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INCOS
Home of Computational Scientists

SOLID MECHANICS

Chapter 5: Formulation – Solution Strategies

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5.1 Review of basic field equations

5.2 Boundary conditions & fundamental problems

5.3 Stress formulation

5.4 Displacement formulation

5.5 Principle of superposition

5.6 Saint-Venant’s principle

5.7 General solution strategies

5.1 Review of basic field equations

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

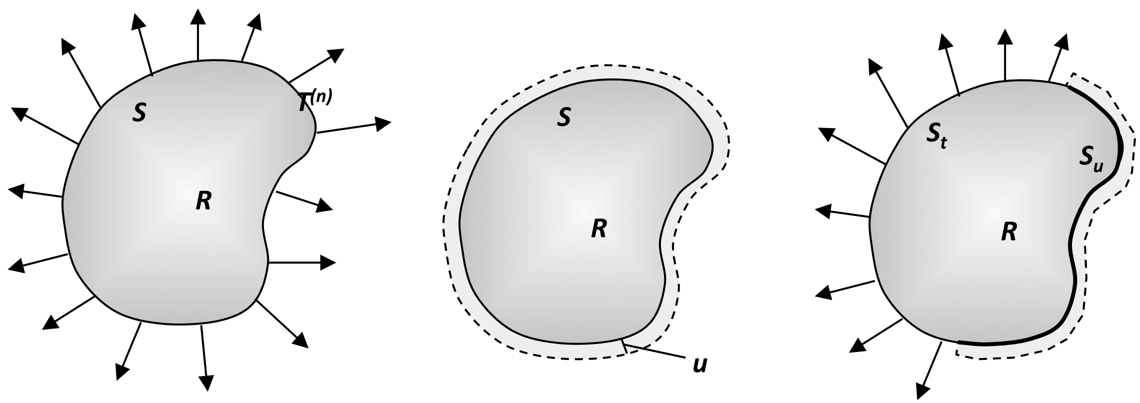
$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0$$ Strain-Displacement Relations

$$\sigma_{ij,j} + F_i = 0$$ Compatibility Relations

$$\sigma_{ij} = (\lambda + \mu) e_{kk}\delta_{ij} + 2\mu e_{ij}$$ Equilibrium Equations

$$e_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$ Hooke's Law

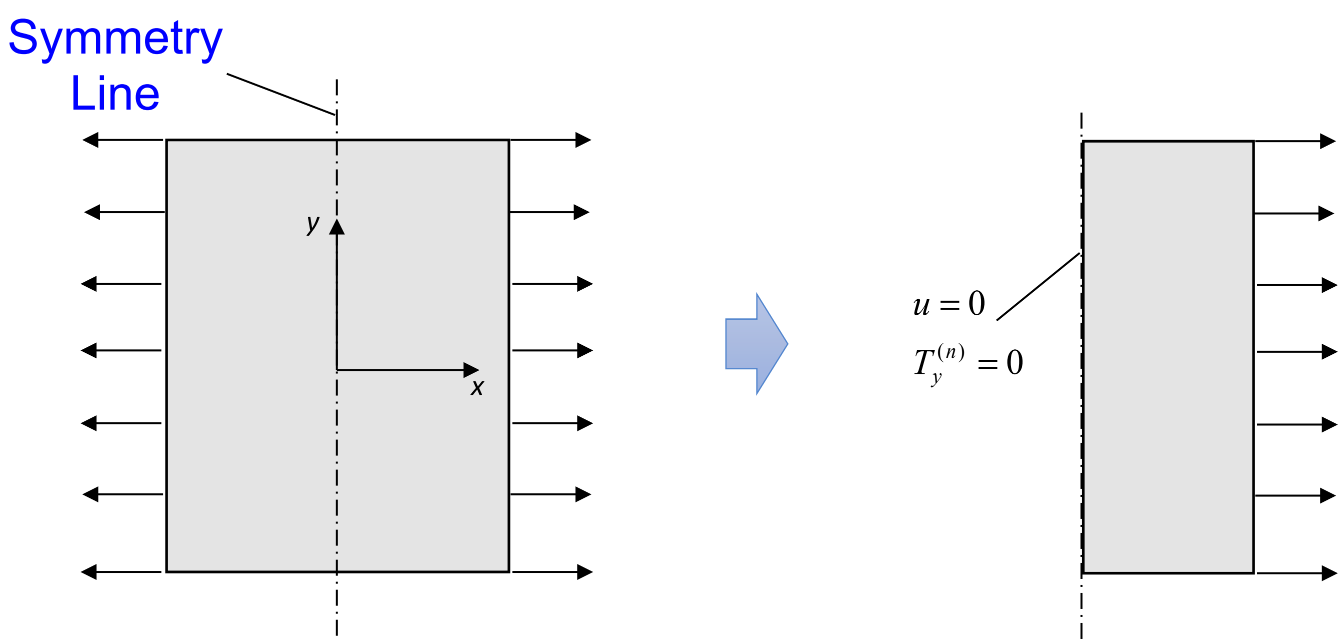
15 Equations for 15 Unknowns σ_{ij} , e_{ij}, u_i



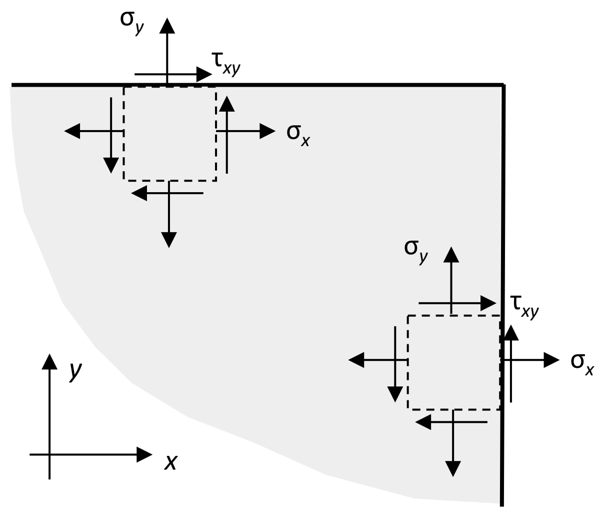
Traction Conditions

Displacement Conditions

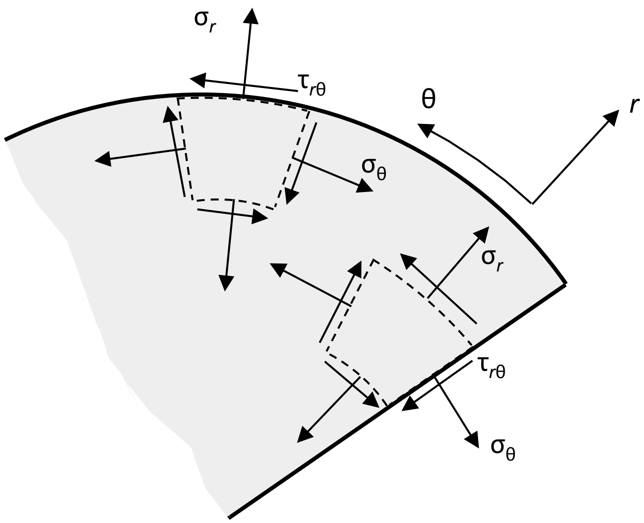
Mixed Conditions



On coordinate surfaces the traction vector reduces to simply particular stress components

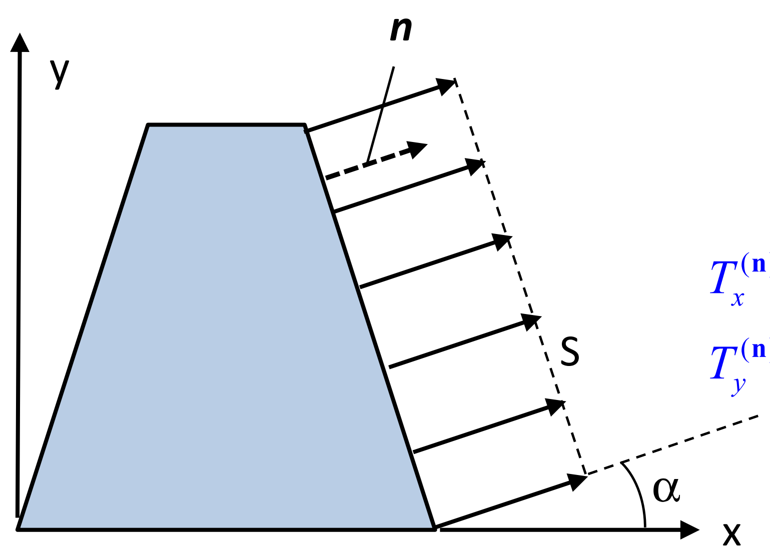


Cartesian Coordinate Boundaries



Polar Coordinate Boundaries

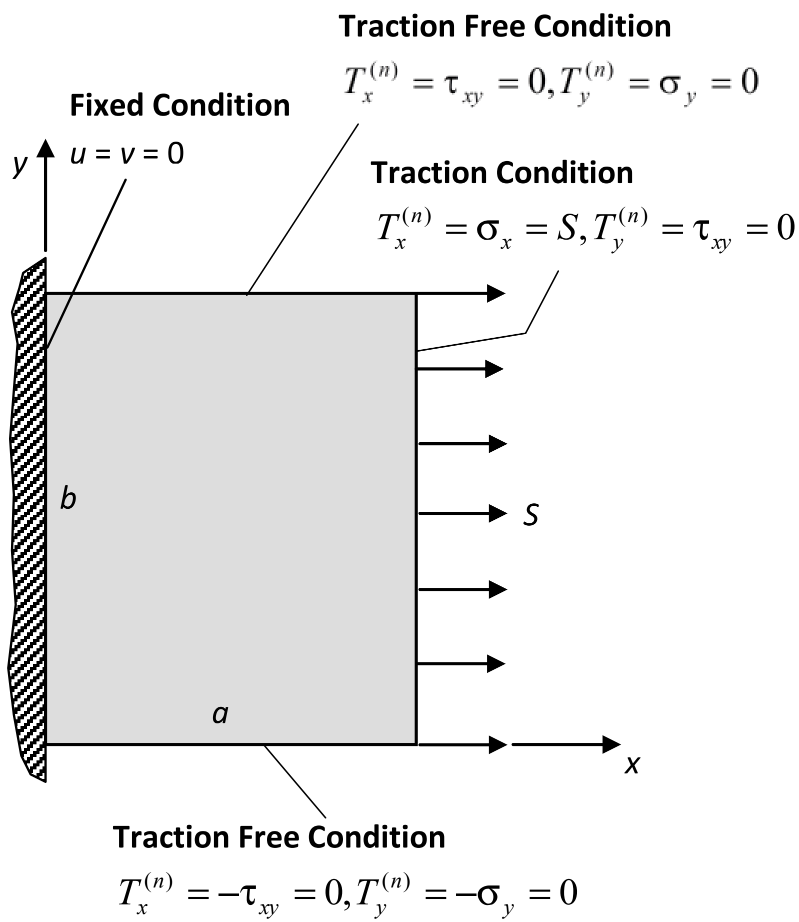
On general non-coordinate surfaces, traction vector will not reduce to individual stress components and general traction vector form must be used



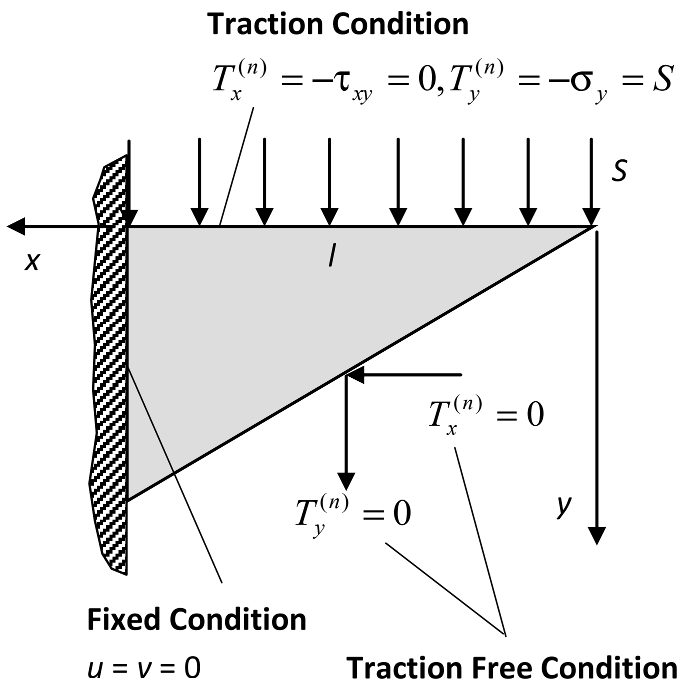
$$T_x^{(\mathbf{n})} = \sigma_x n_x + \tau_{xy} n_y = S\cos\alpha$$
$$T_y^{(\mathbf{n})} = \tau_{xy} n_x + \sigma_y n_y = S\sin\alpha$$

Two-dimensional example

Example boundary conditions

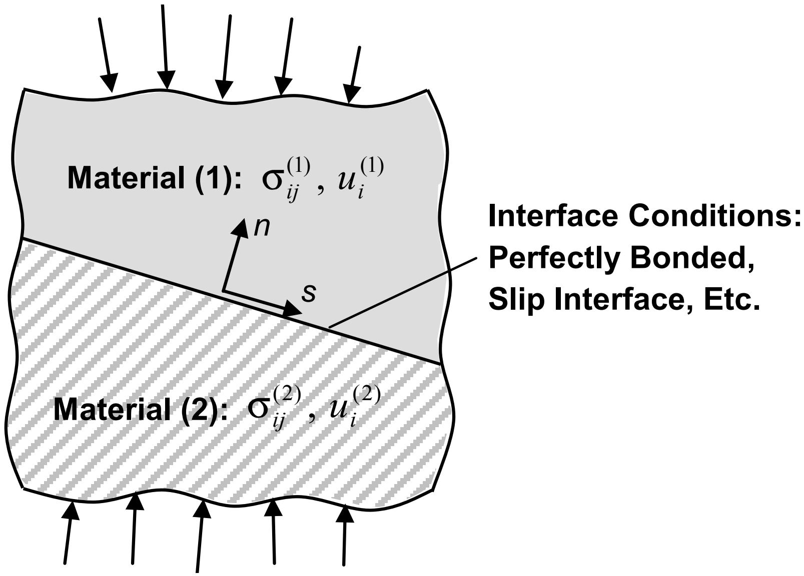


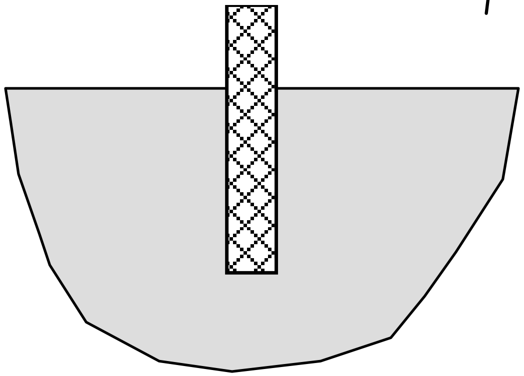
Coordinate Surface Boundaries



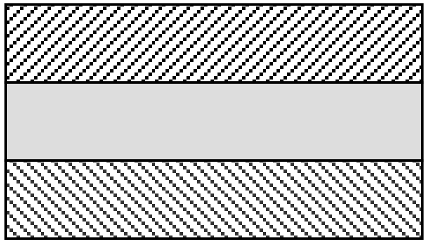
Non-Coordinate Surface

Interface conditions

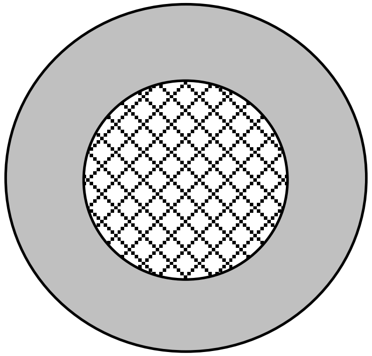




Embedded Fiber or Rod



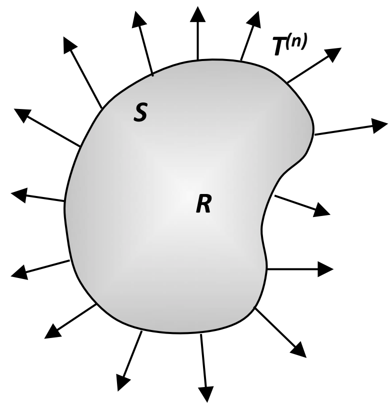
Layered Composite Plate



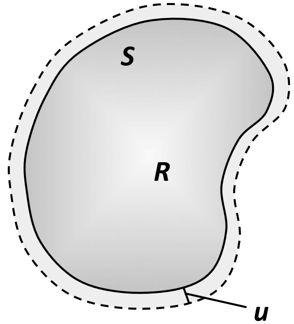
Composite Cylinder or Disk

Fundamental problem classifications

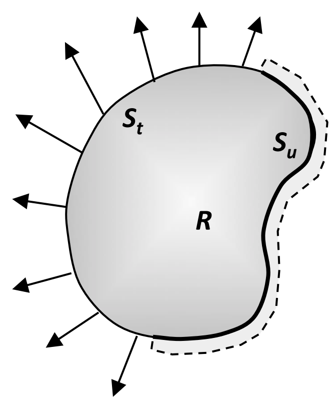
Problem 1 (Traction Problem) Determine the distribution of displacements, strains and stresses in the interior of an elastic body in equilibrium when body forces are given and the distribution of the tractions are prescribed over the surface of the body, $T_i^{(\mathbf{n})}(x_i^{(s)}) = f_i(x_i^{(s)})$



Problem 2 (Displacement Problem) Determine the distribution of displacements, strains and stresses in the interior of an elastic body in equilibrium when body forces are given and the distribution of the displacements are prescribed over the surface of the body, $u_i(x_i^{(s)}) = g_i(x_i^{(s)})$



Problem 3 (Mixed Problem) Determine the distribution of displacements, strains and stresses in the interior of an elastic body in equilibrium when body forces are given and the distribution of the tractions are prescribed as per over the surface S_t and the distribution of the displacements are prescribed as per over the surface S_u of the body (see Figure 5.1).



Eliminate Displacements and Strains from Fundamental Field

Equilibrium Equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0$$

Compatibility in Terms of Stress: Beltrami-Michell Compatibility Equations

$$(1+\nu)\nabla^2\sigma_x + \frac{\partial^2}{\partial x^2}(\sigma_x + \sigma_y + \sigma_z) = 0$$

$$(1+\nu)\nabla^2\sigma_y + \frac{\partial^2}{\partial y^2}(\sigma_x + \sigma_y + \sigma_z) = 0$$

$$(1+\nu)\nabla^2\sigma_z + \frac{\partial^2}{\partial z^2}(\sigma_x + \sigma_y + \sigma_z) = 0$$

$$(1+\nu)\nabla^2\tau_{xy} + \frac{\partial^2}{\partial x \partial y}(\sigma_x + \sigma_y + \sigma_z) = 0$$

$$(1+\nu)\nabla^2\tau_{yz} + \frac{\partial^2}{\partial y \partial z}(\sigma_x + \sigma_y + \sigma_z) = 0$$

$$(1+\nu)\nabla^2\tau_{zx} + \frac{\partial^2}{\partial z \partial x}(\sigma_x + \sigma_y + \sigma_z) = 0$$

6 Equations for 6 Unknown Stresses

Eliminate Stress and Strains from Fundamental Field Equation

Equilibrium Equations in Terms of Displacements:
Navier's/Lame's Equations

$$\mu\nabla^2 u + (\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0$$

$$\mu\nabla^2 v + (\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0$$

$$\mu\nabla^2 w + (\lambda+\mu)\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0$$

3 Equations for 3 Unknown Displacements

Summary of Reduction of Fundamental Elasticity Field

General Field Equation System
(15 Equations, 15 Unknowns:)

$$\Im\{u_i, e_{ij}, \sigma_{ij}; \lambda, \mu, F_i\} = 0$$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$
$$\sigma_{ij,j} + F_i = 0$$
$$\sigma_{ij} = (\lambda + \mu)e_{kk}\delta_{ij} + 2\mu e_{ij}$$
$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0$$

Stress Formulation
(6 Equations, 6 Unknowns:)

$$\Im^{(t)}\{\sigma_{ij}; \lambda, \mu, F_i\}$$

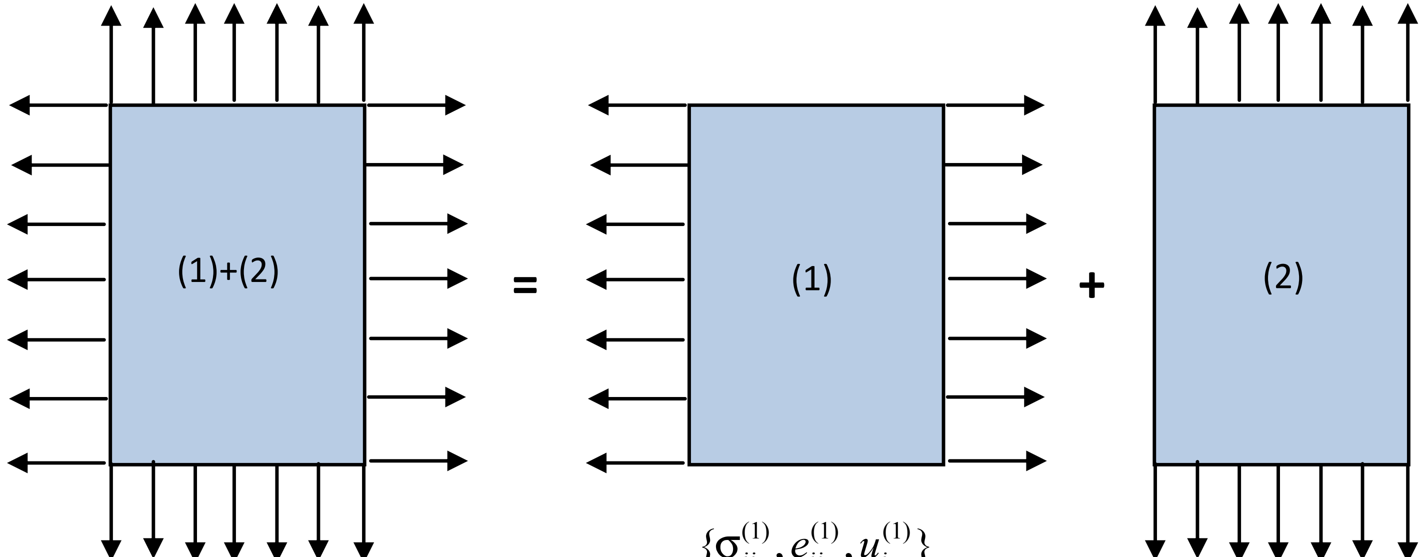
$$\sigma_{ij,j} + F_i = 0$$
$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = -\frac{\nu}{1-\nu}\delta_{ij}F_{k,k} - F_{i,j} - F_{j,i}$$

Displacement Formulation
(3 Equations, 3 Unknowns: u_i)

$$\Im^{(u)}\{u_i; \lambda, \mu, F_i\}$$

$$\mu u_{i,kk} + (\lambda + \mu)u_{k,ki} + F_i = 0$$

For a given problem domain, if the state $\{\sigma_{ij}^{(1)}, e_{ij}^{(1)}, u_i^{(1)}\}$ is a solution to the fundamental elasticity equations with prescribed body forces $F_i^{(1)}$ and surface tractions $T_i^{(1)}$, and the state $\{\sigma_{ij}^{(2)}, e_{ij}^{(2)}, u_i^{(2)}\}$ is a solution to the fundamental equations with prescribed body forces $F_i^{(2)}$ and surface tractions $T_i^{(2)}$, then the state $\{\sigma_{ij}^{(1)}+\sigma_{ij}^{(2)}, e_{ij}^{(1)}+e_{ij}^{(2)}, u_i^{(1)}+u_i^{(2)}\}$ will be a solution to the problem with body forces $F_i^{(1)}+F_i^{(2)}$ and surface tractions $T_i^{(1)}+T_i^{(2)}$.



$\{\sigma_{ij}^{(1)}+\sigma_{ij}^{(2)}, e_{ij}^{(1)}+e_{ij}^{(2)}, u_i^{(1)}+u_i^{(2)}\}$ $\quad$ $\{\sigma_{ij}^{(1)}, e_{ij}^{(1)}, u_i^{(1)}\}$ $\quad$ $\{\sigma_{ij}^{(2)}, e_{ij}^{(2)}, u_i^{(2)}\}$

5.1 Review of basic field equations

5.2 Boundary conditions & fundamental problems

5.3 Stress formulation

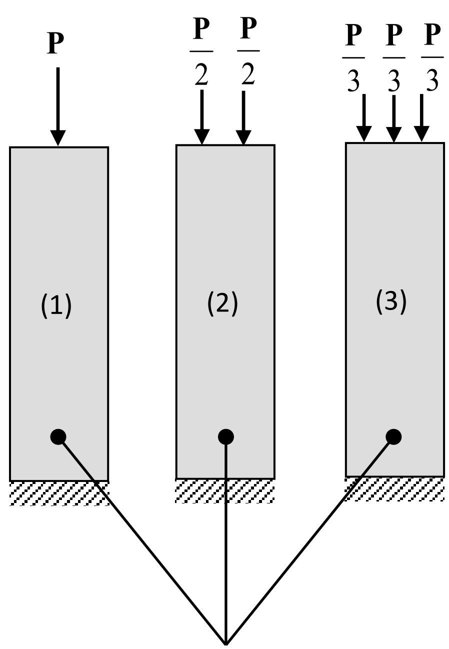
5.4 Displacement formulation

5.5 Principle of superposition

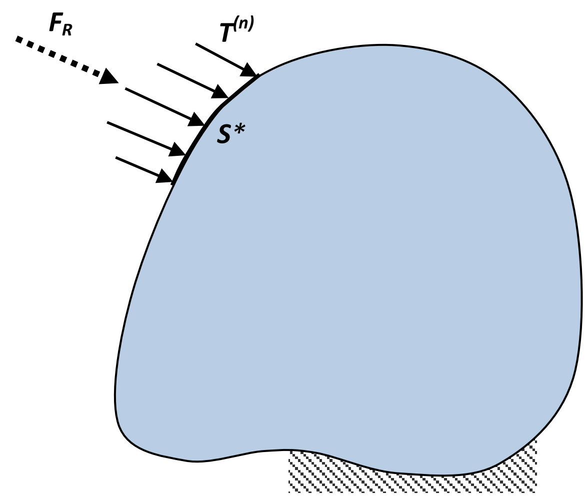
5.6 Saint-Venant's principle

5.7 General solution strategies

The stress, strain and displacement fields due to two different statically equivalent force distributions on parts of the body far away from the loading points are approximately the same



Stresses approximately the same



Boundary loading $T^{(\mathbf{n})}$ would produce detailed and characteristic effects only in vicinity of S^*. Away from S^* stresses would generally depend more on resultant F_R of tractions rather than on exact distribution

5.7 General solution strategies

5.7.1 Direct Method - Solution of field equations by direct integration. Boundary conditions are satisfied exactly. Method normally encounters significant mathematical difficulties thus limiting its application to problems with simple geometry.

5.7.2 Inverse Method - Displacements or stresses are selected that satisfy field equations. A search is then conducted to identify a specific problem that would be solved by this solution field. This amounts to determine appropriate problem geometry, boundary conditions and body forces that would enable the solution to satisfy all conditions on the problem. It is sometimes difficult to construct solutions to a specific problem of practical interest.

5.7.3 Semi-Inverse Method - Part of displacement and/or stress field is specified, while the other remaining portion is determined by the fundamental field equations (normally using direct integration) and the boundary conditions. It is often the case that constructing appropriate displacement and/or stress solution fields can be guided by approximate strength of materials theory. The usefulness of this approach is greatly enhanced by employing Saint-Venant's principle, whereby a complicated boundary condition can be replaced by a simpler statically equivalent distribution.

Example 5-1: Direct Integration Example:

Stretching of Prismatic Bar Under Its Own Weight As an example of a simple direct integration problem, consider the case of a uniform prismatic bar stretched by its own weight, as shown in Figure 5-11. The body forces for this problem are $F_x = F_y = 0$, $F_z = -\rho g$, where ρ is the material mass density and g is the acceleration of gravity

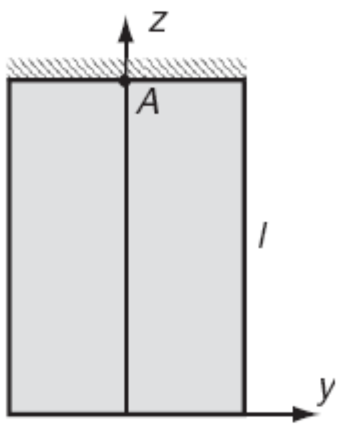


FIGURE 5-11 *Prismatic bar under self-weight.*

Assuming that on each cross-section we have uniform tension produced by the weight of the lower portion of the bar, the stress field would take the from.

$$\sigma_x = \sigma_y = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0 \qquad (5.7.1)$$

The equilibrium equations reduce to the simple result

$$\frac{\partial \sigma_z}{\partial z} = -F_z = \rho g \qquad (5.7.2)$$

This equations can be integrated directly, and applying the boundary condition $\sigma_z = 0$ at $z = 0$ gives the result $\sigma_z(z) = \rho g z$. Next, by using Hooke's law, the strains are easily calculated as

$$e_z = \frac{\rho g z}{E}, e_x = e_y = -\frac{\nu \rho g z}{E} \qquad (5.7.3)$$

$$e_{xy} = e_{yz} = e_{xz} = 0$$

The displacements follow from integrating the strain-displacement relation and for the case with boundary conditions of zeros displacement and rotation at point A ($x = y = 0; z = l$), the final result is

$$u = -\frac{\nu\rho g x z}{E}, v = -\frac{\nu\rho g y z}{E} \qquad (5.7.4)$$

$$w = \frac{\rho g}{2E}\left[z^2 + \nu(x^2 + y^2) - l^2\right]$$

Example 5-2: Inverse Example - Pure Beam Bending

Consider the case of an elasticity problem under zero body forces with following stress field

$$\sigma_x = Ay, \quad \sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0 \qquad (5.7.5)$$

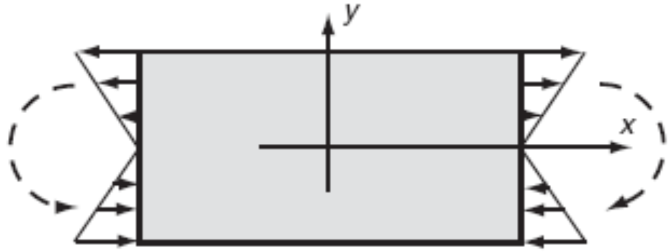


Figure 5.12
pure bending problem

Where A is a constant. It is easily shown that this simple linear stress field satisfies the equations of equilibrium and compatibility, and thus the field is a solution to an elasticity problem.

The equation is, what problem would be solved by such a field? A common scheme to answer this question is to consider some trial domain and investigate the nature of the boundary conditions that would occur using the given stress field. Therefore, consider the tow-dimensional rectangular domain shown in Figure 5-12. Using the field (5.7.5), the tractions (stresses) on each boundary face give zero loadings on the top and bottom and a linear distribution of normal stresses on the right and left side shown. Clearly, this type of boundary loading is related to a *pure bending problem*, whereby the loading on the right and left sides produce no net force and only a pure bending moment.

Example 5-3: Semi-Inverse Example: Torsion of Prismatic Bars

A simple semi-inverse example may be borrowed from the torsion problem that is discussed in detail in Chapter 9. Skipping for now the developmental details, we propose the following displacement field:

$$u = -\alpha yz, \quad v = \alpha xz, \quad w = w(x, y) \qquad (5.7.6)$$

Where α is constant. The assumed field specifies the x and y components of the displacement, while the z component is left to be determined as a function of the indicated spatial variables. By using the strain-displacement relations and Hook's law, the stress field corresponding to (5.7.6) is given by

$$\begin{aligned} &\sigma_x = \sigma_y = \tau_{xy} = 0 \\ &\tau_{xz} = \mu\left(\frac{\partial w}{\partial x} - \alpha y\right) \\ &\tau_{yz} = \mu\left(\frac{\partial w}{\partial y} + \alpha x\right) \end{aligned} \qquad (5.7.7)$$

Using these stresses in the equations of equilibrium gives the following results

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \qquad (5.7.8)$$

Which is actually the form of Navies's equations for this case. This result represents a single equation (Laplace's equation) to determine the unknown part of the assumed solution form. It should be apparent that by assuming part of the solution field, the remaining equations to be solve are greatly simplified. A special domain in the x, y plane along with appropriate boundary conditions is needed to complete the solution to a particular problem. Once this has been accomplished, the assumed solution form (5.7.6) has been shown to satisfy all the field equations of elasticity.

5.7.4. Analytical Solution Procedures

- Power Series Method
- Fourier Method
- Integral Transform Method
- Complex Variable Method

5.7.5 Approximate Solution Procedures

- Ritz Method

5.7.6 Numerical SolutionProcedures

- Finite Difference Method (FDM)
- Finite Element Method (FEM)
- Boundary Element Method (BEM)

See you next week