BÀI GIẢNG MÔN HỌC
TOÁN KỸ THUẬT

Credit: 2

Text book: *Advanced Engineering Mathematics*, Dean G. Duffy, CRC Press LLC, 1998.

NỘI DUNG

CHƯƠNG 1. LÝ THUYẾT TRƯỜNG

THE GRADIENT ∇f

A *scalar point-function* is a scalar quantity, say temperature, that is a function of the coordinates. Consider a scalar point-function f that is

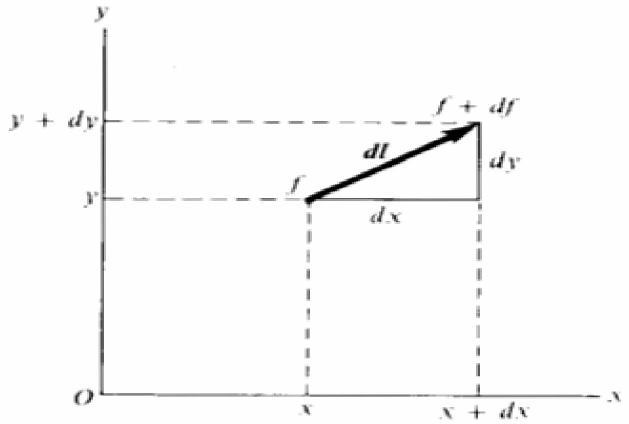

Fig. 1-3. A scalar-point function changes from f to $f + df$ over the distance $\boldsymbol{dl}$.

continuous and differentiable. We wish to know how f changes over the infinitesimal distance dl in Fig. 1-3. The differential

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \tag{1-7}$$

is the scalar product of the two vectors

$$\boldsymbol{dl} = dx\,\hat{\boldsymbol{x}} + dy\,\hat{\boldsymbol{y}} + dz\,\hat{\boldsymbol{z}} \tag{1-8}$$

and

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}. \tag{1-9}$$

The second vector, whose components are the rates of change of f with distance along the coordinate axes, is called the *gradient* of f. The symbol

$$\boldsymbol{\nabla} = \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \tag{1-10}$$

is read "del."

Note the value of the magnitude of the gradient:

$$|\boldsymbol{\nabla} f| = \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right]^{1/2}. \tag{1-11}$$

Thus

$$df = \boldsymbol{\nabla} f \cdot \boldsymbol{dl} = |\boldsymbol{\nabla} f|\,|dl|\cos\theta, \tag{1-12}$$

where θ is the angle between the vectors $\boldsymbol{\nabla} f$ and $\boldsymbol{dl}$.

What direction should one choose for $\boldsymbol{dl}$ to maximize df? That direction is the one for which $\cos\theta = 1$ or $\theta = 0$, that is, the direction of $\boldsymbol{\nabla} f$.

Therefore the gradient of a scalar function at a given point is a vector having the following properties:

(1) Its components are the rates of change of the function along the directions of the coordinate axes.

(2) Its magnitude is the maximum rate of change with distance.

(3) Its direction is that of the maximum rate of change with distance.

(4) It points toward larger values of the function.

The gradient is a vector point-function that derives from a scalar point-function.

Again, we have two definitions: $\boldsymbol{\nabla} f$ is a vector whose magnitude and direction are those of the maximum space rate of change of f, and it is also the vector of Eq. 1-9. It is clear from the first definition that $\boldsymbol{\nabla} f$ is invariant.

INVARIANCE OF THE OPERATOR $\boldsymbol{\nabla}$

We have just seen that $\boldsymbol{\nabla} f$ is invariant. Is the operator $\boldsymbol{\nabla}$ itself also invariant? This requires careful consideration because the components of $\boldsymbol{\nabla}$ are not numbers, but operators.

Let S and S' be any two sets of Cartesian coordinates. Figure 1-5 shows two sets having a common origin, for simplicity. Then a given vector $\boldsymbol{A}$ has the components A_x, A_y, A_z in S, and $A_{x'}$, $A_{y'}$, $A_{z'}$ in S', with

$$A_{x'} = a_{xx}A_x + a_{xy}A_y + a_{xz}A_z, \tag{1-13}$$

$$A_{y'} = a_{yx}A_x + a_{yy}A_y + a_{yz}A_z, \tag{1-14}$$

$$A_{z'} = a_{zx}A_x + a_{zy}A_y + a_{zz}A_z. \tag{1-15}$$

The a coefficients depend only on the orientation of S' with respect to S.

If $\boldsymbol{A}$ is $\boldsymbol{\nabla} f$, then its components are

$$A_x = \frac{\partial f}{\partial x}, \qquad A_y = \frac{\partial f}{\partial y}, \qquad A_z = \frac{\partial f}{\partial z}, \tag{1-16}$$

and

$$\frac{\partial f}{\partial x'} = a_{xx}\frac{\partial f}{\partial x} + a_{xy}\frac{\partial f}{\partial y} + a_{xz}\frac{\partial f}{\partial z}. \tag{1-17}$$

Since this is true for any differentiable f, we know that

$$\frac{\partial}{\partial x'} = a_{xx}\frac{\partial}{\partial x} + a_{xy}\frac{\partial}{\partial y} + a_{xz}\frac{\partial}{\partial z}, \tag{1-18}$$

and similarly for $\partial/\partial y'$ and $\partial/\partial z'$.

The components of $\boldsymbol{\nabla}$ in S', namely $\partial/\partial x'$, $\partial/\partial y'$, and $\partial/\partial z'$, relate to those of $\boldsymbol{\nabla}$ in S, $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, in the same way as the components of any vector $\boldsymbol{A}$ in S' and in S. Therefore $\boldsymbol{\nabla}$ is invariant like any vector, and it transforms as a vector. We shall use this property of $\boldsymbol{\nabla}$ in the following sections.

FLUX

It is often necessary to calculate the flux of a vector quantity through a surface. By definition, the *flux* $d\Phi$ of $\boldsymbol{B}$ through an infinitesimal surface $d\boldsymbol{\mathcal{A}}$ is

$$d\Phi = \boldsymbol{B} \cdot d\boldsymbol{\mathcal{A}}, \tag{1-19}$$

where the vector $d\boldsymbol{\mathcal{A}}$ is normal to the surface. The flux $d\Phi$ is therefore the component of the vector normal to the surface, multiplied by $d\mathcal{A}$. For a surface of finite area $\mathcal{A}$,

$$\Phi = \int_{\mathcal{A}} \boldsymbol{B} \cdot d\boldsymbol{\mathcal{A}}. \tag{1-20}$$

If the surface is closed, the vector $d\boldsymbol{\mathcal{A}}$ points *outward*, by convention.

THE DIVERGENCE $\nabla \cdot \boldsymbol{B}$

The outward flux of a vector through a closed surface can be calculated either from the above equation or as follows. Consider an infinitesimal volume $dx\,dy\,dz$ and a vector $\boldsymbol{B}$, as in Fig. 1-6, whose components B_x, B_y, B_z are functions of x, y, z. The value of B_x at the center of the right-hand face may be taken to be the average value over that face. Through the right-hand face of the volume element, the outgoing flux is

$$d\Phi_R = \left(B_x + \frac{\partial B_x}{\partial x}\frac{dx}{2}\right) dy\,dz, \tag{1-21}$$

since the normal component of $\boldsymbol{B}$ at the right-hand face is the x-component of $\boldsymbol{B}$ at that face. The volume being infinitesimal, we neglect higher-order derivatives of the components of $\boldsymbol{B}$.

At the left-hand face, the outgoing flux is

$$d\Phi_L = -\left(B_x - \frac{\partial B_x}{\partial x}\frac{dx}{2}\right) dy\,dz. \tag{1-22}$$

There is a minus sign before the parenthesis because $B_x\hat{\boldsymbol{x}}$ points inward at this face and $d\boldsymbol{\mathcal{A}}$ outward.

Thus the outward flux through the two faces is

$$d\Phi_L + d\Phi_R = \frac{\partial B_x}{\partial x}\,dx\,dy\,dz = \frac{\partial B_x}{\partial x}\,dv, \tag{1-23}$$

where dv is the volume of the infinitesimal element.

If we calculate the net flux through the other pairs of faces in the same manner, we find that the total outward flux for the element of volume dv is

$$d\Phi_{\text{tot}} = \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) dv. \tag{1-24}$$

Suppose now that we have two adjoining infinitesimal volume elements and that we add the flux emerging through the bounding surface of the first volume to the flux emerging through the bounding surface of the second. At the common face, the fluxes are equal in magnitude but opposite in sign, and they cancel. The sum, then, of the flux from the first volume and that from the second is the flux emerging through the bounding surface of the combined volumes.

To extend this calculation to a finite volume, we sum the individual fluxes for each of the infinitesimal volume elements in the finite volume, and so the total outward flux is

$$\Phi_{\text{tot}} = \int_v \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) dv. \tag{1-25}$$

At any given point in the volume, the quantity

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

is thus the *outgoing* flux per unit volume and is invariant. We call this the *divergence* of $\boldsymbol{B}$ at the point.

The divergence of a vector point-function is a scalar point-function.

According to the rule for the scalar product, we write the *divergence* of $\boldsymbol{B}$ as

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}. \tag{1-26}$$

The divergence is invariant also because both $\boldsymbol{\nabla}$ and the scalar product are invariant.

THE DIVERGENCE THEOREM

Now the total outward flux of a vector $\boldsymbol{B}$ is equal to the surface integral of the normal outward component of $\boldsymbol{B}$. Thus, if we denote by $\mathcal{A}$ the area of the surface bounding v, the total outward flux is

$$\Phi_{\text{tot}} = \int_{\mathcal{A}} \boldsymbol{B} \cdot d\boldsymbol{\mathcal{A}} = \int_v \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) dv = \int_v \boldsymbol{\nabla} \cdot \boldsymbol{B}\, dv. \qquad (1\text{-}27)$$

These relations apply to any continuously differentiable† vector field $\boldsymbol{B}$. Thus

$$\int_{\mathcal{A}} \boldsymbol{B} \cdot d\boldsymbol{\mathcal{A}} = \int_v \boldsymbol{\nabla} \cdot \boldsymbol{B}\, dv. \qquad (1\text{-}28)$$

This is the *divergence theorem,* also called *Green's theorem,* or *Gauss's theorem.* Note that the first integral involves only the values of $\boldsymbol{B}$ on the surface of area $\mathcal{A}$ whereas the second involves the values of $\boldsymbol{B}$ throughout the volume v.

THE LINE INTEGRAL $\int_a^b \boldsymbol{B} \cdot d\boldsymbol{l}$. CONSERVATIVE FIELDS

The integrals

$$\int_a^b \boldsymbol{B} \cdot d\boldsymbol{l}, \qquad \int_a^b \boldsymbol{B} \times d\boldsymbol{l}, \qquad \text{and} \qquad \int_a^b f\, dl,$$

evaluated from the point a to the point b over some specified curve, are examples of *line integrals.*

In the first, which is especially important, the term under the integral sign is the product of an element of length $d\boldsymbol{l}$ on the curve, multiplied by the local value of $\boldsymbol{B}$ according to the rule for the scalar product.

A vector field $\boldsymbol{B}$ is *conservative* if the line integral of $\boldsymbol{B} \cdot d\boldsymbol{l}$ around any closed curve is zero:

$$\oint \boldsymbol{B} \cdot d\boldsymbol{l} = 0. \qquad (1\text{-}29)$$

The circle on the integral sign indicates that the path of integration is closed.

THE CURL $\nabla \times \boldsymbol{B}$

For any given field $\boldsymbol{B}$ and for a closed path situated in the xy-plane,

$$\boldsymbol{B} \cdot \boldsymbol{dl} = B_x\, dx + B_y\, dy \tag{1-30}$$

and

$$\oint \boldsymbol{B} \cdot \boldsymbol{dl} = \oint B_x\, dx + \oint B_y\, dy. \tag{1-31}$$

Now consider the infinitesimal path in Fig. 1-7. There are two contributions to the first integral on the right-hand side of Eq. 1-31, one at $y - dy/2$ and one at $y + dy/2$:

$$\oint B_x\, dx = \left(B_x - \frac{\partial B_x}{\partial y}\frac{dy}{2}\right) dx - \left(B_x + \frac{\partial B_x}{\partial y}\frac{dy}{2}\right) dx. \tag{1-32}$$

There is a minus sign before the second term because the path element at $y + dy/2$ points in the negative x-direction. Therefore, for this infinitesimal path,

$$\oint B_x\, dx = -\frac{\partial B_x}{\partial y}\, dy\, dx. \tag{1-33}$$

Similarly,

$$\oint B_y\, dy = \frac{\partial B_y}{\partial x}\, dx\, dy, \tag{1-34}$$

and

$$\oint \boldsymbol{B} \cdot \boldsymbol{dl} = \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) dx\, dy \tag{1-35}$$

for the infinitesimal path of Fig. 1-7.

If we set

$$g_3 = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}, \tag{1-36}$$

then

$$\oint \boldsymbol{B} \cdot \boldsymbol{dl} = g_3 \, d\mathscr{A}, \tag{1-37}$$

where $d\mathscr{A} = dx\, dy$ is the area enclosed by the infinitesimal path. Note that this is correct only if the line integral runs in the positive direction in the xy-plane, that is, in the direction in which one would turn a right-hand screw to make it advance in the positive direction along the z-axis.

Consider now g_3 and the other two symmetric quantities as the components of a vector

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\hat{\boldsymbol{z}}, \tag{1-38}$$

which may be written as

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} \tag{1-39}$$

This is the *curl* of $\boldsymbol{B}$. The quantity g_3 is its z-component.

If we choose a vector $\boldsymbol{d\mathscr{A}}$ that points in the direction of advance of a right-hand screw turned in the direction chosen for the line integral, then

$$\boldsymbol{d\mathscr{A}} = d\mathscr{A}\, \hat{\boldsymbol{z}} \tag{1-40}$$

and

$$\oint \boldsymbol{B} \cdot \boldsymbol{dl} = (\boldsymbol{\nabla} \times \boldsymbol{B}) \cdot \boldsymbol{d\mathscr{A}}. \tag{1-41}$$

This means that the line integral of $\boldsymbol{B} \cdot \boldsymbol{dl}$ around the edge of the area $\boldsymbol{d\mathscr{A}}$ is equal to the scalar product of the curl of $\boldsymbol{B}$ by this element of area, with the above sign convention.

We have arrived at this result for an element of area $dx\,dy$ in the xy-plane. Is this result general? Does it apply to any small area, whatever its orientation with respect to the coordinate axes? It does if it is invariant. We have already seen that the scalar product is invariant. Thus the above line integral is invariant. We have also seen that the operator $\boldsymbol{\nabla}$ and the vector product are invariant. Therefore $\boldsymbol{\nabla} \times \boldsymbol{B}$ is invariant. This means that $\boldsymbol{\nabla} \times \boldsymbol{B}$ is a vector whose value, defined by Eq. 1-41, is independent of the particular coordinate axes used, as long as they form a right-handed Cartesian system. Then Eq. 1-41 is indeed invariant; it does apply to any element of area $d\boldsymbol{\mathscr{A}}$, and

$$(\boldsymbol{\nabla} \times \boldsymbol{B})_n = \lim_{\mathscr{A} \to 0} \frac{1}{\mathscr{A}} \oint_C \boldsymbol{B} \cdot d\boldsymbol{l}. \tag{1-42}$$

Thus the component of the curl of a vector normal to a small surface of area $\mathscr{A}$ is equal to the line integral of the vector around the periphery C of the surface, divided by $\mathscr{A}$, when this area approaches zero.

In general, $\boldsymbol{\nabla} \times \boldsymbol{B}$ is *not* normal to $\boldsymbol{B}$. See Prob. 1-7.

The curl of a gradient is identically equal to zero:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = 0. \tag{1-43}$$

STOKES'S THEOREM

surface—any finite surface† bounded by the path of integration in question—into elements of area $d\boldsymbol{\mathscr{A}}_1$, $d\boldsymbol{\mathscr{A}}_2$, and so forth, as in Fig. 1-8. For any one of these small areas,

$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{l} = (\boldsymbol{\nabla} \times \boldsymbol{B}) \cdot d\boldsymbol{\mathscr{A}}. \tag{1-46}$$

We add the left-hand sides of these equations for all the $d\boldsymbol{\mathscr{A}}$'s and then all the right-hand sides. The sum of the left-hand sides is the line integral around the external boundary, since there are always two equal and opposite contributions to the sum along every common side between adjacent $d\boldsymbol{\mathscr{A}}$'s. The sum of the right-hand sides is merely the integral of $(\boldsymbol{\nabla} \times \boldsymbol{B}) \cdot d\boldsymbol{\mathscr{A}}$ over the finite surface. Thus

$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{l} = \int_{\mathscr{A}} (\boldsymbol{\nabla} \times \boldsymbol{B}) \cdot d\boldsymbol{\mathscr{A}}, \tag{1-47}$$

where $\mathscr{A}$ is the area of any open surface bounded by the curve C.

THE LAPLACIAN OPERATOR ∇^2

The divergence of the gradient of f is the *Laplacian* of f:

$$\nabla \cdot \nabla f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}, \tag{1-50}$$

where ∇^2 is the *Laplacian operator.*

The Laplacian is invariant because it is the result of two successive invariant operations.

We have defined the Laplacian of a scalar point-function f. It is also useful to define the Laplacian of a vector point-function $\boldsymbol{B}$:

$$\nabla^2 \boldsymbol{B} = \nabla^2 B_{xx} + \nabla^2 B_{yy} + \nabla^2 B_{zz}. \tag{1-51}$$

ORTHOGONAL CURVILINEAR COORDINATES

It is frequently inconvenient, because of the symmetries that exist in certain fields, to use Cartesian coordinates. Of all the other possible coordinate systems, we shall restrict our discussion to cylindrical and spherical polar coordinates, the two most commonly used.

We could calculate the gradient, the divergence, and so on, directly in both cylindrical and spherical coordinates. However, it is easier and more general to introduce first the idea of orthogonal curvilinear coordinates.

Consider the equation

in which q is a constant. This equation defines a family of surfaces in space, each member characterized by a particular value of the parameter q. An obvious example is $x = q$, which defines surfaces parallel to the yz-plane in Cartesian coordinates.

Consider now three equations

$$f_1(x, y, z) = q_1, \qquad f_2(x, y, z) = q_2, \qquad f_3(x, y, z) = q_3 \tag{1-53}$$

defining three families of surfaces that are mutually orthogonal. The intersection of three of these surfaces, one of each family, then defines a point in space, and q_1, q_2, q_3 are the *orthogonal curvilinear coordinates* of that point, as in Fig. 1-9.

Call dl_1 an element of length normal to the surface q_1. This is the distance between the surfaces q_1 and $q_1 + dq_1$ in the infinitesimal region considered. Then

$$dl_1 = h_1\, dq_1, \tag{1-54}$$

where h_1 is, in general, a function of the coordinates q_1, q_2, q_3. Similarly,

$$dl_2 = h_2\, dq_2 \qquad \text{and} \qquad dl_3 = h_3\, dq_3. \tag{1-55}$$

With Cartesian coordinates h_1, h_2, h_3 are all unity.

The unit vectors $\hat{\boldsymbol{q}}_1, \hat{\boldsymbol{q}}_2, \hat{\boldsymbol{q}}_3$ are normal, respectively, to the q_1, q_2, q_3 surfaces and are oriented toward increasing values of these coordinates. We assign the subscripts 1, 2, 3 to the coordinates in order that $\hat{\boldsymbol{q}}_1 \times \hat{\boldsymbol{q}}_2 = \hat{\boldsymbol{q}}_3$.

The orientations of the three unit vectors vary, in general, with q_1, q_2, q_3. Only in Cartesian coordinates do the unit vectors point in fixed directions.

The volume element is

$$dv = dl_1\, dl_2\, dl_3 = h_1 h_2 h_3 (dq_1\, dq_2\, dq_3). \tag{1-56}$$

We can now find the q's, the h's, the elements of length, and the elements of volume for cylindrical and spherical coordinates.

Cylindrical Coordinates

In cylindrical coordinates, as in Fig. 1-10, $q_1 = \rho$, $q_2 = \phi$, $q_3 = z$.

At P there are three mutually orthogonal directions defined by the three unit vectors $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{z}}$. The unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ do *not* maintain the same directions in space as the point P moves about. However, at any given point, the three unit vectors are mutually orthogonal.

The vector that defines the position of P is

$$\boldsymbol{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\boldsymbol{z}}. \tag{1-57}$$

Note that ϕ does not appear explicitly on the right-hand side; it is specified by the orientation of $\hat{\boldsymbol{\rho}}$.

If the coordinates ϕ and z of the point P remain constant while ρ increases by $d\rho$, then P moves by $\boldsymbol{dr} = d\rho\, \hat{\boldsymbol{\rho}}$. If ρ and z remain constant while ϕ increases by $d\phi$, then $\boldsymbol{dr} = \rho\, d\phi\, \hat{\boldsymbol{\phi}}$. Finally, if ρ and ϕ are fixed while z increases by dz, then $\boldsymbol{dr} = dz\, \hat{\boldsymbol{z}}$. For arbitrary increments $d\rho$, $d\phi$, dz, the distance element is thus

$$\boldsymbol{dr} = d\rho\, \hat{\boldsymbol{\rho}} + \rho\, d\phi\, \hat{\boldsymbol{\phi}} + dz\, \hat{\boldsymbol{z}}. \tag{1-58}$$

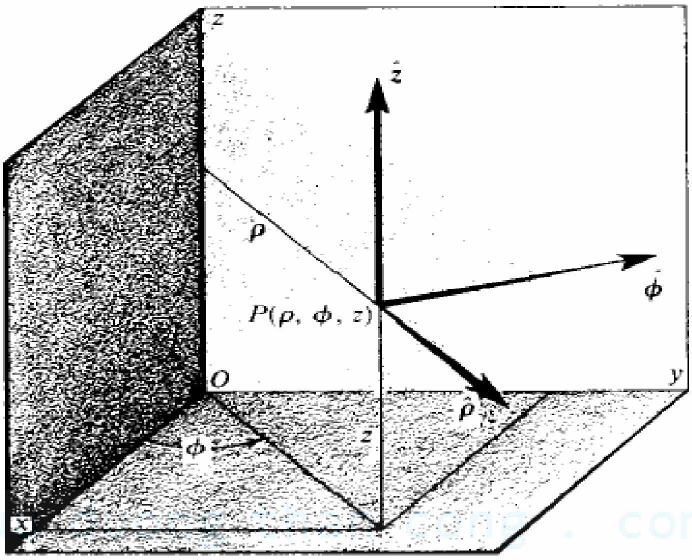

Fig. 1-10. Cylindrical coordinates.

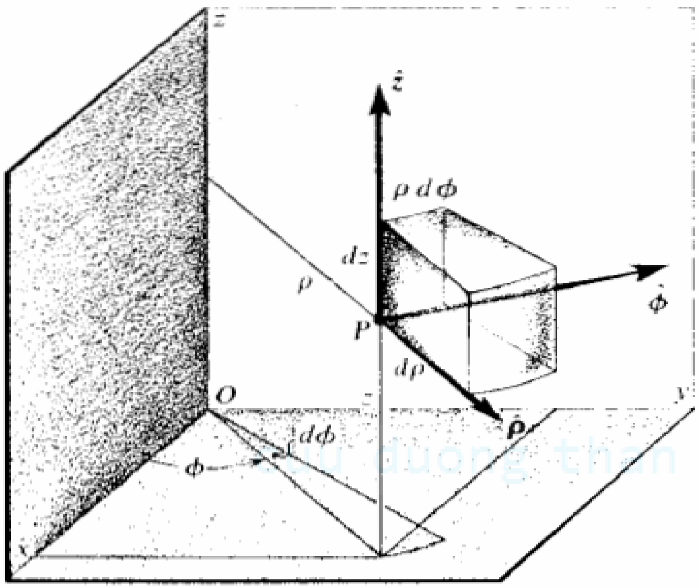

Fig. 1-11. Element of volume in cylindrical coordinates.

Figure 1-11 shows the volume element whose edges are the elements of length corresponding to infinitesimal increments in the coordinates at the point P of Fig. 1-10. The infinitesimal volume is

$$dv = \rho\, d\rho\, d\phi\, dz. \tag{1-59}$$

Spherical Coordinates

In spherical coordinates the position of a point P has the coordinates r, θ, ϕ as in Fig. 1-12. Again, the unit vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$ do not maintain the same orientations in space as P moves about.

The vector $\boldsymbol{r}$ that defines the position of P is now simply $r\hat{\boldsymbol{r}}$, the coordinates θ and ϕ being given by the orientation of $\hat{\boldsymbol{r}}$. Also,

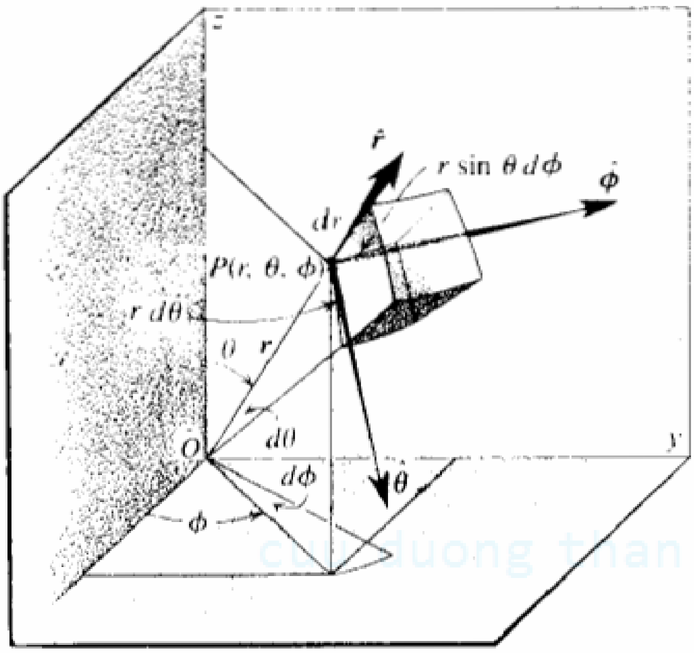

Fig. 1-13. Element of volume in spherical coordinates.

$$\boldsymbol{dr} = dr\,\hat{\boldsymbol{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}. \tag{1-60}$$

The volume element, shown in Fig. 1-13, is

$$dv = r^2 \sin\theta\,dr\,d\theta\,d\phi. \tag{1-61}$$

Table 1-1 shows the correspondence between curvilinear, Cartesian, cylindrical, and spherical coordinates.

Note that the angle ϕ in both cylindrical and spherical coordinates is undefined for points on the z-axis.

With Cartesian coordinates, one uses the operator $\boldsymbol{\nabla}$ for the gradient of a scalar point-function and for the divergence and curl of a vector point-function. A single expression defines $\boldsymbol{\nabla}$, and we obtain the gradient, the divergence, or the curl by performing the appropriate

CURVILINEAR	CARTESIAN	CYLINDRICAL	SPHERICAL
q_1	x	ρ	r
q_2	y	φ	θ
q_3	z	z	φ
h_1	1	1	1
h_2	1	ρ	r
h_3	1	1	$r \sin \theta$
$\hat{\boldsymbol{q}}_1$	$\hat{\boldsymbol{x}}$	$\hat{\boldsymbol{\rho}}$	$\hat{\boldsymbol{r}}$
$\hat{\boldsymbol{q}}_2$	$\hat{\boldsymbol{y}}$	$\hat{\boldsymbol{\varphi}}$	$\hat{\boldsymbol{\theta}}$
$\hat{\boldsymbol{q}}_3$	$\hat{\boldsymbol{z}}$	$\hat{\boldsymbol{z}}$	$\hat{\boldsymbol{\varphi}}$

multiplication. This relatively simple situation is peculiar to the Cartesian coordinate system. With other coordinate systems, the divergence, gradient, and curl do not permit a single definition for $\boldsymbol{\nabla}$ but require more elaborate expressions that we shall now derive.

The Gradient

The gradient is the vector rate of change of a scalar function f:

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial l_1} \hat{\boldsymbol{q}}_1 + \frac{\partial f}{\partial l_2} \hat{\boldsymbol{q}}_2 + \frac{\partial f}{\partial l_3} \hat{\boldsymbol{q}}_3 \tag{1-62}$$

$$= \frac{1}{h_1} \frac{\partial f}{\partial q_1} \hat{\boldsymbol{q}}_1 + \frac{1}{h_2} \frac{\partial f}{\partial q_2} \hat{\boldsymbol{q}}_2 + \frac{1}{h_3} \frac{\partial f}{\partial q_3} \hat{\boldsymbol{q}}_3. \tag{1-63}$$

For cylindrical coordinates, then,

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial \rho} \hat{\boldsymbol{\rho}} + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z} \hat{\boldsymbol{z}}. \tag{1-64}$$

With spherical coordinates,

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial r} \hat{\boldsymbol{r}} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}}. \tag{1-65}$$

On the z-axis, ϕ is undefined and both ρ and $\sin \theta$ are zero, so these two expressions are meaningless.

The Divergence

To find the divergence, consider the volume element of Fig. 1-14. The quantity B_1 is the q_1 component of $\boldsymbol{B}$ at the center, and h_1, h_2, h_3 are the h values at that point. Since the faces are mutually orthogonal, the outward flux through the left-hand face is

$$d\Phi_L = -B_{1L}h_{2L}h_{3L}\, dq_2\, dq_3 \tag{1-66}$$

$$= -\left(B_1 - \frac{\partial B_1}{\partial q_1}\frac{dq_1}{2}\right)\left(h_2 - \frac{\partial h_2}{\partial q_1}\frac{dq_1}{2}\right)\left(h_3 - \frac{\partial h_3}{\partial q_1}\frac{dq_1}{2}\right) dq_2\, dq_3. \tag{1-67}$$

Remember that h_2 and h_3 may be functions of q_1, just as B_1. We may neglect differentials of order higher than the third, and then

$$d\Phi_L = -B_1 h_2 h_3\, dq_2\, dq_3 + \frac{\partial}{\partial q_1}(B_1 h_2 h_3)\frac{dq_1}{2}\, dq_2\, dq_3. \tag{1-68}$$

By a similar argument,

$$d\Phi_R = B_1 h_2 h_3\, dq_2\, dq_3 + \frac{\partial}{\partial q_1}(B_1 h_2 h_3)\frac{dq_1}{2}\, dq_2\, dq_3 \tag{1-69}$$

for the right-hand face. The net flux through this pair of faces is then

$$d\Phi_{LR} = \frac{\partial}{\partial q_1}(B_1 h_2 h_3)\, dq_1\, dq_2\, dq_3. \tag{1-70}$$

If we repeat the calculation for the other pairs of faces to find the net outward flux through the bounding surface and then divide by the volume of the element, we obtain the divergence:

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(B_1 h_2 h_3) + \frac{\partial}{\partial q_2}(B_2 h_3 h_1) + \frac{\partial}{\partial q_3}(B_3 h_1 h_2)\right]. \tag{1-71}$$

In cylindrical coordinates,

$$\nabla \cdot \boldsymbol{B} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\rho) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} \tag{1-72}$$

$$= \frac{B_\rho}{\rho} + \frac{\partial B_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z}. \tag{1-73}$$

In spherical coordinates,

$$\nabla \cdot \boldsymbol{B} = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(B_r r^2 \sin\theta) + \frac{\partial}{\partial \theta}(B_\theta r \sin\theta) + \frac{\partial}{\partial \phi}(B_\phi r)\right] \tag{1-74}$$

$$= \frac{2}{r}B_r + \frac{\partial B_r}{\partial r} + \frac{B_\theta}{r}\cot\theta + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial B_\phi}{\partial \phi}. \tag{1-75}$$

These divergences are also meaningless on the z-axis, where ρ and $\sin\theta$ are both zero.

The Curl

We apply the fundamental definition given in Eq. 1-42:

$$(\nabla \times \boldsymbol{B})_1 = \lim_{\mathscr{A} \to 0} \frac{1}{\mathscr{A}} \oint \boldsymbol{B} \cdot \boldsymbol{dl}, \tag{1-76}$$

where the path of integration C lies in the surface $q_1 = \text{constant}$ and where the direction of integration relates to the direction of the unit vector $\hat{\boldsymbol{q}}_1$ by the right-hand screw rule. For the paths labeled a, b, c, d in Fig. 1-15, we have the following contributions to the line integral:

$$-B_3 h_3\, dq_3,$$

$$\left(B_3 + \frac{\partial B_3}{\partial q_2} dq_2\right)\left(h_3 + \frac{\partial h_3}{\partial q_2} dq_2\right) dq_3,$$

$$+B_2 h_2\, dq_2,$$

$$-\left(B_2 + \frac{\partial B_2}{\partial q_3} dq_3\right)\left(h_2 + \frac{\partial h_2}{\partial q_3} dq_3\right) dq_2.$$

The sum of these four terms, divided by the element of area is equal to the 1-component of the curl of $\boldsymbol{B}$. Neglecting higher-order differentials,

$$(\nabla \times \boldsymbol{B})_1 = \frac{1}{h_2 h_3\, dq_2\, dq_3}\left[\frac{\partial}{\partial q_2}(B_3 h_3)\, dq_2\, dq_3 - \frac{\partial}{\partial q_3}(B_2 h_2)\, dq_2\, dq_3\right] \tag{1-77}$$

$$= \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial q_2}(B_3 h_3) - \frac{\partial}{\partial q_3}(B_2 h_2)\right]. \tag{1-78}$$

Corresponding expressions for the other components of the curl follow by rotating the subscripts. Finally,

$$\nabla \times \boldsymbol{B} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{\boldsymbol{q}}_1 & h_2 \hat{\boldsymbol{q}}_2 & h_3 \hat{\boldsymbol{q}}_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ h_1 B_1 & h_2 B_2 & h_3 B_3 \end{vmatrix}. \tag{1-79}$$

For cylindrical coordinates,

$$\nabla \times \boldsymbol{B} = \frac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \hat{\boldsymbol{z}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ B_\rho & \rho B_\phi & B_z \end{vmatrix}, \tag{1-80}$$

and for spherical coordinates

$$\nabla \times \boldsymbol{B} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{\boldsymbol{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ B_r & rB_\theta & r\sin\theta B_\phi \end{vmatrix}. \tag{1-81}$$

These definitions are not valid on the z-axis.

The Laplacian

We calculate the Laplacian of a scalar function f in curvilinear coordinates by combining the expressions for the divergence and for the gradient:

$$\nabla^2 f = \nabla \cdot \nabla f \tag{1-82}$$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial f}{\partial q_2}\right)\right.$$

$$\left. + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right]. \tag{1-83}$$

For cylindrical coordinates,

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2} \tag{1-84}$$

$$= \frac{1}{\rho}\frac{\partial f}{\partial\rho} + \frac{\partial^2 f}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}, \tag{1-85}$$

except on the z-axis. For spherical coordinates,

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} \tag{1-86}$$

$$= \frac{2}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} + \frac{\cot\theta}{r^2}\frac{\partial f}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}, \tag{1-87}$$

except, again, on the z-axis.

We have already seen in Sec. 1.10 that the Laplacian of a vector $\boldsymbol{B}$ in *Cartesian* coordinates is itself a vector whose components are the Laplacians of B_x, B_y, B_z. Then

$$\nabla\times(\nabla\times\boldsymbol{B}) = \nabla(\nabla\cdot\boldsymbol{B}) - \nabla^2\boldsymbol{B} \tag{1-88}$$

is an identity in Cartesian coordinates.

With other coordinates, $\nabla^2\boldsymbol{B}$ is, by definition, the vector whose components are those of $\nabla(\nabla\cdot\boldsymbol{B}) - \nabla\times(\nabla\times\boldsymbol{B})$, and *not* the sum of the Laplacians of B_1, B_2, B_3:

$$\nabla^2\boldsymbol{B} = \nabla(\nabla\cdot\boldsymbol{B}) - \nabla\times(\nabla\times\boldsymbol{B}). \tag{1-89}$$

CHƯƠNG 2. HÀM BIẾN PHỨC

THE DERIVATIVE IN THE COMPLEX PLANE: THE CAUCHY-RIEMANN EQUATIONS

We have already introduced the complex variable $z = x + iy$, where x and y are variable. We now define another complex variable $w = u + iv$ so that for each value of z there corresponds a value of $w = f(z)$. From all of the possible complex functions that we might invent, we will focus on those functions where for each z there is one, and only one, value of w. These functions are *single-valued*. They differ from functions such

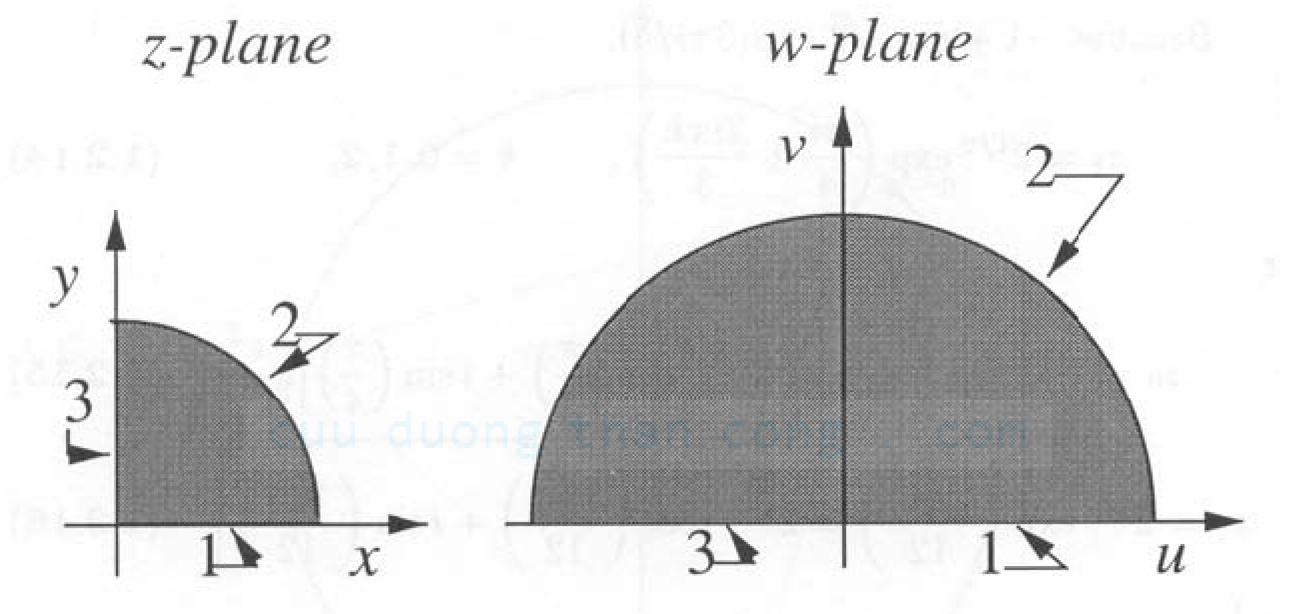

Figure 1.3.1: The complex function $w = z^2$.

as the square root, logarithm, and inverse sine and cosine, where there are multiple answers for each z. These *multivalued functions* do arise in various problems. However, they are beyond the scope of this book and we shall always assume that we are dealing with single-valued functions.

A popular method for representing a complex function involves drawing some closed domain in the z-plane and then showing the corresponding domain in the w-plane. This procedure is called *mapping* and the z-plane illustrates the *domain* of the function while the w-plane illustrates its *image* or *range*. Figure 1.3.1 shows the z-plane and w-plane for $w = z^2$; a pie-shaped wedge in the z-plane maps into a semicircle on the w-plane.

Although the requirement that a complex function be single-valued is important, it is still too general and would cover all functions of two real variables. To have a useful theory, we must introduce additional constraints. Because an important property associated with most functions is the ability to take their derivative, let us examine the derivative in the complex plane.

Following the definition of a derivative for a single real variable, the derivative of a complex function $w = f(z)$ is defined as

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}. \tag{1.3.7}$$

A function of a complex variable that has a derivative at every point within a region of the complex plane is said to be *analytic* (or *regular* or *holomorphic*) over that region. If the function is analytic everywhere in the complex plane, it is *entire*.

Because the derivative is defined as a limit and limits are well behaved with respect to elementary algebraic operations, the following operations carry over from elementary calculus:

$$\frac{d}{dz}\left[cf(z)\right] = cf'(z), \qquad c \text{ a constant} \tag{1.3.8}$$

$$\frac{d}{dz}\left[f(z) \pm g(z)\right] = f'(z) \pm g'(z) \tag{1.3.9}$$

$$\frac{d}{dz}\left[f(z)g(z)\right] = f'(z)g(z) + f(z)g'(z) \tag{1.3.10}$$

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)f'(z) - g'(z)f(z)}{g^2(z)} \tag{1.3.11}$$

$$\frac{d}{dz}\left\{f[g(z)]\right\} = f'[g(z)]g'(z), \qquad \text{the chain rule.} \tag{1.3.12}$$

Another important property that carries over from real variables is l'Hôspital rule: Let $f(z)$ and $g(z)$ be analytic at z_0, where $f(z)$ has a zero[1] of order m and $g(z)$ has a zero of order n. Then, if $m > n$,

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = 0; \tag{1.3.13}$$

if $m = n$,

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f^{(m)}(z_0)}{g^{(m)}(z_0)} \tag{1.3.14}$$

and if $m < n$,

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \infty. \tag{1.3.15}$$

So far we have introduced the derivative and some of its properties. But how do we actually know whether a function is analytic or how do we compute its derivative? At this point we must develop some relationships involving the known quantities $u(x, y)$ and $v(x, y)$.

We begin by returning to the definition of the derivative. Because $\Delta z = \Delta x + i\Delta y$, there is an infinite number of different ways of approaching the limit $\Delta z \to 0$. Uniqueness of that limit requires that (1.3.7) must be independent of the manner in which Δz approaches zero. A simple example is to take Δz in the x-direction so that $\Delta z = \Delta x$; another is to take Δz in the y-direction so that $\Delta z = i\Delta y$. These examples yield

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta x \to 0} \frac{\Delta u + i\Delta v}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{1.3.17}$$

and

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta y \to 0} \frac{\Delta u + i\Delta v}{i\Delta y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \tag{1.3.18}$$

In both cases we are approaching zero from the positive side. For the limit to be unique and independent of path, (1.3.17) must equal (1.3.18), or

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.} \tag{1.3.19}$$

These equations which u and v must both satisfy are the *Cauchy-Riemann* equations. They are necessary but not sufficient to ensure that a function is differentiable. The following example will illustrate this.

Consider the complex function

$$w = \begin{cases} z^5/|z|^4, & z \neq 0 \\ 0, & z = 0. \end{cases} \tag{1.3.20}$$

The derivative at $z = 0$ is given by

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{(\Delta z)^5/|\Delta z|^4 - 0}{\Delta z} = \lim_{\Delta z \to 0} \frac{(\Delta z)^4}{|\Delta z|^4}, \tag{1.3.21}$$

provided that this limit exists. However, this limit does not exist because, in general, the numerator depends upon the path used to approach zero. For example, if $\Delta z = re^{\pi i/4}$ with $r \to 0$, $dw/dz = -1$. On the other hand, if $\Delta z = re^{\pi i/2}$ with $r \to 0$, $dw/dz = 1$.

Are the Cauchy-Riemann equations satisfied in this case? To check this, we first compute

$$u_x(0,0) = \lim_{\Delta x \to 0} \left(\frac{\Delta x}{|\Delta x|}\right)^4 = 1, \tag{1.3.22}$$

$$v_y(0,0) = \lim_{\Delta y \to 0} \left(\frac{i\Delta y}{|\Delta y|}\right)^4 = 1, \tag{1.3.23}$$

$$u_y(0,0) = \lim_{\Delta y \to 0} \operatorname{Re}\left[\frac{(i\Delta y)^5}{\Delta y|\Delta y|^4}\right] = 0 \tag{1.3.24}$$

and

$$v_x(0,0) = \lim_{\Delta x \to 0} \operatorname{Im}\left[\left(\frac{\Delta x}{|\Delta x|}\right)^4\right] = 0. \tag{1.3.25}$$

Hence, the Cauchy-Riemann equations are satisfied at the origin. Thus, even though the derivative is not uniquely defined, (1.3.21) happens to have the same value for paths taken along the coordinate axes so that the Cauchy-Riemann equations are satisfied.

In summary, if a function is differentiable at a point, the Cauchy-Riemann equations hold. Similarly, if the Cauchy-Riemann equations are not satisfied at a point, then the function is not differentiable at that point. This is one of the important uses of the Cauchy-Riemann equations: the location of nonanalytic points. Isolated nonanalytic points of an otherwise analytic function are called *isolated singularities*. Functions that contain isolated singularities are called *meromorphic*.

The Cauchy-Riemann condition can be modified so that it is a sufficient condition for the derivative to exist. Let us require that u_x, u_y, v_x, and v_y be continuous in some region surrounding a point z_0 and satisfy the Cauchy-Riemann conditions there. Then

$$\begin{aligned} f(z) - f(z_0) &= [u(z) - u(z_0)] + i[v(z) - v(z_0)] && (1.3.26) \\ &= [u_x(z_0)(x - x_0) + u_y(z_0)(y - y_0) \\ &\qquad + \epsilon_1(x - x_0) + \epsilon_2(y - y_0)] \\ &+ i[v_x(z_0)(x - x_0) + v_y(z_0)(y - y_0) \\ &\qquad + \epsilon_3(x - x_0) + \epsilon_4(y - y_0)] && (1.3.27) \\ &= [u_x(z_0) + iv_x(z_0)](z - z_0) \\ &+ (\epsilon_1 + i\epsilon_3)(x - x_0) + (\epsilon_2 + i\epsilon_4)(y - y_0), && (1.3.28) \end{aligned}$$

where we have used the Cauchy-Riemann equations and $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \longrightarrow 0$ as $\Delta x, \Delta y \longrightarrow 0$. Hence,

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z) - f(z_0)}{\Delta z} = u_x(z_0) + iv_x(z_0), \tag{1.3.29}$$

because $|\Delta x| \leq |\Delta z|$ and $|\Delta y| \leq |\Delta z|$. Using (1.3.29) and the Cauchy-Riemann equations, we can obtain the derivative from any of the following formulas:

$$\boxed{\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}} \tag{1.3.30}$$

and

$$\boxed{\frac{dw}{dz} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}.} \tag{1.3.31}$$

Furthermore, $f'(z_0)$ is continuous because the partial derivatives are.

Let us show that $\sin(z)$ is an entire function.

$$w = \sin(z) \tag{1.3.32}$$

$$u + iv = \sin(x + iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy) \tag{1.3.33}$$

$$= \sin(x)\cosh(y) + i\cos(x)\sinh(y), \tag{1.3.34}$$

because

$$\cos(iy) = \tfrac{1}{2}\left[e^{i(iy)} + e^{-i(iy)}\right] = \tfrac{1}{2}\left[e^{y} + e^{-y}\right] = \cosh(y) \tag{1.3.35}$$

and

$$\sin(iy) = \tfrac{1}{2i}\left[e^{i(iy)} - e^{-i(iy)}\right] = -\tfrac{1}{2i}\left[e^{y} - e^{-y}\right] = i\sinh(y) \tag{1.3.36}$$

so that

$$u(x,y) = \sin(x)\cosh(y) \quad \text{and} \quad v(x,y) = \cos(x)\sinh(y). \tag{1.3.37}$$

Differentiating both $u(x,y)$ and $v(x,y)$ with respect to x and y, we have that

$$\frac{\partial u}{\partial x} = \cos(x)\cosh(y) \qquad \frac{\partial u}{\partial y} = \sin(x)\sinh(y) \tag{1.3.38}$$

$$\frac{\partial v}{\partial x} = -\sin(x)\sinh(y) \qquad \frac{\partial v}{\partial y} = \cos(x)\cosh(y) \tag{1.3.39}$$

and $u(x,y)$ and $v(x,y)$ satisfy the Cauchy-Riemann equations for all values of x and y. Furthermore, u_x, u_y, v_x, and v_y are continuous for all x and y. Therefore, the function $w = \sin(z)$ is an entire function.

Let us find the derivative of $\sin(z)$.
Using (1.3.30) and (1.3.34),

$$\frac{d}{dz}\left[\sin(z)\right] = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{1.3.46}$$

$$= \cos(x)\cosh(y) - i\sin(x)\sinh(y) \tag{1.3.47}$$

$$= \cos(x+iy) = \cos(z). \tag{1.3.48}$$

Similarly,

$$\frac{d}{dz}\left(\frac{1}{z}\right) = \frac{y^2 - x^2}{(x^2+y^2)^2} + \frac{2ixy}{(x^2+y^2)^2} \tag{1.3.49}$$

$$= -\frac{1}{(x+iy)^2} = -\frac{1}{z^2}. \tag{1.3.50}$$

The results in the above examples are identical to those for z real. As we showed earlier, the fundamental rules of elementary calculus apply to complex differentiation. Consequently, it is usually simpler to apply those rules to find the derivative rather than breaking $f(z)$ down into its real and imaginary parts, applying either (1.3.30) or (1.3.31), and then putting everything back together.

An additional property of analytic functions follows by cross differentiating the Cauchy-Riemann equations or

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2} \quad \text{or} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1.3.51}$$

and

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial y^2} \quad \text{or} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{1.3.52}$$

Any function that has continuous partial derivatives of second order and satisfies Laplace's equation (1.3.51) or (1.3.52) is called a *harmonic function*. Because both $u(x,y)$ and $v(x,y)$ satisfy Laplace's equation if $f(z) = u + iv$ is analytic, $u(x,y)$ and $v(x,y)$ are called *conjugate harmonic functions*.

LINE INTEGRALS

So far, we discussed complex numbers, complex functions, and complex differentiation. We are now ready for integration.

Just as we have integrals involving real variables, we can define an integral that involves complex variables. Because the z-plane is two-dimensional there is clearly greater freedom in what we mean by a complex integral. For example, we might ask whether the integral of some function between points A and B depends upon the curve along which we integrate. (In general it does.) Consequently, an important ingredient in any complex integration is the *contour* that we follow during the integration.

The result of a line integral is a complex number or expression. Unlike its counterpart in real variables, there is no physical interpretation for this quantity, such as area under a curve. Generally, integration in the complex plane is an intermediate process with a physically realizable quantity occurring only after we take its real or imaginary part. For example, in potential fluid flow, the lift and drag are found by taking the real and imaginary part of a complex integral, respectively.

How do we compute $\int_C f(z)\,dz$? Let us deal with the definition; we will illustrate the actual method by examples.

A popular method for evaluating complex line integrals consists of breaking everything up into real and imaginary parts. This reduces the integral to line integrals of real-valued functions which we know how to handle. Thus, we write $f(z) = u(x,y) + iv(x,y)$ as usual, and because $z = x + iy$, formally $dz = dx + i\,dy$. Therefore,

$$\int_C f(z)\,dz = \int_C [u(x,y) + iv(x,y)][dx + i\,dy] \tag{1.4.1}$$

$$= \int_C u(x,y)\,dx - v(x,y)\,dy + i\int_C v(x,y)\,dx + u(x,y)\,dy. \tag{1.4.2}$$

The exact method used to evaluate (1.4.2) depends upon the exact path specified.

From the definition of the line integral, we have the following self-evident properties:

$$\int_C f(z)\,dz = -\int_{C'} f(z)\,dz, \tag{1.4.3}$$

where C' is the contour C taken in the opposite direction of C and

$$\int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz. \tag{1.4.4}$$

THE CAUCHY-GOURSAT THEOREM

In the previous section we showed how to evaluate line integrations by brute-force reduction to real-valued integrals. In general, this direct approach is quite difficult and we would like to apply some of the deeper properties of complex analysis to work smarter. In the remaining portions of this chapter we will introduce several theorems that will do just that.

If we scan over the examples worked in the previous section, we see considerable differences when the function was analytic inside and on the contour and when it was not. We may formalize this anecdotal evidence into the following theorem:

Cauchy-Goursat theorem[2]: *Let $f(z)$ be analytic in a domain D and let C be a simple Jordan curve*[3] *inside D so that $f(z)$ is analytic on and inside of C. Then $\oint_C f(z)\,dz = 0$.*

Proof: Let C denote the contour around which we will integrate $w = f(z)$. We divide the region within C into a series of infinitesimal rectangles. See Figure 1.5.1. The integration around each rectangle equals the product of the average value of w on each side and its length,

$$\begin{aligned}
&\left[w + \frac{\partial w}{\partial x}\frac{dx}{2}\right] dx + \left[w + \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial (iy)}\frac{d(iy)}{2}\right] d(iy) \\
&\quad + \left[w + \frac{\partial w}{\partial x}\frac{dx}{2} + \frac{\partial w}{\partial (iy)}d(iy)\right] (-dx) + \left[w + \frac{\partial w}{\partial (iy)}\frac{d(iy)}{2}\right] d(-iy) \\
&\quad = \left(\frac{\partial w}{\partial x} - \frac{\partial w}{i\partial y}\right)(i\,dx\,dy) \qquad \textbf{(1.5.1)}
\end{aligned}$$

Substituting $w = u + iv$ into (1.5.1),

$$\frac{\partial w}{\partial x} - \frac{\partial w}{i\,\partial y} = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right). \qquad \textbf{(1.5.2)}$$

Because the function is analytic, the right side of (1.5.1) and (1.5.2) equals zero. Thus, the integration around each of these rectangles also equals zero.

We note next that in integrating around adjoining rectangles we transverse each side in opposite directions, the net result being equivalent to integrating around the outer curve C. We therefore arrive at the result $\oint_C f(z)\,dz = 0$, where $f(z)$ is analytic within and on the closed contour. □

The Cauchy-Goursat theorem has several useful implications. Suppose we have a domain where $f(z)$ is analytic. Within this domain let us evaluate a line integral from point A to B along two different contours C_1 and C_2. Then, the integral around the closed contour formed by integrating along C_1 and then back along C_2, only in the opposite direction, is

$$\oint_C f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0 \tag{1.5.3}$$

or

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz. \tag{1.5.4}$$

Because C_1 and C_2 are completely arbitrary, we have the result that if, in a domain, $f(z)$ is analytic, the integral between any two points within the domain is *path independent.*

One obvious advantage of path independence is the ability to choose the contour so that the computations are made easier. This obvious choice immediately leads to

The principle of deformation of contours: *The value of a line integral of an analytic function around any simple closed contour remains unchanged if we deform the contour in such a manner that we do not pass over a nonanalytic point.*

Finally, suppose that we have a function $f(z)$ such that $f(z)$ is analytic in some domain. Furthermore, let us introduce the analytic function $F(z)$ such that $f(z) = F'(z)$. We would like to evaluate $\int_a^b f(z)\,dz$ in terms of $F(z)$.

We begin by noting that we can represent F, f as $F(z) = U + iV$ and $f(z) = u + iv$. From (1.3.30) we have that $u = U_x$ and $v = V_x$.

Therefore,

$$\int_a^b f(z)\,dz = \int_a^b (u+iv)(dx+i\,dy) \tag{1.5.6}$$

$$= \int_a^b U_x\,dx - V_x\,dy + i\int_a^b V_x\,dx + U_x\,dy \tag{1.5.7}$$

$$= \int_a^b U_x\,dx + U_y\,dy + i\int_a^b V_x\,dx + V_y\,dy \tag{1.5.8}$$

$$= \int_a^b dU + i\int_a^b dV = F(b) - F(a) \tag{1.5.9}$$

or

$$\int_a^b f(z)\,dz = F(b) - F(a). \tag{1.5.10}$$

Equation (1.5.10) is the complex variable form of the fundamental theorem of calculus. Thus, if we can find the antiderivative of a function $f(z)$ that is analytic within a specific region, we can evaluate the integral by evaluating the antiderivative at the endpoints for any curves within that region.

CAUCHY'S INTEGRAL FORMULA

In the previous section, our examples suggested that the presence of a singularity within a contour really determines the value of a closed contour integral. Continuing with this idea, let us consider a class of closed contour integrals that explicitly contain a single singularity within the contour, namely $\oint_C g(z)\,dz$, where $g(z) = f(z)/(z-z_0)$ and $f(z)$ is analytic within and on the contour C. We have closed the contour in the *positive sense* where the enclosed area lies to your left as you move along the contour.

We begin by examining a closed contour integral where the closed contour consists of the C_1, C_2, C_3, and C_4 as shown in Figure 1.6.1. The gap or cut between C_2 and C_4 is very small. Because $g(z)$ is analytic within and on the closed integral, we have that

$$\int_{C_1} \frac{f(z)}{z-z_0}\,dz + \int_{C_2} \frac{f(z)}{z-z_0}\,dz + \int_{C_3} \frac{f(z)}{z-z_0}\,dz + \int_{C_4} \frac{f(z)}{z-z_0}\,dz = 0. \tag{1.6.1}$$

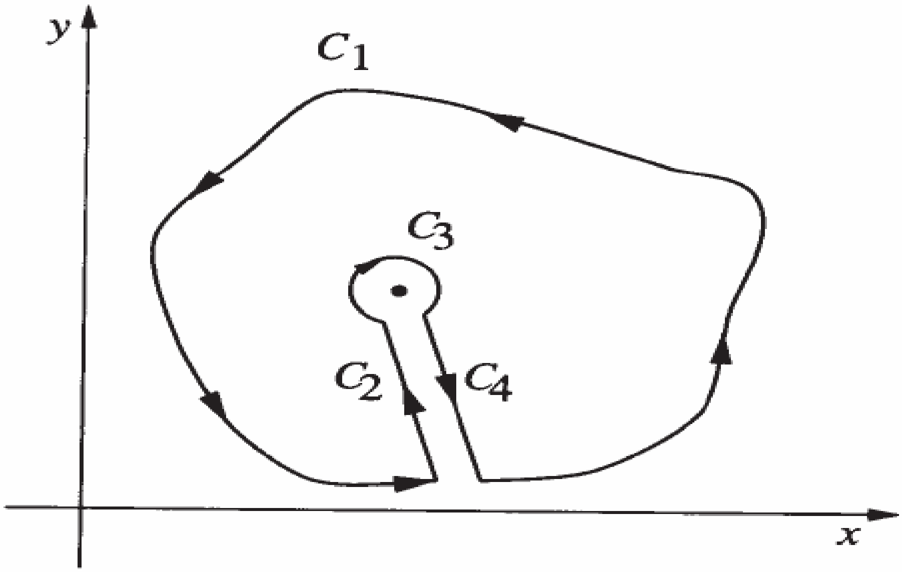

Figure 1.6.1: Diagram used to prove Cauchy's integral formula.

It can be shown that the contribution to the integral from the path C_2 going into the singularity will cancel the contribution from the path C_4 going away from the singularity as the gap between them vanishes. Because $f(z)$ is analytic at z_0, we can approximate its value on C_3 by $f(z) = f(z_0) + \delta(z)$, where δ is a small quantity. Substituting into (1.6.1),

$$\oint_{C_1} \frac{f(z)}{z - z_0}\, dz = -f(z_0) \int_{C_3} \frac{1}{z - z_0}\, dz - \int_{C_3} \frac{\delta(z)}{z - z_0}\, dz. \qquad \textbf{(1.6.2)}$$

Consequently, as the gap between C_2 and C_4 vanishes, the contour C_1 becomes the closed contour C so that (1.6.2) may be written

$$\oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0) + i \int_0^{2\pi} \delta\, d\theta, \qquad \textbf{(1.6.3)}$$

where we have set $z - z_0 = \epsilon e^{\theta i}$ and $dz = i\epsilon e^{\theta i} d\theta$.

Let M denote the value of the integral on the right side of (1.6.3) and Δ equal the greatest value of the modulus of δ along the circle.

Then

$$|M| < \int_0^{2\pi} |\delta| d\theta \le \int_0^{2\pi} \Delta d\theta = 2\pi\Delta. \tag{1.6.4}$$

As the radius of the circle diminishes to zero, Δ also diminishes to zero. Therefore, $|M|$, which is positive, becomes less than any finite quantity, however small, and M itself equals zero. Thus, we have that

$$\boxed{f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz.} \tag{1.6.5}$$

This equation is *Cauchy's integral formula.* By taking n derivatives of (1.6.5), we can extend Cauchy's integral formula[4] to

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz} \tag{1.6.6}$$

for $n = 1, 2, 3, \ldots$. For computing integrals, it is convenient to rewrite (1.6.6) as

$$\oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0). \tag{1.6.7}$$

TAYLOR AND LAURENT EXPANSIONS AND SINGULARITIES

In the previous section we showed what a crucial role singularities play in complex integration. Before we can find the most general way of computing a closed complex integral, our understanding of singularities must deepen. For this, we employ power series.

One reason why power series are so important is their ability to provide locally a general representation of a function even when its arguments are complex. For example, when we were introduced to trigonometric functions in high school, it was in the context of a right triangle and a real angle. However, when the argument becomes complex this geometrical description disappears and power series provide a formalism for defining the trigonometric functions, regardless of the nature of the argument.

Let us begin our analysis by considering the complex function $f(z)$ which is analytic everywhere on the boundary and the interior of a circle whose center is at $z = z_0$. Then, if z denotes any point within the circle, we have from Cauchy's integral formula that

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z_0} \left[\frac{1}{1 - (z - z_0)/(\zeta - z_0)} \right] d\zeta, \tag{1.7.1}$$

where C denotes the closed contour. Expanding the bracketed term as a geometric series, we find that

$$f(z) = \frac{1}{2\pi i} \left[\oint_C \frac{f(\zeta)}{\zeta - z_0}\, d\zeta + (z - z_0) \oint_C \frac{f(\zeta)}{(\zeta - z_0)^2}\, d\zeta + \cdots \right.$$
$$\left. + (z - z_0)^n \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta + \cdots \right]. \tag{1.7.2}$$

Applying Cauchy's integral formula to each integral in (1.7.2), we finally obtain

$$f(z) = f(z_0) + \frac{(z - z_0)}{1!} f'(z_0) + \cdots + \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) + \cdots \tag{1.7.3}$$

or the familiar formula for a Taylor expansion. Consequently, *we can expand any analytic function into a Taylor series.* Interestingly, the radius of convergence[5] of this series may be shown to be the distance between z_0 and the nearest nonanalytic point of $f(z)$.

Consider now the situation where we draw two concentric circles about some arbitrary point z_0; we denote the outer circle by C while we denote the inner circle by C_1. See Figure 1.7.1. Let us assume that $f(z)$ is analytic inside the annulus between the two circles. Outside of this area, the function may or may not be analytic. Within the annulus we pick a point z and construct a small circle around it, denoting the circle by C_2. As the gap or *cut* in the annulus becomes infinitesimally small, the line integrals that connect the circle C_2 to C_1 and C sum to zero, leaving

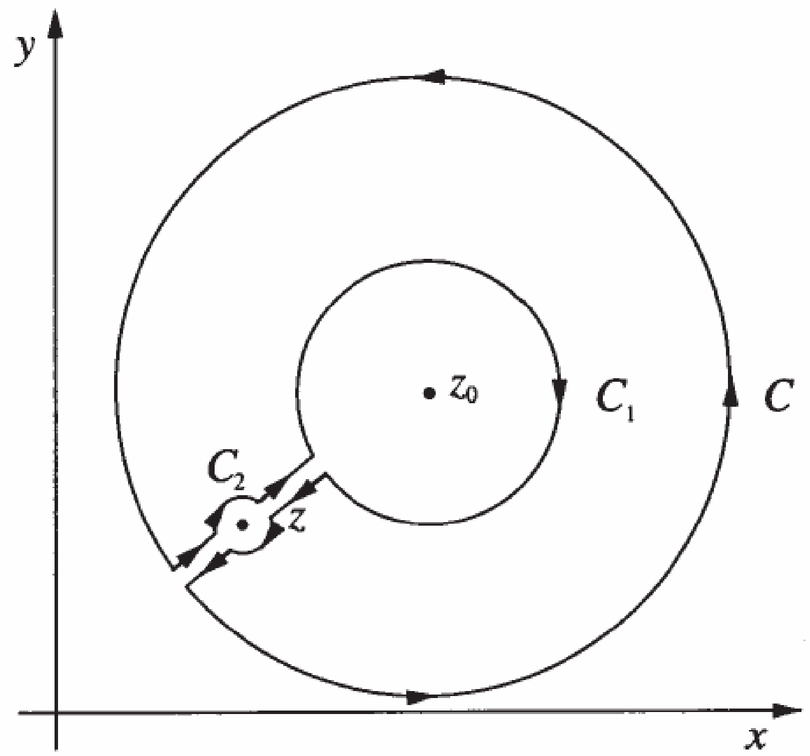

Figure 1.7.1: Contour used in deriving the Laurent expansion.

$$\oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta = \oint_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \oint_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{1.7.8}$$

Because $f(\zeta)$ is analytic everywhere within C_2,

$$2\pi i f(z) = \oint_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{1.7.9}$$

Using the relationship:

$$\oint_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta = -\oint_{C_1} \frac{f(\zeta)}{z - \zeta}\, d\zeta, \tag{1.7.10}$$

(1.7.8) becomes

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i} \oint_{C_1} \frac{f(\zeta)}{z - \zeta}\, d\zeta. \tag{1.7.11}$$

Now,

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - z + z_0} = \frac{1}{\zeta - z_0}\frac{1}{1-(z-z_0)/(\zeta - z_0)} \tag{1.7.12}$$

$$= \frac{1}{\zeta - z_0}\left[1 + \left(\frac{z-z_0}{\zeta - z_0}\right) + \left(\frac{z-z_0}{\zeta - z_0}\right)^2 + \cdots + \left(\frac{z-z_0}{\zeta - z_0}\right)^n + \cdots\right], \tag{1.7.13}$$

where $|z - z_0|/|\zeta - z_0| < 1$ and

$$\frac{1}{z - \zeta} = \frac{1}{z - z_0 - \zeta + z_0} = \frac{1}{z - z_0}\frac{1}{1-(\zeta-z_0)/(z - z_0)} \tag{1.7.14}$$

$$= \frac{1}{z - z_0}\left[1 + \left(\frac{\zeta-z_0}{z - z_0}\right) + \left(\frac{\zeta-z_0}{z - z_0}\right)^2 + \cdots + \left(\frac{\zeta-z_0}{z - z_0}\right)^n + \cdots\right], \tag{1.7.15}$$

where $|\zeta - z_0|/|z - z_0| < 1$. Upon substituting these expressions into (1.7.11),

$$\begin{aligned} f(z) = {} & \left[\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z_0}\,d\zeta + \frac{z - z_0}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z_0)^2}\,d\zeta + \cdots \right.\\ & \left. + \frac{(z-z_0)^n}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,d\zeta + \cdots\right] \\ & + \left[\frac{1}{z-z_0}\frac{1}{2\pi i}\oint_{C_1} f(\zeta)\,d\zeta + \frac{1}{(z-z_0)^2}\frac{1}{2\pi i}\oint_{C_1} f(\zeta)(\zeta - z_0)\,d\zeta + \cdots \right.\\ & \left. + \frac{1}{(z-z_0)^n}\frac{1}{2\pi i}\oint_{C_1} f(\zeta)(\zeta - z_0)^{n-1}\,d\zeta + \cdots\right] \end{aligned} \tag{1.7.16}$$

or

$$\begin{aligned} f(z) = {} & \frac{a_1}{z - z_0} + \frac{a_2}{(z-z_0)^2} + \cdots + \frac{a_n}{(z-z_0)^n} + \cdots \\ & + b_0 + b_1(z - z_0) + \cdots + b_n(z-z_0)^n + \cdots \end{aligned} \tag{1.7.17}$$

Equation (1.7.17) is a *Laurent expansion*.[6] If $f(z)$ is analytic at z_0, then $a_1 = a_2 = \cdots = a_n = \cdots = 0$ and the Laurent expansion reduces to a Taylor expansion. If z_0 is a singularity of $f(z)$, then the Laurent expansion will include both positive and *negative* powers. The coefficient of the $(z - z_0)^{-1}$ term, a_1, is the *residue*, for reasons that will appear in the next section.

Unlike the Taylor series, there is no straightforward method for obtaining a Laurent series. For the remaining portions of this section we will illustrate their construction. These techniques include replacing a function by its appropriate power series, the use of geometric series to expand the denominator, and the use of algebraic tricks to assist in applying the first two method.

Removable Singularity

Pole of order n

For complicated complex functions, it is very difficult to determine the nature of the singularities by finding the complete Laurent expansion and we must try another method. We shall call it "a poor man's Laurent expansion". The idea behind this method is the fact that we generally need only the first few terms of the Laurent expansion to discover its nature. Consequently, we compute these terms through the application of power series where we retain only the leading terms. Consider the following example.

THEORY OF RESIDUES

Having shown that around any singularity we may construct a Laurent expansion, we now use this result in the integration of closed complex integrals. Consider a closed contour in which the function $f(z)$ has a number of isolated singularities. As we did in the case of Cauchy's integral formula, we introduce a new contour C' which excludes all of the singularities because they are isolated. See Figure 1.8.1. Therefore,

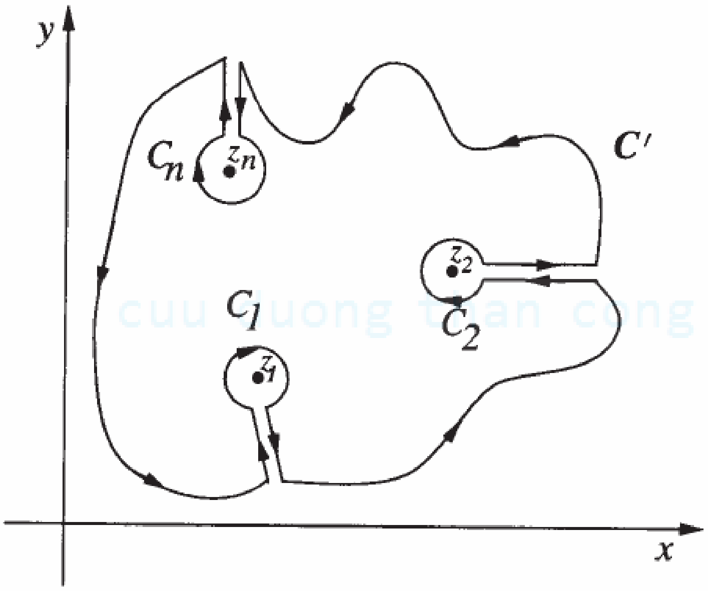

Figure 1.8.1: Contour used in deriving the residue theorem.

$$\oint_C f(z)\,dz - \oint_{C_1} f(z)\,dz - \cdots - \oint_{C_n} f(z)\,dz = \oint_{C'} f(z)\,dz = 0. \quad \textbf{(1.8.1)}$$

Consider now the mth integral, where $1 \leq m \leq n$. Constructing a Laurent expansion for the function $f(z)$ at the isolated singularity $z = z_m$, this integral equals

$$\oint_{C_m} f(z)\,dz = \sum_{k=1}^{\infty} a_k \oint_{C_m} \frac{1}{(z-z_m)^k}\,dz + \sum_{k=0}^{\infty} b_k \oint_{C_m} (z-z_m)^k\,dz. \quad \textbf{(1.8.2)}$$

Because $(z - z_m)^k$ is an entire function if $k \geq 0$, the integrals equal zero for each term in the second summation. We use Cauchy's integral formula to evaluate the remaining terms. The analytic function in the numerator is 1. Because $d^{k-1}(1)/dz^{k-1} = 0$ if $k > 1$, all of the terms vanish except for $k = 1$. In that case, the integral equals $2\pi i a_1$, where a_1 is the value of the residue for that particular singularity. Applying this approach to each of the singularities, we obtain

Cauchy's residue theorem[7]: *If $f(z)$ is analytic inside and on a closed contour C (taken in the positive sense) except at points $z_1, z_2, \ldots, z_n$ where $f(z)$ has singularities, then*

$$\boxed{\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}[f(z); z_j],} \quad \textbf{(1.8.3)}$$

where $\operatorname{Res}[f(z); z_j]$ denotes the residue of the jth isolated singularity of $f(z)$ located at $z = z_j$.

As it presently stands, it would appear that we must always construct a Laurent expansion for each singularity if we wish to use the residue theorem. This becomes increasingly difficult as the structure of the integrand becomes more complicated. In the following paragraphs we will show several techniques that avoid this problem in practice.

We begin by noting that many functions that we will encounter consist of the ratio of two *polynomials*, i.e., rational functions: $f(z) = g(z)/h(z)$. Generally, we can write $h(z)$ as $(z-z_1)^{m_1}(z-z_2)^{m_2}\cdots$. Here we have assumed that we have divided out any common factors between $g(z)$ and $h(z)$ so that $g(z)$ does not vanish at $z_1, z_2, \ldots$. Clearly $z_1, z_2, \ldots,$ are singularities of $f(z)$. Further analysis shows that the nature of the singularities are a pole of order m_1 at $z = z_1$, a pole of order m_2 at $z = z_2$, and so forth.

Having found the nature and location of the singularity, we compute the residue as follows. Suppose we have a pole of order n. Then we know that its Laurent expansion is

$$f(z) = \frac{a_n}{(z-z_0)^n} + \frac{a_{n-1}}{(z-z_0)^{n-1}} + \cdots + b_0 + b_1(z-z_0) + \cdots \quad \textbf{(1.8.6)}$$

Multiplying both sides of (1.8.6) by $(z-z_0)^n$,

$$\begin{aligned} F(z) &= (z-z_0)^n f(z) \\ &= a_n + a_{n-1}(z-z_0) + \cdots + b_0(z-z_0)^n + b_1(z-z_0)^{n+1} + \cdots \end{aligned} \quad \textbf{(1.8.7)}$$

Because $F(z)$ is analytic at $z = z_0$, it has the Taylor expansion

$$F(z) = F(z_0) + F'(z_0)(z-z_0) + \cdots + \frac{F^{(n-1)}(z_0)}{(n-1)!}(z-z_0)^{n-1} + \cdots \quad \textbf{(1.8.8)}$$

Matching powers of $z - z_0$ in (1.8.7) and (1.8.8), the residue equals

$$\mathrm{Res}[f(z); z_0] = a_1 = \frac{F^{(n-1)}(z_0)}{(n-1)!}. \quad \textbf{(1.8.9)}$$

Substituting in $F(z) = (z - z_0)^n f(z)$, we can compute the residue of a pole of order n by

$$\boxed{\operatorname{Res}[f(z); z_j] = \frac{1}{(n-1)!} \lim_{z \to z_j} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_j)^n f(z) \right].}$$

(1.8.10)

For a simple pole (1.8.10) simplifies to

$$\boxed{\operatorname{Res}[f(z); z_j] = \lim_{z \to z_j} (z - z_j) f(z).}$$

(1.8.11)

Quite often, $f(z) = p(z)/q(z)$. From l'Hôspital's rule, it follows that

$$\boxed{\operatorname{Res}[f(z); z_j] = \frac{p(z_j)}{q'(z_j)}.}$$

(1.8.12)

Remember that these formulas work only for finite-order poles. For an essential singularity we must compute the residue from its Laurent expansion; however, essential singularities are very rare in applications.

EVALUATION OF REAL DEFINITE INTEGRALS

One of the important applications of the theory of residues consists in the evaluation of certain types of real definite integrals. Similar techniques apply when the integrand contains a sine or cosine. See Section 3.4.

This example illustrates the basic concepts of evaluating definite integrals by the residue theorem. We introduce a closed contour that includes the real axis and an additional contour. We must then evaluate the integral along this additional contour as well as the closed contour integral. If we have properly chosen our closed contour, this additional integral will vanish. For certain classes of general integrals, we shall now show that this additional contour is a circular arc at infinity.

Theorem: *If, on a circular arc* C_R *with a radius* R *and center at the origin,* $zf(z) \to 0$ *uniformly with* $|z| \in C_R$ *and as* $R \to \infty$*, then*

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0. \tag{1.9.7}$$

This follows from the fact that if $|zf(z)| \leq M_R$, then $|f(z)| \leq M_R/R$. Because the length of C_R is αR, where α is the subtended angle,

$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{M_R}{R}\,\alpha R = \alpha M_R \to 0, \tag{1.9.8}$$

because $M_R \to 0$ as $R \to \infty$. □

CHƯƠNG 3. BIẾN ĐỔI FOURIER

FOURIER SERIES

One of the crowning glories[1] of nineteenth century mathematics was the discovery that the infinite series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \qquad \textbf{(2.1.1)}$$

can represent a function $f(t)$ under certain general conditions. This series, called a *Fourier series*, converges to the value of the function $f(t)$ at every point in the interval $[-L, L]$ with the possible exceptions of the points at any discontinuities and the endpoints of the interval. Because each term has a period of $2L$, the sum of the series also has the same period. The *fundamental* of the periodic function $f(t)$ is the $n = 1$ term while the *harmonics* are the remaining terms whose frequencies are integer multiples of the fundamental.

We must now find some easy method for computing the a_n's and b_n's for a given function $f(t)$. As a first attempt, we integrate (2.1.1) term by term[2] from $-L$ to L. On the right side, all of the integrals multiplied by a_n and b_n vanish because the average of $\cos(n\pi t/L)$ and $\sin(n\pi t/L)$ is zero. Therefore, we are left with

$$\boxed{a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\, dt.} \qquad \textbf{(2.1.2)}$$

Consequently a_0 is twice the mean value of $f(t)$ over one period.

We next multiply each side of (2.1.1) by $\cos(m\pi t/L)$, where m is a fixed integer. Integrating from $-L$ to L,

$$\int_{-L}^{L} f(t)\cos\left(\frac{m\pi t}{L}\right) dt = \frac{a_0}{2}\int_{-L}^{L} \cos\left(\frac{m\pi t}{L}\right) dt + \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos\left(\frac{n\pi t}{L}\right)\cos\left(\frac{m\pi t}{L}\right) dt + \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin\left(\frac{n\pi t}{L}\right)\cos\left(\frac{m\pi t}{L}\right) dt. \quad \textbf{(2.1.3)}$$

The a_0 and b_n terms vanish by direct integration. Finally all of the a_n integrals vanish when $n \neq m$. Consequently, (2.1.3) simplifies to

$$\boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right) dt,} \quad \textbf{(2.1.4)}$$

because $\int_{-L}^{L} \cos^2(n\pi t/L)\, dt = L$. Finally, by multiplying both sides of (2.1.1) by $\sin(m\pi t/L)$ (m is again a fixed integer) and integrating from $-L$ to L,

$$\boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right) dt.} \quad \textbf{(2.1.5)}$$

Although (2.1.2), (2.1.4), and (2.1.5) give us a_0, a_n, and b_n for periodic functions over the interval $[-L, L]$, in certain situations it is convenient to use the interval $[\tau, \tau + 2L]$, where τ is any real number. In that case, (2.1.1) still gives the Fourier series of $f(t)$ and

$$\boxed{\begin{aligned} a_0 &= \frac{1}{L}\int_{\tau}^{\tau+2L} f(t)\,dt, \\ a_n &= \frac{1}{L}\int_{\tau}^{\tau+2L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt, \\ b_n &= \frac{1}{L}\int_{\tau}^{\tau+2L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt. \end{aligned}} \qquad \textbf{(2.1.6)}$$

These results follow when we recall that the function $f(t)$ is a periodic function that extends from minus infinity to plus infinity. The results must remain unchanged, therefore, when we shift from the interval $[-L, L]$ to the new interval $[\tau, \tau + 2L]$.

We now ask the question: what types of functions have Fourier series? Secondly, if a function is discontinuous at a point, what value will the Fourier series give? Dirichlet[3,4] answered these questions in the first half of the nineteenth century. He showed that if any arbitrary function is finite over one period and has a finite number of maxima and minima, then the Fourier series converges. If $f(t)$ is discontinuous at the point t and has two different values at $f(t^-)$ and $f(t^+)$, where t^+ and t^- are points infinitesimally to the right and left of t, the Fourier series converges to the mean value of $[f(t^+) + f(t^-)]/2$. Because *Dirichlet's conditions* are very mild, it is very rare that a convergent Fourier series does not exist for a function that appears in an engineering or scientific problem.

PROPERTIES OF FOURIER SERIES

In the previous section we introduced the Fourier series and showed how to compute one given the function $f(t)$. In this section we examine some particular properties of these series.

Differentiation of a Fourier series

In certain instances we only have the Fourier series representation of a function $f(t)$. Can we find the derivative or the integral of $f(t)$ merely by differentiating or integrating the Fourier series term by term? Is this permitted? Let us consider the case of differentiation first.

Consider a function $f(t)$ of period $2L$ which has the derivative $f'(t)$. Let us assume that we can expand $f'(t)$ as a Fourier series. This implies that $f'(t)$ is continuous except for a finite number of discontinuities and $f(t)$ is continuous over an interval that starts at $t = \tau$ and ends at $t = \tau + 2L$. Then

$$f'(t) = \frac{a_0'}{2} + \sum_{n=1}^{\infty} a_n' \cos\left(\frac{n\pi t}{L}\right) + b_n' \sin\left(\frac{n\pi t}{L}\right), \qquad (2.2.1)$$

where we have denoted the Fourier coefficients of $f'(t)$ with a prime. Computing the Fourier coefficients,

$$a_0' = \frac{1}{L}\int_{\tau}^{\tau+2L} f'(t)\,dt = \frac{1}{L}[f(\tau+2L) - f(\tau)] = 0, \qquad (2.2.2)$$

if $f(\tau + 2L) = f(\tau)$. Similarly, by integrating by parts,

$$a_n' = \frac{1}{L}\int_{\tau}^{\tau+2L} f'(t)\cos\left(\frac{n\pi t}{L}\right) dt \qquad (2.2.3)$$

$$= \frac{1}{L}\left[f(t)\cos\left(\frac{n\pi t}{L}\right)\right]\Bigg|_{\tau}^{\tau+2L} + \frac{n\pi}{L^2}\int_{\tau}^{\tau+2L} f(t)\sin\left(\frac{n\pi t}{L}\right) dt \qquad (2.2.4)$$

$$= \frac{n\pi b_n}{L} \qquad (2.2.5)$$

and

$$b'_n = \frac{1}{L}\int_\tau^{\tau+2L} f'(t)\sin\left(\frac{n\pi t}{L}\right)\,dt \tag{2.2.6}$$

$$= \frac{1}{L}\left[f(t)\sin\left(\frac{n\pi t}{L}\right)\right]\Bigg|_\tau^{\tau+2L} - \frac{n\pi}{L^2}\int_\tau^{\tau+2L} f(t)\cos\left(\frac{n\pi t}{L}\right)\,dt \tag{2.2.7}$$

$$= -\frac{n\pi a_n}{L}. \tag{2.2.8}$$

Consequently, if we have a function $f(t)$ whose derivative $f'(t)$ is continuous except for a finite number of discontinuities and $f(\tau) = f(\tau + 2L)$, then

$$f'(t) = \sum_{n=1}^{\infty} \frac{n\pi}{L}\left[b_n \cos\left(\frac{n\pi t}{L}\right) - a_n \sin\left(\frac{n\pi t}{L}\right)\right]. \tag{2.2.9}$$

That is, the derivative of $f(t)$ is given by a term-by-term differentiation of the Fourier series of $f(t)$.

Integration of a Fourier series

To determine whether we can find the integral of $f(t)$ by term-by-term integration of its Fourier series, consider a form of the antiderivative of $f(t)$:

$$F(t) = \int_0^t \left[f(\tau) - \frac{a_0}{2}\right]\,d\tau. \tag{2.2.14}$$

Now

$$F(t+2L) = \int_0^t \left[f(\tau) - \frac{a_0}{2}\right]\,d\tau + \int_t^{t+2L}\left[f(\tau) - \frac{a_0}{2}\right]\,d\tau \tag{2.2.15}$$

$$= F(t) + \int_{-L}^{L}\left[f(\tau) - \frac{a_0}{2}\right]\,d\tau \tag{2.2.16}$$

$$= F(t) + \int_{-L}^{L} f(\tau)\,d\tau - La_0 = F(t), \tag{2.2.17}$$

so that $F(t)$ has a period of $2L$. Consequently we may expand $F(t)$ as the Fourier series

$$F(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi t}{L}\right) + B_n \sin\left(\frac{n\pi t}{L}\right). \tag{2.2.18}$$

For A_n,

$$A_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt \tag{2.2.19}$$

$$= \frac{1}{L}\left[F(t)\frac{\sin(n\pi t/L)}{n\pi/L}\right]\Bigg|_{-L}^{L} - \frac{1}{n\pi}\int_{-L}^{L}\left[f(t) - \frac{a_0}{2}\right]\sin\left(\frac{n\pi t}{L}\right)dt \tag{2.2.20}$$

$$= -\frac{b_n}{n\pi/L}. \tag{2.2.21}$$

Similarly,

$$B_n = \frac{a_n}{n\pi/L}. \tag{2.2.22}$$

Therefore,

$$\int_0^t f(\tau)\,d\tau = \frac{a_0 t}{2} + \frac{A_0}{2} + \sum_{n=1}^{\infty} \frac{a_n \sin(n\pi t/L) - b_n \cos(n\pi t/L)}{n\pi/L}. \tag{2.2.23}$$

This is identical to a term-by-term integration of the Fourier series for $f(t)$. Thus, we can always find the integral of $f(t)$ by a term-by-term integration of its Fourier series.

Parseval's equality

One of the fundamental quantities in engineering is power. The *power content* of a periodic signal $f(t)$ of period $2L$ is $\int_\tau^{\tau+2L} f^2(t)\,dt/L$. This mathematical definition mirrors the power dissipation I^2R that occurs in a resistor of resistance R where I is the root mean square (RMS) of the current. We would like to compute this power content as simply as possible given the coefficients of its Fourier series.

Assume that $f(t)$ has the Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right). \tag{2.2.27}$$

Then,

$$\frac{1}{L}\int_{\tau}^{\tau+2L} f^2(t)\,dt = \frac{a_0}{2L}\int_{\tau}^{\tau+2L} f(t)\,dt$$
$$+ \sum_{n=1}^{\infty} \frac{a_n}{L}\int_{\tau}^{\tau+2L} f(t)\cos\left(\frac{n\pi t}{L}\right)\,dt$$
$$+ \sum_{n=1}^{\infty} \frac{b_n}{L}\int_{\tau}^{\tau+2L} f(t)\sin\left(\frac{n\pi t}{L}\right)\,dt \tag{2.2.28}$$
$$= \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2). \tag{2.2.29}$$

Equation (2.2.29) is *Parseval's equality.*[9] It allows us to sum squares of Fourier coefficients (which we have already computed) rather than performing the integration $\int_{\tau}^{\tau+2L} f^2(t)\,dt$ analytically or numerically.

Gibbs phenomena

In the actual application of Fourier series, we cannot sum an infinite number of terms but must be content with N terms. If we denote this partial sum of the Fourier series by $S_N(t)$, we have from the definition of the Fourier series:

$$S_N(t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{N} a_n \cos(nt) + b_n \sin(nt) \tag{2.2.34}$$
$$= \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

$$+\frac{1}{\pi}\int_0^{2\pi} f(x)\left[\sum_{n=1}^{N}\cos(nt)\cos(nx)+\sin(nt)\sin(nx)\right]dx \tag{2.2.35}$$

$$S_N(t)=\frac{1}{\pi}\int_0^{2\pi} f(x)\left\{\frac{1}{2}+\sum_{n=1}^{N}\cos[n(t-x)]\right\}dx \tag{2.2.36}$$

$$=\frac{1}{2\pi}\int_0^{2\pi} f(x)\frac{\sin[(N+\frac{1}{2})(x-t)]}{\sin[\frac{1}{2}(x-t)]}\,dx. \tag{2.2.37}$$

The quantity $\sin[(N+\frac{1}{2})(x-t)]/\sin[\frac{1}{2}(x-t)]$ is called a *scanning function*. Over the range $0 \le x \le 2\pi$ it has a very large peak at $x=t$ where the amplitude equals $2N+1$. See Figure 2.2.1. On either side of this peak there are oscillations which decrease rapidly with distance from the peak. Consequently, as $N \to \infty$, the scanning function becomes essentially a long narrow slit corresponding to the area under the large peak at $x=t$. If we neglect for the moment the small area under the minor ripples adjacent to this slit, then the integral (2.2.37) essentially equals $f(t)$ times the area of the slit divided by 2π. If $1/2\pi$ times the area of the slit equals unity, then the value of $S_N(t) \approx f(t)$ to a good approximation for large N.

For a relatively small value of N, the scanning function deviates considerably from its ideal form, and the partial sum $S_N(t)$ only crudely approximates the given function $f(t)$. As the partial sum includes more terms and N becomes relatively large, the form of the scanning function improves and so does the degree of approximation between $S_N(t)$ and $f(t)$. The improvement in the scanning function is due to the large hump becoming taller and narrower. At the same time, the adjacent ripples become larger in number and hence also become narrower in the same proportion as the large hump becomes narrower.

The reason why $S_N(t)$ and $f(t)$ will never become identical, even in the limit of $N \to \infty$, is the presence of the positive and negative side lobes near the large peak. Because

$$\frac{\sin[(N+\frac{1}{2})(x-t)]}{\sin[\frac{1}{2}(x-t)]}=1+2\sum_{n=1}^{N}\cos[n(t-x)], \tag{2.2.38}$$

an integration of the scanning function over the interval 0 to 2π shows that the total area under the scanning function equals 2π. However, from Figure 2.2.1 the net area contributed by the ripples is numerically negative so that the area under the large peak must exceed the value of 2π if the area of the entire function equals 2π. Although the exact value depends upon N, it is important to note that this excess does not become zero as $N \to \infty$.

Thus, the presence of these negative side lobes explains the departure of our scanning function from the idealized slit of area 2π. To illustrate this departure, consider the function:

$$f(t) = \begin{cases} 1, & 0 < t < \pi \\ -1, & \pi < t < 2\pi. \end{cases} \tag{2.2.39}$$

Then,

$$S_N(t) = \frac{1}{2\pi}\int_0^{\pi} \frac{\sin[(N+\frac{1}{2})(x-t)]}{\sin[\frac{1}{2}(x-t)]}\,dx - \frac{1}{2\pi}\int_{\pi}^{2\pi} \frac{\sin[(N+\frac{1}{2})(x-t)]}{\sin[\frac{1}{2}(x-t)]}\,dx \tag{2.2.40}$$

$$= \frac{1}{2\pi}\int_0^{\pi} \left\{ \frac{\sin[(N+\frac{1}{2})(x-t)]}{\sin[\frac{1}{2}(x-t)]}\,dx + \frac{\sin[(N+\frac{1}{2})(x+t)]}{\sin[\frac{1}{2}(x+t)]}\,dx \right\} \tag{2.2.41}$$

$$= \frac{1}{2\pi}\int_{-t}^{\pi-t} \frac{\sin[(N+\frac{1}{2})\theta]}{\sin(\frac{1}{2}\theta)}\,d\theta - \frac{1}{2\pi}\int_{t}^{\pi+t} \frac{\sin[(N+\frac{1}{2})\theta]}{\sin(\frac{1}{2}\theta)}\,d\theta. \tag{2.2.42}$$

The first integral in (2.2.42) gives the contribution to $S_N(t)$ from the jump discontinuity at $t = 0$ while the second integral gives the contribution from $t = \pi$. In Figure 2.2.2 we have plotted the numerical

HALF-RANGE EXPANSIONS

In certain applications, we will find that we need a Fourier series representation for a function $f(x)$ that applies over the interval $(0, L)$ rather than $(-L, L)$. Because we are completely free to define the function over the interval $(-L, 0)$, it is simplest to have a series that consists only of sines or cosines. In this section we shall show how we can obtain these so-called *half-range expansions*.

It is important to remember that half-range expansions are a special case of the general Fourier series. For any $f(x)$ we can construct either a Fourier sine or cosine series over the interval $(-L, L)$. Both of these series will give the correct answer over the interval of $(0, L)$. Which one we choose to use depends upon whether we wish to deal with a cosine or sine series.

COMPLEX FOURIER SERIES

So far in our discussion, we have expressed Fourier series in terms of sines and cosines. We are now ready to reexpress a Fourier series as a series of complex exponentials. There are two reasons for this. First, in certain engineering and scientific applications of Fourier series, the expansion of a function in terms of complex exponentials results in coefficients of considerable simplicity and clarity. Secondly, these complex Fourier series point the way to the development of the Fourier transform in the next chapter.

We begin by introducing the variable

$$\omega_n = \frac{n\pi}{L}, \tag{2.5.1}$$

where $n = 0, \pm 1, \pm 2, \ldots$. Using Euler's formula we can replace the sine and cosine in the Fourier series by exponentials and find that

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{i\omega_n t} + e^{-i\omega_n t}\right) + \frac{b_n}{2i}\left(e^{i\omega_n t} - e^{-i\omega_n t}\right) \tag{2.5.2}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} - \frac{b_n i}{2}\right)e^{i\omega_n t} + \left(\frac{a_n}{2} + \frac{b_n i}{2}\right)e^{-i\omega_n t}. \tag{2.5.3}$$

If we define c_n as

$$c_n = \tfrac{1}{2}(a_n - ib_n), \tag{2.5.4}$$

then

$$c_n = \tfrac{1}{2}(a_n - ib_n) = \frac{1}{2L}\int_{\tau}^{\tau+2L} f(t)[\cos(\omega_n t) - i\sin(\omega_n t)]\,dt \tag{2.5.5}$$

$$= \frac{1}{2L}\int_{\tau}^{\tau+2L} f(t)e^{-i\omega_n t}dt. \tag{2.5.6}$$

Similarly, the complex conjugate of c_n, c_n^*, equals

$$c_n^* = \tfrac{1}{2}(a_n + ib_n) = \frac{1}{2L}\int_{\tau}^{\tau+2L} f(t)e^{i\omega_n t}dt. \tag{2.5.7}$$

To simplify (2.5.3) we note that

$$\omega_{-n} = \frac{(-n)\pi}{L} = -\frac{n\pi}{L} = -\omega_n, \tag{2.5.8}$$

which yields the result that

$$c_{-n} = \frac{1}{2L}\int_{\tau}^{\tau+2L} f(t)e^{-i\omega_{-n}t}dt = \frac{1}{2L}\int_{\tau}^{\tau+2L} f(t)e^{i\omega_n t}dt = c_n^* \quad \mathbf{(2.5.9)}$$

so that we can write (2.5.3) as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n e^{i\omega_n t} + c_n^* e^{-i\omega_n t} = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n e^{i\omega_n t} + c_{-n} e^{-i\omega_n t}. \quad \mathbf{(2.5.10)}$$

Letting $n = -m$ in the second summation on the right side of (2.5.10),

$$\sum_{n=1}^{\infty} c_{-n} e^{-i\omega_n t} = \sum_{m=-1}^{-\infty} c_m e^{-i\omega_{-m} t} = \sum_{m=-\infty}^{-1} c_m e^{i\omega_m t} = \sum_{n=-\infty}^{-1} c_n e^{i\omega_n t}, \quad \mathbf{(2.5.11)}$$

where we have introduced $m = n$ into the last summation in (2.5.11). Therefore,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n e^{i\omega_n t} + \sum_{n=-\infty}^{-1} c_n e^{i\omega_n t}. \quad \mathbf{(2.5.12)}$$

On the other hand,

$$\frac{a_0}{2} = \frac{1}{2L}\int_{\tau}^{\tau+2L} f(t)\,dt = c_0 = c_0 e^{i\omega_0 t}, \quad \mathbf{(2.5.13)}$$

because $\omega_0 = 0\pi/L = 0$. Thus, our final result is

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t},} \quad \mathbf{(2.5.14)}$$

where

$$\boxed{c_n = \frac{1}{2L}\int_{\tau}^{\tau+2L} f(t)e^{-i\omega_n t}\,dt} \quad \mathbf{(2.5.15)}$$

and $n = 0, \pm 1, \pm 2, \ldots$. Note that even though c_n is generally complex, the summation (2.5.14) always gives a *real-valued* function $f(t)$.

Just as we can represent the function $f(t)$ graphically by a plot of t against $f(t)$, we can plot c_n as a function of n, commonly called the frequency *spectrum*. Because c_n is generally complex, it is necessary to make two plots. Typically the plotted quantities are the amplitude spectra $|c_n|$ and the phase spectra φ_n, where φ_n is the phase of c_n. However, we could just as well plot the real and imaginary parts of c_n. Because n is an integer, these plots consist merely of a series of vertical lines representing the ordinates of the quantity $|c_n|$ or φ_n for each n. For this reason we refer to these plots as the *line spectra*.

Because $2c_n = a_n - ib_n$, the c_n's for an even function will be purely real; the c_n's for an odd function are purely imaginary. It is important to note that we lose the advantage of even and odd functions in the sense that we cannot just integrate over the interval 0 to L and then double the result. In the present case we have a line integral of a complex function along the real axis.

The Fourier Transform

FOURIER TRANSFORMS

The Fourier transform is the natural extension of Fourier series to a function $f(t)$ of infinite period. To show this, consider a periodic function $f(t)$ of period $2T$ that satisfies the so-called Dirichlet's conditions.[1] If the integral $\int_a^b |f(t)|\, dt$ exists, this function has the complex Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi t/T}, \tag{3.1.1}$$

where

$$c_n = \frac{1}{2T} \int_{-T}^{T} f(t) e^{-in\pi t/T}\, dt. \tag{3.1.2}$$

Equation (3.1.1) applies only if $f(t)$ is continuous at t; if $f(t)$ suffers from a jump discontinuity at t, then the left side of (3.1.1) equals $\frac{1}{2}[f(t^+) + f(t^-)]$, where $f(t^+) = \lim_{x \to t^+} f(x)$ and $f(t^-) = \lim_{x \to t^-} f(x)$. Substituting (3.1.2) into (3.1.1),

$$f(t) = \frac{1}{2T} \sum_{n=-\infty}^{\infty} e^{in\pi t/T} \int_{-T}^{T} f(x) e^{-in\pi x/T} \, dx. \tag{3.1.3}$$

Let us now introduce the notation $\omega_n = n\pi/T$ so that $\Delta\omega_n = \omega_{n+1} - \omega_n = \pi/T$. Then,

$$f(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(\omega_n) e^{i\omega_n t} \Delta\omega_n, \tag{3.1.4}$$

where

$$F(\omega_n) = \int_{-T}^{T} f(x) e^{-i\omega_n x} \, dx. \tag{3.1.5}$$

As $T \to \infty$, ω_n approaches a continuous variable ω and $\Delta\omega_n$ may be interpreted as the infinitesimal $d\omega$. Therefore, ignoring any possible difficulties.[2]

$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \, d\omega} \tag{3.1.6}$$

and

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt.} \tag{3.1.7}$$

Equation (3.1.7) is the *Fourier transform* of $f(t)$ while (3.1.6) is the *inverse Fourier transform* which converts a Fourier transform back to $f(t)$. Alternatively, we may combine (3.1.6)–(3.1.7) to yield the equivalent real form:

$$f(t) = \frac{1}{\pi} \int_0^\infty \left\{ \int_{-\infty}^\infty f(x) \cos[\omega(t-x)] dx \right\} d\omega. \quad \textbf{(3.1.8)}$$

Hamming[3] has suggested the following analog in understanding the Fourier transform. Let us imagine that $f(t)$ is a light beam. Then the Fourier transform, like a glass prism, breaks up the function into its component frequencies ω, each of intensity $F(\omega)$. In optics, the various frequencies are called colors; by analogy the Fourier transform gives us the color spectrum of a function. On the other hand, the inverse Fourier transform blends a function's spectrum to give back the original function.

Most signals encountered in practice have Fourier transforms because they are absolutely integrable since they are bounded and of finite duration. However, there are some notable exceptions. Examples include the trigonometric functions sine and cosine.

Although this particular example does not show it, the Fourier transform is, in general, a complex function. The most common method of displaying it is to plot its amplitude and phase on two separate graphs for all values of ω. See Figure 3.1.1. Of these two quantities, the amplitude is by far the more popular one and is given the special name of *frequency spectrum*.

From the definition of the inverse Fourier transform,

$$f(t) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{\sin(\omega a)}{\omega} e^{i\omega t} d\omega = \begin{cases} 1, & |t| < a \\ 0, & |t| > a. \end{cases} \quad \textbf{(3.1.12)}$$

An important question is what value does $f(t)$ converge to in the limit as $t \to a$ and $t \to -a$? Because Fourier transforms are an extension of Fourier series, the behavior at a jump is the same as that for a Fourier series. For that reason, $f(a) = \frac{1}{2}[f(a^+) + f(a^-)] = \frac{1}{2}$ and $f(-a) = \frac{1}{2}[f(-a^+) + f(-a^-)] = \frac{1}{2}$.

An important function in transform methods is the *(Heaviside) step function*:

$$H(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0. \end{cases} \tag{3.2.16}$$

In terms of the sign function it can be written

$$H(t) = \tfrac{1}{2} + \tfrac{1}{2}\mathrm{sgn}(t). \tag{3.2.17}$$

Because the Fourier transforms of 1 and $\mathrm{sgn}(t)$ are $2\pi\delta(\omega)$ and $2/i\omega$, respectively, we have that

$$\mathcal{F}[H(t)] = \pi\delta(\omega) + \frac{1}{i\omega}. \tag{3.2.18}$$

These transforms are used in engineering but the presence of the delta function requires extra care to ensure their proper use.

Of the many functions that have a Fourier transform, a particularly important one is the *(Dirac) delta function*.[4] For example, in Section 3.6 we will use it to solve differential equations. We *define* it as the inverse of the Fourier transform $F(\omega) = 1$. Therefore,

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\, d\omega. \tag{3.1.13}$$

To give some insight into the nature of the delta function, consider another band-limited transform:

$$F_\Omega(\omega) = \begin{cases} 1, & |\omega| < \Omega \\ 0, & |\omega| > \Omega, \end{cases} \tag{3.1.14}$$

where Ω is real and positive. Then,

$$f_\Omega(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} e^{i\omega t}\, d\omega = \frac{\Omega}{\pi}\frac{\sin(\Omega t)}{\Omega t}. \tag{3.1.15}$$

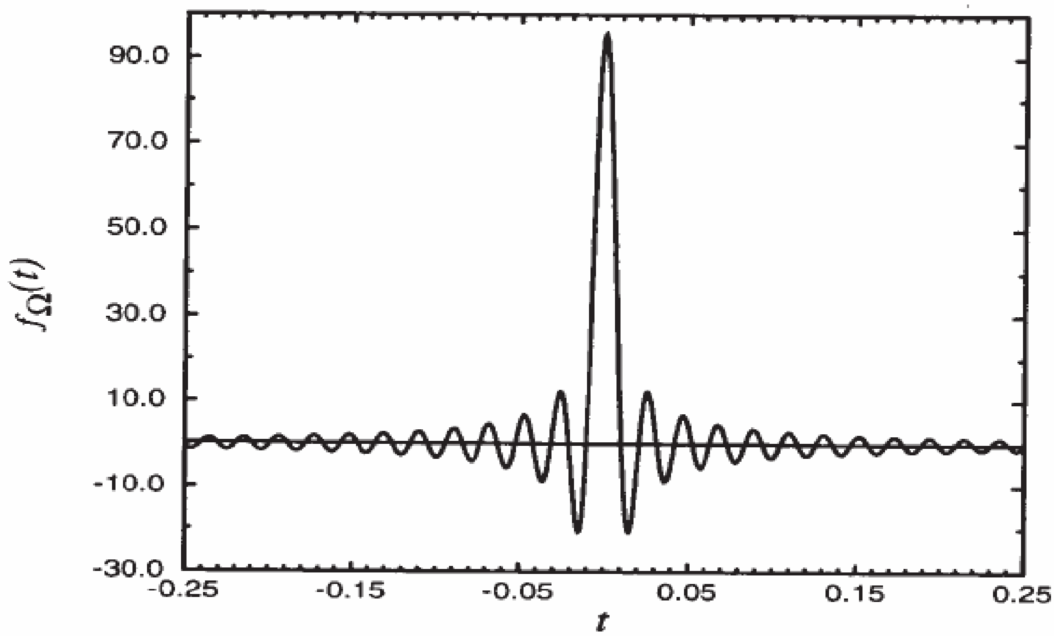

Figure 3.1.2: Graph of the function given in (3.1.15) for $\Omega = 300$.

Figure 3.1.2 illustrates $f_\Omega(t)$ for a large value of Ω. We observe that as $\Omega \to \infty$, $f_\Omega(t)$ becomes very large near $t = 0$ as well as very narrow. On the other hand, $f_\Omega(t)$ rapidly approaches zero as $|t|$ increases. Therefore, we may consider the delta function as the limit:

$$\delta(t) = \lim_{\Omega \to \infty} \frac{\sin(\Omega t)}{\pi t} \tag{3.1.16}$$

or

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0. \end{cases} \tag{3.1.17}$$

Because the Fourier transform of the delta function equals one,

$$\int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} dt = 1. \tag{3.1.18}$$

Since (3.1.18) must hold for any ω, we take $\omega = 0$ and find that

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1. \tag{3.1.19}$$

Thus, the area under the delta function equals unity. Taking (3.1.17) into account, we can also write (3.1.19) as

$$\int_{-a}^{b} \delta(t)\, dt = 1, \qquad a, b > 0. \tag{3.1.20}$$

Finally,

$$\int_{a}^{b} f(t)\delta(t - t_0)\, dt = f(t_0), \tag{3.1.21}$$

if $a < t_0 < b$. This follows from the law of the mean of integrals.

We may also use several other functions with equal validity to represent the delta function. These include the limiting case of the following rectangular or triangular distributions:

$$\delta(t) = \lim_{\epsilon \to 0} \begin{cases} \frac{1}{\epsilon}, & |t| < \frac{\epsilon}{2} \\ 0, & |t| > \frac{\epsilon}{2} \end{cases} \tag{3.1.22}$$

or

$$\delta(t) = \lim_{\epsilon \to 0} \begin{cases} \frac{1}{\epsilon}\left(1 - \frac{|t|}{\epsilon}\right), & |t| < \epsilon \\ 0, & |t| > \epsilon \end{cases} \tag{3.1.23}$$

and the Gaussian function:

$$\delta(t) = \lim_{\epsilon \to 0} \frac{\exp(-\pi t^2/\epsilon)}{\sqrt{\epsilon}}. \tag{3.1.24}$$

Note that the delta function is an even function.

FOURIER TRANSFORMS CONTAINING THE DELTA FUNCTION

In the previous section we stressed the fact that such simple functions as cosine and sine are not absolutely integrable. Does this mean that these functions do not possess a Fourier transform? In this section we shall show that certain functions can still have a Fourier transform even though we cannot compute them directly.

The reason why we can find the Fourier transform of certain functions that are not absolutely integrable lies with the introduction of the delta function because

$$\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{it\omega}\, d\omega = e^{i\omega_0 t} \tag{3.2.1}$$

for all t. Thus, the inverse of the Fourier transform $\delta(\omega - \omega_0)$ is the complex exponential $e^{i\omega_0 t}/2\pi$ or

$$\mathcal{F}\left(e^{i\omega_0 t}\right) = 2\pi\delta(\omega - \omega_0). \tag{3.2.2}$$

	$\mathbf{f(t)},\ \lvert\mathbf{t}\rvert < \infty$	$\mathbf{F}(\omega)$
1.	$e^{-at}H(t), \quad a > 0$	$\dfrac{1}{a+\omega i}$
2.	$e^{at}H(-t), \quad a > 0$	$\dfrac{1}{a-\omega i}$
3.	$te^{-at}H(t), \quad a > 0$	$\dfrac{1}{(a+\omega i)^2}$
4.	$te^{at}H(-t), \quad a > 0$	$\dfrac{-1}{(a-\omega i)^2}$
5.	$t^n e^{-at}H(t),\ \mathrm{Re}(a) > 0,\ n = 1, 2, \ldots$	$\dfrac{n!}{(a+\omega i)^{n+1}}$
6.	$e^{-a\lvert t\rvert}, \quad a > 0$	$\dfrac{2a}{\omega^2+a^2}$
7.	$te^{-a\lvert t\rvert}, \quad a > 0$	$\dfrac{-4a\omega i}{(\omega^2+a^2)^2}$
8.	$\dfrac{1}{1+a^2t^2}$	$\dfrac{\pi}{\lvert a\rvert}e^{-\lvert\omega/a\rvert}$
9.	$\dfrac{\cos(at)}{1+t^2}$	$\frac{\pi}{2}\left(e^{-\lvert\omega-a\rvert}+e^{-\lvert\omega+a\rvert}\right)$
10.	$\dfrac{\sin(at)}{1+t^2}$	$\frac{\pi}{2i}\left(e^{-\lvert\omega-a\rvert}-e^{-\lvert\omega+a\rvert}\right)$
11.	$\begin{cases} 1, & \lvert t\rvert < a \\ 0, & \lvert t\rvert > a \end{cases}$	$\dfrac{2\sin(\omega a)}{\omega}$
12.	$\dfrac{\sin(at)}{at}$	$\begin{cases} \pi/a, & \lvert\omega\rvert < a \\ 0, & \lvert\omega\rvert > a \end{cases}$

This yields immediately the result that

$$\mathcal{F}(1) = 2\pi\delta(\omega), \tag{3.2.3}$$

if we set $\omega_0 = 0$. Thus, the Fourier transform of 1 is an impulse at $\omega = 0$ with weight 2π. Because the Fourier transform equals zero for all $\omega \neq 0$, $f(t) = 1$ does not contain a nonzero frequency and is consequently a DC signal.

Another set of transforms arises from Euler's formula because we have that

$$\begin{aligned}
\mathcal{F}[\sin(\omega_0 t)] &= \tfrac{1}{2i}\left[\mathcal{F}\left(e^{i\omega_0 t}\right) - \mathcal{F}\left(e^{-i\omega_0 t}\right)\right] && (3.2.4)\\
&= \tfrac{\pi}{i}\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right] && (3.2.5)\\
&= -\pi i\delta(\omega - \omega_0) + \pi i\delta(\omega + \omega_0) && (3.2.6)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}[\cos(\omega_0 t)] &= \tfrac{1}{2}\left[\mathcal{F}\left(e^{i\omega_0 t}\right) + \mathcal{F}\left(e^{-i\omega_0 t}\right)\right] && (3.2.7)\\
&= \pi\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]. && (3.2.8)
\end{aligned}$$

Note that although the amplitude spectra of $\sin(\omega_0 t)$ and $\cos(\omega_0 t)$ are the same, their phase spectra are different.

Let us consider the Fourier transform of any arbitrary periodic function. Recall that any such function $f(t)$ with period $2L$ can be rewritten as the complex Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}, \tag{3.2.9}$$

where $\omega_0 = \pi/L$. The Fourier transform of $f(t)$ is

$$F(\omega) = \mathcal{F}[f(t)] = \sum_{n=-\infty}^{\infty} 2\pi c_n \delta(\omega - n\omega_0). \tag{3.2.10}$$

Therefore, the Fourier transform of any arbitrary periodic function is a sequence of impulses with weight $2\pi c_n$ located at $\omega = n\omega_0$ with $n = 0, \pm 1, \pm 2, \ldots$. Thus, the Fourier series and transform of a periodic function are closely related.

Linearity

If $f(t)$ and $g(t)$ are functions with Fourier transforms $F(\omega)$ and $G(\omega)$, respectively, then

$$\mathcal{F}[c_1 f(t) + c_2 g(t)] = c_1 F(\omega) + c_2 G(\omega), \tag{3.3.1}$$

where c_1 and c_2 are (real or complex) constants.

This result follows from the integral definition:

$$\mathcal{F}[c_1 f(t) + c_2 g(t)] = \int_{-\infty}^{\infty} [c_1 f(t) + c_2 g(t)] e^{-i\omega t}\, dt \tag{3.3.2}$$

$$= c_1 \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt + c_2 \int_{-\infty}^{\infty} g(t) e^{-i\omega t}\, dt \tag{3.3.3}$$

$$= c_1 F(\omega) + c_2 G(\omega). \tag{3.3.4}$$

Time shifting

If $f(t)$ is a function with a Fourier transform $F(\omega)$, then $\mathcal{F}[f(t - \tau)] = e^{-i\omega\tau} F(\omega)$.

This follows from the definition of the Fourier transform:

$$\mathcal{F}[f(t-\tau)] = \int_{-\infty}^{\infty} f(t-\tau) e^{-i\omega t}\, dt = \int_{-\infty}^{\infty} f(x) e^{-i\omega(x+\tau)}\, dx \tag{3.3.5}$$

$$= e^{-i\omega\tau} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx = e^{-i\omega\tau} F(\omega). \tag{3.3.6}$$

Scaling factor

Let $f(t)$ be a function with a Fourier transform $F(\omega)$ and k be a real, nonzero constant. Then $\mathcal{F}[f(kt)] = F(\omega/k)/|k|$.

From the definition of the Fourier transform:

$$\mathcal{F}[f(kt)] = \int_{-\infty}^{\infty} f(kt)e^{-i\omega t}dt = \frac{1}{|k|}\int_{-\infty}^{\infty} f(x)e^{-i(\omega/k)x}dx = \frac{1}{|k|}F\left(\frac{\omega}{k}\right). \tag{3.3.9}$$

Symmetry

If the function $f(t)$ has the Fourier transform $F(\omega)$, then $\mathcal{F}[F(t)] = 2\pi f(-\omega)$.

From the definition of the inverse Fourier transform,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x)e^{ixt}dx. \tag{3.3.11}$$

Then

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(x)e^{-i\omega x}dx = \int_{-\infty}^{\infty} F(t)e^{-i\omega t}dt = \mathcal{F}[F(t)]. \tag{3.3.12}$$

Derivatives of functions

Let $f^{(k)}(t), k = 0, 1, 2, \ldots, n-1$, be continuous and $f^{(n)}(t)$ be piece-wise continuous. Let $|f^{(k)}(t)| \leq Ke^{-bt}, b > 0, 0 \leq t < \infty; |f^{(k)}(t)| \leq Me^{at}, a > 0, -\infty < t \leq 0, k = 0, 1, .., n$. Then, $\mathcal{F}[f^{(n)}(t)] = (i\omega)^n F(\omega)$.

We begin by noting that if the transform $\mathcal{F}[f'(t)]$ exists, then

$$\mathcal{F}[f'(t)] = \int_{-\infty}^{\infty} f'(t)e^{-i\omega t}\,dt \tag{3.3.15}$$

$$= \int_{-\infty}^{\infty} f'(t)e^{\omega_i t}[\cos(\omega_r t) - i\sin(\omega_r t)]\,dt \tag{3.3.16}$$

$$= (-\omega_i + i\omega_r)\int_{-\infty}^{\infty} f(t)e^{\omega_i t}[\cos(\omega_r t) - i\sin(\omega_r t)]\,dt \tag{3.3.17}$$

$$= i\omega \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt = i\omega F(\omega). \tag{3.3.18}$$

Finally,

$$\mathcal{F}[f^{(n)}(t)] = i\omega\mathcal{F}[f^{(n-1)}(t)] = (i\omega)^2\mathcal{F}[f^{(n-2)}(t)] = \cdots = (i\omega)^n F(\omega). \tag{3.3.19}$$

Modulation

In communications a popular method of transmitting information is by *amplitude modulation* (AM). In this process some signal is carried according to the expression $f(t)e^{i\omega_0 t}$, where ω_0 is the *carrier frequency* and $f(t)$ is some arbitrary function of time whose amplitude spectrum peaks at some frequency that is usually small compared to ω_0. We now want to show that the Fourier transform of $f(t)e^{i\omega_0 t}$ is $F(\omega - \omega_0)$, where $F(\omega)$ is the Fourier transform of $f(t)$.

We begin by using the definition of the Fourier transform:

$$\mathcal{F}[f(t)e^{i\omega_0 t}] = \int_{-\infty}^{\infty} f(t)e^{i\omega_0 t}e^{-i\omega t}\,dt = \int_{-\infty}^{\infty} f(t)e^{-i(\omega-\omega_0)t}\,dt \tag{3.3.22}$$

$$= F(\omega - \omega_0). \tag{3.3.23}$$

Therefore, if we have the spectrum of a particular function $f(t)$, then the Fourier transform of the modulated function $f(t)e^{i\omega_0 t}$ is the same as that for $f(t)$ except that it is now centered on the frequency ω_0 rather than on the zero frequency.

Parseval's equality

In applying Fourier methods to practical problems we may encounter a situation where we are interested in computing the energy of a system. Energy is usually expressed by the integral $\int_{-\infty}^{\infty} |f(t)|^2\,dt$. Can we compute this integral if we only have the Fourier transform of $F(\omega)$?

From the definition of the inverse Fourier transform

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega, \tag{3.3.37}$$

we have that

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\left[\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega\right] dt. \tag{3.3.38}$$

Interchanging the order of integration on the right side of (3.3.38),

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\left[\int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt\right] d\omega. \tag{3.3.39}$$

However,

$$F^*(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt. \tag{3.3.40}$$

Therefore,

$$\boxed{\int_{-\infty}^{\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega.} \tag{3.3.41}$$

This is *Parseval's equality*[6] as it applies to Fourier transforms. The quantity $|F(\omega)|^2$ is called the *power spectrum*.

Poisson's summation formula

If $f(x)$ is integrable over $(-\infty, \infty)$, there exists a relationship between the function and its Fourier transform, commonly called *Poisson's summation formula.*[7]

We begin by inventing a periodic function $g(x)$ defined by

$$g(x) = \sum_{k=-\infty}^{\infty} f(x + 2\pi k). \tag{3.3.44}$$

Because $g(x)$ is a periodic function of 2π, it can be represented by the complex Fourier series:

$$g(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{3.3.45}$$

or

$$g(0) = \sum_{k=-\infty}^{\infty} f(2\pi k) = \sum_{n=-\infty}^{\infty} c_n. \tag{3.3.46}$$

Computing c_n, we find that

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} f(x+2k\pi)e^{-inx}\,dx \tag{3.3.47}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi} f(x+2k\pi)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-inx}\,dx \tag{3.3.48}$$

$$= \frac{F(n)}{2\pi}, \tag{3.3.49}$$

where $F(\omega)$ is the Fourier transform of $f(x)$. Substituting (3.3.49) into (3.3.46), we obtain

$$\sum_{k=-\infty}^{\infty} f(2\pi k) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} F(n) \tag{3.3.50}$$

or

$$\boxed{\sum_{k=-\infty}^{\infty} f(\alpha k) = \frac{1}{\alpha} \sum_{n=-\infty}^{\infty} F\left(\frac{2\pi n}{\alpha}\right).} \tag{3.3.51}$$

INVERSION OF FOURIER TRANSFORMS

Having focused on the Fourier transform in the previous sections, we consider in detail the inverse Fourier transform in this section. Recall that the improper integral (3.1.6) defines the inverse. Consequently one method of inversion is direct integration.

Another method for inverting Fourier transforms is rewriting the Fourier transform using partial fractions so that we can use transform tables. The following example illustrates this technique.

Although we may find the inverse by direct integration or partial fractions, in many instances the Fourier transform does not lend itself to these techniques. On the other hand, if we view the inverse Fourier transform as a line integral along the real axis in the complex ω-plane, then perhaps some of the techniques that we developed in Chapter 1 might be applicable to this problem. To this end, we rewrite the inversion integral (3.1.6) as

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{it\omega}\,d\omega = \frac{1}{2\pi}\oint_C F(z)e^{itz}\,dz - \frac{1}{2\pi}\int_{C_R} F(z)e^{itz}\,dz, \tag{3.4.9}$$

where C denotes a closed contour consisting of the entire real axis plus a new contour C_R that joins the point $(\infty, 0)$ to $(-\infty, 0)$. There are countless possibilities for C_R. For example, it could be the loop $(\infty, 0)$ to (∞, R) to $(-\infty, R)$ to $(-\infty, 0)$ with $R > 0$. However, any choice of C_R must be such that we can compute $\int_{C_R} F(z)e^{itz}\,dz$. When we take that constraint into account, the number of acceptable contours decrease to just a few. The best is given by *Jordan's lemma*:[8]

Jordan's lemma: *Suppose that, on a circular arc* C_R *with radius* R *and center at the origin,* $f(z) \to 0$ *uniformly as* $R \to \infty$. *Then*

$$(1) \qquad \lim_{R\to\infty} \int_{C_R} f(z)e^{imz}\,dz = 0, \qquad (m > 0) \qquad \textbf{(3.4.10)}$$

if C_R *lies in the first and/or second quadrant;*

$$(2) \qquad \lim_{R\to\infty} \int_{C_R} f(z)e^{-imz}\,dz = 0, \qquad (m > 0) \qquad \textbf{(3.4.11)}$$

if C_R *lies in the third and/or fourth quadrant;*

$$(3) \qquad \lim_{R\to\infty} \int_{C_R} f(z)e^{mz}\,dz = 0, \qquad (m > 0) \qquad \textbf{(3.4.12)}$$

if C_R *lies in the second and/or third quadrant; and*

$$(4) \qquad \lim_{R\to\infty} \int_{C_R} f(z)e^{-mz}\,dz = 0, \qquad (m > 0) \qquad \textbf{(3.4.13)}$$

if C_R *lies in the first and/or fourth quadrant.*

Technically, only (1) is actually Jordan's lemma while the remaining points are variations.

Proof: We shall prove the first part; the remaining portions follow by analog. We begin by noting that

$$|I_R| = \left|\int_{C_R} f(z)e^{imz}\,dz\right| \le \int_{C_R} |f(z)|\left|e^{imz}\right||dz|. \qquad \textbf{(3.4.14)}$$

Now

$$|dz| = R\,d\theta, \quad |f(z)| \le M_R, \qquad \textbf{(3.4.15)}$$

$$\left|e^{imz}\right| = \left|\exp(imRe^{\theta i})\right| = |\exp\{imR[\cos(\theta) + i\sin(\theta)]\}| = e^{-mR\sin(\theta)}. \qquad \textbf{(3.4.16)}$$

Therefore,

$$|I_R| \le RM_R \int_{\theta_0}^{\theta_1} \exp[-mR\sin(\theta)]\, d\theta, \qquad \textbf{(3.4.17)}$$

where $0 \le \theta_0 < \theta_1 \le \pi$. Because the integrand is positive, the right side of (3.4.17) is largest if we take $\theta_0 = 0$ and $\theta_1 = \pi$. Then

$$|I_R| \le RM_R \int_0^{\pi} e^{-mR\sin(\theta)} d\theta = 2RM_R \int_0^{\pi/2} e^{-mR\sin(\theta)} d\theta. \qquad \textbf{(3.4.18)}$$

We cannot evaluate the integrals in (3.4.18) as they stand. However, because $\sin(\theta) \ge 2\theta/\pi$ if $0 \le \theta \le \pi/2$, we can bound the value of the integral by

$$|I_R| \le 2RM_R \int_0^{\pi/2} e^{-2mR\theta/\pi} d\theta = \frac{\pi}{m} M_R \left(1 - e^{-mR}\right). \qquad \textbf{(3.4.19)}$$

If $m > 0$, $|I_R|$ tends to zero with M_R as $R \to \infty$. □

CONVOLUTION

The most important property of Fourier transforms is convolution. We shall use it extensively in the solution of differential equations and the design of filters because it yields in time or space the effect of multiplying two transforms together.

The convolution operation is

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(x)g(t-x)\, dx = \int_{-\infty}^{\infty} f(t-x)g(x)\, dx. \qquad \textbf{(3.5.1)}$$

Then,

$$\mathcal{F}[f(t) * g(t)] = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} \left[\int_{-\infty}^{\infty} g(t-x)e^{-i\omega(t-x)} dt\right] dx \qquad \textbf{(3.5.2)}$$

$$= \int_{-\infty}^{\infty} f(x)G(\omega)e^{-i\omega x} dx = F(\omega)G(\omega). \qquad \textbf{(3.5.3)}$$

Thus, the Fourier transform of the convolution of two functions equals the product of the Fourier transforms of each of the functions.

CHƯƠNG 4. BIẾN ĐỔI LAPLACE

DEFINITION AND ELEMENTARY PROPERTIES

Consider a function $f(t)$ such that $f(t) = 0$ for $t < 0$. Then the *Laplace integral*

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,dt \qquad \textbf{(4.1.1)}$$

defines the Laplace transform of $f(t)$, which we shall write $\mathcal{L}[f(t)]$ or $F(s)$. The Laplace transform converts a function of t into a function of the transform variable s.

Not all functions have a Laplace transform because the integral (4.1.1) may fail to exist. For example, the function may have infinite discontinuities. For this reason, $f(t) = \tan(t)$ does *not* have a Laplace transform. We may avoid this difficulty by requiring that $f(t)$ be *piece-wise continuous*. That is, we can divide a finite range into a finite number of intervals in such a manner that $f(t)$ is continuous inside each interval and approaches finite values as we approach either end of any interval from the interior.

Another unacceptable function is $f(t) = 1/t$ because the integral (4.1.1) fails to exist. This leads to the requirement that the product $t^n|f(t)|$ is bounded near $t = 0$ for some number $n < 1$.

Finally $|f(t)|$ cannot grow too rapidly or it could overwhelm the e^{-st} term. To express this, we introduce the concept of functions of *exponential order*. By exponential order we mean that there exists some constants, M and k, for which

$$|f(t)| \leq Me^{kt} \qquad \textbf{(4.1.2)}$$

for all $t > 0$. Then, the Laplace transform of $f(t)$ exists if s, or just the real part of s, is greater than k.

In summary, the Laplace transform of $f(t)$ exists, for sufficiently large s, provided $f(t)$ satisfies the following conditions:

- $f(t) = 0$ for $t < 0$,
- $f(t)$ is continuous or piece-wise continuous in every interval,
- $t^n|f(t)| < \infty$ as $t \to 0$ for some number n, where $n < 1$,
- $e^{-s_0 t}|f(t)| < \infty$ as $t \to \infty$, for some number s_0. The quantity s_0 is called the *abscissa of convergence*.

The Laplace transform inherits two important properties from its integral definition. First, the transform of a sum equals the sum of the transforms:

$$\boxed{\mathcal{L}[c_1 f(t) + c_2 g(t)] = c_1 \mathcal{L}[f(t)] + c_2 \mathcal{L}[g(t)].} \qquad \textbf{(4.1.11)}$$

This linearity property holds with complex numbers and functions as well.

The second important property deals with derivatives. Suppose $f(t)$ is continuous and has a piece-wise continuous derivative $f'(t)$. Then

$$\mathcal{L}[f'(t)] = \int_0^\infty f'(t)e^{-st}dt = e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}dt \qquad \textbf{(4.1.17)}$$

by integration by parts. If $f(t)$ is of exponential order, $e^{-st}f(t)$ tends to zero as $t \to \infty$, for large enough s, so that

$$\mathcal{L}[f'(t)] = sF(s) - f(0). \qquad \textbf{(4.1.18)}$$

Similarly, if $f(t)$ and $f'(t)$ are continuous, $f''(t)$ is piece-wise continuous, and all three functions are of exponential order, then

$$\mathcal{L}[f''(t)] = s\mathcal{L}[f'(t)] - f'(0) = s^2F(s) - sf(0) - f'(0). \qquad \textbf{(4.1.19)}$$

In general,

$$\boxed{\mathcal{L}[f^{(n)}(t)] = s^nF(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)} \qquad \textbf{(4.1.20)}$$

on the assumption that $f(t)$ and its first $n-1$ derivatives are continuous, $f^{(n)}(t)$ is piece-wise continuous, and all are of exponential order so that the Laplace transform exists.

Table 4.1.1: The Laplace Transforms of Some Commonly Encountered Functions.

	$f(t),\ t \geq 0$	$F(s)$
1.	1	$\frac{1}{s}$
2.	e^{-at}	$\frac{1}{s+a}$
3.	$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$
4.	$\frac{1}{a-b}(e^{-bt}-e^{-at})$	$\frac{1}{(s+a)(s+b)}$
5.	$\frac{1}{b-a}(be^{-bt}-ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
6.	$\sin(at)$	$\frac{a}{s^2+a^2}$
7.	$\cos(at)$	$\frac{s}{s^2+a^2}$
8.	$\sinh(at)$	$\frac{a}{s^2-a^2}$
9.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
10.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
11.	$1-\cos(at)$	$\frac{a^2}{s(s^2+a^2)}$
12.	$at-\sin(at)$	$\frac{a^3}{s^2(s^2+a^2)}$
13.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
14.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$
15.	$t\sinh(at)$	$\frac{2as}{(s^2-a^2)^2}$
16.	$t\cosh(at)$	$\frac{s^2+a^2}{(s^2-a^2)^2}$
17.	$at\cosh(at)-\sinh(at)$	$\frac{2a^3}{(s^2-a^2)^2}$

Table 4.1.1 (contd.): The Laplace Transforms of Some Commonly Encountered Functions.

	$\mathbf{f(t),\ t \geq 0}$	$\mathbf{F(s)}$
18.	$e^{-bt}\sin(at)$	$\dfrac{a}{(s+b)^2+a^2}$
19.	$e^{-bt}\cos(at)$	$\dfrac{s+b}{(s+b)^2+a^2}$
20.	$(1+a^2t^2)\sin(at)-\cos(at)$	$\dfrac{8a^3s^2}{(s^2+a^2)^3}$
21.	$\sin(at)\cosh(at)-\cos(at)\sinh(at)$	$\dfrac{4a^3}{s^4+4a^4}$
22.	$\sin(at)\sinh(at)$	$\dfrac{2a^2s}{s^4+4a^4}$
23.	$\sinh(at)-\sin(at)$	$\dfrac{2a^3}{s^4-a^4}$
24.	$\cosh(at)-\cos(at)$	$\dfrac{2a^2s}{s^4-a^4}$
25.	$\dfrac{a\sin(at)-b\sin(bt)}{a^2-b^2}, a^2 \neq b^2$	$\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$
26.	$\dfrac{b\sin(at)-a\sin(bt)}{ab(b^2-a^2)}, a^2 \neq b^2$	$\dfrac{1}{(s^2+a^2)(s^2+b^2)}$
27.	$\dfrac{\cos(at)-\cos(bt)}{b^2-a^2}, a^2 \neq b^2$	$\dfrac{s}{(s^2+a^2)(s^2+b^2)}$
28.	$t^n, n \geq 0$	$\dfrac{n!}{s^{n+1}}$
29.	$\dfrac{t^{n-1}e^{-at}}{(n-1)!}, n > 0$	$\dfrac{1}{(s+a)^n}$
30.	$\dfrac{(n-1)-at}{(n-1)!}t^{n-2}e^{-at}, n > 1$	$\dfrac{s}{(s+a)^n}$
31.	$t^n e^{-at}, n \geq 0$	$\dfrac{n!}{(s+a)^{n+1}}$
32.	$\dfrac{2^n t^{n-(1/2)}}{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}, n \geq 1$	$s^{-[n+(1/2)]}$
33.	$J_0(at)$	$\dfrac{1}{\sqrt{s^2+a^2}}$

Table 4.1.1 (contd.): The Laplace Transforms of Some Commonly Encountered Functions.

	$\mathbf{f(t),\ t \geq 0}$	$\mathbf{F(s)}$
34.	$I_0(at)$	$\dfrac{1}{\sqrt{s^2-a^2}}$
35.	$\dfrac{1}{\sqrt{a}}\operatorname{erf}(\sqrt{at})$	$\dfrac{1}{s\sqrt{s+a}}$
36.	$\dfrac{1}{\sqrt{\pi t}}e^{-at}+\sqrt{a}\operatorname{erf}(\sqrt{at})$	$\dfrac{\sqrt{s+a}}{s}$
37.	$\dfrac{1}{\sqrt{\pi t}}-ae^{a^2t}\operatorname{erfc}(a\sqrt{t})$	$\dfrac{1}{a+\sqrt{s}}$
38.	$e^{at}\operatorname{erfc}(\sqrt{at})$	$\dfrac{1}{s+\sqrt{as}}$
39.	$\dfrac{1}{2\sqrt{\pi t^3}}\left(e^{bt}-e^{at}\right)$	$\sqrt{s-a}-\sqrt{s-b}$
40.	$\dfrac{1}{\sqrt{\pi t}}+ae^{a^2t}\operatorname{erf}(a\sqrt{t})$	$\dfrac{\sqrt{s}}{s-a^2}$
41.	$\dfrac{1}{\sqrt{\pi t}}e^{at}(1+2at)$	$\dfrac{s}{(s-a)\sqrt{s-a}}$
42.	$\dfrac{1}{a}e^{a^2t}\operatorname{erf}(a\sqrt{t})$	$\dfrac{1}{(s-a^2)\sqrt{s}}$
43.	$\sqrt{\dfrac{a}{\pi t^3}}e^{-a/t}, a>0$	$e^{-2\sqrt{as}}$
44.	$\dfrac{1}{\sqrt{\pi t}}e^{-a/t}, a\geq 0$	$\dfrac{1}{\sqrt{s}}e^{-2\sqrt{as}}$
45.	$\operatorname{erfc}\left(\sqrt{\dfrac{a}{t}}\right), a\geq 0$	$\dfrac{1}{s}e^{-2\sqrt{as}}$
46.	$2\sqrt{\dfrac{t}{\pi}}\exp\left(-\dfrac{a^2}{4t}\right)-a\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right),\ a\geq 0$	$\dfrac{e^{-a\sqrt{s}}}{s\sqrt{s}}$
47.	$-e^{b^2t+ab}\operatorname{erfc}\left(b\sqrt{t}+\frac{a}{2\sqrt{t}}\right)+\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right), a\geq 0$	$\dfrac{be^{-a\sqrt{s}}}{s(b+\sqrt{s})}$
48.	$e^{ab}e^{b^2t}\operatorname{erfc}\left(b\sqrt{t}+\dfrac{a}{2\sqrt{t}}\right), a\geq 0$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}(b+\sqrt{s})}$

Notes: Error function: $\operatorname{erf}(x)=\dfrac{2}{\pi}\displaystyle\int_0^x e^{-y^2}\,dy$

Complementary error function: $\operatorname{erfc}(x)=1-\operatorname{erf}(x)$

The converse of (4.1.20) is also of some importance. If

$$u(t) = \int_0^t f(\tau)\,d\tau, \tag{4.1.21}$$

then

$$\mathcal{L}[u(t)] = \int_0^\infty e^{-st}\left[\int_0^t f(\tau)\,d\tau\right] dt \tag{4.1.22}$$

$$= -\frac{e^{-st}}{s}\int_0^t f(\tau)\,d\tau\bigg|_0^\infty + \frac{1}{s}\int_0^\infty f(t)e^{-st}\,dt \tag{4.1.23}$$

and

$$\boxed{\mathcal{L}\left[\int_0^t f(\tau)\,d\tau\right] = \frac{F(s)}{s},} \tag{4.1.24}$$

where $u(0) = 0$.

THE HEAVISIDE STEP AND DIRAC DELTA FUNCTIONS

Change can occur abruptly. We throw a switch and electricity suddenly flows. In this section we introduce two functions, the Heaviside step and Dirac delta, that will give us the ability to construct complicated discontinuous functions to express these changes.

Heaviside step function

We define the *Heaviside step function* as

$$H(t-a) = \begin{cases} 1, & t > a \\ 0, & t < a, \end{cases} \tag{4.2.1}$$

where $a \geq 0$. From this definition,

$$\mathcal{L}[H(t-a)] = \int_a^\infty e^{-st}dt = \frac{e^{-as}}{s}, \qquad s > 0. \tag{4.2.2}$$

Note that this transform is identical to that for $f(t) = 1$ if $a = 0$. This should not surprise us. As pointed out earlier, the function $f(t)$ is zero for all $t < 0$ by definition. Thus, when dealing with Laplace transforms $f(t) = 1$ and $H(t)$ are identical. Generally we will take 1 rather than $H(t)$ as the inverse of $1/s$.

The Heaviside step function is essentially a bookkeeping device that gives us the ability to "switch on" and "switch off" a given function. For example, if we want a function $f(t)$ to become nonzero at time $t = a$, we represent this process by the product $f(t)H(t-a)$. On the other hand, if we only want the function to be "turned on" when $a < t < b$, the desired expression is then $f(t)[H(t-a) - H(t-b)]$. For $t < a$, both step functions in the brackets have the value of zero. For $a < t < b$, the first step function has the value of unity and the second step function has the value of zero, so that we have $f(t)$. For $t > b$, both step functions equal unity so that their difference is zero.

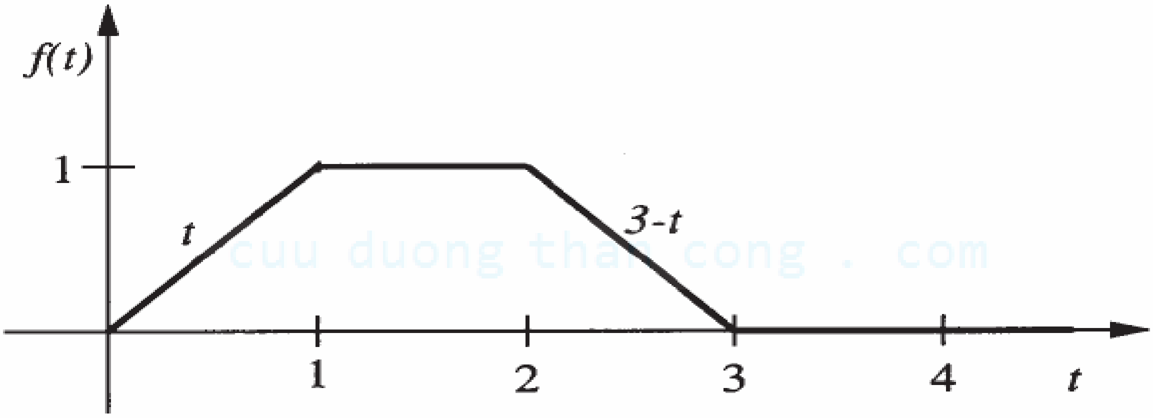

Figure 4.2.2: Graphical representation of (4.2.5).

Consider Figure 4.2.2. We would like to express this graph in terms of Heaviside step functions. We begin by introducing step functions at each point where there is a kink (discontinuity in the first derivative) or jump in the graph – in the present case at $t = 0$, $t = 1$, $t = 2$, and $t = 3$. Thus,

$$f(t) = a_0(t)H(t) + a_1(t)H(t-1) + a_2(t)H(t-2) + a_3(t)H(t-3), \quad \textbf{(4.2.3)}$$

where the coefficients $a_0(t), a_1(t), \ldots$ are yet to be determined. Proceeding from left to right in Figure 4.2.2, the coefficient of each step function equals the mathematical expression that we want after the kink or jump minus the expression before the kink or jump. Thus, in the present example,

$$f(t) = (t-0)H(t) + (1-t)H(t-1) + [(3-t)-1]H(t-2) + [0-(3-t)]H(t-3) \quad \textbf{(4.2.4)}$$

or

$$f(t) = tH(t) - (t-1)H(t-1) - (t-2)H(t-2) + (t-3)H(t-3). \quad \textbf{(4.2.5)}$$

Dirac delta function

The second special function is the *Dirac delta function* or *impulse function*. We define it by

$$\delta(t-a) = \begin{cases} \infty, & t = a \\ 0, & t \neq a, \end{cases} \qquad \int_0^\infty \delta(t-a)\,dt = 1, \tag{4.2.9}$$

where $a \geq 0$.

A popular way of visualizing the delta function is as a very narrow rectangular pulse:

$$\delta(t-a) = \lim_{\epsilon \to 0} \begin{cases} 1/\epsilon, & 0 < |t-a| < \epsilon/2 \\ 0, & |t-a| > \epsilon/2, \end{cases} \tag{4.2.10}$$

where $\epsilon > 0$ is some small number and $a > 0$. This pulse has a width ϵ, height $1/\epsilon$, and centered at $t = a$ so that its area is unity. Now as this pulse shrinks in width ($\epsilon \to 0$), its height increases so that it remains centered at $t = a$ and its area equals unity. If we continue this process, always keeping the area unity and the pulse symmetric about $t = a$, eventually we obtain an extremely narrow, very large amplitude pulse at $t = a$. If we proceed to the limit, where the width approaches zero and the height approaches infinity (but still with unit area), we obtain the delta function $\delta(t-a)$.

The delta function was introduced earlier during our study of Fourier transforms. So what is the difference between the delta function introduced then and the delta function now? Simply put, the delta function can now only be used on the interval $[0, \infty)$. Outside of that, we shall use it very much as we did with Fourier transforms.

Using (4.2.10), the Laplace transform of the delta function is

$$\mathcal{L}[\delta(t-a)] = \int_0^\infty \delta(t-a)e^{-st}\,dt = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{a-\epsilon/2}^{a+\epsilon/2} e^{-st}\,dt \tag{4.2.11}$$

$$= \lim_{\epsilon\to 0}\quad \frac{1}{\epsilon s}\left(e^{-as+\epsilon s/2} - e^{-as-\epsilon s/2}\right) \tag{4.2.12}$$

$$= \lim_{\epsilon\to 0}\frac{e^{-as}}{\epsilon s}\left(1+\frac{\epsilon s}{2}+\frac{\epsilon^2 s^2}{8}+\cdots - 1 + \frac{\epsilon s}{2} - \frac{\epsilon^2 s^2}{8}+\cdots\right) \tag{4.2.13}$$

$$= e^{-as}. \tag{4.2.14}$$

In the special case when $a = 0$, $\mathcal{L}[\delta(t)] = 1$, a property that we will use in Section 4.9. Note that this is exactly the result that we obtained for the Fourier transform of the delta function.

If we integrate the impulse function,

$$\int_0^t \delta(\tau - a)\,d\tau = \begin{cases} 0, & t < a \\ 1, & t > a, \end{cases} \tag{4.2.15}$$

according to whether the impulse does or does not come within the range of integration. This integral gives a result that is precisely the definition of the Heaviside step function so that we can rewrite (4.2.15)

$$\int_0^t \delta(\tau - a)\,d\tau = H(t-a). \tag{4.2.16}$$

Consequently the delta function behaves like the derivative of the step function or

$$\frac{d}{dt}\Big[H(t-a)\Big] = \delta(t-a). \tag{4.2.17}$$

SOME USEFUL THEOREMS

First shifting theorem

Consider the transform of the function $e^{-at}f(t)$, where a is any real number. Then, by definition,

$$\mathcal{L}\left[e^{-at}f(t)\right] = \int_0^\infty e^{-st}e^{-at}f(t)\,dt = \int_0^\infty e^{-(s+a)t}f(t)\,dt, \qquad \textbf{(4.3.1)}$$

or

$$\mathcal{L}\left[e^{-at}f(t)\right] = F(s+a). \qquad \textbf{(4.3.2)}$$

That is, if $F(s)$ is the transform of $f(t)$ and a is a constant, then $F(s+a)$ is the transform of $e^{-at}f(t)$.

Second shifting theorem

The *second shifting theorem* states that if $F(s)$ is the transform of $f(t)$, then $e^{-bs}F(s)$ is the transform of $f(t-b)H(t-b)$, where b is real and positive. To show this, consider the Laplace transform of $f(t-b)H(t-b)$. Then, from the definition,

$$\mathcal{L}[f(t-b)H(t-b)] = \int_0^\infty f(t-b)H(t-b)e^{-st}dt \qquad \textbf{(4.3.8)}$$

$$= \int_b^\infty f(t-b)e^{-st}dt = \int_0^\infty e^{-bs}e^{-sx}f(x)\,dx \qquad \textbf{(4.3.9)}$$

$$= e^{-bs}\int_0^\infty e^{-sx}f(x)\,dx \qquad \textbf{(4.3.10)}$$

or

$$\mathcal{L}[f(t-b)H(t-b)] = e^{-bs}F(s), \qquad \textbf{(4.3.11)}$$

where we have set $x = t-b$. This theorem is of fundamental importance because it allows us to write down the transforms for "delayed" time functions. That is, functions which "turn on" b units after the initial time.

Laplace transform of $t^n f(t)$

In addition to the shifting theorems, there are two other particularly useful theorems that involve the derivative and integral of the transform $F(s)$. For example, if we write

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t) e^{-st}\, dt \tag{4.3.20}$$

and differentiate with respect to s, then

$$F'(s) = \int_0^\infty -t f(t) e^{-st}\, dt = -\mathcal{L}[t f(t)]. \tag{4.3.21}$$

In general, we have that

$$F^{(n)}(s) = (-1)^n \mathcal{L}[t^n f(t)]. \tag{4.3.22}$$

Laplace transform of $f(t)/t$

Consider the following integration of the Laplace transform $F(s)$:

$$\int_s^\infty F(z)\, dz = \int_s^\infty \left[\int_0^\infty f(t) e^{-zt}\, dt \right] dz. \tag{4.3.23}$$

Upon interchanging the order of integration, we find that

$$\int_s^\infty F(z)\, dz = \int_0^\infty f(t) \left[\int_s^\infty e^{-zt}\, dz \right] dt \tag{4.3.24}$$

$$= -\int_0^\infty f(t) \left. \frac{e^{-zt}}{t} \right|_s^\infty dt = \int_0^\infty \frac{f(t)}{t} e^{-st}\, dt. \tag{4.3.25}$$

Therefore,

$$\int_s^\infty F(z)\, dz = \mathcal{L}\left[\frac{f(t)}{t} \right]. \tag{4.3.26}$$

Initial-value theorem

Let $f(t)$ and $f'(t)$ possess Laplace transforms. Then, from the definition of the Laplace transform,

$$\int_0^\infty f'(t)e^{-st}dt = sF(s) - f(0). \tag{4.3.32}$$

Because s is a parameter in (4.3.32) and the existence of the integral is implied by the derivative rule, we can let $s \to \infty$ before we integrate. In that case, the left side of (4.3.32) vanishes to zero, which leads to

$$\lim_{s\to\infty} sF(s) = f(0). \tag{4.3.33}$$

This is the *initial-value theorem.*

Final-value theorem

Let $f(t)$ and $f'(t)$ possess Laplace transforms. Then, in the limit of $s \to 0$, (4.3.32) becomes

$$\int_0^\infty f'(t)\,dt = \lim_{t\to\infty}\int_0^t f'(\tau)\,d\tau = \lim_{t\to\infty} f(t) - f(0) = \lim_{s\to 0} sF(s) - f(0). \tag{4.3.34}$$

Because $f(0)$ is not a function of t or s, the quantity $f(0)$ cancels from the (4.3.34), leaving

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s). \tag{4.3.35}$$

Equation (4.3.35) is the *final-value theorem.* It should be noted that this theorem assumes that $\lim_{t\to\infty} f(t)$ exists. For example, it does not apply to sinusoidal functions. Thus, we must restrict ourselves to Laplace transforms that have singularities in the left half of the s-plane unless they occur at the origin.

THE LAPLACE TRANSFORM OF A PERIODIC FUNCTION

Periodic functions frequently occur in engineering problems and we shall now show how to calculate their transform. They possess the property that $f(t+T) = f(t)$ for $t > 0$ and equal zero for $t < 0$, where T is the period of the function.

For convenience let us define a function $x(t)$ which equals zero except over the interval $(0, T)$ where it equals $f(t)$:

$$x(t) = \begin{cases} f(t), & 0 < t < T \\ 0, & t > T. \end{cases} \tag{4.4.1}$$

By definition

$$F(s) = \int_0^\infty f(t) e^{-st}\, dt \tag{4.4.2}$$

$$= \int_0^T f(t) e^{-st}\, dt + \int_T^{2T} f(t) e^{-st}\, dt + \cdots + \int_{kT}^{(k+1)T} f(t) e^{-st}\, dt + \cdots \tag{4.4.3}$$

Now let $z = t - kT$, where $k = 0, 1, 2, \ldots$, in the kth integral and $F(s)$ becomes

$$F(s) = \int_0^T f(z) e^{-sz}\, dz + \int_0^T f(z+T) e^{-s(z+T)}\, dz + \cdots$$
$$+ \int_0^T f(z+kT) e^{-s(z+kT)}\, dz + \cdots \tag{4.4.4}$$

However,

$$x(z) = f(z) = f(z+T) = \ldots = f(z+kT) = \ldots, \tag{4.4.5}$$

because the range of integration in each integral is from 0 to T. Thus, $F(s)$ becomes

$$F(s) = \int_0^T x(z) e^{-sz}\, dz + e^{-sT} \int_0^T x(z) e^{-sz}\, dz + \cdots$$
$$+ e^{-ksT} \int_0^T x(z) e^{-sz}\, dz + \cdots \tag{4.4.6}$$

or

$$F(s) = (1 + e^{-sT} + e^{-2sT} + \cdots + e^{-ksT} + \cdots) X(s). \tag{4.4.7}$$

The first term on the right side of (4.4.7) is a geometric series with common ratio e^{-sT}. If $|e^{-sT}| < 1$, then the series converges and

$$\boxed{F(s) = \frac{X(s)}{1 - e^{-sT}}.} \tag{4.4.8}$$

INVERSION BY PARTIAL FRACTIONS: HEAVISIDE'S EXPANSION THEOREM

In the previous sections, we have devoted our efforts to calculating the Laplace transform of a given function. Obviously we must have a method for going the other way. Given a transform, we must find the corresponding function. This is often a very formidable task. In the next few sections we shall present some general techniques for the inversion of a Laplace transform.

The first technique involves transforms that we can express as the ratio of two polynomials: $F(s) = q(s)/p(s)$. We shall assume that the order of $q(s)$ is *less* than $p(s)$ and we have divided out any common factor between them. In principle we know that $p(s)$ has n zeros, where n is the order of the $p(s)$ polynomial. Some of the zeros may be complex, some of them may be real, and some of them may be duplicates of other zeros. In the case when $p(s)$ has n simple zeros (nonrepeating roots), a simple method exists for inverting the transform.

We want to rewrite $F(s)$ in the form:

$$F(s) = \frac{a_1}{s - s_1} + \frac{a_2}{s - s_2} + \cdots + \frac{a_n}{s - s_n} = \frac{q(s)}{p(s)}, \tag{4.5.1}$$

where $s_1, s_2, \ldots, s_n$ are the n simple zeros of $p(s)$. We now multiply both sides of (4.5.1) by $s - s_1$ so that

$$\frac{(s - s_1)q(s)}{p(s)} = a_1 + \frac{(s - s_1)a_2}{s - s_2} + \cdots + \frac{(s - s_1)a_n}{s - s_n}. \tag{4.5.2}$$

If we set $s = s_1$, the right side of (4.5.2) becomes simply a_1. The left side takes the form 0/0 and there are two cases. If $p(s) = (s - s_1)g(s)$, then $a_1 = q(s_1)/g(s_1)$. If we cannot explicitly factor out $s - s_1$, l'Hôspital's rule gives

$$a_1 = \lim_{s \to s_1} \frac{(s - s_1)q(s)}{p(s)} = \lim_{s \to s_1} \frac{(s - s_1)q'(s) + q(s)}{p'(s)} = \frac{q(s_1)}{p'(s_1)}. \quad (4.5.3)$$

In a similar manner, we can compute all of the a_k's, where $k = 1, 2, \ldots, n$. Therefore,

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{q(s)}{p(s)}\right] = \mathcal{L}^{-1}\left(\frac{a_1}{s - s_1} + \frac{a_2}{s - s_2} + \cdots + \frac{a_n}{s - s_n}\right) \quad (4.5.4)$$

$$= a_1 e^{s_1 t} + a_2 e^{s_2 t} + \cdots + a_n e^{s_n t}. \quad (4.5.5)$$

This is *Heaviside's expansion theorem*, applicable when $p(s)$ has only simple poles.

Let us now find the expansion when we have multiple roots, namely

$$F(s) = \frac{q(s)}{p(s)} = \frac{q(s)}{(s - s_1)^{m_1}(s - s_2)^{m_2} \cdots (s - s_n)^{m_n}}, \quad (4.5.14)$$

where the order of the denominator, $m_1 + m_2 + \cdots + m_n$, is greater than that for the numerator. Once again we have eliminated any common factor between the numerator and denominator. Now we can write $F(s)$ as

$$F(s) = \sum_{k=1}^{n} \sum_{j=1}^{m_k} \frac{a_{kj}}{(s - s_k)^{m_k - j + 1}}. \quad (4.5.15)$$

Multiplying (4.5.15) by $(s - s_k)^{m_k}$,

$$\frac{(s - s_k)^{m_k} q(s)}{p(s)} = a_{k1} + a_{k2}(s - s_k) + \cdots + a_{km_k}(s - s_k)^{m_k - 1}$$
$$+ (s - s_k)^{m_k}\left[\frac{a_{11}}{(s - s_1)^{m_1}} + \cdots + \frac{a_{nm_n}}{s - s_n}\right], \quad (4.5.16)$$

where we have grouped together into the square-bracketed term all of the terms except for those with a_{kj} coefficients. Taking the limit as $s \longrightarrow s_k$,

$$a_{k1} = \lim_{s \to s_k} \frac{(s - s_k)^{m_k} q(s)}{p(s)}. \tag{4.5.17}$$

Let us now take the derivative of (4.5.16),

$$\begin{aligned} \frac{d}{ds}\left[\frac{(s - s_k)^{m_k} q(s)}{p(s)}\right] & \\ &= a_{k2} + 2a_{k3}(s - s_k) + \cdots + (m_k - 1)a_{km_k}(s - s_k)^{m_k - 2} \\ &+ \frac{d}{ds}\left\{(s - s_k)^{m_k}\left[\frac{a_{11}}{(s - s_1)^{m_1}} + \cdots + \frac{a_{nm_n}}{s - s_n}\right]\right\}. \end{aligned} \tag{4.5.18}$$

Taking the limit as $s \longrightarrow s_k$,

$$a_{k2} = \lim_{s \to s_k} \frac{d}{ds}\left[\frac{(s - s_k)^{m_k} q(s)}{p(s)}\right]. \tag{4.5.19}$$

In general,

$$a_{kj} = \lim_{s \to s_k} \frac{1}{(j-1)!} \frac{d^{j-1}}{ds^{j-1}}\left[\frac{(s - s_k)^{m_k} q(s)}{p(s)}\right] \tag{4.5.20}$$

and by direct inversion,

$$f(t) = \sum_{k=1}^{n} \sum_{j=1}^{m_k} \frac{a_{kj}}{(m_k - j)!} t^{m_k - j} e^{s_k t}. \tag{4.5.21}$$

CONVOLUTION

In this section we turn to a fundamental concept in Laplace transforms: convolution. We shall restrict ourselves to its use in finding the inverse of a transform when that transform consists of the *product* of two simpler transforms. In subsequent sections we will use it to solve ordinary differential equations.

We begin by formally introducing the mathematical operation of the *convolution product*:

$$f(t) * g(t) = \int_0^t f(t - x) g(x)\, dx = \int_0^t f(x) g(t - x)\, dx. \tag{4.6.1}$$

In most cases the operations required by (4.6.1) are straightforward.

INTEGRAL EQUATIONS

An *integral equation* contains the dependent variable under an integral sign. The convolution theorem provides an excellent tool for solving a very special class of these equations, *Volterra equation of the second kind* :[6]

$$f(t) - \int_0^t K[t, x, f(x)]\, dx = g(t), \qquad 0 \le t \le T. \tag{4.7.1}$$

These equations appear in history-dependent problems, such as epidemics,[7] vibration problems,[8] and viscoelasticity.[9]

SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

For the engineer, as it was for Oliver Heaviside, the primary use of Laplace transforms is the solution of ordinary, constant coefficient, linear differential equations. These equations are important not only because they appear in many engineering problems but also because they may serve as approximations, even if locally, to ordinary differential equations with nonconstant coefficients or to nonlinear ordinary differential equations.

For all of these reasons, we wish to solve the *initial-value problem*

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = f(t), \quad t > 0 \tag{4.8.1}$$

by Laplace transforms, where $a_1, a_2, \ldots$ are constants and we know the value of $y, y', \ldots, y^{(n-1)}$ at $t = 0$. The procedure is as follows. Applying the derivative rule (4.1.20) to (4.8.1), we reduce the *differential* equation to an *algebraic* one involving the constants $a_1, a_2, \ldots, a_n$, the parameter s, the Laplace transform of $f(t)$, and the values of the initial conditions. We then solve for the Laplace transform of $y(t)$, $Y(s)$. Finally, we apply one of the many techniques of inverting a Laplace transform to find $y(t)$.

Similar considerations hold with *systems* of ordinary differential equations. The Laplace transform of the system of ordinary differential equations results in an algebraic set of equations containing $Y_1(s), Y_2(s), \ldots, Y_n(s)$. By some method we solve this set of equations and invert each transform $Y_1(s), Y_2(s), \ldots, Y_n(s)$ in turn to give $y_1(t), y_2(t), \ldots, y_n(t)$.

TRANSFER FUNCTIONS, GREEN'S FUNCTION, AND INDICIAL ADMITTANCE

One of the drawbacks of using Laplace transforms to solve ordinary differential equations with a forcing term is its lack of generality. Each new forcing function requires a repetition of the entire process. In this section we give some methods for finding the solution in a somewhat more general manner for stationary systems where the forcing, not any initially stored energy (i.e., nonzero initial conditions), produces the total output. Unfortunately, the solution must be written as an integral.

In Example 4.8.3 we solved the linear differential equation

$$y'' + 2y' + y = f(t) \tag{4.9.1}$$

subject to the initial conditions $y(0) = y'(0) = 0$. At that time we wrote the Laplace transform of $y(t)$, $Y(s)$, as the product of two Laplace transforms:

$$Y(s) = \frac{1}{(s+1)^2} F(s). \tag{4.9.2}$$

One drawback in using (4.9.2) is its dependence upon an unspecified Laplace transform $F(s)$. Is there a way to eliminate this dependence and yet retain the essence of the solution?

One way of obtaining a quantity that is independent of the forcing is to consider the ratio:

$$\frac{Y(s)}{F(s)} = G(s) = \frac{1}{(s+1)^2}. \tag{4.9.3}$$

This ratio is called the *transfer function* because we can transfer the input $F(s)$ into the output $Y(s)$ by multiplying $F(s)$ by $G(s)$. It depends only upon the properties of the system.

Let us now consider a related problem to (4.9.1), namely

$$g'' + 2g' + g = \delta(t), \qquad t > 0 \tag{4.9.4}$$

with $g(0) = g'(0) = 0$. Because the forcing equals the Dirac delta function, $g(t)$ is called the *impulse response* or *Green's function*.[23] Computing $G(s)$,

$$G(s) = \frac{1}{(s+1)^2}. \tag{4.9.5}$$

From (4.9.3) we see that $G(s)$ is also the transfer function. Thus, an alternative method for computing the transfer function is to subject the system to impulse forcing and the Laplace transform of the response is the transfer function.

From (4.9.3),

$$Y(s) = G(s)F(s) \tag{4.9.6}$$

or

$$y(t) = g(t) * f(t). \tag{4.9.7}$$

That is, the convolution of the impulse response with the particular forcing gives the response of the system. Thus, we may describe a stationary system in one of two ways: (1) in the transform domain we have the transfer function, and (2) in the time domain there is the impulse response.

Despite the fundamental importance of the impulse response or Green's function for a given linear system, it is often quite difficult to determine, especially experimentally, and a more convenient practice is to deal with the response to the unit step $H(t)$. This response is called the *indicial admittance* or *step response*, which we shall denote by $a(t)$.

Because $\mathcal{L}[H(t)] = 1/s$, we can determine the transfer function from the indicial admittance because $\mathcal{L}[a(t)] = G(s)\mathcal{L}[H(t)]$ or $sA(s) = G(s)$. Furthermore, because

$$\mathcal{L}[g(t)] = G(s) = \frac{\mathcal{L}[a(t)]}{\mathcal{L}[H(t)]}, \tag{4.9.8}$$

then

$$g(t) = \frac{da(t)}{dt} \tag{4.9.9}$$

from (4.1.18).

INVERSION BY CONTOUR INTEGRATION

In Sections 4.5 and 4.6 we showed how we may use partial fractions and convolution to find the inverse of the Laplace transform $F(s)$. In many instances these methods fail simply because of the complexity of the transform to be inverted. In this section we shall show how we may invert transforms through the powerful method of contour integration. Of course, the student must be proficient in the use of complex variables.

Consider the piece-wise differentiable function $f(x)$ which vanishes for $x < 0$. We can express the function $e^{-cx}f(x)$ by the complex Fourier representation of

$$f(x)e^{-cx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\left[\int_0^{\infty} e^{-ct}f(t)e^{-i\omega t}\,dt\right]d\omega, \tag{4.10.1}$$

for any value of the real constant c, where the integral

$$I = \int_0^{\infty} e^{-ct}|f(t)|\,dt \tag{4.10.2}$$

exists. By multiplying both sides of (4.10.1) by e^{cx} and bringing it inside the first integral,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(c+\omega i)x}\left[\int_0^{\infty} f(t)e^{-(c+\omega i)t}\,dt\right]d\omega. \tag{4.10.3}$$

With the substitution $z = c + \omega i$, where z is a new, complex variable of integration,

$$f(x) = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} e^{zx}\left[\int_0^{\infty} f(t)e^{-zt}\,dt\right]dz. \tag{4.10.4}$$

The quantity inside the square brackets is the Laplace transform $F(z)$. Therefore, we can express $f(t)$ in terms of its transform by the complex contour integral:

$$\boxed{f(t) = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} F(z)e^{tz}\,dz.} \tag{4.10.5}$$