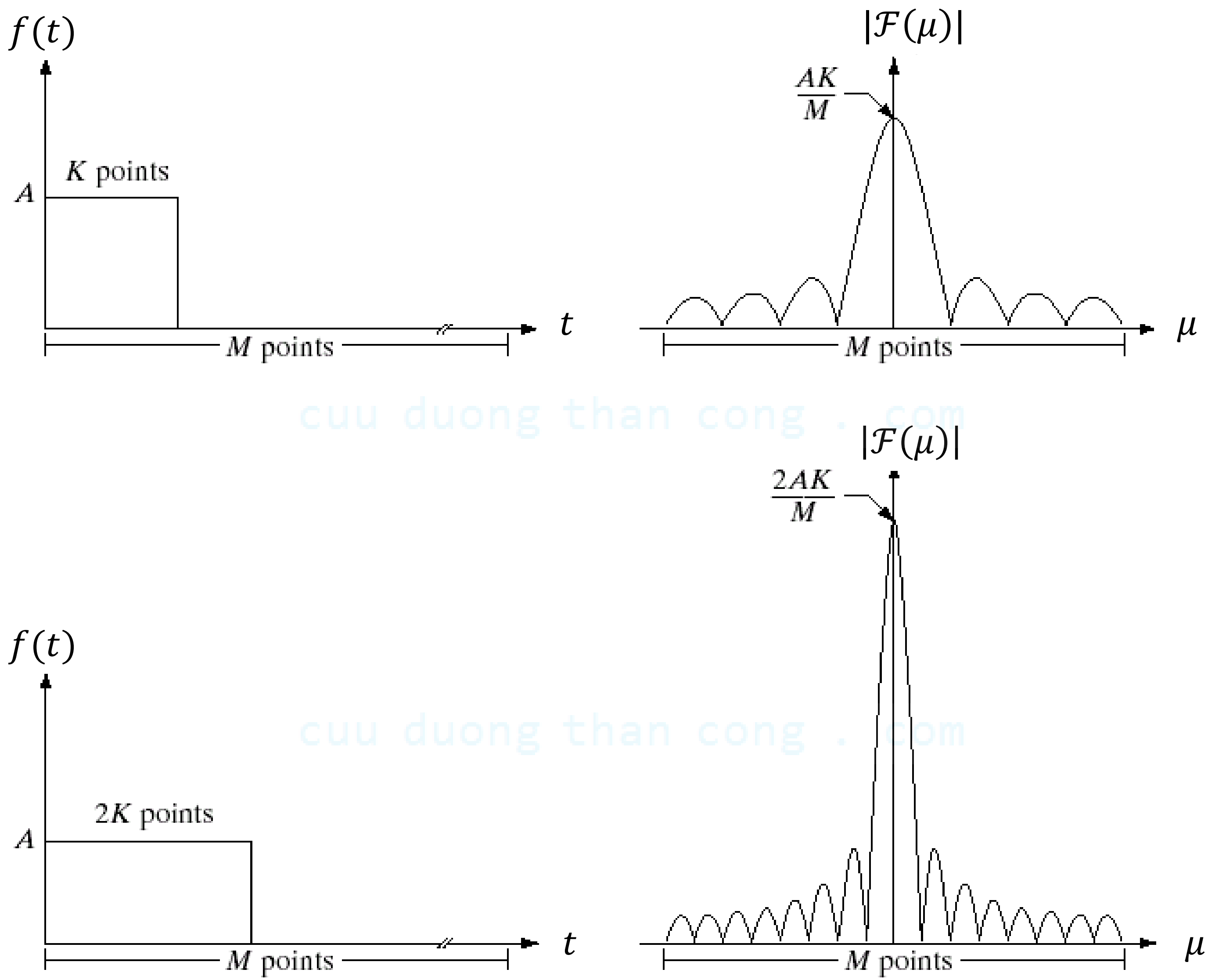


Relation between Δx and Δμ

- Given a signal $f(x)$ with M sample points.
- Let spatial resolution Δx be space between samples in $f(x)$ and frequency resolution $\Delta\mu$ be space between frequencies components in $\mathcal{F}(\mu)$
- The relation between Δx and $\Delta\mu$ is $\Delta\mu = \frac{1}{M\Delta x}$
- Example:
  - A signal $f(x)$ with sampling period 0.5 sec, 100 point
  - The frequency resolution $\Delta\mu = \frac{1}{M\Delta x} = \frac{1}{0.5 \times 100} = 0.02\ Hz$
  - In $\mathcal{F}(\mu)$ we can distinguish 2 frequencies that are apart by 0.02 Hertz or more

a b
c d

(a) A discrete function of M points, and (b) its Fourier spectrum. (c) A discrete function with twice the number of nonzero points and (d) its Fourier spectrum



2-D Discrete Fourier Transform

- Given an image $f(x, y)$ of size $M \times N$ pixels
- The Fourier Transform pair

$$\boldsymbol{\mathcal{F}(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}}$$

$$u = 0, 1, 2, \dots, M-1,$$
$$v = 0, 1, 2, \dots, N-1$$

$$\boldsymbol{f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} \mathcal{F}(u,v)e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}}$$

$$x = 0, 1, 2, \dots, M-1,$$
$$y = 0, 1, 2, \dots, N-1$$

2-D Discrete Fourier Transform

- $\mathcal{F}(u, v)$ can be written as

$$\mathcal{F}(u, v) = R(u, v) + jI(u, v)$$

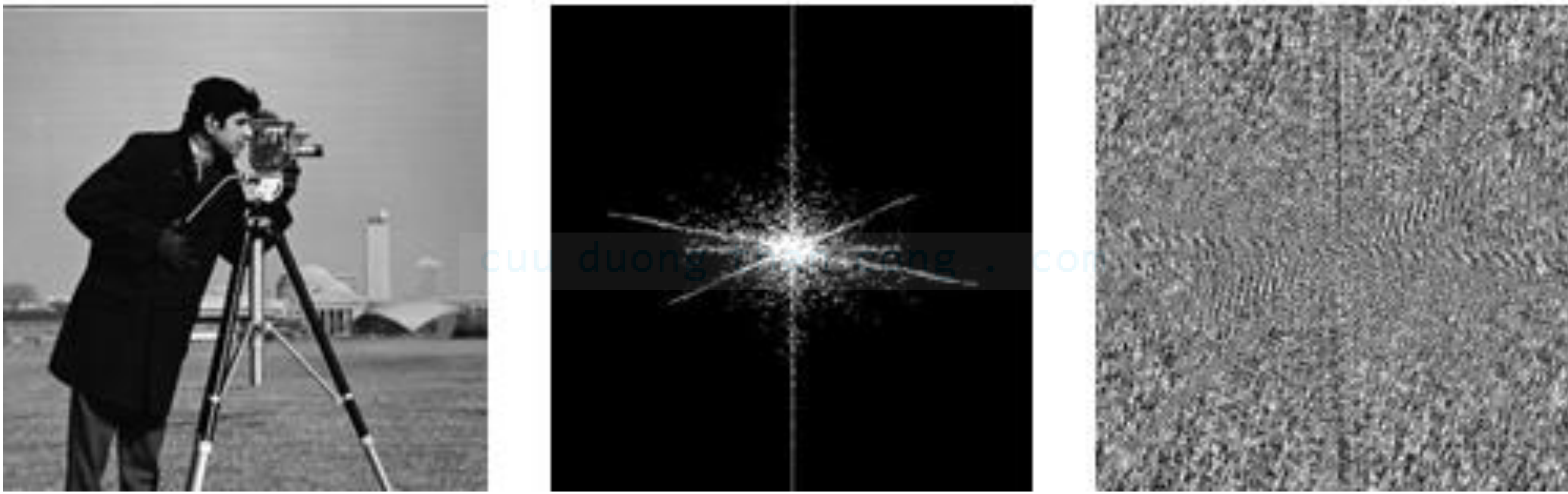
or

$$\mathcal{F}(u, v) = |\mathcal{F}(u, v)|e^{-j\phi(u,v)}$$

  - Magnitude $|\boldsymbol{\mathcal{F}(u, v)}| = \sqrt{\boldsymbol{R(u, v)^2 + I(u, v)^2}}$
  - Phase angle $\phi(u, v) = \arctan\left(\frac{I(u,v)}{R(u,v)}\right)$

- For the purpose of viewing, display only the Magnitude part of $\mathcal{F}(u, v)$

a b c (a) The original image. (b) Fourier spectrum of (a). (c) Phase angle of (a)



Property	Expression(s)
Fourier transform	$F(u, v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$
Inverse Fourier transform	$f(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$
Polar representation	$F(u, v) = \lvert F(u, v)\rvert e^{-j\phi(u,v)}$
Spectrum	$\lvert F(u, v)\rvert = \left[R^2(u, v) + I^2(u, v)\right]^{1/2}$, $R = \text{Real}(F)$ and $I = \text{Imag}(F)$
Phase angle	$\phi(u, v) = \tan^{-1}\left[\frac{I(u, v)}{R(u, v)}\right]$
Power spectrum	$P(u, v) = \lvert F(u, v)\rvert^2$
Average value	$\bar{f}(x, y) = F(0, 0) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)$
Translation	$f(x, y) e^{j2\pi(u_0 x/M + v_0 y/N)} \Leftrightarrow F(u - u_0, v - v_0)$ $f(x - x_0, y - y_0) \Leftrightarrow F(u, v) e^{-j2\pi(ux_0/M + vy_0/N)}$ When $x_0 = u_0 = M/2$ and $y_0 = v_0 = N/2$, then $f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$ $f(x - M/2, y - N/2) \Leftrightarrow F(u, v)(-1)^{u+v}$

Conjugate symmetry	$F(u, v) = F^*(-u, -v)$ $\lvert F(u, v)\rvert = \lvert F(-u, -v)\rvert$
Differentiation	$\dfrac{\partial^n f(x, y)}{\partial x^n} \Leftrightarrow (ju)^n F(u, v)$ $(-jx)^n f(x, y) \Leftrightarrow \dfrac{\partial^n F(u, v)}{\partial u^n}$
Laplacian	$\nabla^2 f(x, y) \Leftrightarrow -(u^2 + v^2)F(u, v)$
Distributivity	$\Im[f_1(x, y) + f_2(x, y)] = \Im[f_1(x, y)] + \Im[f_2(x, y)]$ $\Im[f_1(x, y) \cdot f_2(x, y)] \neq \Im[f_1(x, y)] \cdot \Im[f_2(x, y)]$
Scaling	$af(x, y) \Leftrightarrow aF(u, v), f(ax, by) \Leftrightarrow \dfrac{1}{\lvert ab\rvert} F(u/a, v/b)$
Rotation	$x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$ $f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$
Periodicity	$F(u, v) = F(u + M, v) = F(u, v + N) = F(u + M, v + N)$ $f(x, y) = f(x + M, y) = f(x, y + N) = f(x + M, y + N)$
Separability	See Eqs. (4.6-14) and (4.6-15). Separability implies that we can compute the 2-D transform of an image by first computing 1-D transforms along each row of the image, and then computing a 1-D transform along each column of this intermediate result. The reverse, columns and then rows, yields the same result.

Property	Expression(s)
Computation of the inverse Fourier transform using a forward transform algorithm	$\frac{1}{MN} f^*(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F^*(u, v) e^{-j2\pi(ux/M + vy/N)}$ This equation indicates that inputting the function $F^*(u, v)$ into an algorithm designed to compute the forward transform (right side of the preceding equation) yields $f^*(x, y)/MN$. Taking the complex conjugate and multiplying this result by MN gives the desired inverse.
Convolution†	$f(x, y) * h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) h(x - m, y - n)$
Correlation†	$f(x, y) \circ h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m, n) h(x + m, y + n)$
Convolution theorem†	$f(x, y) * h(x, y) \Leftrightarrow F(u, v) H(u, v)$; $f(x, y) h(x, y) \Leftrightarrow F(u, v) * H(u, v)$
Correlation theorem†	$f(x, y) \circ h(x, y) \Leftrightarrow F^*(u, v) H(u, v)$; $f^*(x, y) h(x, y) \Leftrightarrow F(u, v) \circ H(u, v)$

Some useful FT pairs:

Impulse $\delta(x, y) \Leftrightarrow 1$

Gaussian $A\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2(x^2+y^2)} \Leftrightarrow Ae^{-(u^2+v^2)/2\sigma^2}$

Rectangle $\text{rect}[a, b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$

Cosine $\cos(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow$

$$\frac{1}{2}\left[\delta(u + u_0, v + v_0) + \delta(u - u_0, v - v_0)\right]$$

Sine $\sin(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow$

$$j\frac{1}{2}\left[\delta(u + u_0, v + v_0) - \delta(u - u_0, v - v_0)\right]$$

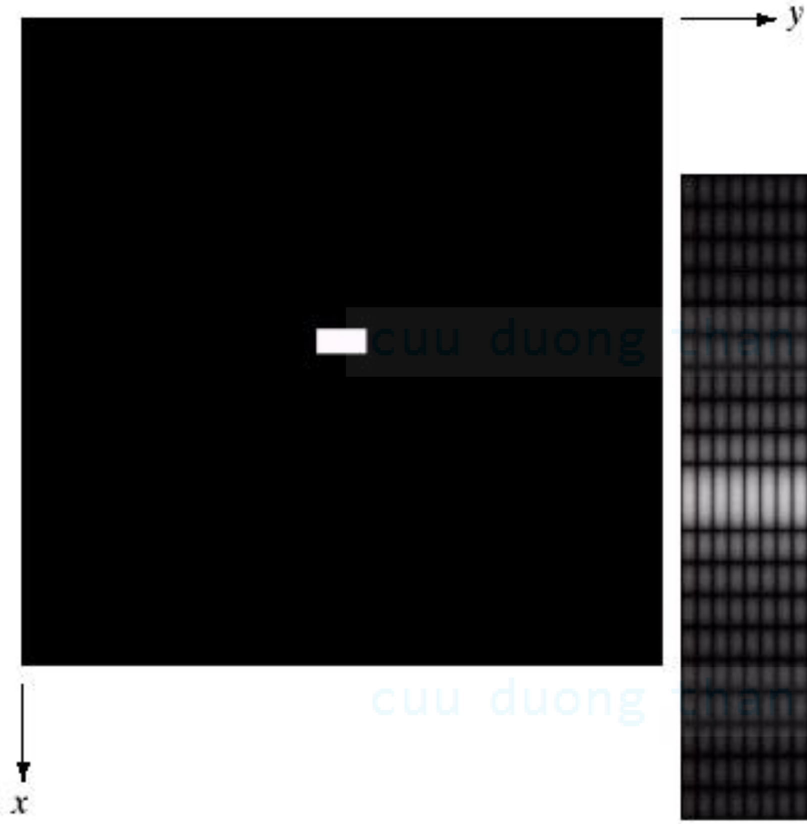
† Assumes that functions have been extended by zero padding.

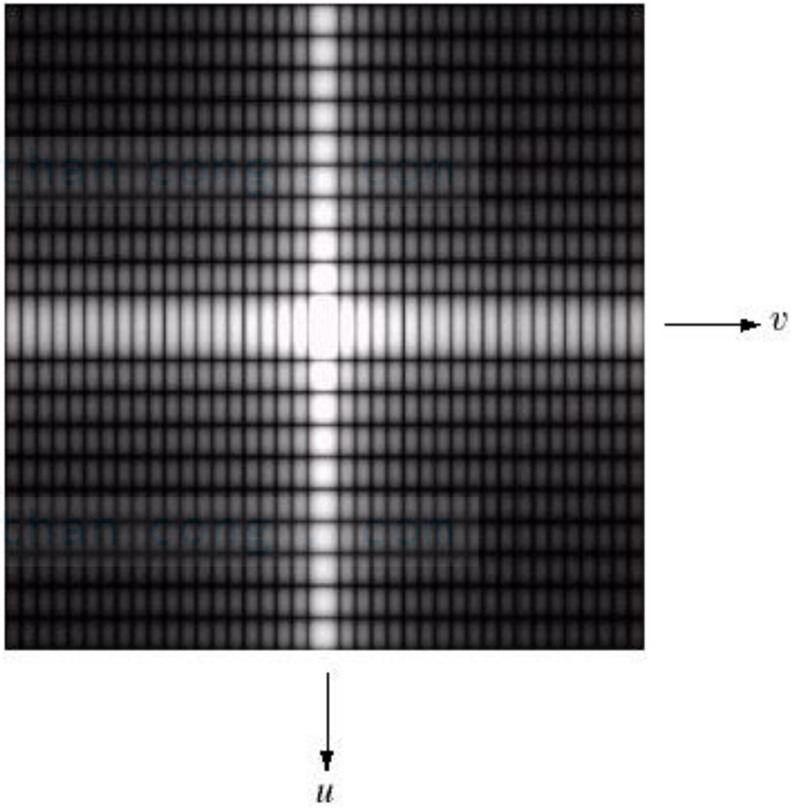
Relation between Spatial and Frequency resolutions

- Given an image $f(x, y)$ of size $M \times N$
- Let Δx and Δy be the spatial resolution in x-direction and y-direction, respectively.
- Let Δu and Δv be the frequency resolution in x-direction and y-direction, respectively
- The relation between spatial and frequency resolution is

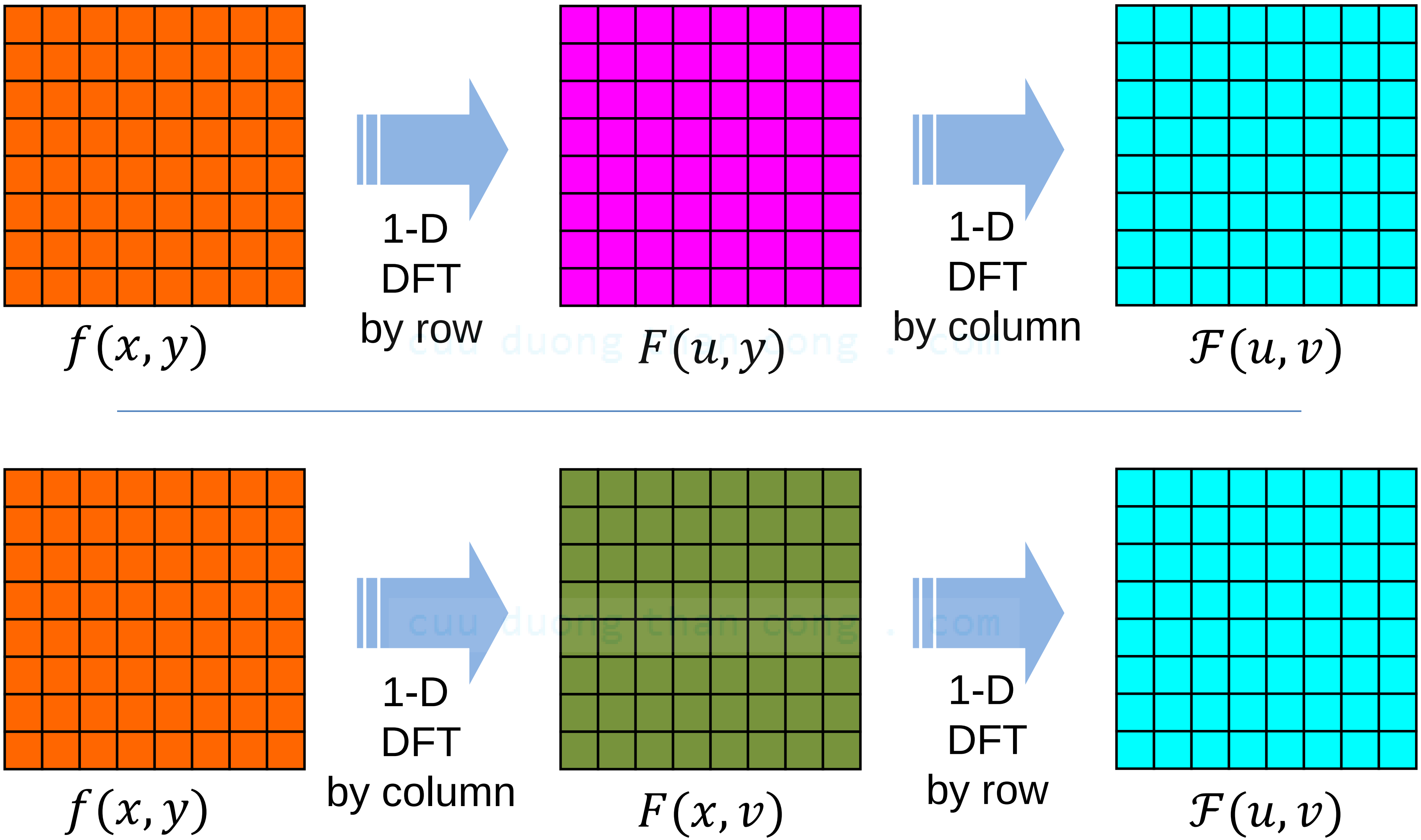
$$\Delta u = \frac{1}{M \Delta x} \qquad \Delta v = \frac{1}{N \Delta y}$$

a b (a) Image of a 20 × 40 white rectangle on black background of size 512 × 512. (b) Centered Fourier Spectrum shown after application of the log transformation.



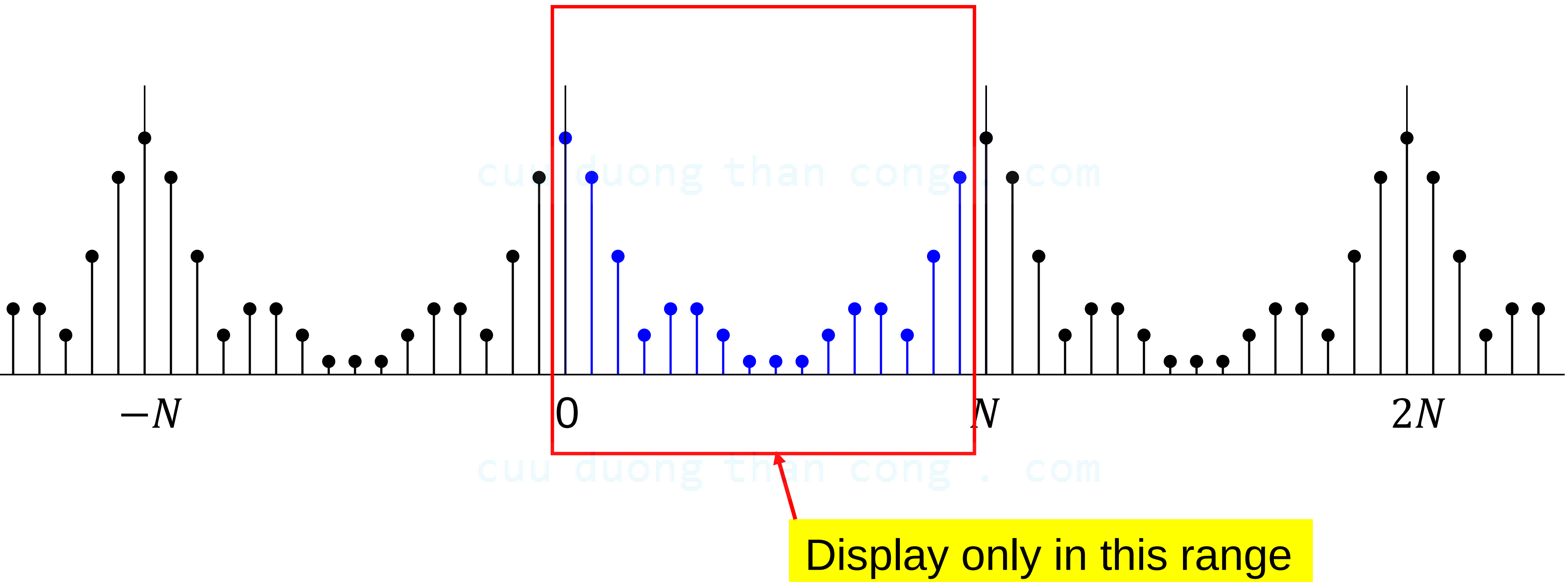


Performing 2-D DFT using 1-D DFT

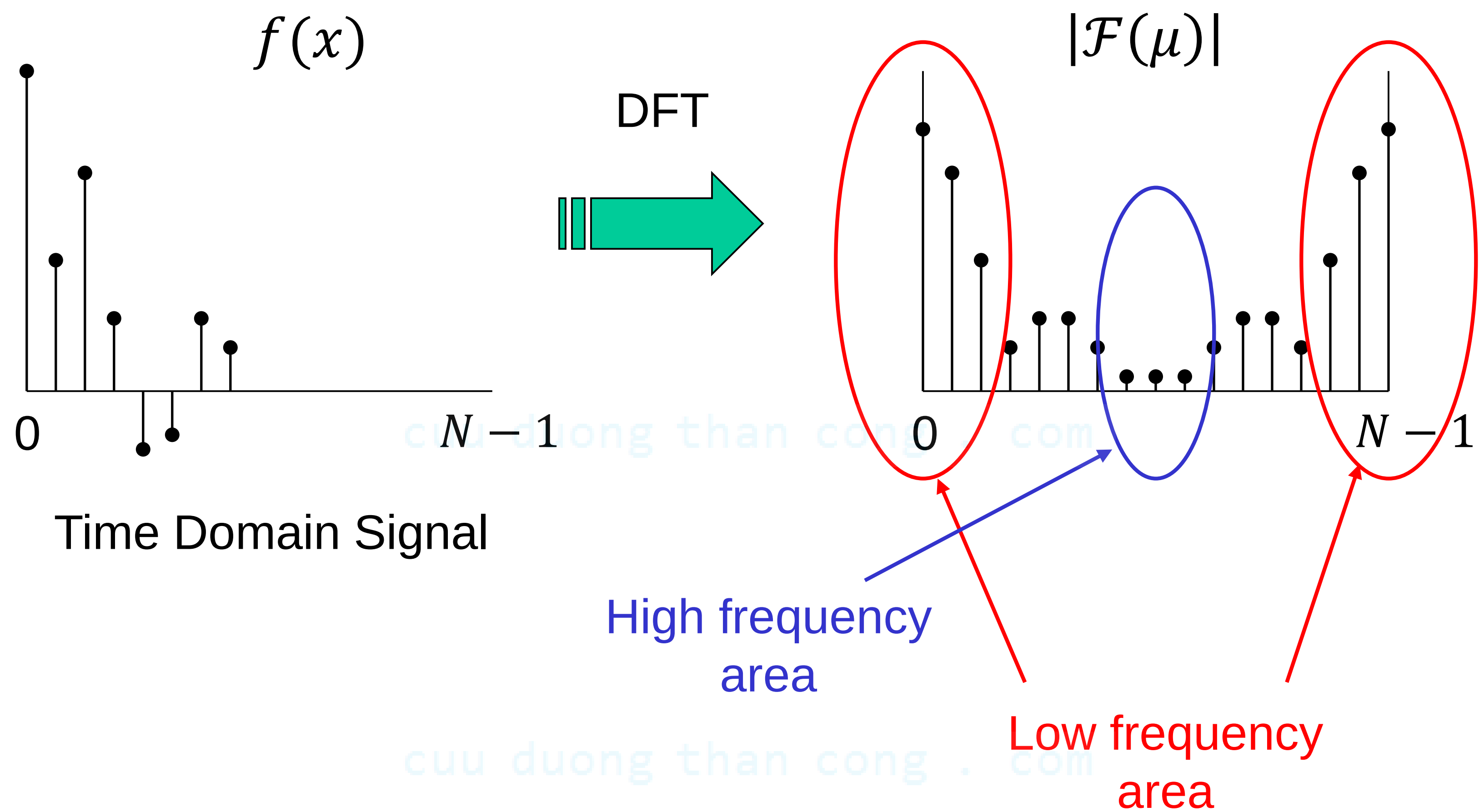


Periodicity of 1-D DFT

- DFT repeats itself every N points but it is usually displayed for $n = 0, \ldots, N-1$



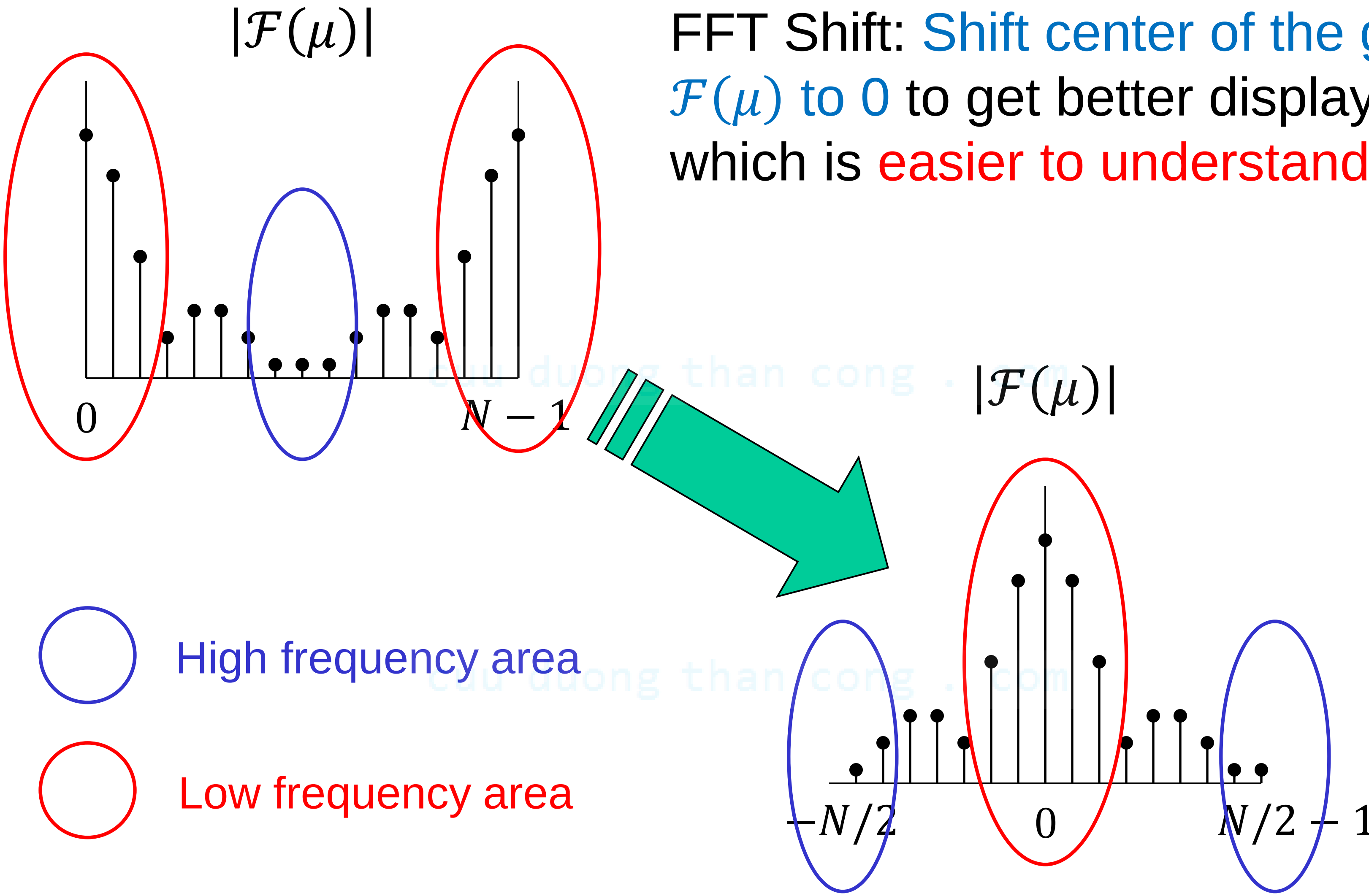
Conventional display for 1-D DFT



The graph $|\mathcal{F}(\mu)|$ is not easy to understand!

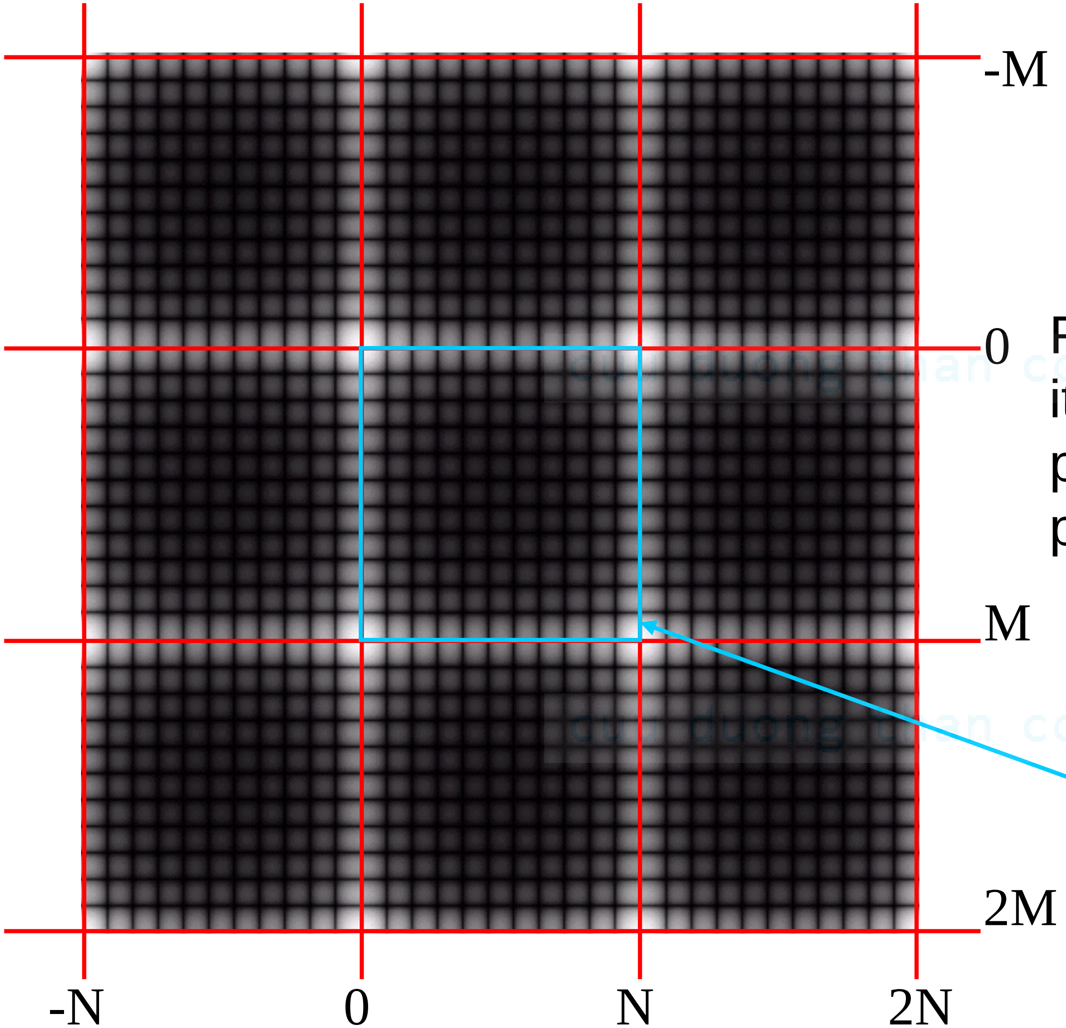
Conventional display for 1-D DFT: FFT Shift

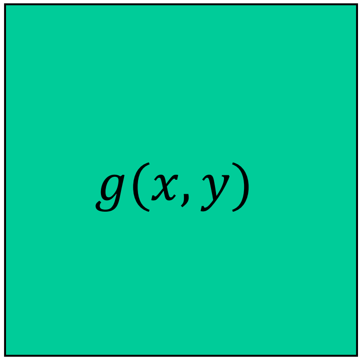
FFT Shift: Shift center of the graph $\mathcal{F}(\mu)$ to 0 to get better display which is easier to understand



Periodicity of 2-D DFT

$$\mathcal{F}(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

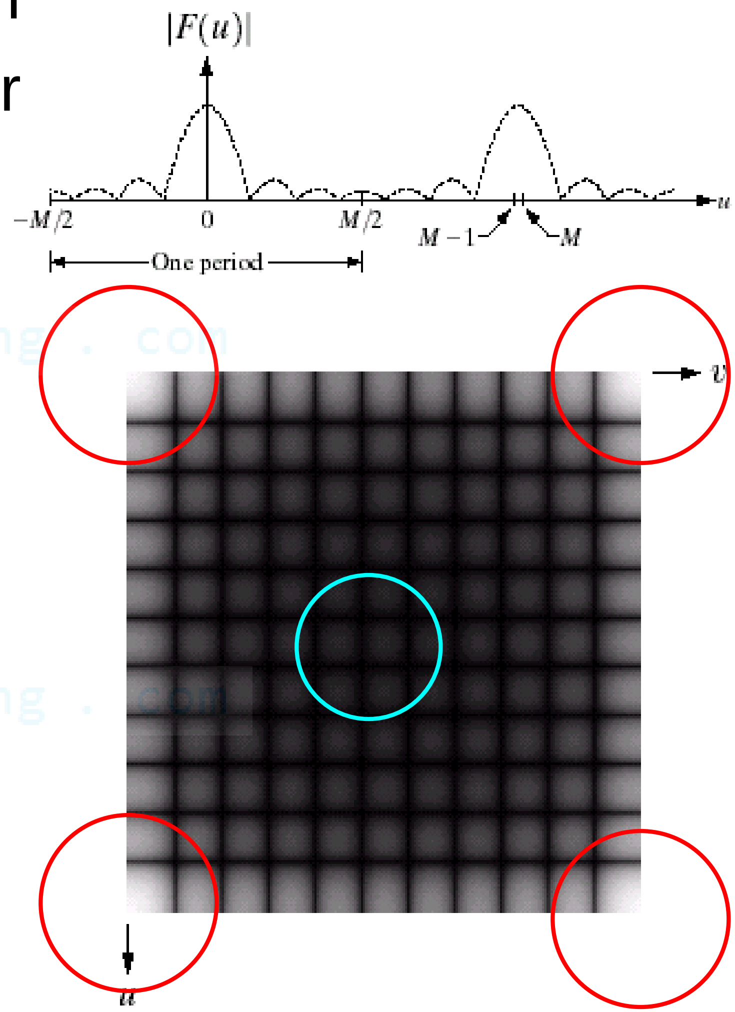




For an image of size $M \times N$ pixels, its 2-D DFT repeats itself every M points in *x*-direction and every N points in *y*-direction.

Conventional display for 2-D DFT

- $\mathcal{F}(u,v)$ has low frequency areas at corners of the image while high frequency areas are at the center $\rightarrow$ inconvenient to interpret

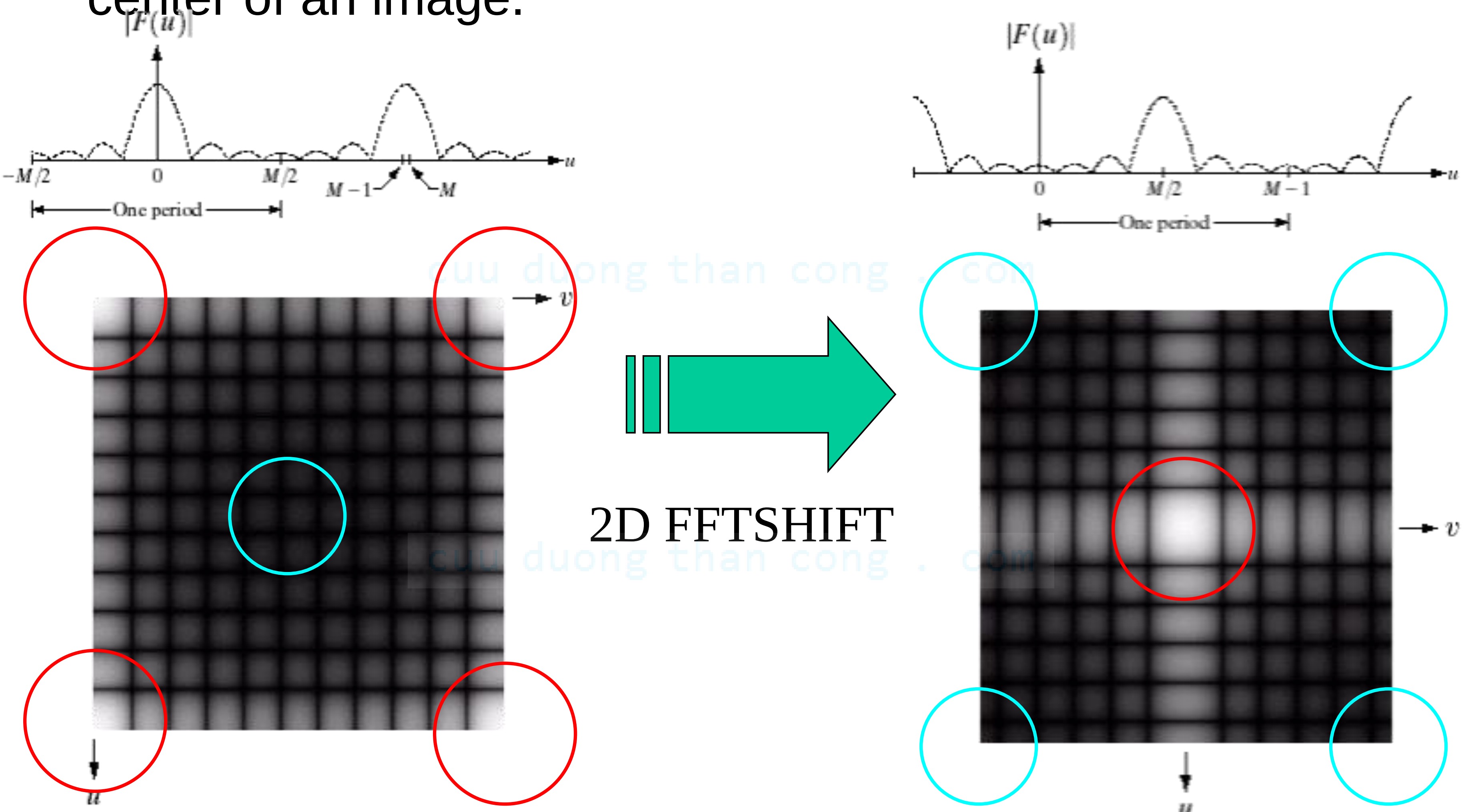


High frequency area

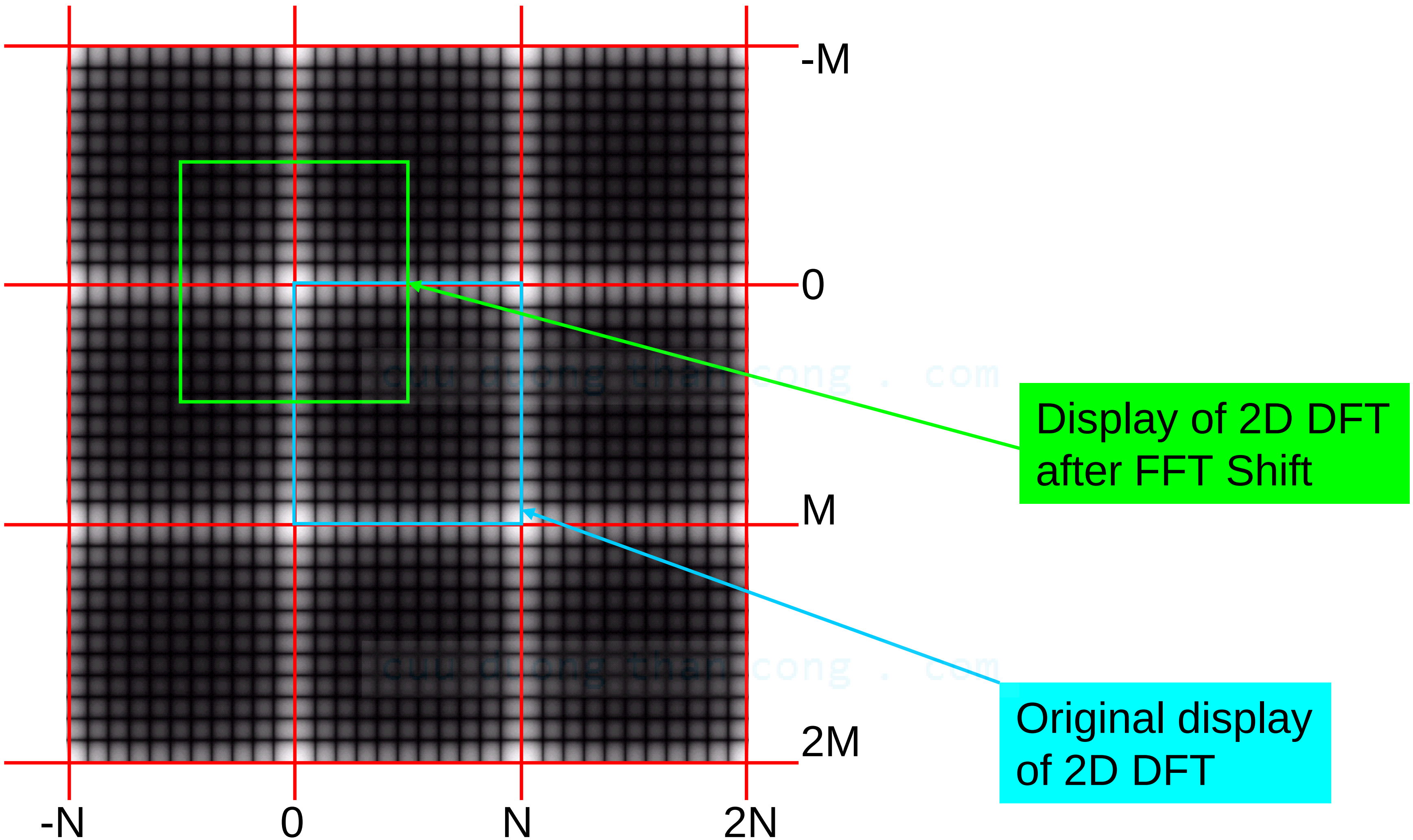
Low frequency area

Conventional display for 2-D DFT: FFT Shift

- **2-D FFT Shift:** Shift the zero frequency of $\mathcal{F}(u,v)$ to the center of an image.

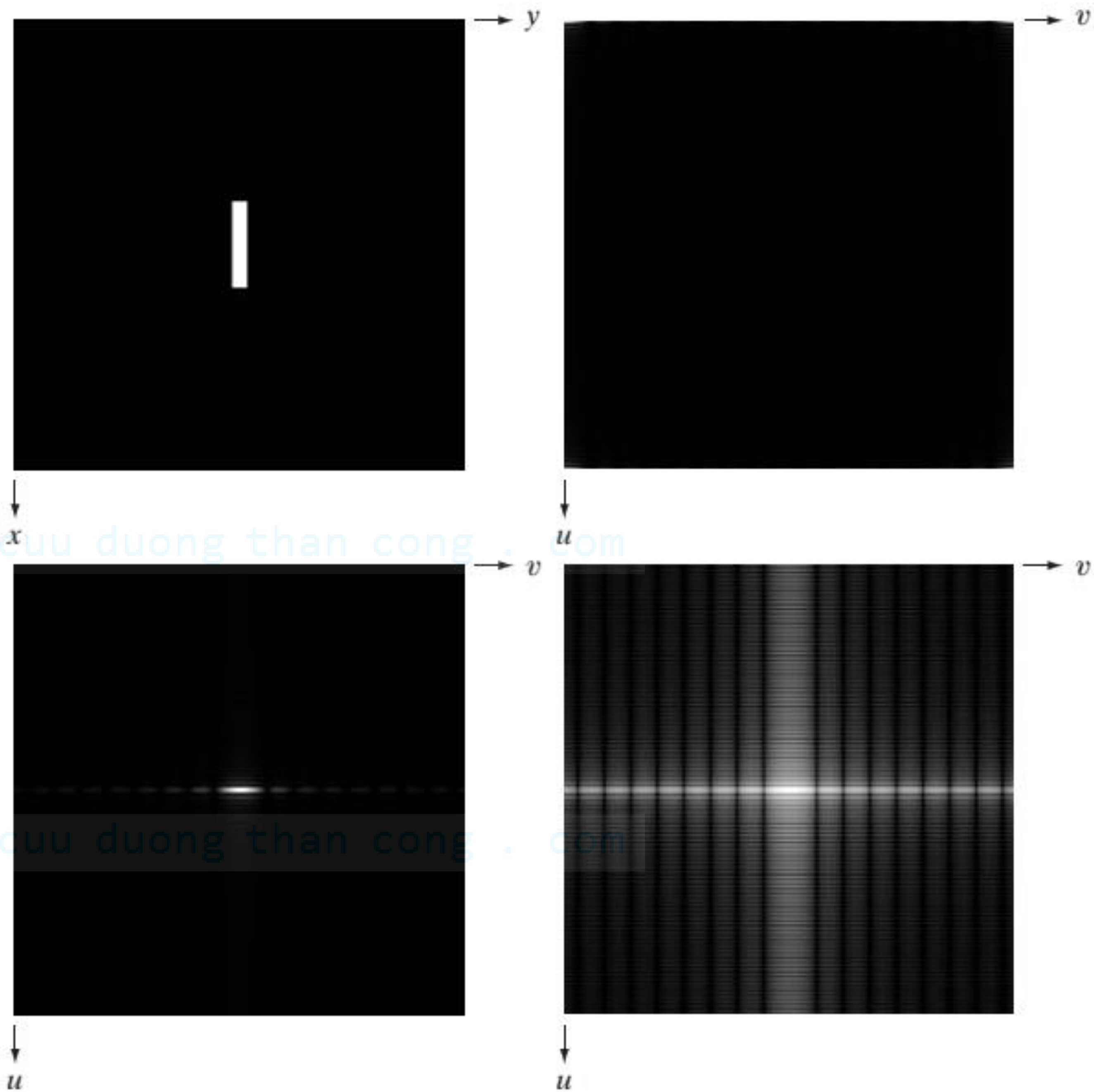


Conventional display for 2-D DFT: FFT Shift



a b
c d

(a) Image. (b) Spectrum showing bright spots in the four corners. (c) Centered spectrum. (d) Result showing increased detail after a log transformation



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- Rafael C. Gonzalez, Richard E. Woods, “Digital Image Processing”, 3rd edition, 2008. Chapter 4
- Fourier Transform: http://www.thefouriertransform.com/
- Discrete Fourier Transform

  http://www.robots.ox.ac.uk/~sjrob/Teaching/SP/l7.pdf
- gear.kku.ac.th/~nawapak/178353/Chapter04.ppt
- Images are obtained from the above materials and Google