Hypothesis Test

“ **Hypothesis Test**”: A procedure for deciding between two hypotheses (null hypothesis – alternative hypothesis) on the basis of observations in a random sample

One – sample Hypothesis test

- **Compare proportion to a given value of rate**

- **Compare mean value to a given value of expectation**

Test 1. *Compare proportion to a given rate*

$(X_1, X_2, \ldots, X_n)$ - a sample of ***n*** independent observations collected from a binary variable ***X*** taking value 1 with (unknown) probability p $(0 < p < 1)$ and value 0 with probability $1 - p$ → Given a number q, how to have a conclusion comparing p with q based on information of the sample?

(Null) Hypothesis

H: *p* = *q*

Alternative Hypothesis

K: *p differs from q*

(Two tails Hypothesis Test)

Solution

By Moivre-Laplace Theorem, for large sample size, sample proportion $m(p)/n$ of appearance of number 1 has distribution approximate to normal distribution with expectation p and variance $p \,.\, (1\text{-}p) \,/\, n$. Then a testing procedure can be as follows:

Step 1. Estimate a sample proportion by

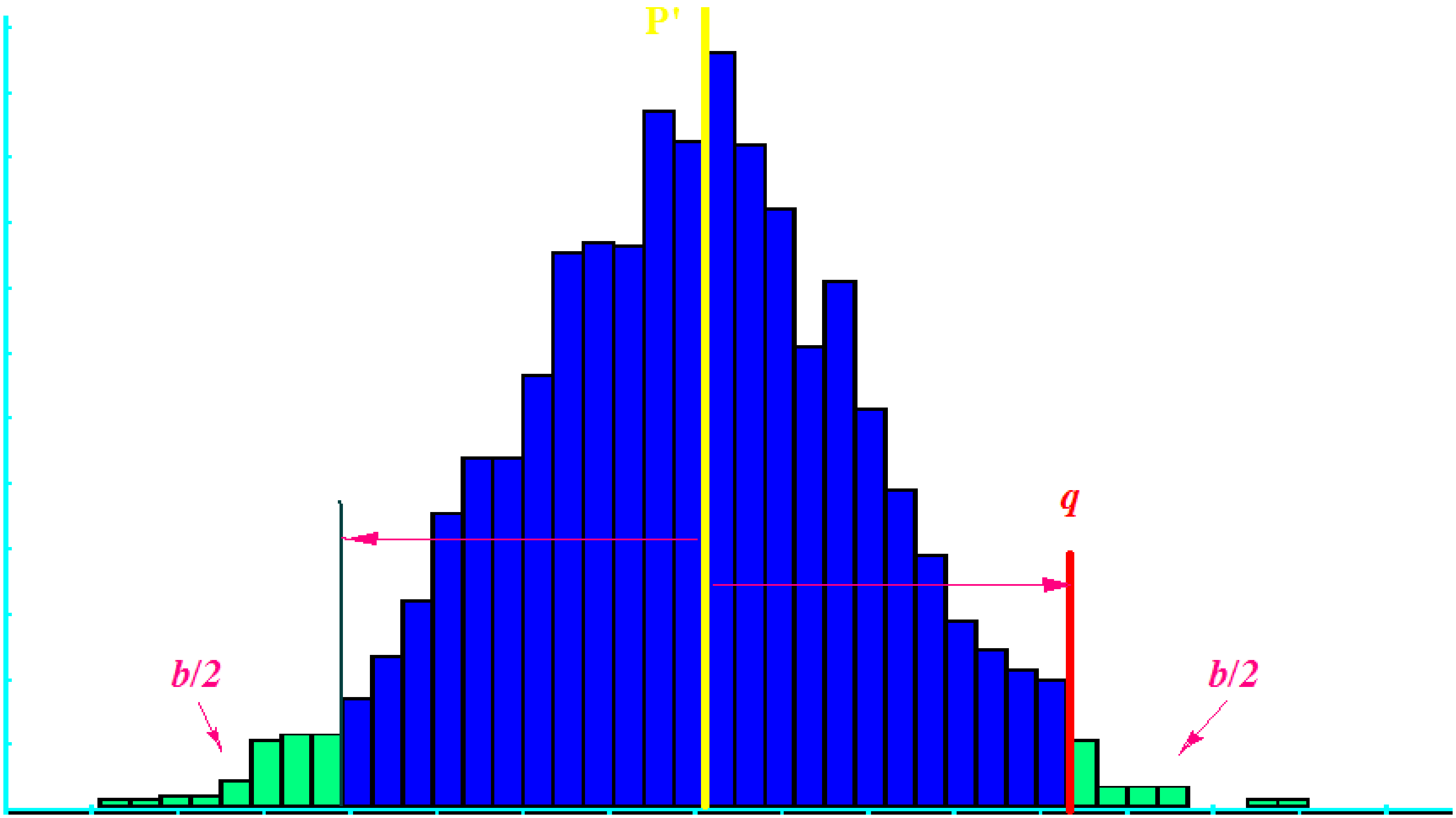
$$p' = m(p) \,/\, n$$

Step 2.

Version A (by computer): Calculate the probability (using normal distribution with expectation ***p'*** and variance ***p' . (1-p') / n***) such that a estimate point should appear at a location with distance to the ***p'*** longer than

$$| q - p' |$$

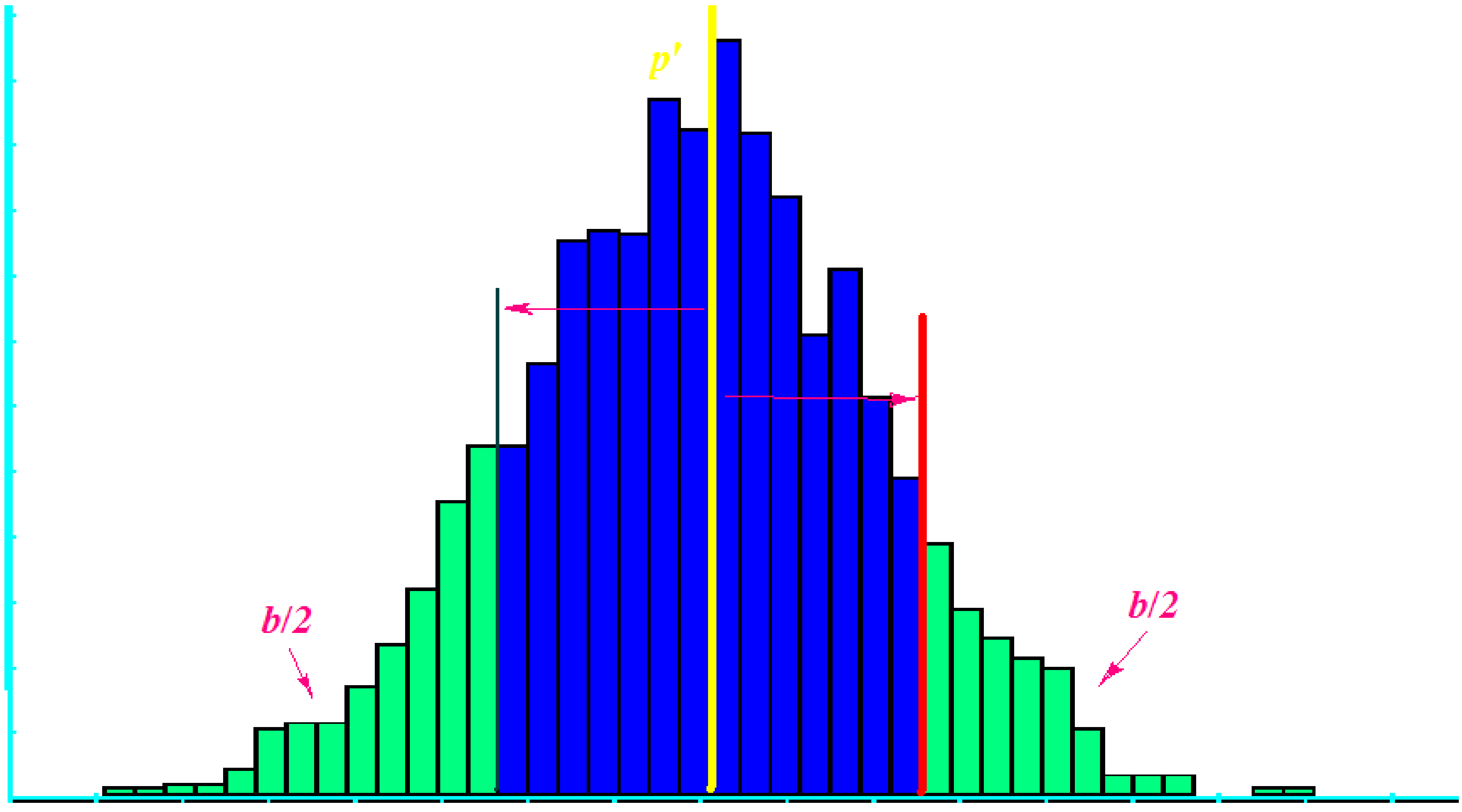
= Probability ***b*** (called ***p-value***) of wrong decision of excluding estimation value ***q*** (saying that ***q*** differs from true value of ***p***) when this value should be a "good" value of estimation


P'
q
b/2
b/2

Step 3. Compare **b** with a given confidence level **alpha** (**5%,** 1%, 0.5% or 0.1%)

- If **b** < **alpha** → ***reject the hypothesis*** H, conclude that ***q*** differs from ***p*** , because possibility of getting mistake in decision is "very small"

* If **b** > **alpha** → ***accept the hypothesis*** H, confirm ***q* = *p*** , because possibility of having mistake by rejecting the hypothesis is too large


p'
b/2
b/2

Version B. (Calculate by hand, using critical value)

Using Table of Normal Distribution to have a ***critical value Z(alpha/2)*** with given confidence level ***alpha*** (***5%, 1%*** or ***0.5%,*** for ***alpha = 5%*** we have ***Z(alpha/2) = 1.96***) and calculate the value

$$U = |p' - q| / \left[\sqrt{p'.(1 - p')/n}\right]$$

Decide

Reject the Hypothesis **H** if ***U > Z(alpha/2)***

Accept the Hypothesis **H** if ***U =< Z(alpha/2)***

Version C. Using confidence intervals

With ***confidence level*** of **5%**, we can use confidence intervals for hypothesis testing:

$$\left[p' - 1.96 * \sqrt{p'.(1 - p')/n} ; p' + 1.96 * \sqrt{p.(1 - p)/n} \right]$$

Decide

- ***Reject the Hypothesis*** **H** if the confidence interval does not contain the point ***q***
- ***Accept the Hypothesis*** **H** if the confidence interval contains the point ***q***

Test 1A. *Compare proportion to a given value - One-tail test*

With a sample of n independent observations collected from a binary variable X taking value 1 with (unknown) probability p $(0 < p < 1)$ and value 0 with probability $1 - p$ → Given a number q , can we conclude that $p < q$?

Null Hypothesis

$$H: \ q = p$$

Alternative Hypothesis

$$K: \ q > p$$

Steps of testing

Step 1. Estimate sample proportion by

$$p' = m(p) / n$$

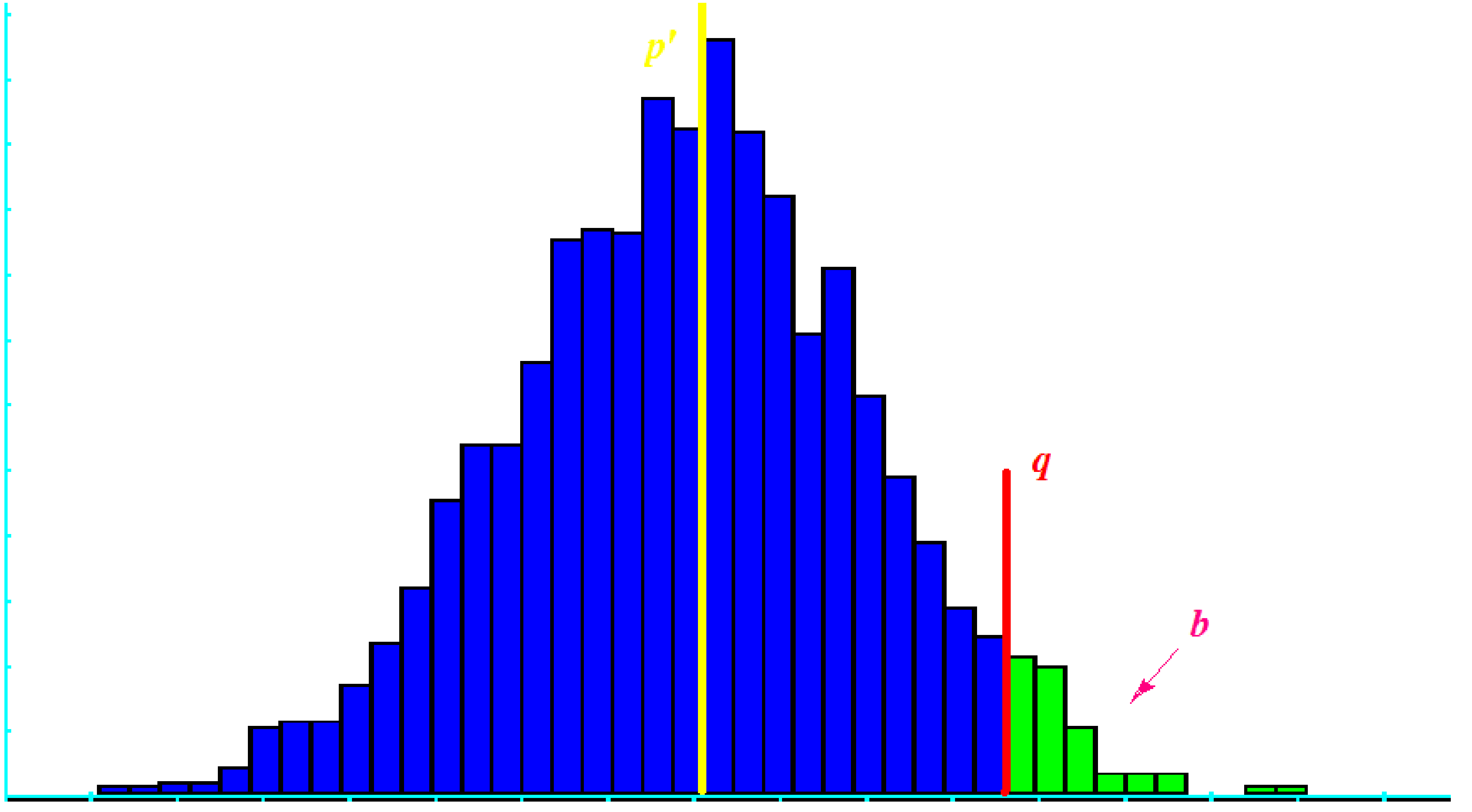
Step 2. Using normal distribution (with expectation p' and variance $p' . (1-p') / n$) calculate the probability of estimation value being greater than q .

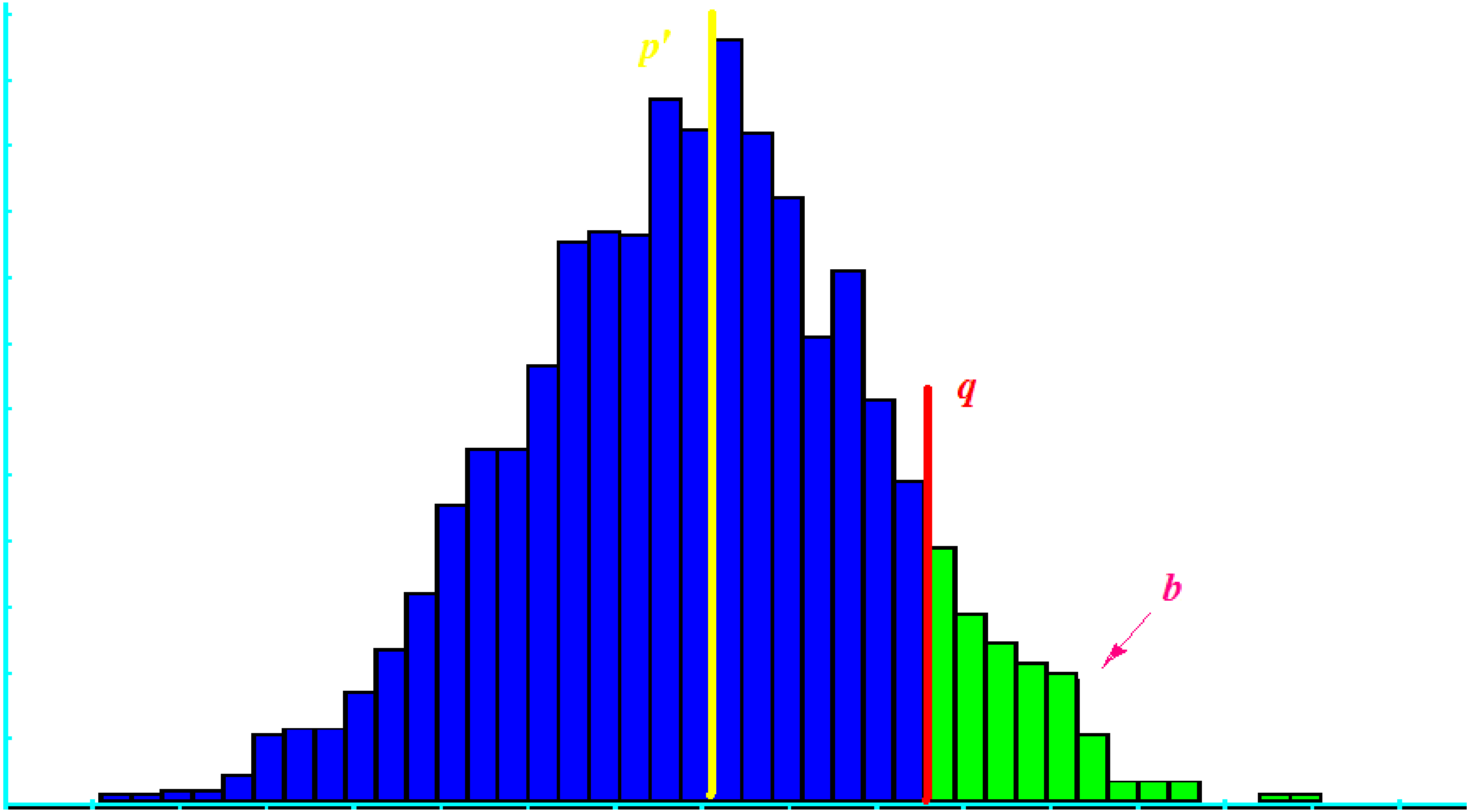
= probability **b** of those estimated values which should be rejected by chance

Step 3 Compare b to a given confidence level alpha (5%, 1%, 0.5% or 0.1%)

- If **b < alpha** → reject the hypothesis H and confirming **q > p** (because probability to get wrong conclusion is small enough)

* If **b > alpha** → accept the hypothesis H, confirming **q = p** , (because possibility to meet mistake accepting **q > p** should be too large)


p'
q
b


p'
q
b

Note 1

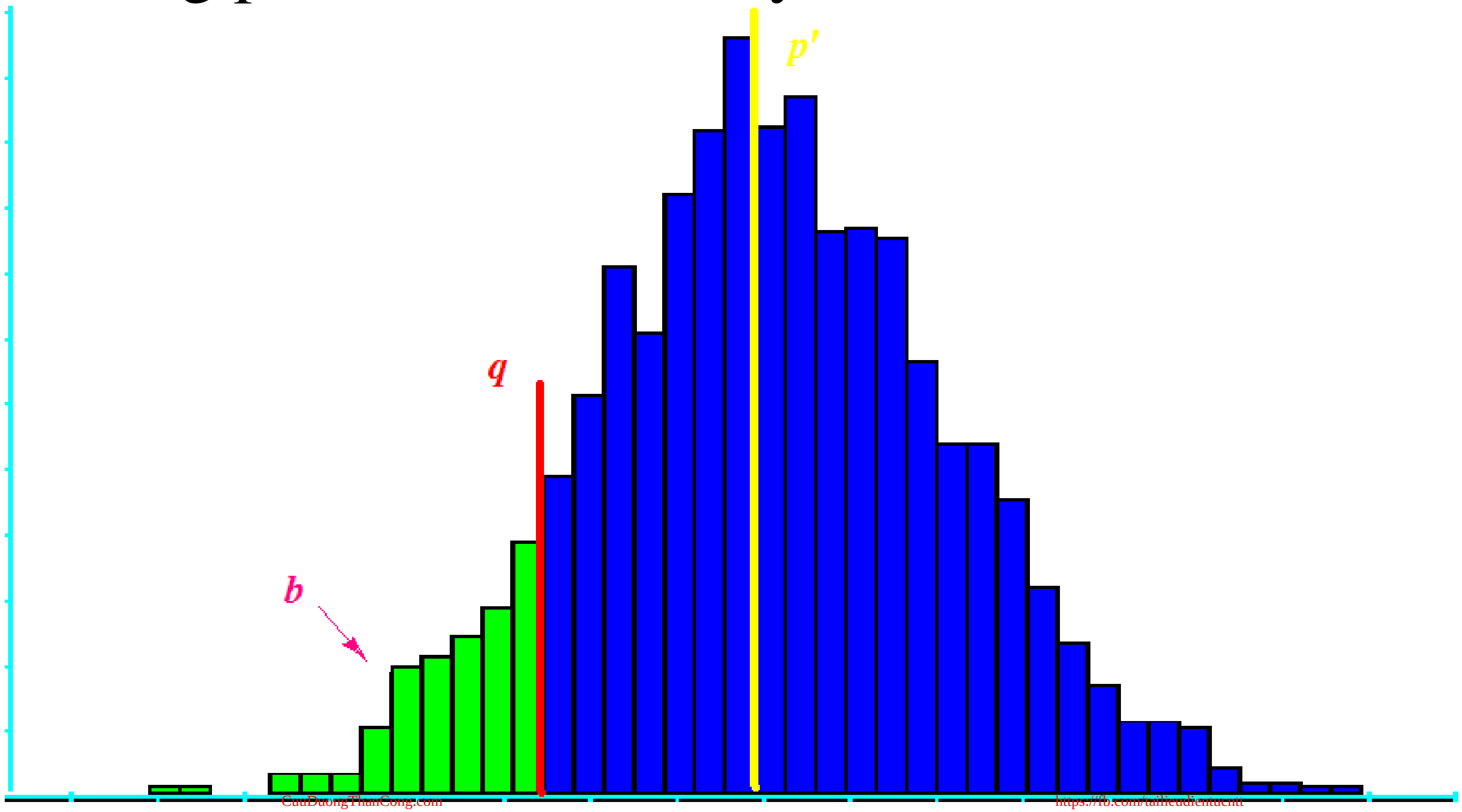
For Hypothesis

H: q = p

with Alternative Hypothesis

K: q < p

the testing procedure is exactly the same



Test 2. Compare mean value to a given value of expectation

Problem: Taking a sample from a variable X with normal distribution (or sample size be large), we need to compare the sample mean Mean(X) to a given value **a**. Then there are 3 types of test

A. **Two-tail Test:** **Hypothesis**

H: Mean(X) = **a**

Alternative Hypothesis

K: Mean(X) differs from **a**

B. "Right hand side" **One-tail Test:**

Hypothesis

H: Mean(X) = **a**

Alternative Hypothesis

K: Mean(X) > **a**

C. "Left hand side" **One-tail Test**: **Hypothesis**

H: Mean(X) = **a**

Alternative Hypothesis

K: Mean(X) < **a**

For testing the above hypothesis, the distribution of sample mean value must be known. Meantime the variance of the variable X is unknown and must be estimated. Then the following theorem can be applied:

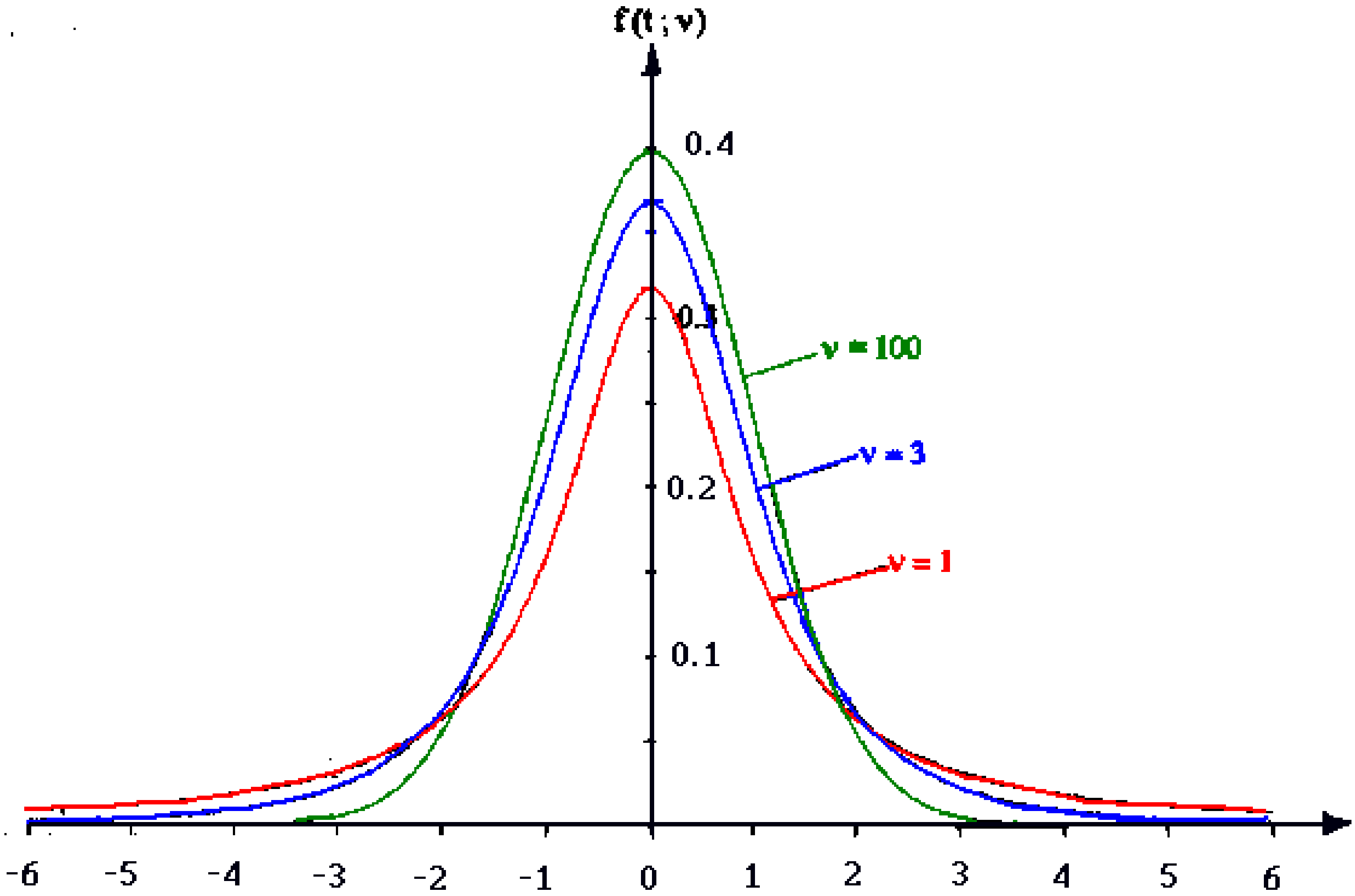
Theorem. Let $(X_1, X_2, \ldots, X_n)$ be a sample of *n* independent observations taken from a normal distributed variable X with expectation μ, $\overline{X}$ is sample mean value and S^2 is sample variance. Then the (new) variable

$$t = \frac{\sqrt{n-1}}{S}.(\overline{X} - \mu)$$

has T-Student distribution with *(n-1)* degrees of freedom.

Remark. By Central Limit Theorem, when sample size is large, distribution of sample mean value is approximate to normal distribution. Then the above theorem can be applied also for testing hypothesis comparing mean value of variable with non-normal distribution

Student (T) distribution



Parameter of Student Distribution: “Degree of Fredom” ν

Steps of Testing

Step 1. Estimate sample Mean Value Mean(X)
Standard Deviation SD(X)

Step 2. Calculate the statistic

$$t(a) = \frac{\sqrt{n-1}.(Mean(X)-a)}{SD(X)}$$

where ***n*** is sample size and ***a*** is the given value to which the mean value has be compared

Step 3 (Version A – by computer). Taking a Student distribution variable ***T(n-1)*** with (n-1) degree of freedom and calculate the probability

$$b = P\{\ |T(n-1)|\ >=\ |t(a)|\ \}$$

for two-tails test,

$$b = P\ \{\ T(n-1)\ >=\ t(a)\ \}$$

for right side one-tail test, and

$$b = P\ \{\ T(n-1)\ =<\ t(a)\ \}$$

for left side one-tail test (then **t(a)** < **0**)

Step 4. Compare the probability ***b*** with a given ahead level of significance ***alpha*** (= 5%, 1%, 0.5%, 0.1%, etc.):

If ***b* >= *alpha*** → accept the hypothesis **H** , conclude

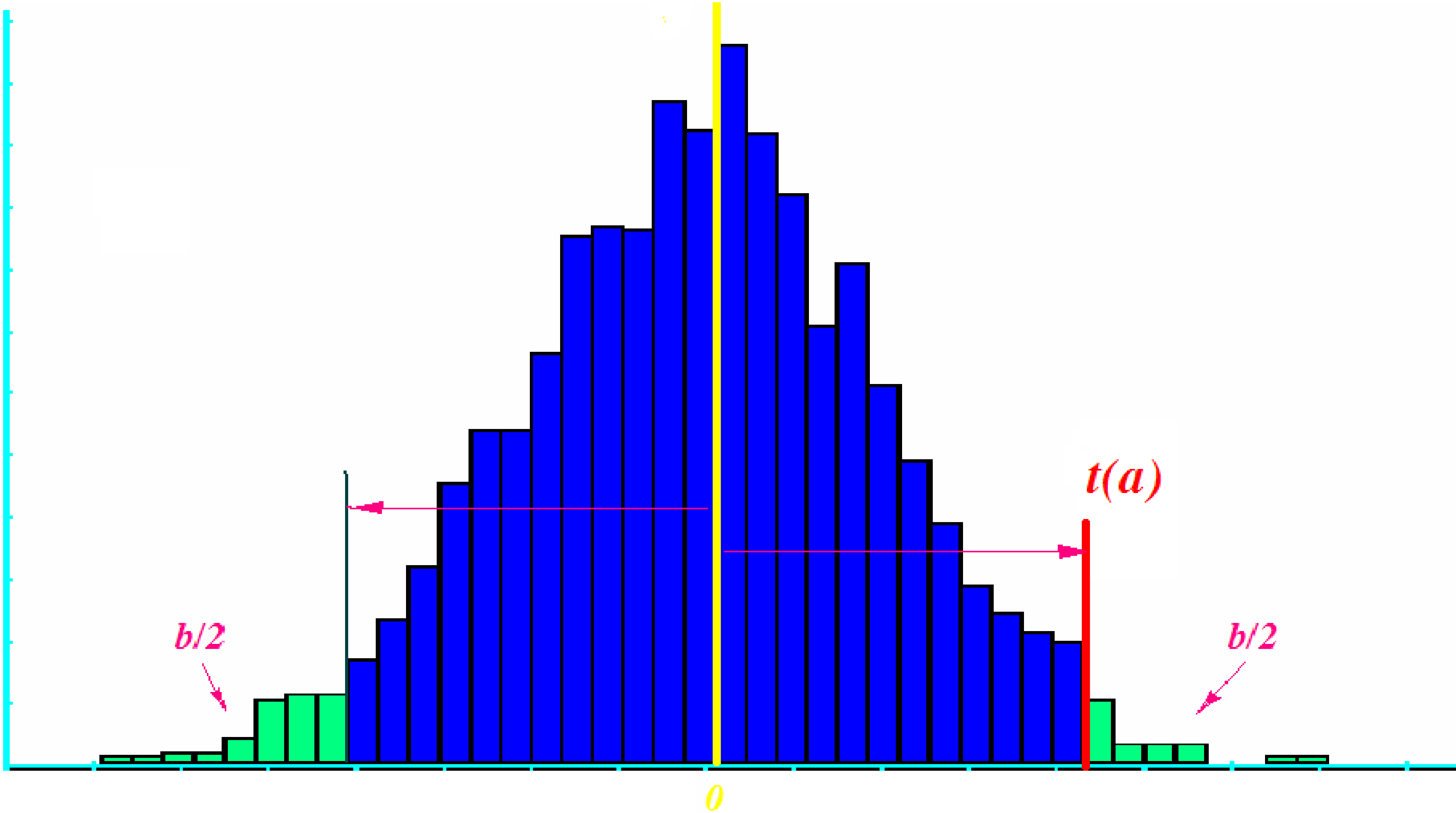
Mean(X)* = *a

If ***b*** < ***alpha*** → reject the hypothesis **H** :

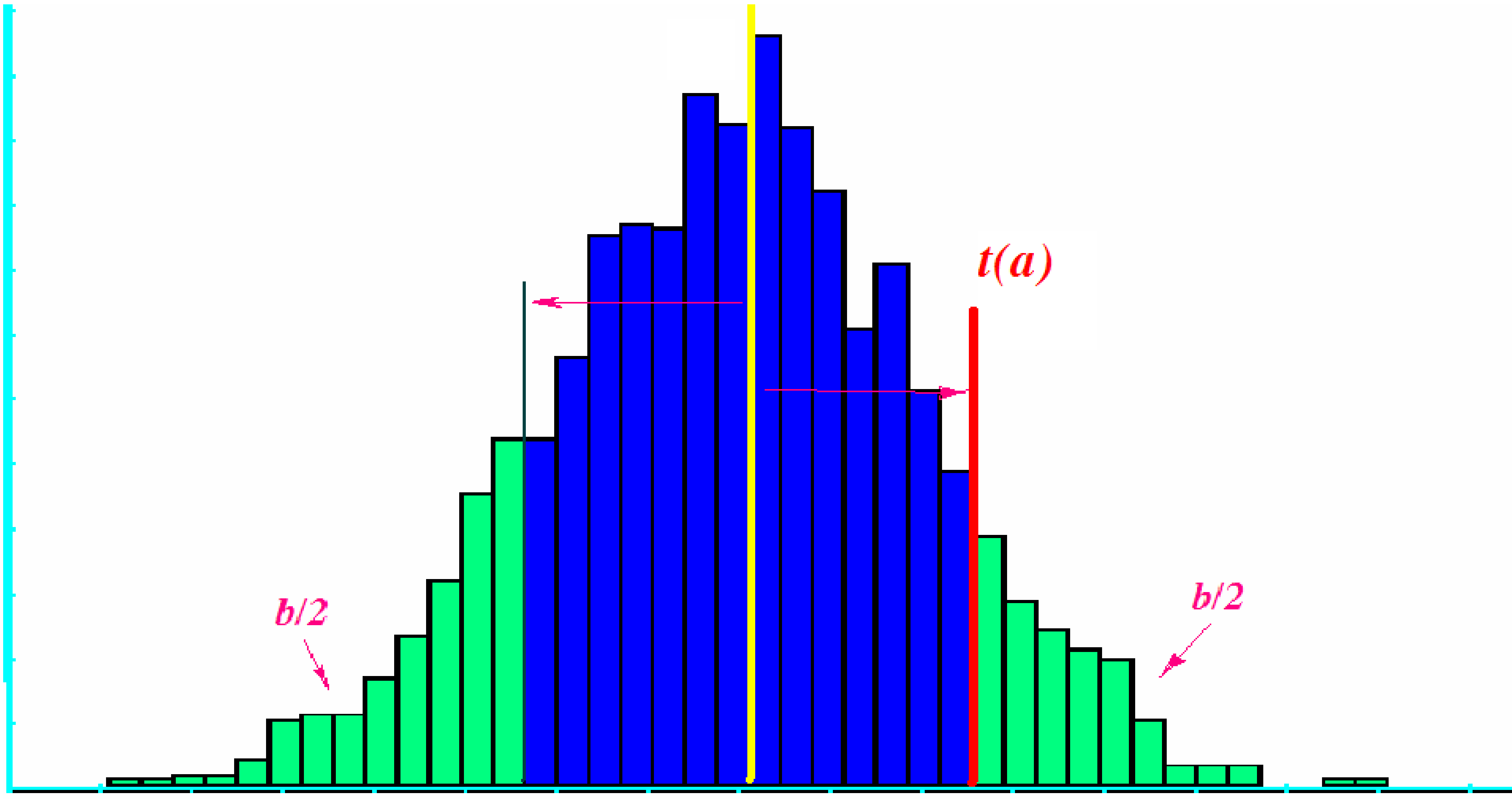
- Declare ***Mean(X)*** ***differs from a*** (for the two+tails test)

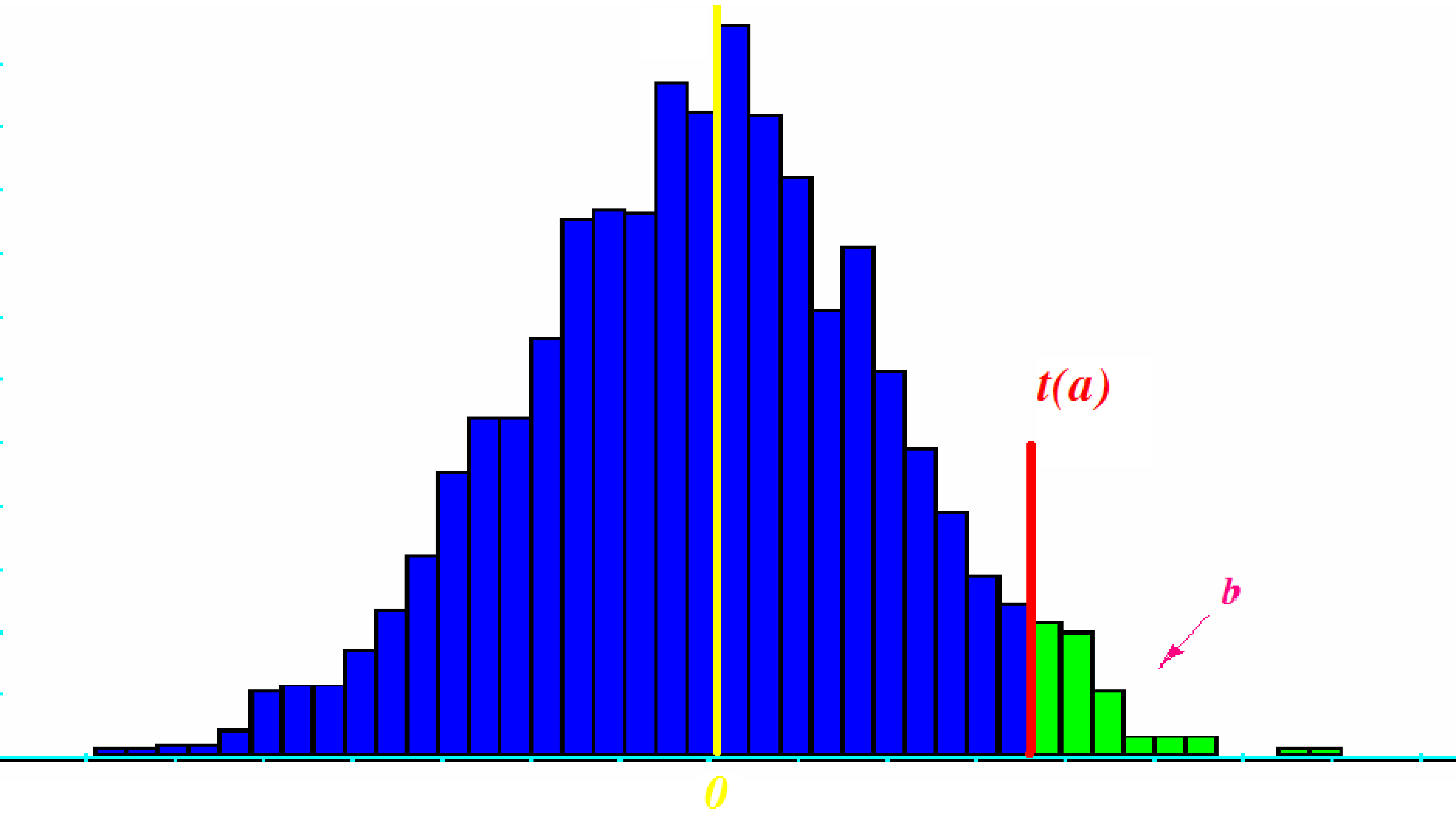
- Declare ***Mean(X)*** > ***a*** (for right side one-tail test)

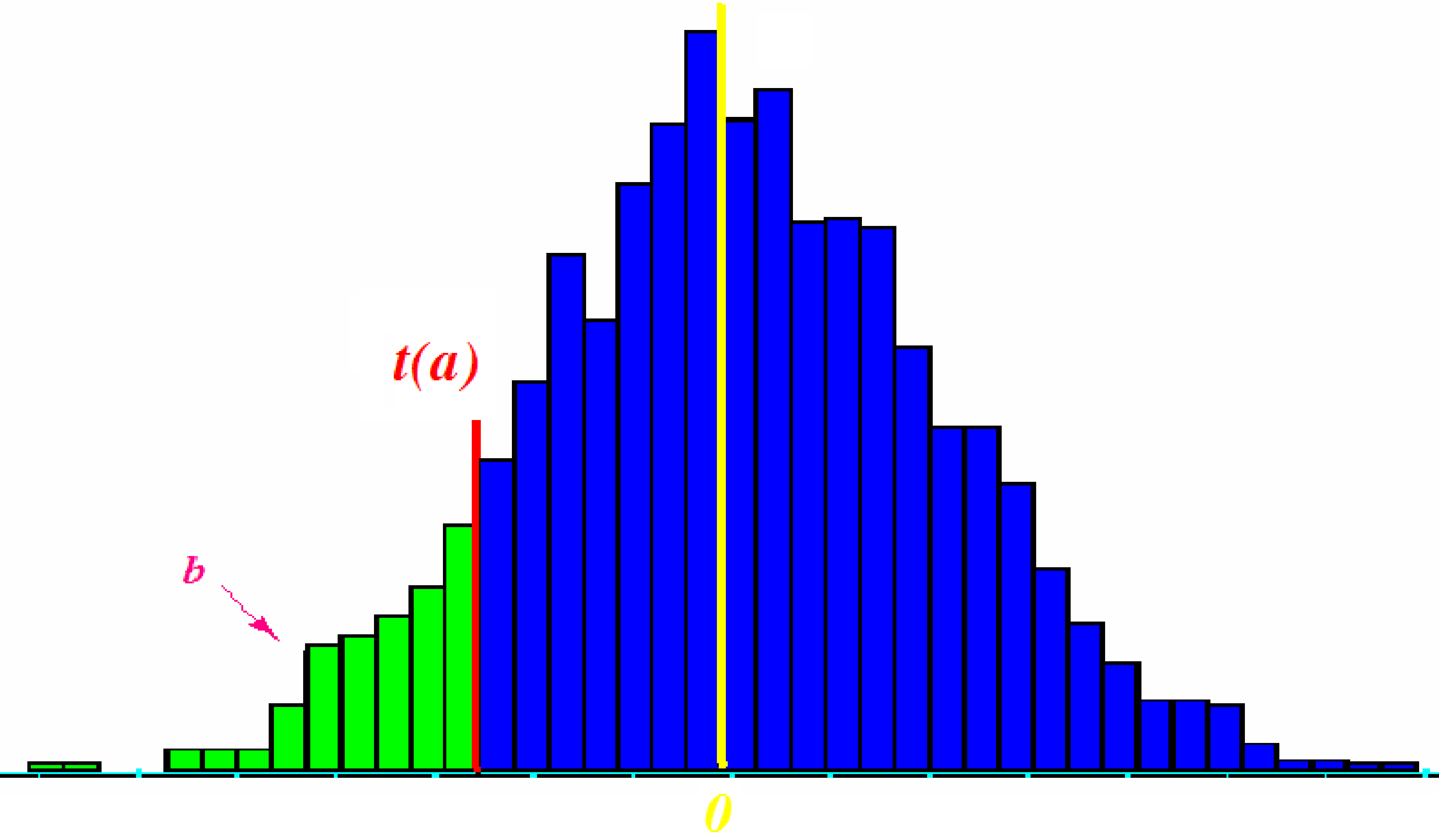
- Declare ***Mean(X)*** < ***a*** (for left side one-tail test)



t(a)
b/2
b/2
0


t(a)
b/2
b/2


t(a)
b
0


t(a)
b
0

Version B. Using table of distribution (for calculation by hand)

Looking in the table of Student distribution for critical value *T(n,alpha/2)* with *n* is degree of freedom and alpha is a given ahead significance level (5%, 1% or 0.5%)

Decide

Reject the hypothesis **H: =** if

t(a) > T(n,alpha/2)

Accept the hypothesis **H: =** if

t(a) =< T(n,alpha/2)

Version C. Using confidence interval

When the sample size n is large, Student distribution approximates to Normal distribution. Then with significance level of 5%, we can use 95% confidence interval for testing the hypothesis:

$$\left[Mean(X) - 1.96 * SD(X)/\sqrt{n} ; Mean(X) + 1.96 * SD(X)/\sqrt{n} \right]$$

Decide

- Reject the hypothesis **H: =** if a is found outsides the interval

- Accept the hypothesis **H: =** if a is a inside point of the interval