SOLID MECHANICS

Chapter 3: Stress & Equilibrium

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3.1 Body and Surface Forces

3.2 Traction Vector and Stress Tensor

3.3 Stress Transformation

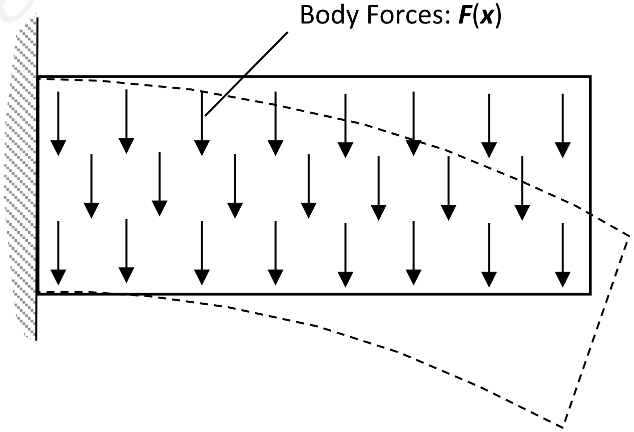
3.4 Principal Stresses & Directions

3.5 Spherical, Deviatoric, Octahedral and Von Mises Stresses

3.6 Equilibrium Equations

3.7 Relations in Cylindrical and Spherical Coordinates

- Body forces are proportional to the body's mass and are reacted with an agent outside of the body. Example: gravitational-weight forces, magnetic forces, inertial forces.
- By using continuum mechanics principles, a body force density (force per unit volume) **F(x)** can be defined such that the total resultant body force of an entire solid can be written as a volume integral over the body.



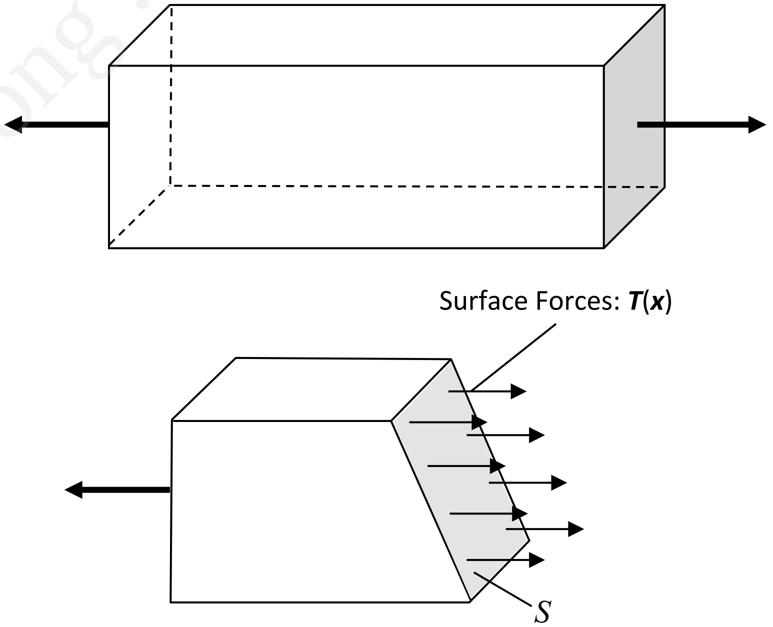
(a) Cantilever Beam Under Self-Weight Loading

$$\mathbf{F}_R = \iiint_V \mathbf{F}(\mathbf{x})\,\mathrm{d}V$$

- Surface forces always act on a surface and result from physical contact with another body.
- The resultant surface force over the entire surface S can be expressed as the integral of a surface force density function $\mathbf{T}^{\mathbf{n}}(\mathbf{x})$

$$\mathbf{F}_S = \iint_S \mathbf{T}^{\mathbf{n}}(\mathbf{x})\,\mathrm{d}S$$

- The surface force density is normally referred to as the traction vector

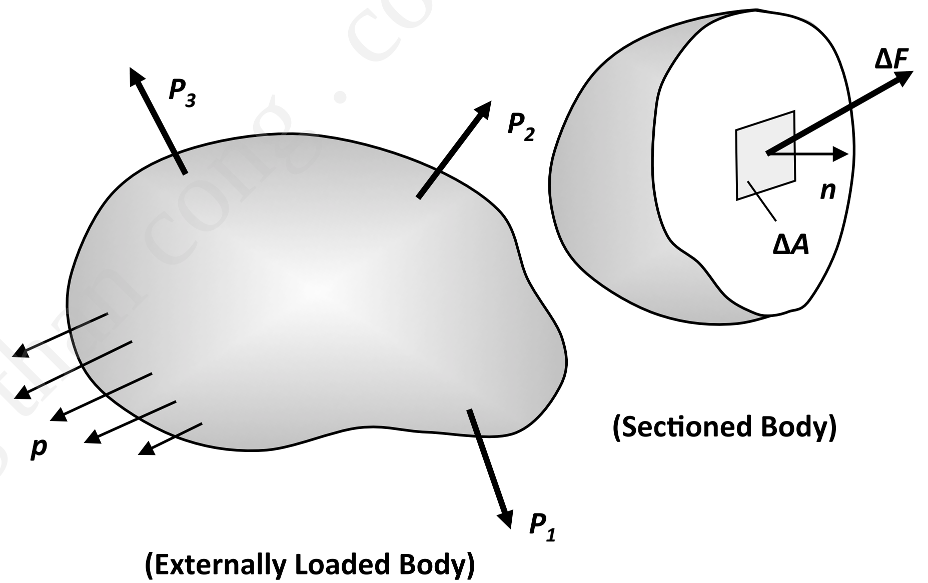


(b) Sectioned Axially Loaded Beam

- The stress or traction vector is defined by

$$\mathbf{T}^{\mathbf{n}}(\mathbf{x},\mathbf{n}) = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{F}}{\Delta A}$$

- Notice that the stress vector depends on both the spatial location and the unit normal vector to the surface under study.



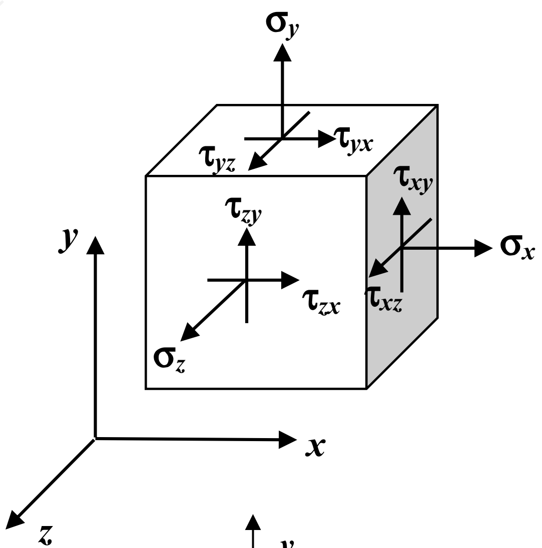
- In order to define the stress tensor, we consider 3 special cases in which 3 unit normal vectors of ΔA point along the positive coordinate axes. For these cases, the traction vectors on each face are

$$\mathbf{T}^{\mathbf{n}}(\mathbf{x},\mathbf{n}=\mathbf{e}_1) = \sigma_x \mathbf{e}_1 + \tau_{xy}\mathbf{e}_2 + \tau_{xz}\mathbf{e}_3$$
$$\mathbf{T}^{\mathbf{n}}(\mathbf{x},\mathbf{n}=\mathbf{e}_2) = \tau_{yx} \mathbf{e}_1 + \sigma_{y}\mathbf{e}_2 + \tau_{yz}\mathbf{e}_3$$
$$\mathbf{T}^{\mathbf{n}}(\mathbf{x},\mathbf{n}=\mathbf{e}_3) = \tau_{zx} \mathbf{e}_1 + \tau_{zy}\mathbf{e}_2 + \sigma_{z}\mathbf{e}_3$$

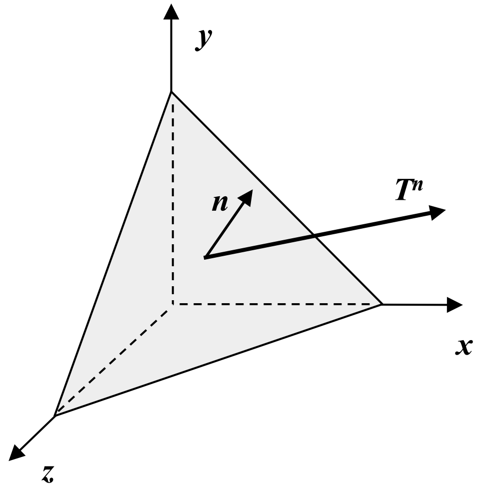
σ is called the stress tensor (2rd order)

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

These 9 components are called the stress components.



σ_x is normal stress, τ_{xy} is shearing stress where x shows plane of action and y shows direction of stress



Traction on an Oblique Plane

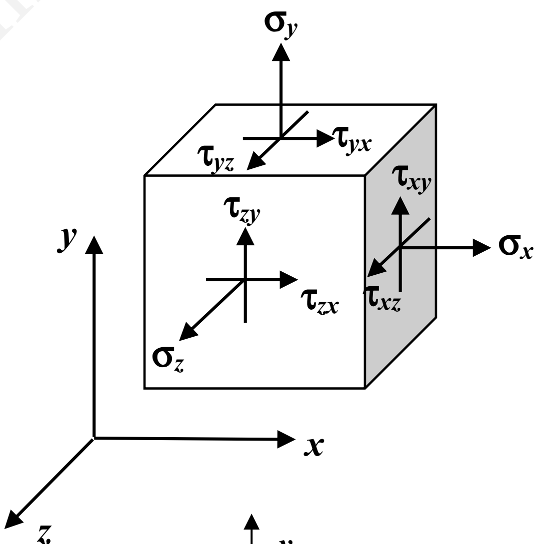
- Consider the traction vector on an oblique plane with arbitrary orientation. The unit normal to the surface is

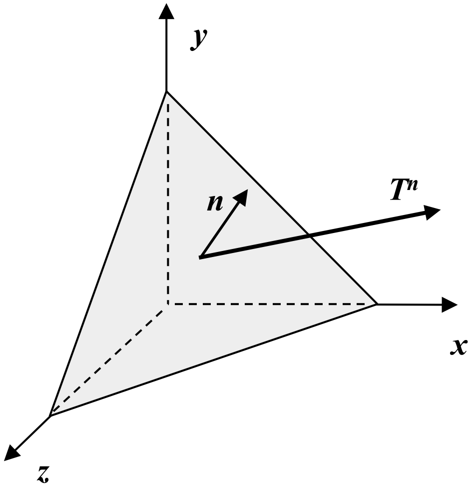
- Using the force balance between tractions on the oblique and coordinate faces gives

$$\mathbf{T^n} = n_x\mathbf{T^n}(\mathbf{n}=\mathbf{e}_1) + n_y\mathbf{T^n}(\mathbf{n}=\mathbf{e}_2) + n_z\mathbf{T^n}(\mathbf{n}=\mathbf{e}_3)$$

$$\mathbf{T^n} = (\sigma_x n_x + \tau_{yx} n_y + \tau_{zx} n_z)\mathbf{e}_1 + (\tau_{xy} n_x + \sigma_y n_y + \tau_{zy} n_z)\mathbf{e}_2 + (\tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z)\mathbf{e}_3$$

$$T_i^{\mathbf{n}} = \sigma_{ji} n_j$$





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$$\sigma'_{ij} = Q_{ip} Q_{jq} \sigma_{pq} \quad ; \quad Q_{ij} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

$$\sigma'_x = \sigma_x l_1^2 + \sigma_y m_1^2 + \sigma_z n_1^2 + 2(\tau_{xy} l_1 m_1 + \tau_{yz} m_1 n_1 + \tau_{zx} n_1 l_1)$$

$$\sigma'_y = \sigma_x l_2^2 + \sigma_y m_2^2 + \sigma_z n_2^2 + 2(\tau_{xy} l_2 m_2 + \tau_{yz} m_2 n_2 + \tau_{zx} n_2 l_2)$$

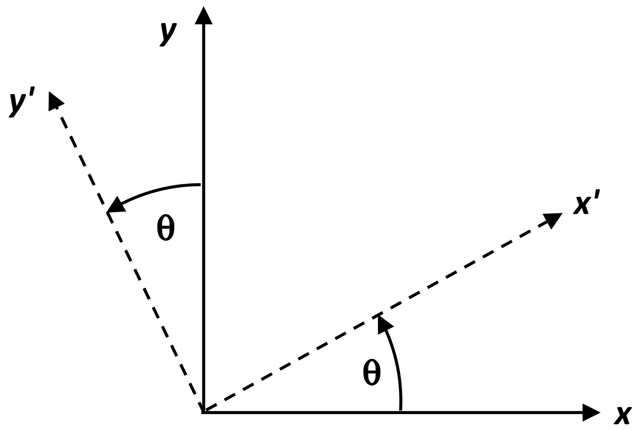
$$\sigma'_z = \sigma_x l_3^2 + \sigma_y m_3^2 + \sigma_z n_3^2 + 2(\tau_{xy} l_3 m_3 + \tau_{yz} m_3 n_3 + \tau_{zx} n_3 l_3)$$

$$\tau'_{xy} = \sigma_x l_1 l_2 + \sigma_y m_1 m_2 + \sigma_z n_1 n_2 + \tau_{xy}(l_1 m_2 + m_1 l_2) + \tau_{yz}(m_1 n_2 + n_1 m_2) + \tau_{zx}(n_1 l_2 + l_1 n_2)$$

$$\tau'_{yz} = \sigma_x l_2 l_3 + \sigma_y m_2 m_3 + \sigma_z n_2 n_3 + \tau_{xy}(l_2 m_3 + m_2 l_3) + \tau_{yz}(m_2 n_3 + n_2 m_3) + \tau_{zx}(n_2 l_3 + l_2 n_3)$$

$$\tau'_{zx} = \sigma_x l_3 l_1 + \sigma_y m_3 m_1 + \sigma_z n_3 n_1 + \tau_{xy}(l_3 m_1 + m_3 l_1) + \tau_{yz}(m_3 n_1 + n_3 m_1) + \tau_{zx}(n_3 l_1 + l_3 n_1)$$

Two-Dimensional Stress Transformation



$$\sigma'_y = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta$$

$$\sigma'_y = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\sigma_{xy}\sin\theta\cos\theta$$

$$\sigma'_{xy} = -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta)$$

$$Q_{ij} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$

$$\sigma'_x = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \sigma_{xy}\sin 2\theta$$

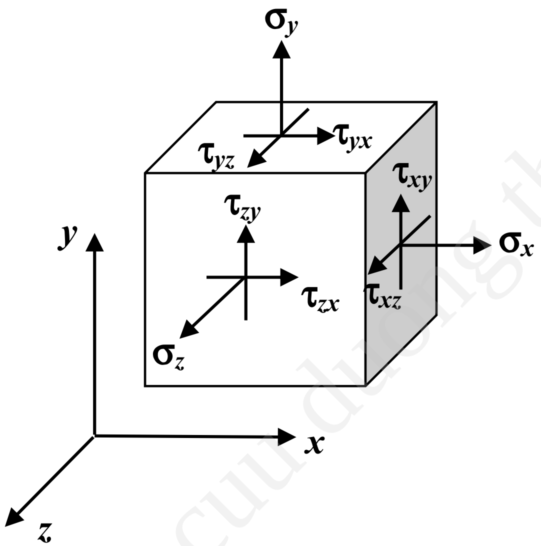
$$\sigma'_y = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \sigma_{xy}\sin 2\theta$$

$$\sigma'_{xy} = \frac{\sigma_y - \sigma_x}{2}\sin 2\theta + \sigma_{xy}\cos 2\theta$$

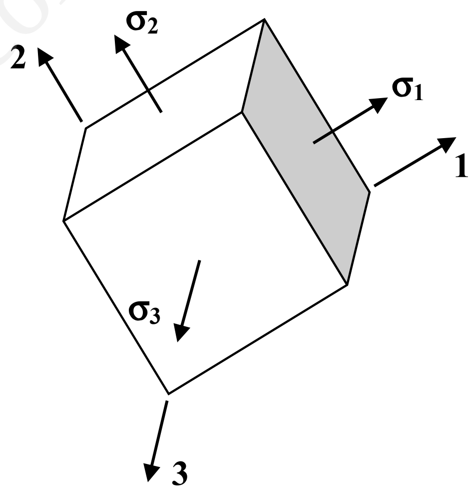
3.4 Principal Stresses & Directions

$$\det\left[\sigma_{ij} - \sigma\delta_{ij}\right] = -\sigma^3 + I_1\sigma^2 - I_2\sigma + I_3 = 0 \quad \Rightarrow \quad \sigma_1, \sigma_2, \sigma_3$$

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 \;;\; I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \;;\; I_3 = \sigma_1\sigma_2\sigma_3$$



(General Coordinate System)

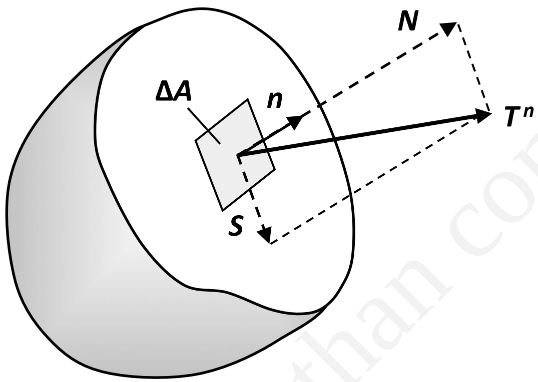


(Principal Coordinate System)

Traction Vector Components

$$N = \mathbf{T}^{\mathbf{n}} \cdot \mathbf{n}$$

$$S = \left(|\mathbf{T}^{\mathbf{n}}|^2 - N^2 \right)^{1/2}$$



$$N = \mathbf{T}^{\mathbf{n}} \cdot \mathbf{n} = T_i^{\mathbf{n}} n_i = \sigma_{ji} n_j n_i$$

$$= \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2$$

$$|\mathbf{T}^{\mathbf{n}}|^2 = \mathbf{T}^{\mathbf{n}} \cdot \mathbf{T}^{\mathbf{n}} = T_i^{\mathbf{n}} T_i^{\mathbf{n}} = \sigma_{ji} n_j \sigma_{ki} n_k$$

$$= \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2$$

$$N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2$$

$$S^2 + N^2 = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2$$

$$1 = n_1^2 + n_2^2 + n_3^2$$

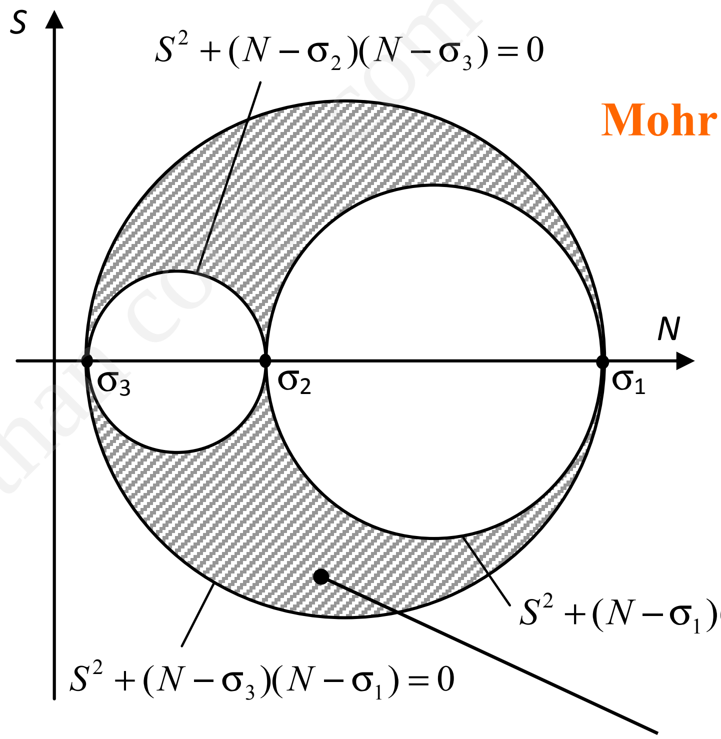
$\Rightarrow$

$$n_1^2 = \frac{S^2 + (N - \sigma_2)(N - \sigma_3)}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)}$$

$$n_2^2 = \frac{S^2 + (N - \sigma_3)(N - \sigma_1)}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)}$$

$$n_3^2 = \frac{S^2 + (N - \sigma_1)(N - \sigma_2)}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)}$$

- Without loss in generality, we can rank the principal stresses as $\sigma_1 > \sigma_2 > \sigma_3$. And applying the conditions the positivity of square of unit normal vectors , we get



Mohr's Circles of Stress

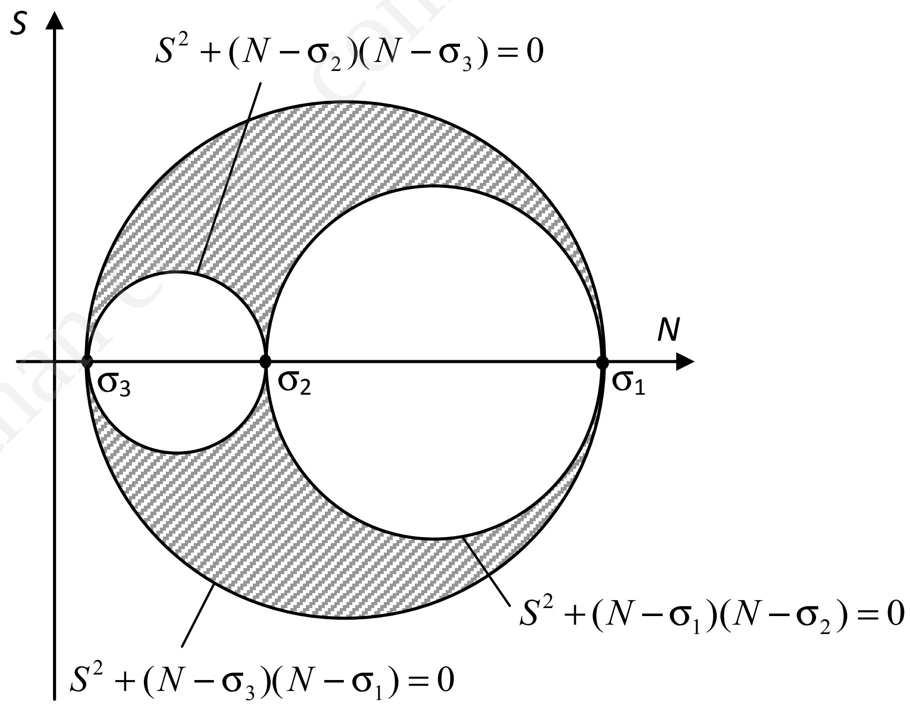
$$S^2 + (N - \sigma_2)(N - \sigma_3) \geq 0$$
$$S^2 + (N - \sigma_3)(N - \sigma_1) \leq 0$$
$$S^2 + (N - \sigma_1)(N - \sigma_2) \geq 0$$

Admissible N and S values lie in the shaded area

For the equality, the above equations represent three circles in an *S*-*N* coordinate system which is called Mohr's circles of stress.

Three above inequalities imply that all admissible values of N and S lie in the shaded regions bounded by three circles.

Note that for the ranked principal stresses, the largest shear is easily determined as



Example 3-1 Stress Transformation

For the given state of stress below, determine the principal stresses and directions and find the traction vector on a plane with unit normal $\boldsymbol{n} = (0,1,1)/\sqrt{2}$.

$$\sigma_{ij} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

The principal stress problem is started by calculating the three invariants, giving the result $I_1 = 3$, $I_2 = -6$, $I_3 = -8$. This yields the following characteristic equation

$$-\sigma^3 + 3\sigma^2 + 6\sigma - 8 = 0$$

The roots of this equation are found to be $\sigma = 4, 1, -2$. Back-substituting the first root into the fundamental system (1.6.1) gives

$$\begin{aligned} -n_1^{(1)} + n_2^{(1)} + n_3^{(1)} &= 0 \\ n_1^{(1)} - 4n_2^{(1)} + 2n_3^{(1)} &= 0 \\ n_1^{(1)} + 2n_2^{(1)} - 4n_3^{(1)} &= 0 \end{aligned}$$

Solving this system, the normalized principal direction is found to be $\boldsymbol{n}^{(1)} = (2, 1, 1)/\sqrt{6}$. In similar fashion the other two principal directions are $\boldsymbol{n}^{(2)} = (-1, 1, 1)/\sqrt{3}$, $\boldsymbol{n}^{(3)} = (0, -1, 1)/\sqrt{2}$. The traction vector on the specified plane is calculated using the relation

$$T_i^n = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} \\ 2/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix}$$

3.5 Spherical, Deviatoric, Octahedral and Von Mises Stresses

$$\tilde{\sigma}_{ij} = \frac{1}{3}\sigma_{kk}\delta_{ij} \quad ; \quad \hat{\sigma}_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \quad ; \quad \sigma_{ij} = \tilde{\sigma}_{ij} + \hat{\sigma}_{ij}$$

Consider the normal and shear stresses (tractions) that act on a special plane whose normal makes equal angles with three principal axes. This plane is referred to as the octohedral plane. The unit normal vector to the octohedral plane is

$$n_i = \pm\frac{1}{\sqrt{3}}(1,1,1) \Rightarrow$$

$$N = \sigma_{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}\sigma_{kk} = \frac{1}{3}I_1$$

$$S = \tau_{oct} = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$

$$= \frac{1}{3}\left(2I_1^2 - 6I_2\right)^{1/2}$$

The octahedral shear stress τ_{oct} is directly related to the distortional strain energy.

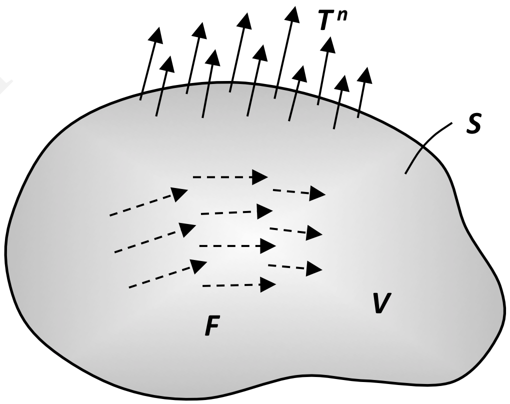
The effective or von Mises stress is given by

$$\sigma_e = \sigma_{vonMises} = \frac{1}{\sqrt{2}}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$

$$\sigma_e = \sigma_{vonMises} = \frac{1}{\sqrt{2}}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_x - \sigma_z)^2 + 3\left(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2\right)\right]^{1/2}$$

$$\sum \mathbf{F} = 0 \Rightarrow \iint_S T_i^{\mathbf{n}} \mathrm{d}S + \iiint_V F_i \mathrm{d}V = 0$$

$$\iint_S \sigma_{ji} n_j \mathrm{d}S + \iiint_V F_i \mathrm{d}V = 0$$



Applying the divergence theorem (1.8.7), then

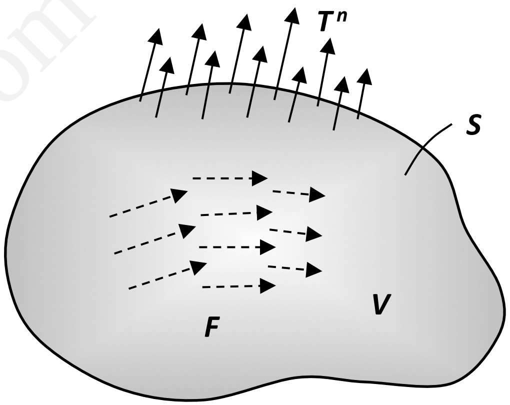
$$\iiint_V \left(\sigma_{ji,j} + F_i\right) \mathrm{d}V = 0$$

Because the region V is arbitrary, and the integrand is continuous, then by the zero-value theorem (1.8.12), the integrand must vanish

$$\sigma_{ji,j} + F_i = 0 \quad \Rightarrow \quad \begin{aligned} &\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = 0 \\ &\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + F_y = 0 \\ &\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z = 0 \end{aligned}$$

$$\sum \mathbf{r}\times\mathbf{F}=0=\iint_S \varepsilon_{ijk}x_jT_k^{\mathbf{n}}\mathrm{d}S+\iiint_V \varepsilon_{ijk}x_jF_k\mathrm{d}V=0$$

$$\iint_S \varepsilon_{ijk}x_j\sigma_{lk}n_l\mathrm{d}S+\iiint_V \varepsilon_{ijk}x_jF_k\mathrm{d}V=0$$



Applying the divergence theorem (1.8.7), then

$$\iiint_V \left[\left(\varepsilon_{ijk}x_j\sigma_{lk}\right)_{,l}+\varepsilon_{ijk}x_jF_k\right]\mathrm{d}V=0$$

$$\iiint_V \left[\varepsilon_{ijk}x_{j,l}\sigma_{lk}+\varepsilon_{ijk}x_j\sigma_{lk,l}+\varepsilon_{ijk}x_jF_k\right]\mathrm{d}V=0$$

$$\iiint_V \left[\varepsilon_{ijk}\delta_{jl}\sigma_{lk}+\varepsilon_{ijk}x_j\sigma_{lk,l}+\varepsilon_{ijk}x_jF_k\right]\mathrm{d}V=0$$

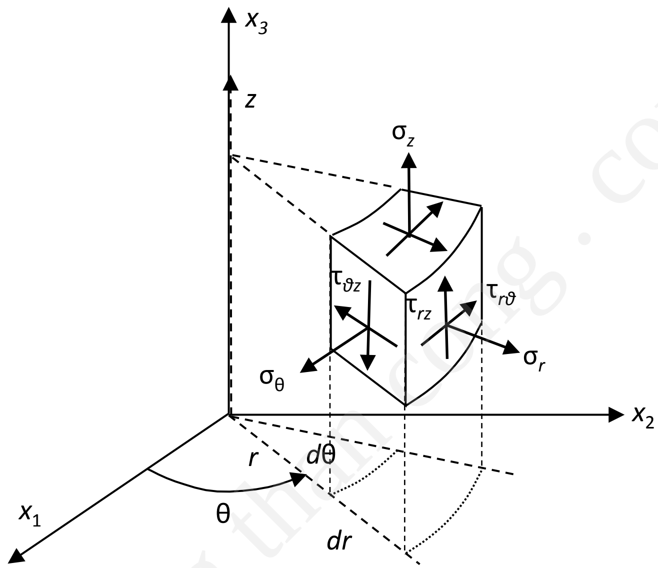
Using equilibrium equations (3.6.4)

$$\iiint_V \left[\varepsilon_{ijk}\sigma_{jk}-\varepsilon_{ijk}x_jF_k+\varepsilon_{ijk}x_jF_k\right]\mathrm{d}V=\iiint_V \varepsilon_{ijk}\sigma_{jk}\mathrm{d}V=0$$

$$\sum \mathbf{r}\times\mathbf{F}=0\Rightarrow \varepsilon_{ijk}\sigma_{jk}=0\Rightarrow \sigma_{ij}=\sigma_{ji}\Rightarrow \begin{matrix}\tau_{xy}=\tau_{yx}\\ \tau_{yz}=\tau_{zy}\\ \tau_{zx}=\tau_{xz}\end{matrix}\Rightarrow \sigma_{ij,j}+F_i=0$$

3.7 Relations in Cylindrical and Spherical Coordinates

Cylindrical Coordinates



$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_r & \tau_{r\theta} & \tau_{rz} \\ \tau_{r\theta} & \sigma_\theta & \tau_{\theta z} \\ \tau_{rz} & \tau_{\theta z} & \sigma_z \end{bmatrix}$$

$$\mathbf{T}_r = \sigma_r \mathbf{e}_r + \tau_{r\theta}\mathbf{e}_\theta + \tau_{rz}\mathbf{e}_z$$

$$\mathbf{T}_\theta = \tau_{r\theta}\mathbf{e}_r + \sigma_\theta \mathbf{e}_\theta + \tau_{\theta z}\mathbf{e}_z$$

$$\mathbf{T}_z = \tau_{rz}\mathbf{e}_r + \tau_{\theta z}\mathbf{e}_\theta + \sigma_z \mathbf{e}_z$$

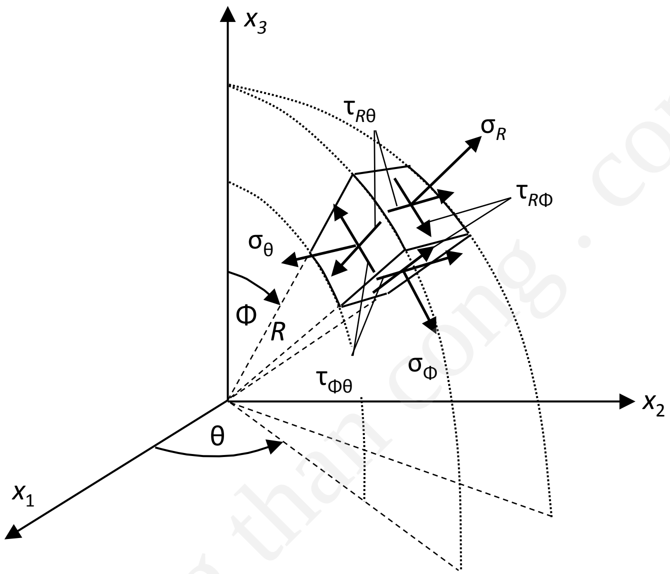
Equilibrium Equations

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{1}{r}\left[\sigma_r - \sigma_\theta\right] + F_r = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2}{r}\tau_{r\theta} + F_\theta = 0$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{1}{r}\tau_{rz} + F_z = 0$$

Spherical Coordinates



$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_R & \tau_{R\varphi} & \tau_{R\theta} \\ \tau_{R\varphi} & \sigma_\varphi & \tau_{\varphi\theta} \\ \tau_{R\theta} & \tau_{\varphi\theta} & \sigma_\theta \end{bmatrix}$$

$$\mathbf{T}_R = \sigma_R \mathbf{e}_R + \tau_{R\varphi}\mathbf{e}_\varphi + \tau_{R\theta}\mathbf{e}_\theta$$

$$\mathbf{T}_\varphi = \tau_{R\varphi} \mathbf{e}_R + \sigma_{\varphi}\mathbf{e}_\varphi + \tau_{\varphi\theta}\mathbf{e}_\theta$$

$$\mathbf{T}_\theta = \tau_{R\theta} \mathbf{e}_R + \tau_{\varphi\theta}\mathbf{e}_\varphi + \sigma_{\theta}\mathbf{e}_\theta$$

Equilibrium Equations

$$\frac{\partial \sigma_{\mathrm{R}}}{\partial R} + \frac{1}{R}\frac{\partial \tau_{R\varphi}}{\partial \varphi} + \frac{1}{R\sin\varphi}\frac{\partial \tau_{\mathrm{R}\theta}}{\partial \theta} + \frac{1}{R}\left(2\sigma_R - \sigma_\varphi - \sigma_\theta + \tau_{R\varphi}\cot\varphi\right) + F_R = 0$$

$$\frac{\partial \tau_{\mathrm{r}\varphi}}{\partial R} + \frac{1}{R}\frac{\partial \sigma_{\varphi}}{\partial \varphi} + \frac{1}{R\sin\varphi}\frac{\partial \tau_{\varphi\theta}}{\partial \theta} + \frac{1}{R}\left[\left(\sigma_\varphi - \sigma_\theta\right)\cot\varphi + 3\tau_{R\varphi}\right] + F_\varphi = 0$$

$$\frac{\partial \tau_{\mathrm{r}\theta}}{\partial R} + \frac{1}{R}\frac{\partial \tau_{\varphi\theta}}{\partial \varphi} + \frac{1}{R\sin\varphi}\frac{\partial \sigma_{\theta}}{\partial \theta} + \frac{1}{R}\left(2\tau_{\varphi\theta}\cot\varphi + 3\tau_{R\theta}\right) + F_\theta = 0$$

THANK YOU