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TON DUC THANG UNIVERSITY

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INCOS
Home of Computational Scientists

SOLID MECHANICS

Chapter 4: Material Behavior – Linear elastic solid

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TDT University - 2015

4.1 Material characterization

4.2 Linear elastic material – Hooke's law

4.3 Physical meaning of elastic module

4.4 Thermo-elastic constitutive relations

4.1 Material characterization

4.1 Material characterization

So far, we studied

$$\begin{cases} \varepsilon_x = \dfrac{\partial u}{\partial x}, \quad \varepsilon_y = \dfrac{\partial v}{\partial y}, \quad \varepsilon_z = \dfrac{\partial w}{\partial z} \\ \gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}, \quad \gamma_{yz} = \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}, \\ \gamma_{zx} = \dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z} \end{cases}$$

+ 6 strain-displacement equations

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} + \dfrac{\partial \tau_{zx}}{\partial z} + F_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{zy}}{\partial z} + F_y = 0 \\ \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} + F_z = 0 \end{cases}$$

+ 3 equilibrium equations

$$\begin{cases} \dfrac{\partial^2 e_x}{\partial y^2} + \dfrac{\partial^2 e_y}{\partial x^2} = 2\dfrac{\partial^2 e_{xy}}{\partial x \partial y}; & \dfrac{\partial^2 e_x}{\partial y \partial z} = \dfrac{\partial}{\partial x}\left(-\dfrac{\partial e_{yz}}{\partial x} + \dfrac{\partial e_{zx}}{\partial y} + \dfrac{\partial e_{xy}}{\partial z} \right) \\ \dfrac{\partial^2 e_y}{\partial z^2} + \dfrac{\partial^2 e_z}{\partial y^2} = 2\dfrac{\partial^2 e_{yz}}{\partial y \partial z}; & \dfrac{\partial^2 e_y}{\partial z \partial x} = \dfrac{\partial}{\partial y}\left(-\dfrac{\partial e_{zx}}{\partial y} + \dfrac{\partial e_{xy}}{\partial z} + \dfrac{\partial e_{yz}}{\partial x} \right) \\ \dfrac{\partial^2 e_z}{\partial x^2} + \dfrac{\partial^2 e_x}{\partial z^2} = 2\dfrac{\partial^2 e_{zx}}{\partial z \partial x}; & \dfrac{\partial^2 e_z}{\partial x \partial y} = \dfrac{\partial}{\partial z}\left(-\dfrac{\partial e_{xy}}{\partial z} + \dfrac{\partial e_{yz}}{\partial x} + \dfrac{\partial e_{zx}}{\partial y} \right) \end{cases}$$

+ 6 compatibility equation

4.1 Material characterization

In which, 6 compatibility equations represent only 3 independent relations, and these equations are needed only ensure that a given strain field will produce single-valued continuous displacements. => No need for the general problems

Excluding the compatibility relations, it is found that we have 9 field equations. The unknowns in these equations include 3 displacement components, 6 components of strain, and 6 stress components => total 15 unknowns.

So far, 9 equations are not sufficient to solve for 15 unknowns.

- We need additional field equations
- The material response => the relationship between the strains and stresses.

4.1 Material characterization

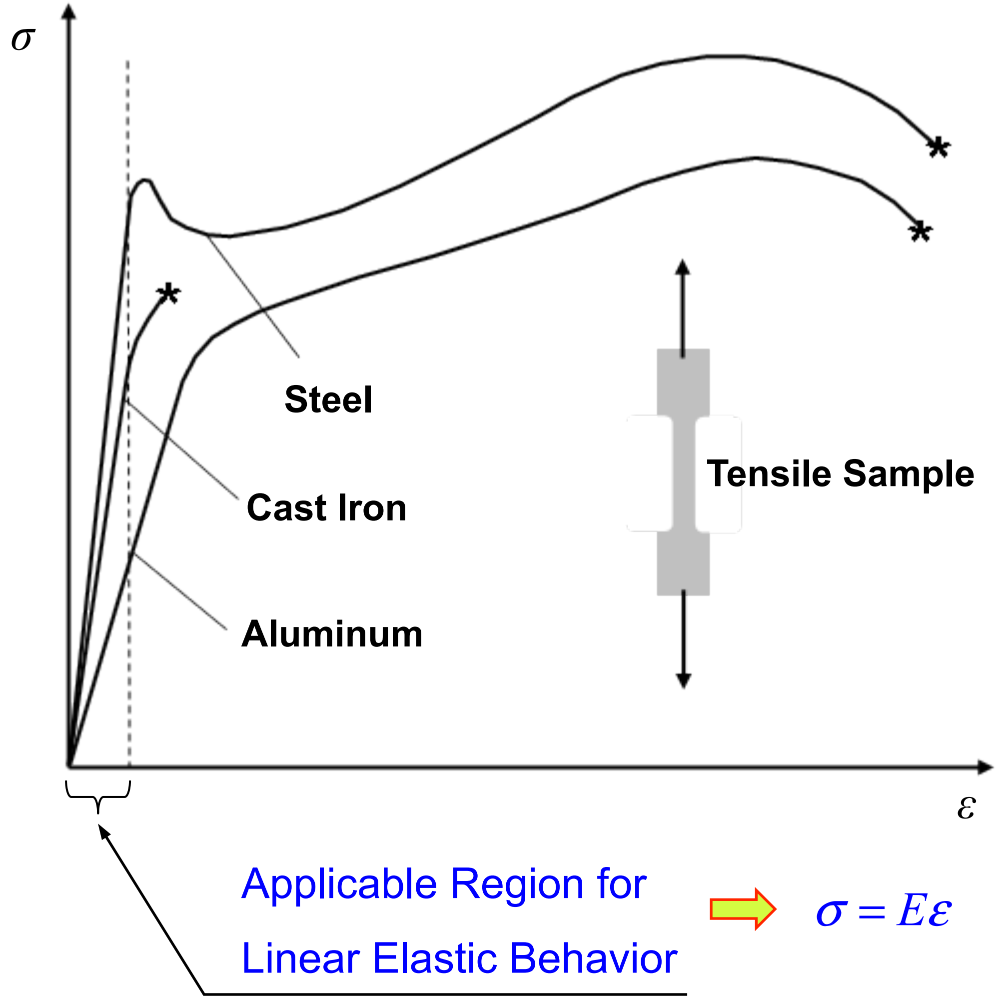
Mechanical behavior of solids is normally defined by constitutive stress-strain relations. Commonly, these relations express the stress as a function of the strain, strain rate, strain history, temperature, and material properties. Here, we use the Linear Elastic Constitutive Solid Model in which the Stress-Strain Relations are under the Assumptions:

- Solid Recovers Original Configuration When Loads Are Removed
- Linear Relation Between Stress and Strain
- Neglect Rate and History Dependent Behavior
- Include Only Mechanical Loadings
- Thermal, Electrical, Pore-Pressure, and Other Loadings Can Also Be Included As Special Cases

4.1 Material characterization

Typical One-Dimensional

Stress-Strain Behavior



4.2 Linear elastic material – Hooke's law

$$\sigma_{ij} = C_{ijkm}\varepsilon_{km} \qquad \sigma_K = C_{KM}\varepsilon_M \qquad (K, M = 1,2,3,4,5,6)$$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1112} & C_{1113} & C_{1211} & C_{1212} & C_{1213} & C_{1311} & C_{1312} & C_{1313} \\ C_{1121} & C_{1122} & C_{1123} & C_{1221} & C_{1222} & C_{1223} & C_{1321} & C_{1322} & C_{1323} \\ C_{1131} & C_{1132} & C_{1133} & C_{1231} & C_{1232} & C_{1233} & C_{1331} & C_{1332} & C_{1333} \\ C_{2111} & C_{2112} & C_{2113} & C_{2211} & C_{2212} & C_{2213} & C_{2311} & C_{2312} & C_{2313} \\ C_{2121} & C_{2122} & C_{2123} & C_{2221} & C_{2222} & C_{2223} & C_{2321} & C_{2322} & C_{2323} \\ C_{2131} & C_{2132} & C_{2133} & C_{2231} & C_{2232} & C_{2233} & C_{2331} & C_{2332} & C_{2333} \\ C_{3111} & C_{3112} & C_{3113} & C_{3211} & C_{3212} & C_{3213} & C_{3311} & C_{3312} & C_{3313} \\ C_{3121} & C_{3122} & C_{3123} & C_{3221} & C_{3222} & C_{3223} & C_{3321} & C_{3322} & C_{3323} \\ C_{3131} & C_{3132} & C_{3133} & C_{3231} & C_{3232} & C_{3233} & C_{3331} & C_{3332} & C_{3333} \end{bmatrix} : \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1113} & C_{1123} \\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2213} & C_{2223} \\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3313} & C_{3323} \\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1213} & C_{1223} \\ C_{1311} & C_{1322} & C_{1333} & C_{1312} & C_{1313} & C_{1323} \\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2313} & C_{2323} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix}$$

4.2 Linear elastic material – Hooke's law

$$\sigma_{ij} = C_{ijkl} e_{kl} \Rightarrow \begin{cases} \sigma_x = C_{11}e_x + C_{12}e_y + C_{13}e_z + 2C_{14}e_{xy} + 2C_{15}e_{yz} + 2C_{16}e_{zx} \\ \sigma_y = C_{21}e_x + C_{22}e_y + C_{23}e_z + 2C_{24}e_{xy} + 2C_{25}e_{yz} + 2C_{26}e_{zx} \\ \sigma_z = C_{31}e_x + C_{32}e_y + C_{33}e_z + 2C_{34}e_{xy} + 2C_{35}e_{yz} + 2C_{36}e_{zx} \\ \tau_{xy} = C_{41}e_x + C_{42}e_y + C_{43}e_z + 2C_{44}e_{xy} + 2C_{45}e_{yz} + 2C_{46}e_{zx} \\ \tau_{yz} = C_{51}e_x + C_{52}e_y + C_{53}e_z + 2C_{54}e_{xy} + 2C_{55}e_{yz} + 2C_{56}e_{zx} \\ \tau_{zx} = C_{61}e_x + C_{62}e_y + C_{63}e_z + 2C_{64}e_{xy} + 2C_{65}e_{yz} + 2C_{66}e_{zx} \end{cases}$$

with

$$C_{ijkl} = C_{jikl}; C_{ijkl} = C_{ijlk}$$

(Due to the symmetry of stress and strain tensors)

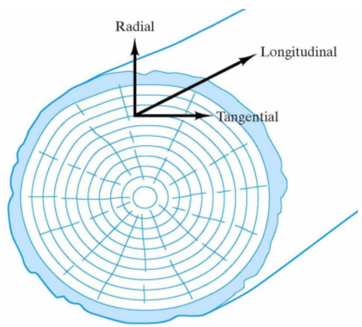
or

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & \cdot & \cdot & \cdot & C_{16} \\ C_{21} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ C_{61} & \cdot & \cdot & \cdot & \cdot & C_{66} \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ e_z \\ 2e_{xy} \\ 2e_{yz} \\ 2e_{zx} \end{bmatrix} \Rightarrow$$

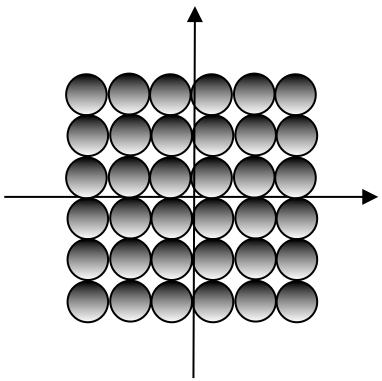
36 Independent Elastic Constants

4.2 Linear elastic material – Hooke's law

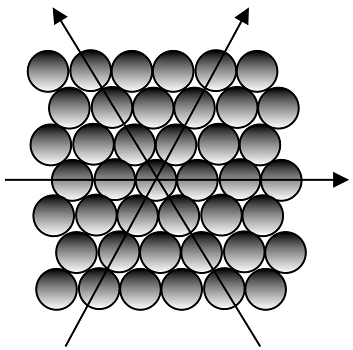
Anisotropy - Differences in material properties under different directions. Materials like wood, crystalline minerals, fiber-reinforced composites have such behavior.



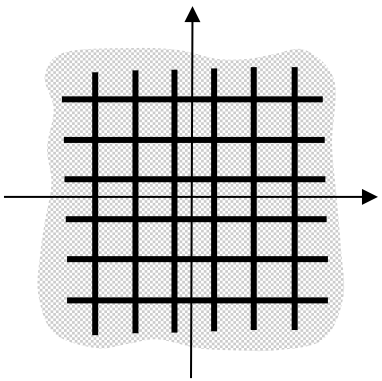
Typical Wood Structure



(Body-Centered Crystal)

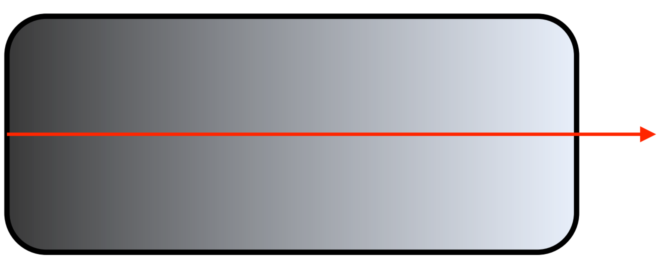


(Hexagonal Crystal)



(Fiber Reinforced Composite)

Nonhomogeneity - Spatial differences in material properties. Soil materials in the earth vary with depth, and new functionally graded materials (FGM's) are now being developed with deliberate spatial variation in elastic properties to produce desirable behaviors.



Gradation Direction

4.2 Linear elastic material – Hooke's law

Isotropic Materials

Although many materials exhibit non-homogeneous and anisotropic behavior, we will primarily restrict our study to isotropic solids. For this case, material response is independent of coordinate rotation

$$C_{ijkl} = Q_{im}Q_{jn}Q_{kp}Q_{lq}C_{mnpq} \Rightarrow C_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$ (prove)

$$\sigma_{ij} = C_{ijkl}e_{kl} \Rightarrow \sigma_{ij} = \left(\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}\right)e_{kl} = \alpha e_{kk}\delta_{ij} + \beta e_{ij} + \gamma e_{ij}$$

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}$$ Generalized Hooke's Law

$$\begin{cases} \sigma_x = \lambda\left(e_x + e_y + e_z\right) + 2\mu e_x \ ; \ \sigma_y = \lambda\left(e_x + e_y + e_z\right) + 2\mu e_y \\ \sigma_z = \lambda\left(e_x + e_y + e_z\right) + 2\mu e_z \\ \tau_{xy} = 2\mu e_{xy} \ ; \ \tau_{yz} = 2\mu e_{yz} \ ; \ \tau_{zx} = 2\mu e_{zx} \end{cases}$$

λ - Lamé's constant
μ - shear modulus or *modulus of rigidity*

4.2 Linear elastic material – Hooke's law

Prove

$$C_{ijkl} = Q_{im}Q_{jn}Q_{kp}Q_{lq}C_{mnpq} \Rightarrow C_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

$$\alpha'\delta'_{ij}\delta'_{kl} + \beta'\delta'_{ik}\delta'_{jl} + \gamma'\delta'_{il}\delta'_{jk} = Q_{im}Q_{jn}Q_{kp}Q_{lq}\left(\alpha\delta_{mn}\delta_{pq} + \beta\delta_{mp}\delta_{nq} + \gamma\delta_{mq}\delta_{np}\right)$$
$$= \alpha Q_{im}Q_{jm}Q_{kp}Q_{lp} + \beta Q_{im}Q_{jn}Q_{km}Q_{ln} + \gamma Q_{im}Q_{jn}Q_{kn}Q_{lm}$$
$$= \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

$$C'_{ijkl} = \alpha'\delta'_{ij}\delta'_{kl} + \beta'\delta'_{ik}\delta'_{jl} + \gamma'\delta'_{il}\delta'_{jk}$$

$$C_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

4.2 Linear elastic material – Hooke's law

Isotropic Materials

Inverted Form - Strain in Terms of Stress

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \Rightarrow \sigma_{kk} = (3\lambda + 2\mu)e_{kk}$$

$$\Downarrow$$

$$e_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\sigma_{kk}\delta_{ij}\right)$$

$$\Downarrow$$

$$e_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \Rightarrow$$

$$e_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$e_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_z + \sigma_x)\right]$$

$$e_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$

$$e_{xy} = \frac{1+\nu}{E}\tau_{xy} = \frac{1}{2\mu}\tau_{xy}$$

$$e_{yz} = \frac{1+\nu}{E}\tau_{yz} = \frac{1}{2\mu}\tau_{yz}$$

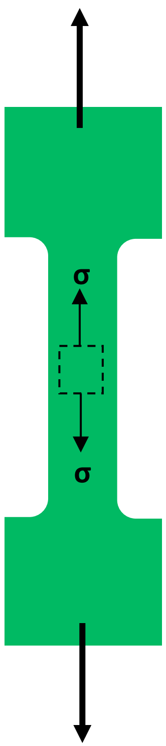
$$e_{zx} = \frac{1+\nu}{E}\tau_{zx} = \frac{1}{2\mu}\tau_{zx}$$

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$ Young's modulus or modulus of elasticity

$$\nu = \frac{\lambda}{2(\lambda + \mu)}$$ Poisson's ratio

4.3 Physical meaning of elastic module

Simple Tension



Consider the simple tension with a sample subjected to tension in the x-direction. The state of stress is represented by the one-dimensional field

$$\sigma_{ij}=\begin{bmatrix}\sigma & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \Rightarrow e_{ij}=\begin{bmatrix}\frac{\sigma}{E} & 0 & 0\\ 0 & -\frac{\nu}{E}\sigma & 0\\ 0 & 0 & -\frac{\nu}{E}\sigma\end{bmatrix}$$

$$E=\sigma/e_x$$

Slope of the stress-strain curve or Elastic module in the x-direction

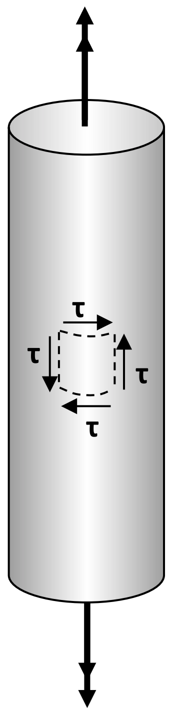
$$\nu=-e_y/e_x$$
$$=-e_z/e_x$$

Ratio of the transverse strain to the axial strain

Standard measurement systems can easily collect axial stress and transverse and axial strain data, and thus through this, one type of test both elastic constants E and ν can be determined for material of interest.

4.3 Physical meaning of elastic module

Pure Shear



If a thin-walled cylinder is subjected to torsion loading, the state of stress on the surface of the cylindrical sample is given by

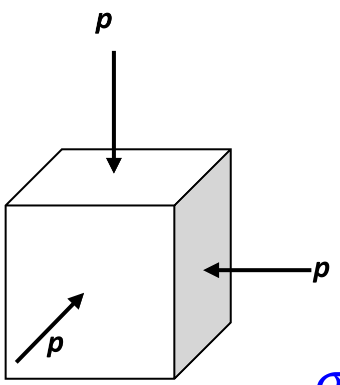
$$\sigma_{ij} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow e_{ij} = \begin{bmatrix} 0 & \tau/2\mu & 0 \\ \tau/2\mu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \mu = \tau / 2e_{xy} = \tau / \gamma_{xy}$$

Shear modulus which is the slope of the shear stress-shear strain curve

4.3 Physical meaning of elastic module

Hydrostatic Compression



The final example is associated with the uniform compression (or tension) loading of a cubical specimen. This type of test can be realizable if the sample was placed in a high-pressure compression chamber. The state of stress for this case is given by

$$\sigma_{ij} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} \Rightarrow e_{ij} = \begin{bmatrix} -\frac{1-2\nu}{E}p & 0 & 0 \\ 0 & -\frac{1-2\nu}{E}p & 0 \\ 0 & 0 & -\frac{1-2\nu}{E}p \end{bmatrix}$$

$$\Rightarrow p = -ke_{kk} = -k\vartheta$$

$$k = \frac{E}{3(1-2\nu)}$$

Bulk Modulus

Elastic constant k represents the ratio of pressure to the dilatation (which represents the change in material volume)

Note that when Poisson's ratio approaches 0.5, the bulk modulus becomes unbounded and the material does not undergo any volumetric deformation and hence is incompressible.

4.3 Physical meaning of elastic module

- Our discussion of elastic modulus for isotropic materials has led to the definition of five constants λ, μ, E, v and k. However, keep in mind that only two of these are needed to characterize the material.
- In can be shown that all five elastic constants are interrelated, and if any two are given, the remaining three can be determined by using simple formulae. Results of these relations are conveniently summarized in Table 4.1.
- In addition, nominal values of elastic constants for particular engineering materials are given in Table 4.2.

4.3 Physical meaning of elastic module

Table 4.1: Relations Among Elastic Constants

	E	ν	k	μ	λ
E,ν	E	ν	$\frac{E}{3(1-2\nu)}$	$\frac{E}{2(1+\nu)}$	$\frac{E\nu}{(1+\nu)(1-2\nu)}$
E,k	E	$\frac{3k-E}{6k}$	k	$\frac{3kE}{9k-E}$	$\frac{3k(3k-E)}{9k-E}$
E,μ	E	$\frac{E-2\mu}{2\mu}$	$\frac{\mu E}{3(3\mu-E)}$	μ	$\frac{\mu(E-2\mu)}{3\mu-E}$
E,λ	E	$\frac{2\lambda}{E+\lambda+R}$	$\frac{E+3\lambda+R}{6}$	$\frac{E-3\lambda+R}{4}$	λ
ν,k	$3k(1-2\nu)$	ν	k	$\frac{3k(1-2\nu)}{2(1+\nu)}$	$\frac{3k\nu}{1+\nu}$
ν,μ	$2\mu(1+\nu)$	ν	$\frac{2\mu(1+\nu)}{3(1-2\nu)}$	μ	$\frac{2\mu\nu}{1-2\nu}$
ν,λ	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	ν	$\frac{\lambda(1+\nu)}{3\nu}$	$\frac{\lambda(1-2\nu)}{2\nu}$	λ
k,μ	$\frac{9k\mu}{6k+\mu}$	$\frac{3k-2\mu}{6k+2\mu}$	k	μ	$k-\frac{2}{3}\mu$
k,λ	$\frac{9k(k-\lambda)}{3k-\lambda}$	$\frac{\lambda}{3k-\lambda}$	k	$\frac{3}{2}(k-\lambda)$	λ
μ,λ	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$	$\frac{3\lambda+2\mu}{3}$	μ	λ

$$R=\sqrt{E^2+9\lambda^2+2E\lambda}$$

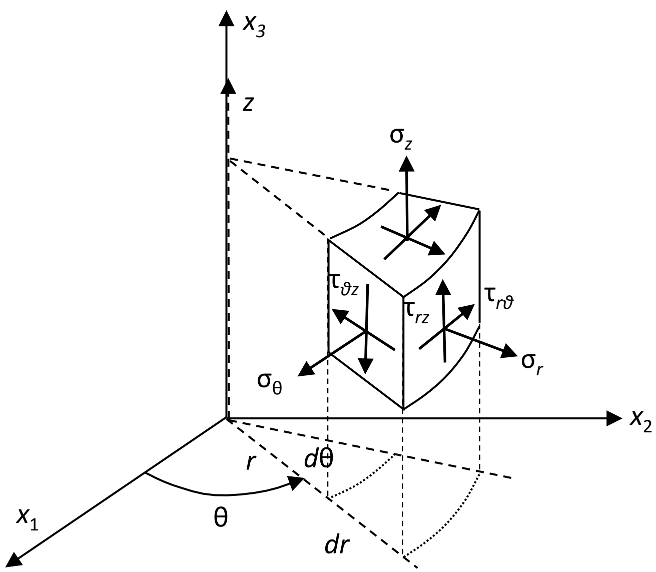
4.3 Physical meaning of elastic module

Table 4.2: Typical Values of Elastic Moduli for Common Engineering Materials

	E (GPa)	ν	$\mu(GPa)$	$\lambda(GPa)$	$k(GPa)$	$\alpha(10^{-6}/{}^{o}C)$
Aluminum	68.9	0.34	25.7	54.6	71.8	25.5
Concrete	27.6	0.20	11.5	7.7	15.3	11
Cooper	89.6	0.34	33.4	71	93.3	18
Glass	68.9	0.25	27.6	27.6	45.9	8.8
Nylon	28.3	0.40	10.1	4.04	47.2	102
Rubber	0.0019	0.499	$0.654\text{x}10^{-3}$	0.326	0.326	200
Steel	207	0.29	80.2	111	164	13.5

4.3 Physical meaning of elastic module

Hooke's Law in Cylindrical Coordinates

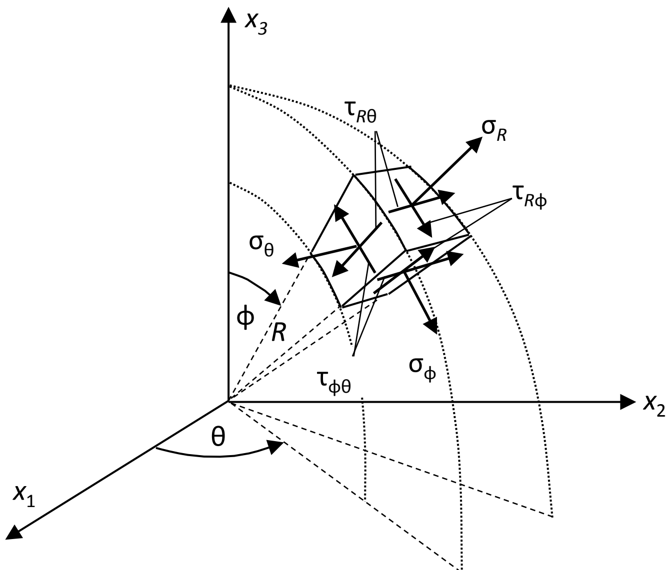


$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_r & \tau_{r\theta} & \tau_{rz} \\ \tau_{r\theta} & \sigma_\theta & \tau_{\theta z} \\ \tau_{rz} & \tau_{\theta z} & \sigma_z \end{bmatrix}$$

$$\sigma_r = \lambda(e_r + e_\theta + e_z) + 2\mu e_r$$
$$\sigma_\theta = \lambda(e_r + e_\theta + e_z) + 2\mu e_\theta$$
$$\sigma_z = \lambda(e_r + e_\theta + e_z) + 2\mu e_z$$
$$\tau_{r\theta} = 2\mu e_{r\theta}$$
$$\tau_{\theta z} = 2\mu e_{\theta z}$$
$$\tau_{zr} = 2\mu e_{zr}$$

4.3 Physical meaning of elastic module

Hooke's Law in Spherical Coordinates



$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_R & \tau_{R\varphi} & \tau_{R\theta} \\ \tau_{R\varphi} & \sigma_\varphi & \tau_{\varphi\theta} \\ \tau_{R\theta} & \tau_{\varphi\theta} & \sigma_\theta \end{bmatrix}$$

$$\sigma_R = \lambda(e_R + e_\varphi + e_\theta) + 2\mu e_R$$

$$\sigma_\varphi = \lambda(e_R + e_\varphi + e_\theta) + 2\mu e_\varphi$$

$$\sigma_\theta = \lambda(e_R + e_\varphi + e_\theta) + 2\mu e_\theta$$

$$\tau_{R\varphi} = 2\mu e_{R\varphi}$$

$$\tau_{\varphi\theta} = 2\mu e_{\varphi\theta}$$

$$\tau_{\theta R} = 2\mu e_{\theta R}$$

4.4 Thermo-elastic constitutive relations

- It is well known that a temperature change in an unrestrained elastic solid produces deformation. Thus a general strain field results from both mechanical and thermal effects. Within the context of linear small deformation theory, the total strain can be decomposed into the sum of mechanical and thermal components as

$$e_{ij} = e_{ij}^{(M)} + e_{ij}^{(T)}$$

- If T_0 is taken as the reference temperature and T as an arbitrary temperature, the thermal strains in an unrestrained solid can be written in the linear form

$$e_{ij}^{(T)} = \alpha_{ij}\left(T - T_0\right)$$

where α_{ij} is the coefficient of thermal expansion tensor. Notice that it is the temperature difference that creates thermal strain. If the material is taken as isotropic, then e_{ij} must be an isotropic second-order tensor, and

$$e_{ij}^{(T)} = \alpha\left(T - T_0\right)\delta_{ij}$$

where α is the coefficient of thermal expansion. Table 4.2 provides typical values of this constant for some common materials.

4.4 Thermo-elastic constitutive relations

- Notice that for isotropic materials, no shear strains are created by temperature change. This result can be combined with the mechanical relation to give

$$e_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha\left(T - T_0\right)\delta_{ij} \qquad (4.4.4)$$

- The corresponding results for the stress in terms of strain can be written as

$$\sigma_{ij} = C_{ijkl}e_{kl} - \beta_{ij}\left(T - T_0\right)$$

where β_{ij} is a second-order tensor containing thermo-elastic modulus. This result is sometimes referred to as the Duhamel-Neumann thermo-elastic constitutive law. The isotropic case can be found by simply inverting relation (4.4.4) to get

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \left(3\lambda + 2\mu\right)\alpha\left(T - T_0\right)\delta_{ij}$$

- Having developed the necessary 6 constitutive relations, the elasticity field equation system is now complete with 15 equations (6 strain-displacement, 3 equilibrium, 6 Hooke's law) for 15 unknowns (3 displacements, 6 strains and 6 stresses)

THANK YOU