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INCOS
Home of Computational Scientists

SOLID MECHANICS

Chapter 1: Mathematical Preliminaries

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1.1 Common Variable Types in Elasticity

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Elasticity theory is a mathematical model of material deformation. Using principles of continuum mechanics, it is formulated in terms of many different types of field variables specified at spatial points in the body under study. Some examples include:

Scalars - Single magnitude

mass density ρ, temperature *T*, modulus of elasticity *E*, . . .

Vectors – Three components in three dimensions

displacement vector $\mathbf{u} = u\mathbf{e}_1 + v\mathbf{e}_2 + w\mathbf{e}_3$ $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are unit basis vectors

Matrices – Nine components in three dimensions

stress matrix

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

Other – Variables with more than nine components

With the wide variety of variables, elasticity formulation makes use of a *tensor formalism* using index notation. This enables efficient representation of all variables and governing equations using a single standardized method.

Index notation is a shorthand scheme whereby a whole set of numbers or components can be represented by a single symbol with subscripts

$$a_i = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

In general a symbol $a_{ij\ldots k}$ with N distinct indices represents 3^N distinct numbers

Addition, subtraction, multiplication and equality of index symbols are defined in the normal fashion; e.g.

$$a_i \pm b_i = \begin{bmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ a_3 \pm b_3 \end{bmatrix}, \quad a_{ij} \pm b_{ij} = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & a_{23} \pm b_{23} \\ a_{31} \pm b_{31} & a_{32} \pm b_{32} & a_{33} \pm b_{33} \end{bmatrix}$$

$$\lambda a_i = \begin{bmatrix} \lambda a_1 \\ \lambda a_2 \\ \lambda a_3 \end{bmatrix}, \quad \lambda a_{ij} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \\ \lambda a_{31} & \lambda a_{32} & \lambda a_{33} \end{bmatrix} \qquad a_i b_j = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

Summation Convention - if a subscript appears twice in the same term, then *summation* over that subscript from one to three is implied; for example

$$a_{ii} = \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33}$$

$$a_{ij}b_j = \sum_{j=1}^{3} a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3$$

A symbol $a_{i\ldots j\ldots m\ldots n\ldots}$ is said to be *symmetric* with respect to index pair *mn* if

$$a_{i\ldots j\ldots m\ldots n\ldots} = a_{\ldots n\ldots m\ldots j\ldots i}$$

A symbol $a_{i\ldots j\ldots m\ldots n\ldots}$ is said to be *antisymmetric* with respect to index pair *mn* if

$$a_{i\ldots j\ldots m\ldots n\ldots} = -a_{\ldots n\ldots m\ldots j\ldots i}$$

If $a_{i\ldots j\ldots m\ldots n\ldots}$ is *symmetric* in *mn* while $b_{p\ldots q\ldots m\ldots n\ldots}$ is antisymetric in *mn*, then the product is zero

$$a_{i\ldots j\ldots m\ldots n\ldots} b_{p\ldots q\ldots m\ldots n\ldots} = 0$$

Useful Identity $a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji}) = a_{(ij)} + a_{[ij]}$

$a_{(ij)} = \frac{1}{2}(a_{ij} + a_{ji}) \ldots$ symmetric $\quad a_{[ij]} = \frac{1}{2}(a_{ij} - a_{ji}) \ldots$ antisymmetric

Example 1-1: Index Notation Examples

The matrix a_{ij} and vector b_i are specified by

$$a_{ij} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 3 \\ 2 & 1 & 2 \end{bmatrix}, \; b_i = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

Determine the following quantities:

$a_{ii}, a_{ij}a_{ij}, a_{ij}a_{jk}, a_{ij}b_j, a_{ij}b_ib_j, b_ib_i, b_ib_j, a_{(ij)}, a_{[ij]}$

Indicate whether they are a scalar, vector or matrix.

Following the standard definitions given in section 1.2,

$$a_{ii} = a_{11} + a_{22} + a_{33} = 7 \text{ (scalar)}$$

$$a_{ij}a_{ij} = a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} + a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} + a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33}$$
$$= 1 + 4 + 0 + 0 + 16 + 9 + 4 + 1 + 4 = 39 \text{ (scalar)}$$

$$a_{ij}a_{jk} = a_{i1}a_{1k} + a_{i2}a_{2k} + a_{i3}a_{3k} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 2 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 8 & 6 \\ 0 & 16 & 12 \\ 0 & 4 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 6 & 3 & 6 \\ 4 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 10 & 6 \\ 6 & 19 & 18 \\ 6 & 10 & 7 \end{bmatrix} \text{ (matrix)}$$

$$a_{ij}b_j = a_{i1}b_1 + a_{i2}b_2 + a_{i3}b_3 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 8 \\ 16 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 16 \\ 8 \end{bmatrix} \text{ (vector)}$$

$$a_{ij}b_ib_j = a_{11}b_1b_1 + a_{12}b_1b_2 + a_{13}b_1b_3 + a_{21}b_2b_1 + \cdots = 84 \text{ (scalar)}$$

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Indicate whether they are a scalar, vector or matrix.

Following the standard definitions given in section 1.2,

$$b_ib_i = b_1b_1 + b_2b_2 + b_3b_3 = 4 + 16 + 0 = 20 \text{ (scalar)}$$

$$b_ib_j = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 0 \\ 8 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ (matrix)}$$

$$a_{(ij)} = \frac{1}{2}(a_{ij} + a_{ji}) = \frac{1}{2}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 3 \\ 2 & 1 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 1 \\ 0 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 2 \end{bmatrix} \text{ (matrix)}$$

$$a_{[ij]} = \frac{1}{2}(a_{ij} - a_{ji}) = \frac{1}{2}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 3 \\ 2 & 1 & 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 1 \\ 0 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \text{ (matrix)}$$

Kronecker Delta

$$\delta_{ij} = \begin{cases} 1\,, \text{ if } i = j \,(\text{no sum}) \\ 0\,, \text{ if } i \neq j \end{cases} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Properties:

$$\delta_{ij} = \delta_{ji},\ \delta_{ii} = 3,$$
$$\delta_{ij} a_j = a_i\,,\ \delta_{ij} a_i = a_j$$
$$\delta_{ij} a_{jk} = a_{ik}\,,\ \delta_{jk} a_{ik} = a_{ij}$$
$$\delta_{ij} a_{ij} = a_{ii}\,,\ \delta_{ij}\delta_{ij} = 3$$

Alternating or Permutation Symbol

$$\varepsilon_{ijk} = \begin{cases} +1\,, \text{ if } ijk \text{ is an even permutation of } 1,2,3 \\ -1\,, \text{ if } ijk \text{ is an odd permutation of } 1,2,3 \\ 0\,, \text{ otherwise} \end{cases}$$

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$$
$$\varepsilon_{321} = \varepsilon_{132} = \varepsilon_{213} = -1$$
$$\varepsilon_{112} = \varepsilon_{131} = \varepsilon_{222} = \ldots = 0$$

$$\delta_{ij} = \begin{cases} 1, \text{ if } i = j \,(\text{no sum}) \\ 0, \text{ if } i \neq j \end{cases}$$

$$\varepsilon_{ijk} = \begin{cases} +1, \text{ if } ijk \text{ is an even permutation of } 1,2,3 \\ -1, \text{ if } ijk \text{ is an odd permutation of } 1,2,3 \\ 0, \text{ otherwise} \end{cases}$$

Useful in evaluating determinants
and vector cross-products

$$\det\left[a_{ij}\right] = \left|a_{ij}\right| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k} = \varepsilon_{ijk} a_{i1} a_{j2} a_{k3}$$

If we use the property

$$\varepsilon_{ijk}\varepsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix}$$

$$\Rightarrow \det\left[a_{ij}\right] = \frac{1}{6}\varepsilon_{ijk}\varepsilon_{pqr} a_{ip} a_{jq} a_{kr}$$

1.4 Coordinate Transformations

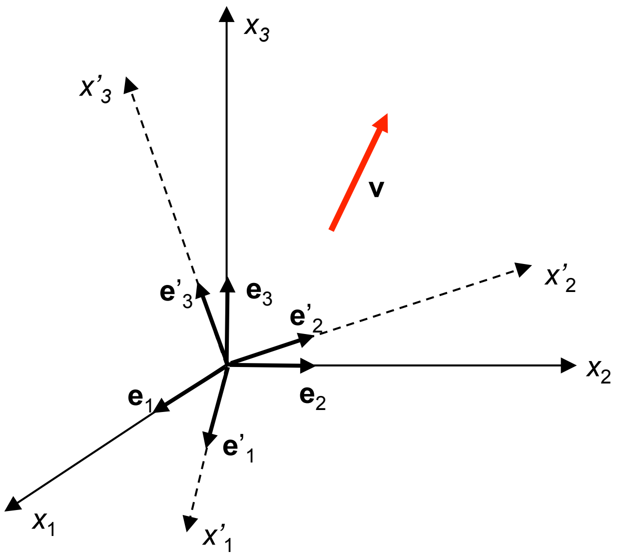

Fig 1. Change of Cartersian coordinate frames

We wish to express elasticity variables in different coordinate systems. This requires development of transformation rules for scalar, vector, matrix and higher order variables – a concept connected with basic definitions of tensor variables. The two Cartesian frames (x_1, x_2, x_3) and (x_1', x_2', x_3') differ only by orientation

Using Rotation Matrix $Q_{ij} = \cos(x_i', x_j)$

$$\mathbf{e}_1' = Q_{11}\mathbf{e}_1 + Q_{12}\mathbf{e}_2 + Q_{13}\mathbf{e}_3$$
$$\mathbf{e}_2' = Q_{21}\mathbf{e}_1 + Q_{22}\mathbf{e}_2 + Q_{23}\mathbf{e}_3$$
$$\mathbf{e}_3' = Q_{31}\mathbf{e}_1 + Q_{32}\mathbf{e}_2 + Q_{33}\mathbf{e}_3$$

$$\mathbf{e}_i' = Q_{ij}\mathbf{e}_j \qquad (1.4.3)$$
$$\mathbf{e}_i = Q_{ji}\mathbf{e}_j' \qquad (1.4.4)$$

$$\begin{aligned}\mathbf{v} &= v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 = v_i\mathbf{e}_i \\ &= v_1'\mathbf{e}_1' + v_2'\mathbf{e}_2' + v_3'\mathbf{e}_3' = v_i'\mathbf{e}_i'\end{aligned} \qquad (1.4.5)$$

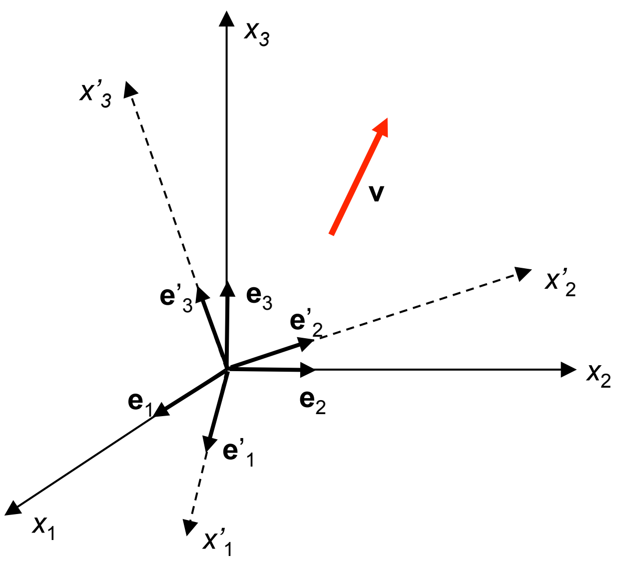

Fig 1. Change of Cartersian coordinate frames

Substitute (1.4.4) into $(1.4.5)_1$, gives

$$\mathbf{v} = v_i Q_{ji} \mathbf{e}'_j$$

And from $(1.4.5)_2$, $\mathbf{v} = v'_i \mathbf{e}'_i$, we find that

$$v'_i = Q_{ij} v_j \qquad (1.4.6)$$

Similarly, we find

$$v_i = Q_{ji} v'_j \qquad (1.4.7)$$

Relations (1.4.6) and (1.4.7) constitute the transformation laws for Cartesian vector components under a change of rectangular Cartesian coordinate frame.

1.5 Cartesian Tensors General Transformation Laws

Scalars, vectors, matrices, and higher order quantities can be represented by an index notational scheme, and thus all quantities may then be referred to as tensors of different orders. The transformation properties of a vector can be used to establish the general transformation properties of these tensors. Restricting the transformations to those only between Cartesian coordinate systems, the general set of transformation relations for various orders are:

$$a' = a \text{, zero order (scalar)}$$

$$a'_i = Q_{ip} a_p \text{ , first order (vector)}$$

$$a'_{ij} = Q_{ip} Q_{jq} a_{pq} \text{ , second order (matrix)}$$

$$a'_{ijk} = Q_{ip} Q_{jq} Q_{kr} a_{pqr} \text{ , third order}$$

$$a'_{ijkl} = Q_{ip} Q_{jq} Q_{kr} Q_{ls} a_{pqrs} \text{ , fourth order}$$

M

$$a'_{ijk\ldots m} = Q_{ip} Q_{jq} Q_{kr} \cdots Q_{mt} a_{pqr\ldots t} \quad \text{general order}$$

Example 1-2 Transformation Examples

The components of a first and second order tensor in a particular coordinate frame are given by

$$a_i = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix},\ a_{ij} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix}$$

Determine the components of each tensor in a new coordinate system found through a rotation of 60° (π/6 radians) about the x_3-axis. Choose a counterclockwise rotation when viewing down the negative x_3-axis, see Figure 1-2.

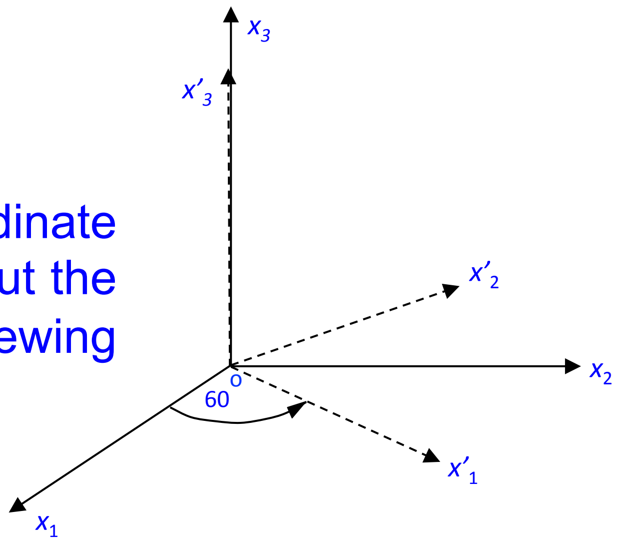

Fig 2. Coordinate transformation

The original and primed coordinate systems are shown in Figure 1-2. The solution starts by determining the rotation matrix for this case

$$Q_{ij} = \begin{bmatrix} \cos 60^\circ & \cos 30^\circ & \cos 90^\circ \\ \cos 150^\circ & \cos 60^\circ & \cos 90^\circ \\ \cos 90^\circ & \cos 90^\circ & \cos 0^\circ \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transformation for the vector quantity follows from equation $(1.5.1)_2$

Example 1-2 Transformation Examples

The components of a first and second order tensor in a particular coordinate frame are given by

$$a_i = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \quad a_{ij} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix}$$

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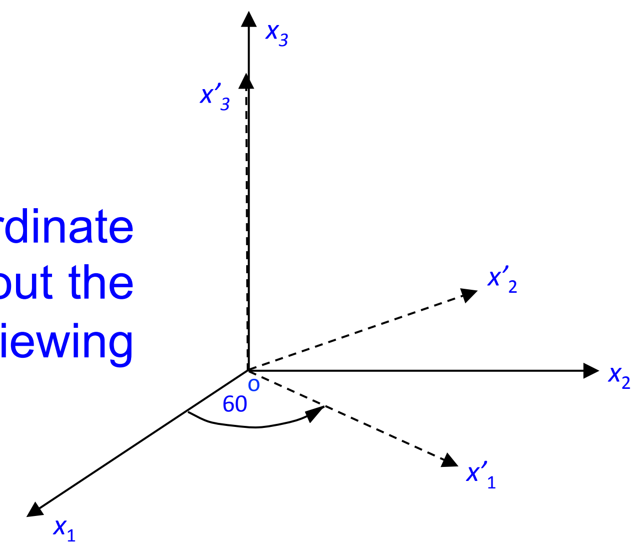


Fig 2. Coordinate transformation

The transformation for the vector quantity follows from equation $(1.5.1)_2$

$$a'_i = Q_{ij}a_j = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 + 2\sqrt{3} \\ 2 - \sqrt{3}/2 \\ 2 \end{bmatrix}$$

and the second order tensor (matrix) transforms according to $(1.5.1)_3$

$$a'_{ij} = Q_{ip}Q_{jq}a_{pq} = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 7/4 & \sqrt{3}/4 & 3/2 + \sqrt{3} \\ \sqrt{3}/4 & 5/4 & 1 - 3\sqrt{3}/2 \\ 3/2 + \sqrt{3} & 1 - 3\sqrt{3}/2 & 4 \end{bmatrix}$$

The direction determined by unit vector ***n*** is said to be a *principal direction* or *eigenvector* of the symmetric second order tensor a_{ij} if there exists a parameter λ (*principal value* or *eigenvalue*) such that

$$a_{ij}n_j = \lambda n_i \;\Rightarrow\; \left(a_{ij} - \lambda\delta_{ij}\right)n_j = 0$$

which is a homogeneous system of three linear algebraic equations in the unknowns n_1, n_2, n_3. The system possesses nontrivial solution if and only if determinant of coefficient matrix vanishes

$$\det\left[a_{ij} - \lambda\delta_{ij}\right] = -\lambda^3 + I_a\lambda^2 - II_a\lambda + III_a = 0$$

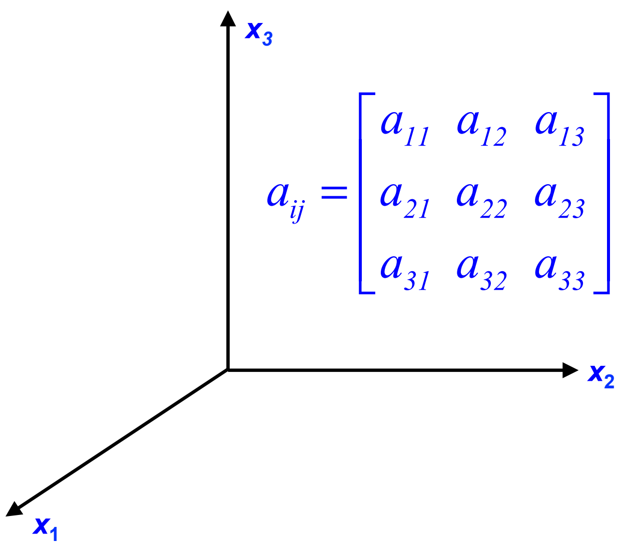
scalars I_a, II_a and III_a are called the *fundamental invariants* of the tensor a_{ij}

$$I_a = a_{ii} = a_{11} + a_{22} + a_{33}$$

$$II_a = \frac{1}{2}(a_{ii}a_{jj} - a_{ij}a_{ij}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

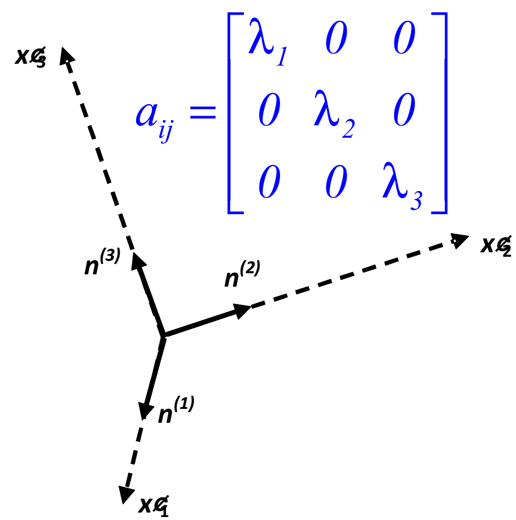
$$III_a = \det\left[a_{ij}\right]$$

It is always possible to identify a right-handed Cartesian coordinate system such that each axes lie along principal directions of any given symmetric second order tensor. Such axes are called the *principal axes* of the tensor, and the basis vectors are the principal directions $\{\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)}\}$



$$a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Fig 3. Original given axes



$$a_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Fig 4. Principle axes

Example 1-3 Principal Value Problem

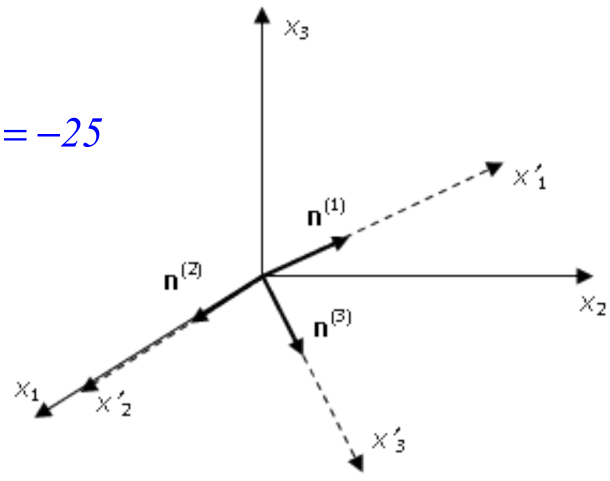
Determine the invariants, and principal values and directions of

$$a_{ij} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}$$

First determine the principal invariants

$$I_a = a_{ii} = 2+3-3 = 2 \,,\; II_a = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 4 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & -3 \end{vmatrix} = 6-25-6 = -25$$

$$III_a = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{vmatrix} = 2(-9-16) = -50$$



The characteristic equation then becomes

$$\det\left[a_{ij} - \lambda\delta_{ij}\right] = -\lambda^3 + 2\lambda^2 + 25\lambda - 50 = 0 \Rightarrow (\lambda-2)\left(\lambda^2-25\right) = 0$$

$$\therefore \lambda_1 = 5, \lambda_2 = 2, \lambda_3 = -5$$

Example 1-3 Principal Value Problem

Determine the invariants, and principal values and directions of

$$a_{ij} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}$$

Thus for this case all principal values are distinct

For the $\lambda_1 = 5$ root, equation (1.6.1) gives the system

$$\begin{aligned} -3n_1^{(1)} &= 0 \\ -2n_2^{(1)} + 4n_3^{(1)} &= 0 \\ 4n_2^{(1)} - 8n_3^{(1)} &= 0 \end{aligned}$$

which gives a normalized solution $\mathbf{n}^{(1)} = \pm\frac{1}{\sqrt{5}}(2\mathbf{e}_2 + \mathbf{e}_3)$

In similar fashion the other two principal directions are found to be

$$\mathbf{n}^{(2)} = \pm\mathbf{e}_1, \mathbf{n}^{(3)} = \pm\frac{1}{\sqrt{5}}(\mathbf{e}_2 - 2\mathbf{e}_3)$$

It is easily verified that these directions are mutually orthogonal.

Note for this case, the transformation matrix Q_{ij} defined by (1.4.1) becomes

$$Q_{ij} = \begin{bmatrix} 0 & 2/\sqrt{5} & 1/\sqrt{5} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix} \Rightarrow a'_{ij} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

1.7 Vector, Matrix and Tensor Algebra

Scalar or Dot Product $\mathbf{a}\cdot\mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i$

Vector or Cross Product

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \varepsilon_{ijk} a_j b_k \mathbf{e}_i$$

$$\mathbf{a}.\mathbf{b}\times\mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \varepsilon_{ijk} a_i b_j c_k$$

Common Matrix Products

$$\mathbf{Aa} = [\mathrm{A}]\{\mathrm{a}\} = A_{ij} a_j = a_j A_{ij}$$

$$\mathbf{a}^{\mathrm{T}}\mathbf{A} = \{\mathrm{a}\}^T [\mathrm{A}] = a_i A_{ij} = A_{ij} a_i$$

$$\mathbf{AB} = [\mathrm{A}][\mathrm{B}] = A_{ij} B_{jk}$$

$$\mathbf{AB}^{\mathrm{T}} = A_{ij} B_{kj}$$

$$\mathbf{A}^{\mathrm{T}}\mathbf{B} = A_{ji} B_{jk}$$

$$tr(\mathbf{AB}) = A_{ij} B_{ji}$$

$$tr(\mathbf{AB}^{\mathrm{T}}) = tr(\mathbf{A}^{\mathrm{T}}\mathbf{B}) = A_{ij} B_{ij}$$

$$A_{ij}^T = A_{ji} \;\; ; \;\; tr\mathbf{A} = A_{ii} = A_{11} + A_{22} + A_{33}$$

Second Order Transformation Law

$$a'_{ij} = Q_{ip} Q_{jq} a_{pq}$$

$$\Rightarrow \mathbf{a}' = \mathbf{QaQ}^{\mathrm{T}}$$

1.8 Calculus of Cartesian Tensors

Field concept for tensor components

$$a = a(x_1, x_2, x_3) = a(x_i) = a(\mathbf{x})$$
$$a_i = a_i(x_1, x_2, x_3) = a_i(x_i) = a_i(\mathbf{x})$$
$$a_{ij} = a_{ij}(x_1, x_2, x_3) = a_{ij}(x_i) = a_{ij}(\mathbf{x})$$
$$\vdots$$

Comma notation for partial differentiation

$$a_{,i} = \frac{\partial}{\partial x_i} a \;,\; a_{i,j} = \frac{\partial}{\partial x_j} a_i \;,\; a_{ij,k} = \frac{\partial}{\partial x_k} a_{ij} \;,\ldots$$

If differentiation index is distinct, order of the tensor will be increased by one; e.g. derivative operation on a vector produces a second order tensor or matrix

$$a_{i,j} = \begin{bmatrix} \frac{\partial a_1}{\partial x_1} & \frac{\partial a_1}{\partial x_2} & \frac{\partial a_1}{\partial x_3} \\ \frac{\partial a_2}{\partial x_1} & \frac{\partial a_2}{\partial x_2} & \frac{\partial a_2}{\partial x_3} \\ \frac{\partial a_3}{\partial x_1} & \frac{\partial a_3}{\partial x_2} & \frac{\partial a_3}{\partial x_3} \end{bmatrix}$$

Directional Derivative of Scalar Field $$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} = \mathbf{n}\cdot\nabla f$$

$$\mathbf{n} = \text{unit normal vector in direction of } s = \frac{dx}{ds}\mathbf{e}_1 + \frac{dy}{ds}\mathbf{e}_2 + \frac{dz}{ds}\mathbf{e}_3$$

$$\nabla = \text{vector differential operator } = \mathbf{e}_1\frac{\partial}{\partial x} + \mathbf{e}_2\frac{\partial}{\partial y} + \mathbf{e}_3\frac{\partial}{\partial z}$$

$$\nabla f = \mathbf{grad}\ f = \text{gradient of scalar function } f = \mathbf{e}_1\frac{\partial f}{\partial x} + \mathbf{e}_2\frac{\partial f}{\partial y} + \mathbf{e}_3\frac{\partial f}{\partial z}$$

Common Differential Operations

Gradient of a Scalar $\nabla\varphi = \varphi_{,i}\mathbf{e}_i$

Gradient of a Vector $\nabla\mathbf{u} = u_{i,j}\mathbf{e}_i\mathbf{e}_j$

Laplacian of a Scalar $\nabla^2\varphi = \nabla\cdot\nabla\varphi = \varphi_{,ii}$

Divergence of a Vector $\nabla\cdot\mathbf{u} = u_{i,i}$

Curl of a Vector $\nabla\times\mathbf{u} = \varepsilon_{ijk}u_{k,j}\mathbf{e}_i$

Laplacian of a Vector $\nabla^2\mathbf{u} = u_{i,kk}\mathbf{e}_i$

Example 1-4: Scalar/Vector Field Example

Scalar and vector field functions are given by $\phi = x^2 - y^2, \mathbf{u} = 2x\mathbf{e}_1 + 3yz\mathbf{e}_2 + xy\mathbf{e}_3$

Calculate the following expressions, $\nabla\phi, \nabla^2\phi, \nabla\cdot\mathbf{u}, \nabla\mathbf{u}, \nabla\times\mathbf{u}$

Using the basic relations

$$\nabla\phi = 2x\mathbf{e}_1 - 2y\mathbf{e}_2$$

$\nabla^2\phi = 2 - 2 = 0$ - (satisfies Laplace equation)

$$\nabla\cdot\mathbf{u} = 2 + 3z + 0 = 2 + 3z$$

$$\nabla\mathbf{u} = u_{i,j} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3z & 3y \\ y & x & 0 \end{bmatrix}$$

$$\nabla\times\mathbf{u} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x & 3yz & xy \end{vmatrix}$$

$$= (x - 3y)\mathbf{e}_1 - y\mathbf{e}_2$$

Note vector field $\nabla\phi$ is orthogonal to ϕ-contours, a result true in general for all scalar fields

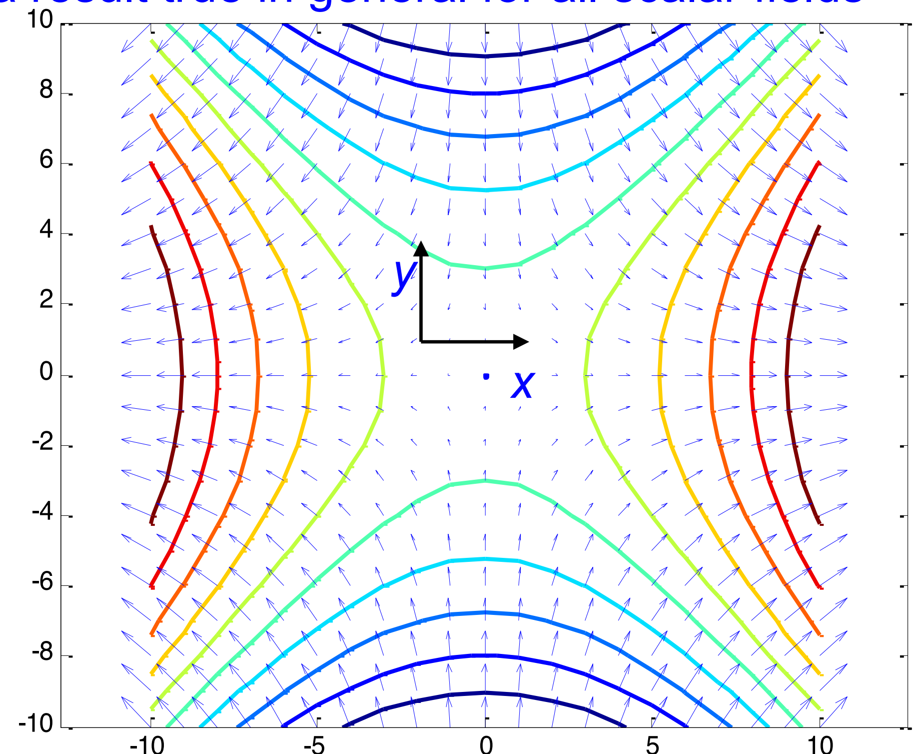


Fig 5. Contours ϕ=constant and vector distributions of $\nabla\phi$

Divergence Theorem

$$\iint_S \mathbf{u}\cdot\mathbf{n}\, dS = \iiint_V \nabla\cdot\mathbf{u}\, dV \Rightarrow \iint_S a_{ij\ldots k} n_k\, dS = \iiint_V a_{ij\ldots k,k}\, dV$$

Stokes Theorem

$$\int_C \mathbf{u}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{u})\cdot\mathbf{n}\, dS \Rightarrow \int_C a_{ij\ldots} dx_t = \iint_S \varepsilon_{rst} a_{ij\ldots k,s} n_r\, dS$$

Green's Theorem in the Plane

Apply Stoke theorem to a planar domain S with the vector field selected as $\mathbf{u} = f\mathbf{e}_1 + g\mathbf{e}_2$

$$\iint_S \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy = \int_C (fdx + gdy) \Rightarrow \iint_S \frac{\partial g}{\partial x} dxdy = \int_C g n_x ds \,,\ \iint_S \frac{\partial f}{\partial y} dxdy = \int_C f n_y ds$$

Zero-Value Theorem

$$\iiint_V f_{ij\ldots k}\, dV = 0 \Rightarrow f_{ij\ldots k} = 0 \in V$$

1.9 Orthogonal Curvilinear Coordinate Systems

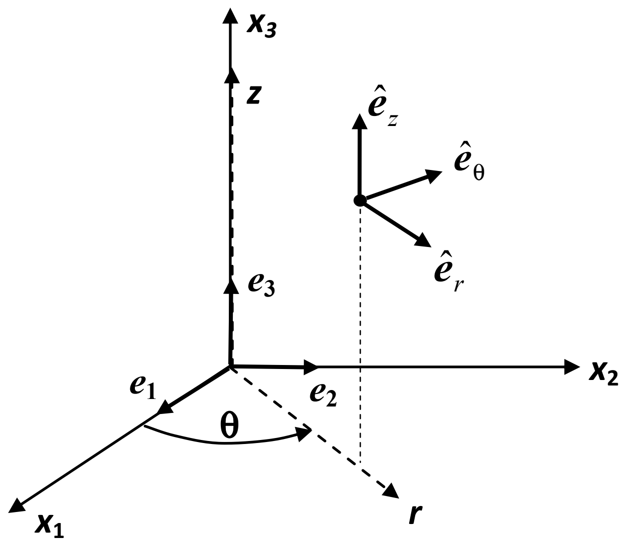


Fig 6. Cylindrical Coordinate System (*r*,*θ*,*z*)

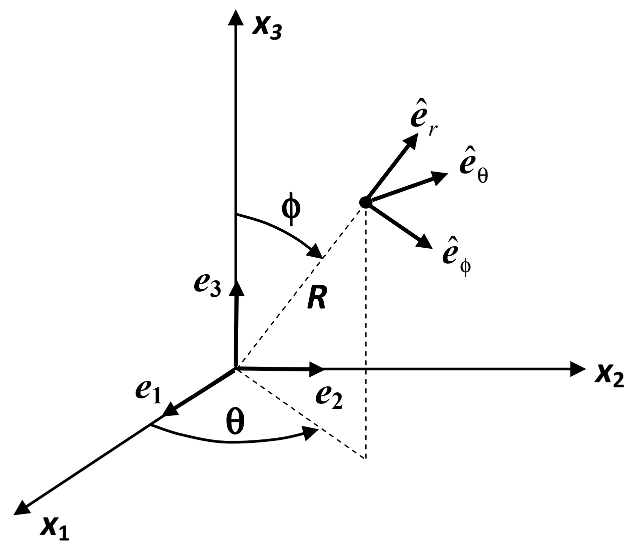


Fig 7. Spherical Coordinate System (*R*,*ϕ*,*θ*)

$$x_1 = r\cos\theta, x_2 = \sin\theta, x_3 = z$$

$$r = \sqrt{x_1^2 + x_2^2}, \theta = \tan^{-1}\frac{x_2}{x_1}, z = x_3$$

$$x_1 = R\cos\theta\sin\varphi,\ x_2 = R\sin\theta\sin\varphi,\ x_3 = R\cos\varphi$$

$$R = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$\varphi = \cos^{-1}\frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$\theta = \tan^{-1}\frac{x_2}{x_1},$$

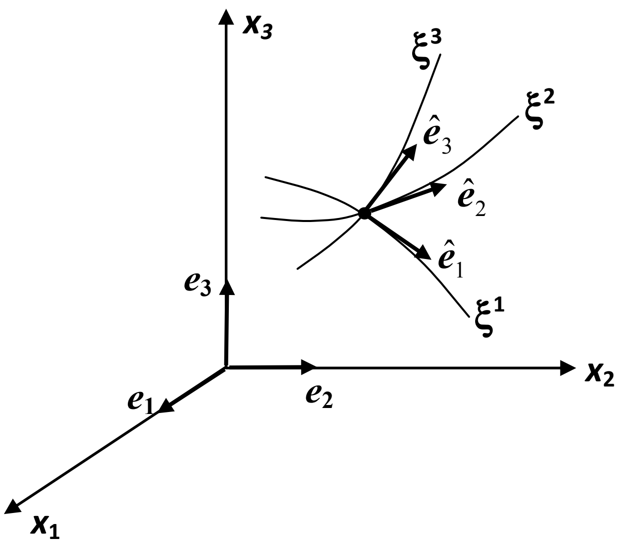


Fig 8. Curvilinear coordinates

$$\xi^m = \xi^m(x^1, x^2, x^3)\,,\; x^m = x^m(\xi^1, \xi^2, \xi^3)$$

$$(ds)^2 = (h_1 d\xi^1)^2 + (h_2 d\xi^2)^2 + (h_3 d\xi^3)^2$$

Common Differential Forms

$$\nabla = \hat{e}_1 \frac{1}{h_1}\frac{\partial}{\partial \xi^1} + \hat{e}_2 \frac{1}{h_2}\frac{\partial}{\partial \xi^2} + \hat{e}_3 \frac{1}{h_3}\frac{\partial}{\partial \xi^3} = \sum_i \hat{e}_i \frac{1}{h_i}\frac{\partial}{\partial \xi^i}$$

$$\nabla f = \hat{e}_1 \frac{1}{h_1}\frac{\partial f}{\partial \xi^1} + \hat{e}_2 \frac{1}{h_2}\frac{\partial f}{\partial \xi^2} + \hat{e}_3 \frac{1}{h_3}\frac{\partial f}{\partial \xi^3} = \sum_i \hat{e}_i \frac{1}{h_i}\frac{\partial f}{\partial \xi^i}$$

$$\nabla \cdot \mathbf{u} = \frac{1}{h_1 h_2 h_3}\sum_i \frac{\partial}{\partial \xi^i}\left(\frac{h_1 h_2 h_3}{h_i} u_{<i>}\right)$$

$$\nabla^2 \varphi = \frac{1}{h_1 h_2 h_3}\sum_i \frac{\partial}{\partial \xi^i}\left(\frac{h_1 h_2 h_3}{(h_i)^2}\frac{\partial \varphi}{\partial \xi^i}\right)$$

$$\nabla \times \mathbf{u} = \sum_i \sum_j \sum_k \frac{\varepsilon_{ijk}}{h_j h_k}\frac{\partial}{\partial \xi^j}(u_{<k>} h_k)\hat{e}_i$$

$$\nabla \mathbf{u} = \sum_i \sum_j \frac{\hat{e}_i}{h_i}\left(\frac{\partial u_{<j>}}{\partial \xi^i}\hat{e}_j + u_{<j>}\frac{\partial \hat{e}_j}{\partial \xi^i}\right)$$

$$\nabla^2 \mathbf{u} = \left(\sum_i \frac{\hat{e}_i}{h_i}\frac{\partial}{\partial \xi^i}\right)\cdot\left(\sum_j \sum_k \frac{\hat{e}_k}{h_k}\left[\frac{\partial u_{<j>}}{\partial \xi^k}\hat{e}_j + u_{<j>}\frac{\partial \hat{e}_j}{\partial \xi^k}\right]\right)$$

From relations (1.9.5) or simply using the geometry shown in Figure

$$\hat{e}_r = \cos\theta e_1 + \sin\theta e_2, \quad \hat{e}_\theta = -\sin\theta e_1 + \cos\theta e_2 \;\Rightarrow\; \frac{\partial \hat{e}_r}{\partial\theta} = \hat{e}_\theta,\; \frac{\partial \hat{e}_\theta}{\partial\theta} = -\hat{e}_r,\; \frac{\partial \hat{e}_r}{\partial r} = \frac{\partial \hat{e}_\theta}{\partial r} = 0$$

The basic vector differential operations then follow to be

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta}$$

$$\nabla\varphi = \hat{e}_r \frac{\partial\varphi}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial\varphi}{\partial\theta}$$

$$\nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}$$

$$\nabla^2\varphi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}$$

$$\nabla\times\mathbf{u} = \left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\hat{e}_z$$

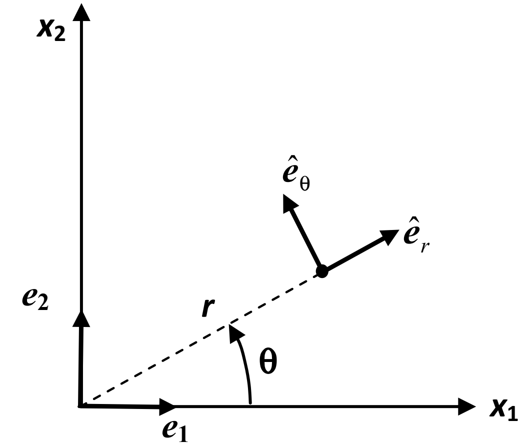


Fig 9. Polar coordinate system

$$(ds)^2 = (dr)^2 + (r d\theta)^2 \;\Rightarrow\; h_1 = 1,\; h_2 = r$$

$$\nabla\mathbf{u} = \frac{\partial u_r}{\partial r}\hat{e}_r\hat{e}_r + \frac{\partial u_\theta}{\partial r}\hat{e}_r\hat{e}_\theta + \frac{1}{r}\left(\frac{\partial u_r}{\partial\theta} - u_\theta\right)\hat{e}_\theta\hat{e}_r + \frac{1}{r}\left(\frac{\partial u_\theta}{\partial\theta} - u_r\right)\hat{e}_\theta\hat{e}_\theta$$

$$\nabla^2\mathbf{u} = \left(\nabla^2 u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta} - \frac{u_r}{r^2}\right)\hat{e}_r + \left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2}\right)\hat{e}_\theta$$

where $\mathbf{u} = u_r\hat{\mathbf{e}}_r + u_\theta\hat{\mathbf{e}}_\theta,\; \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_\theta$

THANK YOU