

EXAMPLE 5 If $R = [0, \pi/2] \times [0, \pi/2]$, then, by Equation 5,

$$\begin{aligned} \iint_R \sin x \cos y \, dA &= \int_0^{\pi/2} \sin x \, dx \int_0^{\pi/2} \cos y \, dy \\ &= [-\cos x]_0^{\pi/2} [\sin y]_0^{\pi/2} = 1 \cdot 1 = 1 \end{aligned}$$

The function $f(x, y) = \sin x \cos y$ in Example 5 is positive on R , so the integral represents the volume of the solid that lies above R and below the graph of f shown in Figure 6.

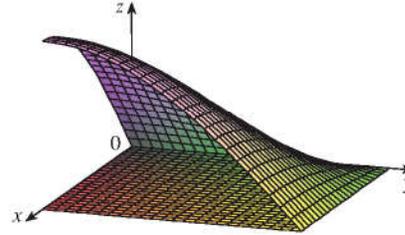


FIGURE 6

15.2 Exercises

1–2 Find $\int_0^5 f(x, y) \, dx$ and $\int_0^1 f(x, y) \, dy$.

1. $f(x, y) = 12x^2y^3$ 2. $f(x, y) = y + xe^y$

3–14 Calculate the iterated integral.

3. $\int_1^4 \int_0^2 (6x^2y - 2x) \, dy \, dx$ 4. $\int_0^1 \int_1^2 (4x^3 - 9x^2y^2) \, dy \, dx$
 5. $\int_0^2 \int_0^4 y^3 e^{2x} \, dy \, dx$ 6. $\int_{\pi/6}^{\pi/2} \int_{-1}^5 \cos y \, dx \, dy$
 7. $\int_{-3}^3 \int_0^{\pi/2} (y + y^2 \cos x) \, dx \, dy$ 8. $\int_1^3 \int_1^5 \frac{\ln y}{xy} \, dy \, dx$
 9. $\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x} \right) \, dy \, dx$ 10. $\int_0^1 \int_0^3 e^{x+3y} \, dx \, dy$
 11. $\int_0^1 \int_0^1 v(u + v^2)^4 \, du \, dv$ 12. $\int_0^1 \int_0^1 xy\sqrt{x^2 + y^2} \, dy \, dx$
 13. $\int_0^2 \int_0^\pi r \sin^2 \theta \, d\theta \, dr$ 14. $\int_0^1 \int_0^1 \sqrt{s+t} \, ds \, dt$

15–22 Calculate the double integral.

15. $\iint_R \sin(x - y) \, dA$, $R = \{(x, y) \mid 0 \leq x \leq \pi/2, 0 \leq y \leq \pi/2\}$
 16. $\iint_R (y + xy^{-2}) \, dA$, $R = \{(x, y) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}$
 17. $\iint_R \frac{xy^2}{x^2 + 1} \, dA$, $R = \{(x, y) \mid 0 \leq x \leq 1, -3 \leq y \leq 3\}$

18. $\iint_R \frac{1+x^2}{1+y^2} \, dA$, $R = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$

19. $\iint_R x \sin(x + y) \, dA$, $R = [0, \pi/6] \times [0, \pi/3]$

20. $\iint_R \frac{x}{1+xy} \, dA$, $R = [0, 1] \times [0, 1]$

21. $\iint_R ye^{-xy} \, dA$, $R = [0, 2] \times [0, 3]$

22. $\iint_R \frac{1}{1+x+y} \, dA$, $R = [1, 3] \times [1, 2]$

23–24 Sketch the solid whose volume is given by the iterated integral.

23. $\int_0^1 \int_0^1 (4 - x - 2y) \, dx \, dy$

24. $\int_0^1 \int_0^1 (2 - x^2 - y^2) \, dy \, dx$

25. Find the volume of the solid that lies under the plane $4x + 6y - 2z + 15 = 0$ and above the rectangle $R = \{(x, y) \mid -1 \leq x \leq 2, -1 \leq y \leq 1\}$.

26. Find the volume of the solid that lies under the hyperbolic paraboloid $z = 3y^2 - x^2 + 2$ and above the rectangle $R = [-1, 1] \times [1, 2]$.

27. Find the volume of the solid lying under the elliptic paraboloid $x^2/4 + y^2/9 + z = 1$ and above the rectangle $R = [-1, 1] \times [-2, 2]$.
28. Find the volume of the solid enclosed by the surface $z = 1 + e^x \sin y$ and the planes $x = \pm 1, y = 0, y = \pi,$ and $z = 0$.
29. Find the volume of the solid enclosed by the surface $z = x \sec^2 y$ and the planes $z = 0, x = 0, x = 2, y = 0,$ and $y = \pi/4$.
30. Find the volume of the solid in the first octant bounded by the cylinder $z = 16 - x^2$ and the plane $y = 5$.
31. Find the volume of the solid enclosed by the paraboloid $z = 2 + x^2 + (y - 2)^2$ and the planes $z = 1, x = 1, x = -1, y = 0,$ and $y = 4$.

 32. Graph the solid that lies between the surface $z = 2xy/(x^2 + 1)$ and the plane $z = x + 2y$ and is bounded by the planes $x = 0, x = 2, y = 0,$ and $y = 4$. Then find its volume.

 33. Use a computer algebra system to find the exact value of the integral $\iint_R x^5 y^3 e^{xy} dA$, where $R = [0, 1] \times [0, 1]$. Then use the CAS to draw the solid whose volume is given by the integral.

 34. Graph the solid that lies between the surfaces $z = e^{-x^2} \cos(x^2 + y^2)$ and $z = 2 - x^2 - y^2$ for $|x| \leq 1, |y| \leq 1$. Use a computer algebra system to approximate the volume of this solid correct to four decimal places.

35–36 Find the average value of f over the given rectangle.

35. $f(x, y) = x^2 y, R$ has vertices $(-1, 0), (-1, 5), (1, 5), (1, 0)$

36. $f(x, y) = e^y \sqrt{x + e^y}, R = [0, 4] \times [0, 1]$

37–38 Use symmetry to evaluate the double integral.

37. $\iint_R \frac{xy}{1 + x^4} dA, R = \{(x, y) \mid -1 \leq x \leq 1, 0 \leq y \leq 1\}$

38. $\iint_R (1 + x^2 \sin y + y^2 \sin x) dA, R = [-\pi, \pi] \times [-\pi, \pi]$

 39. Use your CAS to compute the iterated integrals

$$\int_0^1 \int_0^1 \frac{x - y}{(x + y)^3} dy dx \quad \text{and} \quad \int_0^1 \int_0^1 \frac{x - y}{(x + y)^3} dx dy$$

Do the answers contradict Fubini's Theorem? Explain what is happening.

40. (a) In what way are the theorems of Fubini and Clairaut similar?
 (b) If $f(x, y)$ is continuous on $[a, b] \times [c, d]$ and

$$g(x, y) = \int_a^x \int_c^y f(s, t) dt ds$$

for $a < x < b, c < y < d$, show that $g_{xy} = g_{yx} = f(x, y)$.

15.3 Double Integrals over General Regions

For single integrals, the region over which we integrate is always an interval. But for double integrals, we want to be able to integrate a function f not just over rectangles but also over regions D of more general shape, such as the one illustrated in Figure 1. We suppose that D is a bounded region, which means that D can be enclosed in a rectangular region R as in Figure 2. Then we define a new function F with domain R by

$$\mathbf{1} \quad F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

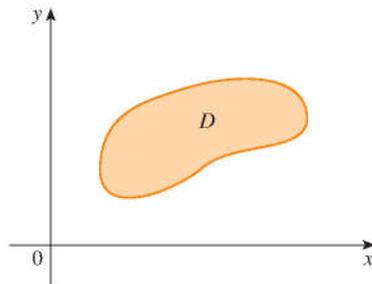


FIGURE 1

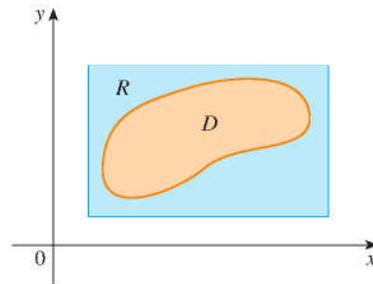


FIGURE 2