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Home of Computational Scientists

SOLID MECHANICS

Chapter 6: Strain energy and related principles

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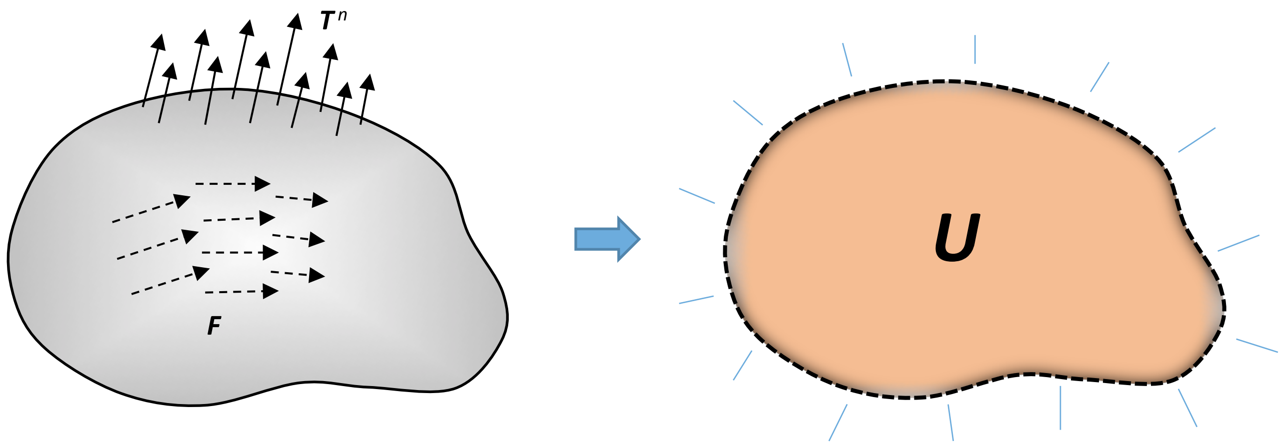
6.1 Review of Strain energy and related principles

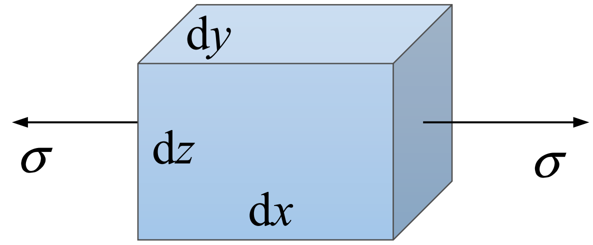
6.1 Review of Strain energy and related principles

Work done by surface and body forces on elastic solids is stored inside the body in the form of strain energy

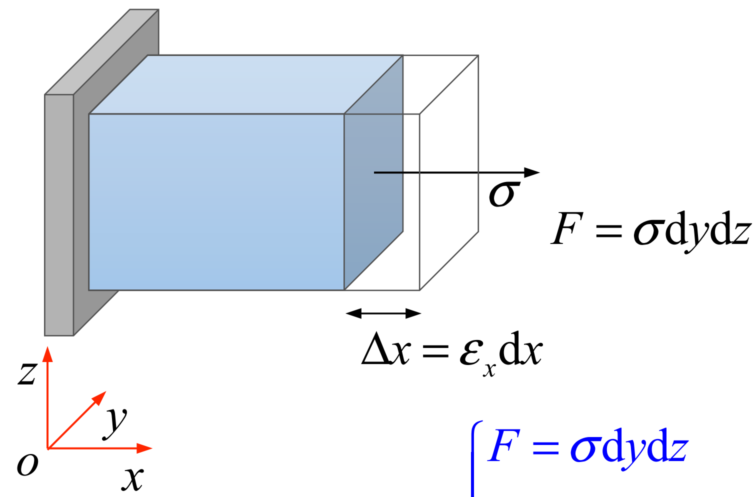


6.2 Strain energy

- Consider first the simple uniform uniaxial deformation case with no body forces

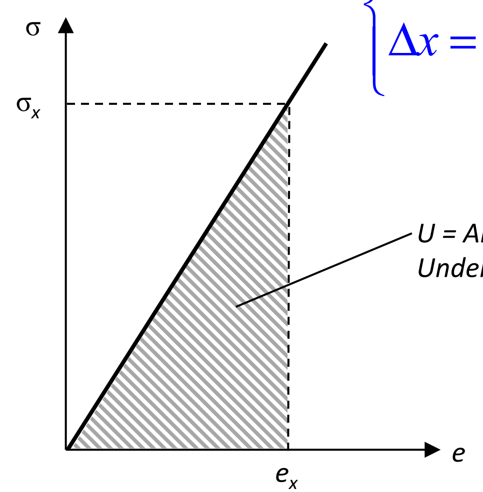


- The cubical element of dimensions dx, dy, dz is under the action of a uniform normal stress σ in the x – direction.



$$\begin{cases} F = \sigma \mathrm{d}y\mathrm{d}z \\ \Delta x = \varepsilon_x \mathrm{d}x = \dfrac{\sigma}{E}\mathrm{d}x \end{cases}$$

we assume that the stress increases slowly from zero to σ_x



$$\mathrm{d}U = \int_0^{\sigma_x} F\mathrm{d}\Delta x = \int_0^{\sigma_x} \sigma \mathrm{d}y\mathrm{d}z \frac{\mathrm{d}\sigma}{E}\mathrm{d}x = \frac{1}{E}\int_0^{\sigma_x} (\sigma \mathrm{d}\sigma)\mathrm{d}y\mathrm{d}z\mathrm{d}x$$

$$= \frac{1}{2E}\sigma_x^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{2}\sigma_x\varepsilon_x \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$U = \frac{\mathrm{d}U}{V} = \frac{1}{2}\sigma_x\varepsilon_x$$ . . . Strain Energy Density

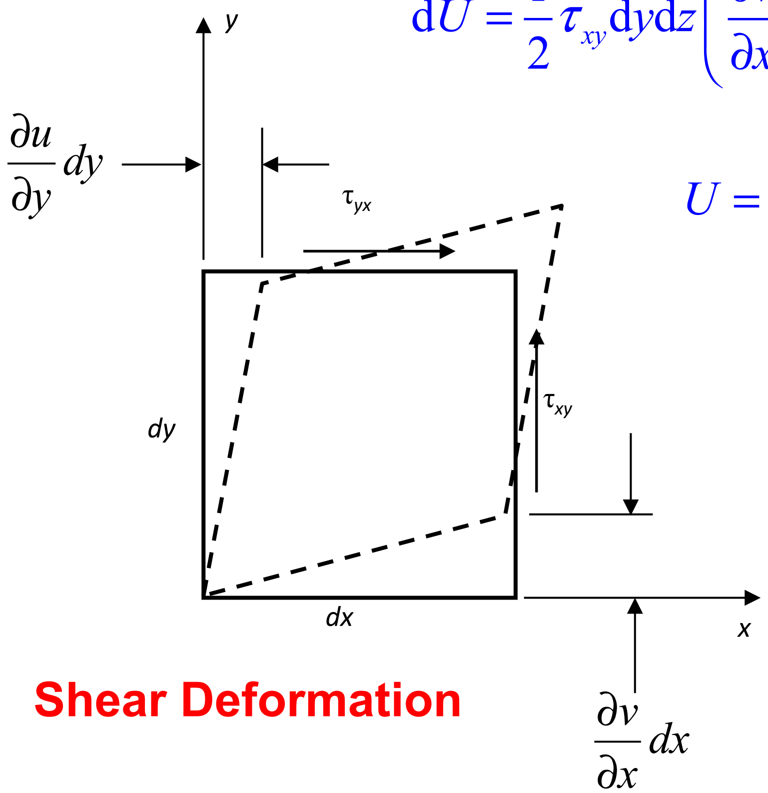
6.2 Strain energy

We next investigate the strain energy caused by the action of uniform shear stress.

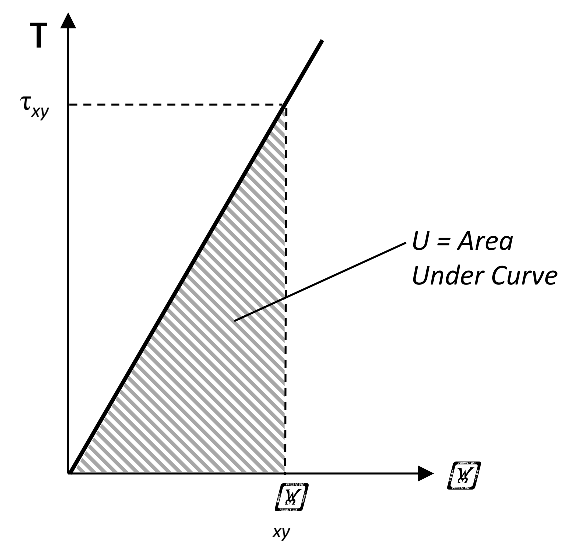
Choosing the same cubical element as previously analyzed, consider the case under uniform τ_{xy} and τ_{yx} loading.

$$dU = \frac{1}{2}\tau_{xy}dydz\left(\frac{\partial v}{\partial x}dx\right) + \frac{1}{2}\tau_{yx}dxdz\left(\frac{\partial u}{\partial y}dy\right) = \frac{1}{2}\tau_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)dxdydz$$

$$U = \frac{1}{2}\tau_{xy}\gamma_{xy} = \frac{\tau_{xy}^2}{2\mu} = \frac{\mu\gamma_{xy}^2}{2}$$ . . . Strain Energy Density



Shear Deformation



6.2 Strain energy

General Deformation Case $U=\frac{1}{2}\left(\sigma_x e_x+\sigma_y e_y+\sigma_z e_z+\tau_{xy}\gamma_{xy}+\tau_{yz}\gamma_{yz}+\tau_{zx}\gamma_{zx}\right)=\frac{1}{2}\sigma_{ij}e_{ij}$

Total strain energy $U_T=\iiint_V U\mathrm{d}x\mathrm{d}y\mathrm{d}z$

In Terms of Strain

$$U(\mathbf{e})=\frac{1}{2}\lambda e_{jj}e_{kk}+\mu e_{ij}e_{ij}$$
$$=\frac{1}{2}\lambda(e_x+e_y+e_z)^2+\mu(e_x^2+e_y^2+e_z^2+\frac{1}{2}\gamma_{xy}^2+\frac{1}{2}\gamma_{yz}^2+\frac{1}{2}\gamma_{zx}^2)$$

In Terms of Stress

$$U(\boldsymbol{\sigma})=\frac{1+\nu}{2E}\sigma_{ij}\sigma_{ij}-\frac{\nu}{2E}\sigma_{jj}\sigma_{kk}$$
$$=\frac{1+\nu}{2E}\left(\sigma_x^2+\sigma_y^2+\sigma_z^2+2\tau_{xy}^2+2\tau_{yz}^2+2\tau_{zx}^2\right)-\frac{\nu}{2E}\left(\sigma_x+\sigma_y+\sigma_z\right)^2$$

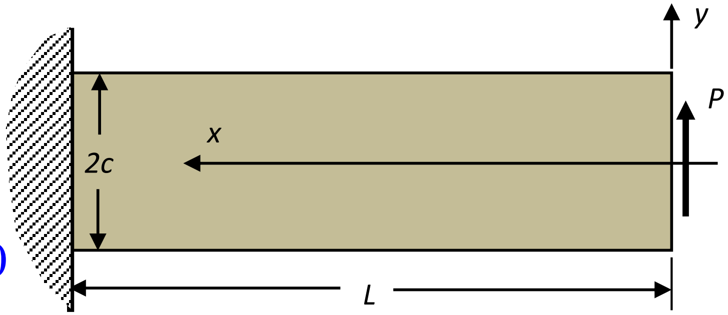
Note Strain Energy Is *Positive Definite Quadratic Form* ⇨ $U\geq 0$

Relation $U\geq 0$ is valid for all elastic materials, including both isotropic and anisotropic solids.

6.2 Strain energy

Example Problem

Let us consider stress field



$$\sigma_x = -\frac{3P}{2c^3}xy, \tau_{xy} = -\frac{3P}{4c}\left(1-\frac{y^2}{c^2}\right), \sigma_y = \sigma_z = \tau_{yz} = \tau_{zx} = 0$$

$$U = \frac{1+\nu}{2E}\left(\sigma_x^2 + 2\tau_{xy}^2\right) - \frac{\nu}{2E}\sigma_x^2 = \frac{1}{2E}\sigma_x^2 + \frac{1+\nu}{E}\tau_{xy}^2$$

$$U_T = \iiint U \mathrm{d}V = \int_0^1\int_{-c}^{c}\int_0^L\left(\frac{1}{2E}\sigma_x^2 + \frac{1+\nu}{E}\tau_{xy}^2\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \int_{-c}^{c}\int_0^L\left(\frac{1}{2E}\sigma_x^2 + \frac{1+\nu}{E}\tau_{xy}^2\right)\mathrm{d}x\mathrm{d}y$$

$$= \frac{1}{2E}\int_{-c}^{c}\int_0^L \frac{9P^2}{4c^6}x^2y^2\mathrm{d}x\mathrm{d}y + \frac{1+\nu}{E}\int_{-c}^{c}\int_0^L \frac{9P^2}{16c^2}\left(1-\frac{y^2}{c^2}\right)^2\mathrm{d}x\mathrm{d}y$$

$$= \frac{P^2L^2}{4Ec^3} + \frac{9P^2L(1+\nu)}{Ec}$$

6.2 Strain energy

Derivative Operations on Strain Energy

For the Uniaxial Deformation Case:

$$\frac{\partial U(\mathbf{e})}{\partial e_x} = \frac{\partial}{\partial e_x}\left(\frac{E e_x^2}{2}\right) = E e_x = \sigma_x \qquad \frac{\partial U(\boldsymbol{\sigma})}{\partial \sigma_x} = \frac{\partial}{\partial \sigma_x}\left(\frac{\sigma_x^2}{2E}\right) = \frac{\sigma_x}{E} = e_x$$

For the General Deformation Case:

$$\sigma_{ij} = \frac{\partial U(\mathbf{e})}{\partial e_{ij}}, \; e_{ij} = \frac{\partial U(\boldsymbol{\sigma})}{\partial \sigma_{ij}}$$

$$\frac{\partial \sigma_{ij}}{\partial e_{kl}} = \frac{\partial \sigma_{kl}}{\partial e_{ij}}$$

$$\frac{\partial e_{ij}}{\partial \sigma_{kl}} = \frac{\partial e_{kl}}{\partial \sigma_{ij}}$$

$$\Rightarrow \quad C_{ijkl} = C_{klij}$$

Therefore $C_{ij} = C_{ji}$, and thus there are only 21 independent elastic constants for general anisotropic elastic materials

6.2 Strain energy

Decomposition of Strain Energy

Strain Energy May Be Decomposed into Two Parts Associated With

Volumetric Deformation U_v , and ***Distortional Deformation***, U_d

$$U = U_v + U_d$$

$$U_v = \frac{1}{2}\tilde{\sigma}_{ij}\tilde{e}_{ij} = \frac{1}{6}\sigma_{jj}e_{kk} = \frac{1-2\nu}{6E}\sigma_{jj}\sigma_{kk} = \frac{1-2\nu}{6E}\left(\sigma_x + \sigma_y + \sigma_z\right)^2$$

$$U_d = \frac{1}{12\mu}\left[\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + \left(\sigma_z - \sigma_x\right)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)\right]$$

Failure Theories of Solids Incorporate Strain Energy of Distortion by Proposing That Material Failure or Yielding Will Initiate When U_d Reaches a Critical Value

6.3 Uniqueness of the elasticity Boundary-Value Problem

Consider the general mixed boundary-value problem in which tractions are specified over the boundary S_t and displacements are prescribed over the remaining part S_u. Assume that there exist two different solutions to the same problem:

$$\left\{\sigma_{ij}^{(1)}, e_{ij}^{(1)}, u_i^{(1)}\right\} \text{ and } \left\{\sigma_{ij}^{(2)}, e_{ij}^{(2)}, u_i^{(2)}\right\}$$

Define the difference solution

$$\sigma_{ij} = \sigma_{ij}^{(1)} - \sigma_{ij}^{(2)} \quad ; \quad e_{ij} = e_{ij}^{(1)} - e_{ij}^{(2)} \quad ; \quad u_i = u_i^{(1)} - u_i^{(2)}$$

Because the solutions $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$ have the same body force, the difference solution must satisfy the equilibrium equation

$$\sigma_{ij,j} = 0$$

Likewise, the boundary conditions satisfied by the difference solution are given by

$$T_i^n = \sigma_{ij} n_j = 0 \ \text{ on } \ S_t$$

$$u_i = 0 \ \text{ on } \ S_u$$

6.3 Uniqueness of the elasticity Boundary-Value Problem

Starting with the definition of strain energy, we may write

$$2\int_V U\mathrm{d}V = \int_V \sigma_{ij}\varepsilon_{ij}\mathrm{d}V = \int_V \sigma_{ij}\left(u_{i,j}-\omega_{ij}\right)\mathrm{d}V = \int_V \sigma_{ij}u_{i,j}\mathrm{d}V$$
$$= \int_V \left(\sigma_{ij}u_i\right)_{,j}\mathrm{d}V - \int_V \sigma_{ij,j}u_i\mathrm{d}V = \int_S \sigma_{ij}n_j u_i\mathrm{d}S - \int_V \sigma_{ij,j}u_i\mathrm{d}V$$

Using the conditions, $\sigma_{ij,j}=0$ and $T_i^n=\sigma_{ij}n_j=0 \text{ on } S_t$, $u_i=0 \text{ on } S_u$ we get $2\int_V U\mathrm{d}V=0$

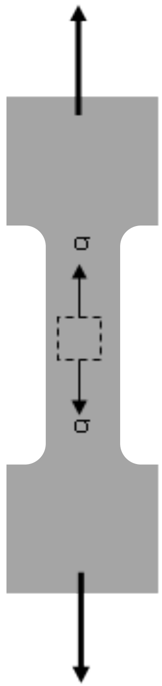
which implies that U must vanish in the region V, and since the strain energy is a positive definite quadratic form, the associated strains and stresses also vanish; that is $\sigma_{ij}=\varepsilon_{ij}=0$. If the strain field vanishes, then the corresponding displacements u_i can be at most rigid-body mode. However, if $u_i=0$ on S_u, then the displacement field must vanish everywhere. Thus, we have shown that

$$\sigma_{ij}^{(1)}=\sigma_{ij}^{(2)}\ ,\ e_{ij}^{(1)}=e_{ij}^{(2)}\ ,\ u_i^{(1)}=u_i^{(2)}$$

and therefore the problem solution is unique.

6.4 Bounds on Elastic Constants

Simple Tension

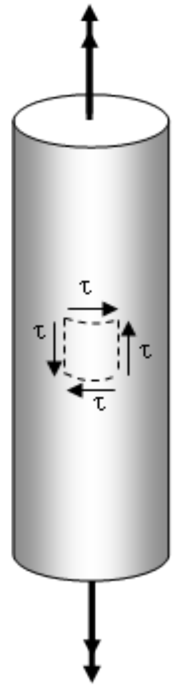


$$\sigma_{ij} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$U = \frac{1+\nu}{2E}\sigma^2 - \frac{\nu}{2E}\sigma^2 = \frac{\sigma^2}{2E}$$

$$E > 0$$

Pure Shear

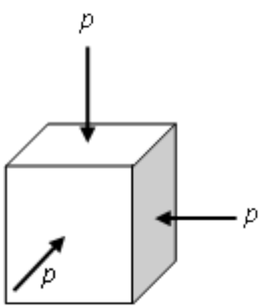


$$\sigma_{ij} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$U = \frac{1+\nu}{2E}(2\tau^2) = \frac{\tau^2}{E}(1+\nu)$$

$$1+\nu > 0 \Rightarrow \nu > -1$$

Hydrostatic Compression



$$\sigma_{ij} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} = -p\delta_{ij}$$

$$U = \frac{1+\nu}{2E}3p^2 - \frac{\nu}{2E}(-3p)^2 = \frac{3p^2}{2E}(1-2\nu)$$

$$1-2\nu > 0 \Rightarrow \nu < \frac{1}{2}$$

$$\Rightarrow \quad -1 < \nu < \frac{1}{2} \qquad k > 0\ ,\ \mu > 0$$

6.5 Related Integral Theorems

Clapeyron's Theorem The strain energy of an elastic solid in static equilibrium is equal to one-half the work done by the external body forces F_i and surface tractions T_i^n

$$2\int_V U\mathrm{d}V = \int_S T_i^n u_i \mathrm{d}S + \int_V F_i u_i \mathrm{d}V$$

Betti/Rayleigh Reciprocal Theorem If an elastic body is subject to two body and surface force systems, then the work done by the first system of forces $\{\mathbf{T}^{(1)}, \mathbf{F}^{(1)}\}$ acting through the displacements $\mathbf{u}^{(2)}$ of the second system is equal to the work done by the second system of forces $\{\mathbf{T}^{(2)}, \mathbf{F}^{(2)}\}$ acting through the displacements $\mathbf{u}^{(1)}$ of the first system

$$\int_S T_i^{(1)} u_i^{(2)} \mathrm{d}S + \int_V F_i^{(1)} u_i^{(2)} \mathrm{d}V = \int_S T_i^{(2)} u_i^{(1)} \mathrm{d}S + \int_V F_i^{(2)} u_i^{(1)} \mathrm{d}V$$

6.5 Related Integral Theorems

$$\int_S T_i^{(1)} u_i^{(2)} \mathrm{d}S + \int_V F_i^{(1)} u_i^{(2)} \mathrm{d}V = \int_S T_i^{(2)} u_i^{(1)} \mathrm{d}S + \int_V F_i^{(2)} u_i^{(1)} \mathrm{d}V$$

$$\int_V \sigma_{ij}^{(1)} e_{ij}^{(2)} \mathrm{d}V = \int_S \sigma_{ij}^{(1)} n_j^{(1)} u_i^{(2)} \mathrm{d}S - \int_V \sigma_{ij,j}^{(1)} u_i^{(2)} \mathrm{d}V = \int_S T_i^{(1)} u_i^{(2)} \mathrm{d}S + \int_V F_i^{(1)} u_i^{(2)} \mathrm{d}V$$

$$\int_V \sigma_{ij}^{(2)} e_{ij}^{(1)} \mathrm{d}V = \int_S \sigma_{ij}^{(2)} n_j^{(2)} u_i^{(1)} \mathrm{d}S - \int_V \sigma_{ij,j}^{(2)} u_i^{(1)} \mathrm{d}V = \int_S T_i^{(2)} u_i^{(1)} \mathrm{d}S + \int_V F_i^{(2)} u_i^{(1)} \mathrm{d}V$$

Now $\sigma_{ij}^{(1)} e_{ij}^{(2)} = C_{ijkl} e_{kl}^{(1)} e_{ij}^{(2)} = C_{klij} e_{kl}^{(1)} e_{ij}^{(2)} = C_{klij} e_{ij}^{(2)} e_{kl}^{(1)} = \sigma_{kl}^{(2)} e_{kl}^{(1)}$ then $\sigma_{ij}^{(1)} e_{ij}^{(2)} = \sigma_{kl}^{(2)} e_{kl}^{(1)}$

Combining these results then proves the theorem. The reciprocal theorem can yield useful applications by special selection of two systems. One such application follows

6.6 Principle of Virtual Work

Based on work and energy principles, several additional solution methods can be developed. The *principle of virtual work* provides the foundation for many of these methods.

The *virtual displacement* of a material point is a fictitious displacement such that the forces acting on the point remain unchanged.

The work done by these forces during the virtual displacement is called the *virtual work*.

For an object in static equilibrium, the virtual work is zero because the resultant force vanishes on every portion of an equilibrated body. The converse is also true that if the virtual work is zero, then the material point must be in equilibrium.

6.6 Principle of Virtual Work

Introduce some notions:

- The virtual displacements of an elastic solid are denoted by $\delta u_i = \{\delta u, \delta v, \delta w\}$, and the corresponding virtual strains are then expressed as $\delta e_{ij} = ½(\delta u_{i,j} + \delta u_{j,i})$.

- Consider the standard elasticity boundary-value problem of a solid in equilibrium under the action of surface tractions over the boundary S_t with displacement conditions over the remaining boundary S_u. Now, imagine that the body undergoes a virtual displacement δu_i from its equilibrium configuration. The virtual displacement is arbitrary except that it must not violate the *kinematic displacement boundary condition*, and thus $\delta u_i = 0$ on S_u.

- The virtual work done by the surface and body forces can be written as

$$\delta W = \int_{S_t} T_i^n \delta u_i \mathrm{d}S + \int_V F_i \, \delta u_i \mathrm{d}V$$

Now, because the virtual displacement vanishes on S_u, the integration domain of the first integral can be changed into S.

6.6 Principle of Virtual Work

The surface integral can be changed to a volume integral and combined with the body force term. These steps are summarized as

$$
\begin{aligned}
\delta W &= \int_S T_i^n \delta u_i \mathrm{d}S + \int_V F_i\, \delta u_i \mathrm{d}V \\
&= \int_S \sigma_{ij} n_j \delta u_i \mathrm{d}S + \int_V F_i\, \delta u_i \mathrm{d}V \\
&= \int_V \left(\sigma_{ij}\delta u_i\right)_{,j} \mathrm{d}V + \int_V F_i\, \delta u_i \mathrm{d}V \\
&= \int_V \left(\sigma_{ij,j}\delta u_i + \sigma_{ij}\delta u_{i,j}\right)\mathrm{d}V + \int_V F_i\, \delta u_i \mathrm{d}V \\
&= \int_V \left(-F_i\delta u_i + \sigma_{ij}\delta e_{ij}\right)\mathrm{d}V + \int_V F_i\, \delta u_i \mathrm{d}V \\
&= \int_V \sigma_{ij}\delta e_{ij} \mathrm{d}V
\end{aligned}
$$

$$
\int_V \sigma_{ij}\delta e_{ij}\mathrm{d}V = \int_V \left(\sigma_x\delta e_x + \sigma_y\delta e_y + \sigma_z\delta e_z + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx}\right)\mathrm{d}V
$$

Notice that the virtual strain energy does not contain the factor of ½ . It is because the stresses are constant during the virtual displacement.

6.6 Principle of Virtual Work

The external forces are unchanged during the virtual displacements and the region *V* is fixed

$$\delta\left(\int_V U \mathrm{d}V - \int_{S_t} T_i^n u_i \mathrm{d}S - \int_V F_i u_i \mathrm{d}V\right) = \delta\left(U_T - W\right) = 0$$

This is one of the statements of the principle of virtual work for an elastic solid. The quantity $(U_T - W)$ actually represents the *total potential energy* of the elastic solid, and thus the change in potential energy during a virtual displacement from equilibrium is zero.

6.6 Principle of Virtual Work

The principle of virtual work provides a convenient method for deriving equilibrium equations and associated boundary conditions for various special theories of elastic bodies

$$\int_V \sigma_{ij}\delta e_{ij}\mathrm{d}V - \int_{S_t} T_i^n \delta u_i \mathrm{d}S - \int_V F_i \delta u_i \mathrm{d}V = 0$$

The integrand of the first term can be reduced as

$$\sigma_{ij}\delta e_{ij} = \frac{1}{2}\sigma_{ij}\left(\delta u_{i,j} + \delta u_{j,i}\right) = \sigma_{ij}\delta u_{i,j} = \left(\sigma_{ij}\delta u_i\right)_{,j} - \sigma_{ij,j}\delta u_i$$

$$\Rightarrow \int_V \left[\left(\sigma_{ij}\delta u_i\right)_{,j} - \sigma_{ij,j}\delta u_i\right]\mathrm{d}V - \int_{S_t} T_i^n \delta u_i \mathrm{d}S - \int_V F_i \delta u_i \mathrm{d}V = 0$$

$$\int_V \left(\sigma_{ij,j} + F_i\right)\delta u_i \mathrm{d}V + \int_S \left(T_i^n - \sigma_{ij}n_j\right)\delta u_i \mathrm{d}S = 0$$

For arbitrary δu_i

$$\sigma_{ij,j} + F_i = 0 \quad \text{on } V$$

and either

$$\delta u_i = 0 \quad \text{on } S_u \quad or \quad T_i^n = \sigma_{ij}n_j \quad \text{on } S_t$$

6.7 Principles of Minimum potential & complementary energy

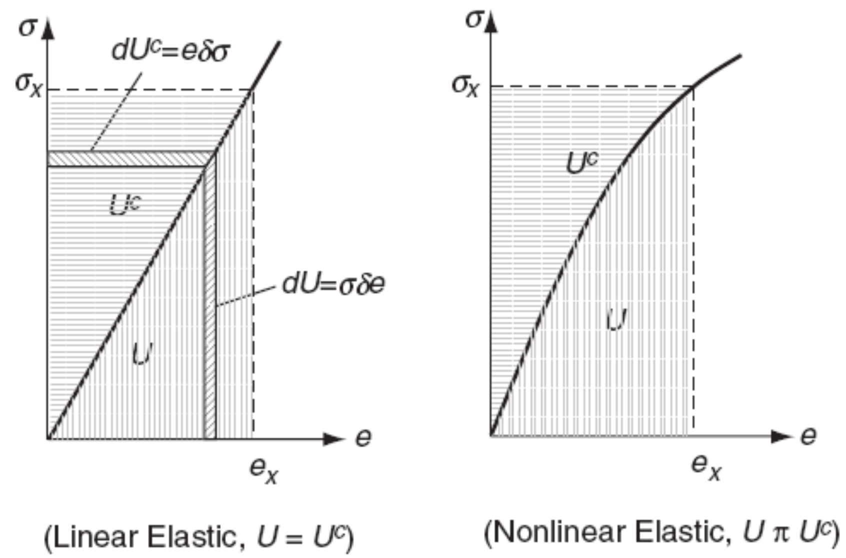
Principle of minimum potential energy:

Of all displacement satisfying the given boundary conditions of an elastic solid, those that satisfy the equilibrium equations make the potential energy a local minimum

An additional minimum principle can be developed by reversing the nature of the variation. Thus, consider the variation of stresses while holding the displacements constant.

Principle of minimum complementary energy:

Of all elastic stress satisfying the given boundary conditions of an elastic solid, those that satisfy the equilibrium equations make the complementary energy a local minimum



Strain and complementary energy for linear and nonlinear elastic materials.

6.7 Principles of Minimum potential & complementary energy

Example 6.1: Euler-Bernoulli Beam theory

In order to demonstrate the utility of energy principles, consider an application dealing with the bending of an elastic beam, as shown in Figure 6-5. The external distributed Loading q will induce internal bending moments M and shear forces V at each section of the beam. According to classical Euler-Bernoulli theory, the bending stress σ_x and moment-curvature and moment-shear relations are given by

$$\sigma_x = -\frac{My}{I},\ M = EI\frac{d^2w}{dx^2},\ V = \frac{dM}{dx}$$

Where $I = \iint_A y^2 dA$ is the area moment of inertia of the cross-section about the neutral axis, and w is the beam deflection (positive in y direction).

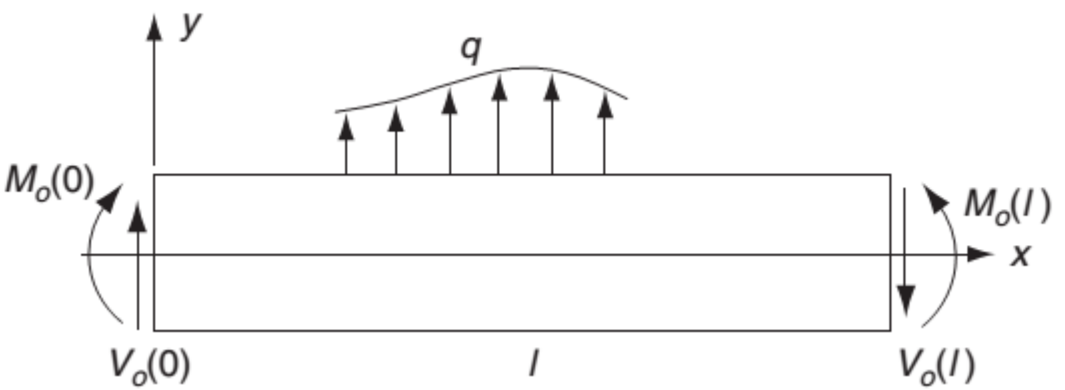


FIGURE 6-5 *Euler-Bernoulli beam geometry.*

6.7 Principles of Minimum potential & complementary energy

Example 6.1: Euler-Bernoulli Beam theory

Considering only the strain energy caused by the bending stresses

$$U = \frac{\sigma_x^2}{2E} = \frac{M^2 y^2}{2EI^2} = \frac{E}{2}\left(\frac{d^2 w}{\mathrm{d}x^2}\right)^2 y^2$$

And thus the total strain energy in a beam of length l is

$$U = \int_0^l \left[\iint_A \frac{E}{2}\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 y^2 \mathrm{d}A\right]\mathrm{d}x = \int_0^l \frac{EI}{2}\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x$$

Now the work done by the external forces (tractions) includes contributions from the distributed loading q and the loading at the ends $x = 0$ and l

$$W = \int_0^l qw\mathrm{d}x - \left[V_0 w - M_0 \frac{\mathrm{d}w}{\mathrm{d}x}\right]_0^l \qquad (6.6.11)$$

6.7 Principles of Minimum potential & complementary energy

Example 6.1: Euler-Bernoulli Beam theory

This result is simply the differential equilibrium equations for the beam, and thus the stationary value for the potential energy leads directly to the governing equilibrium equation in term of displacement and the associated boundary conditions. Of course, this entire formulation is based on the simplifying assumption found in Euler-Bernoulli beam theory, and resulting solutions would not match with the more exact theory of elasticity results.

6.8 Rayleigh-Ritz method

6.8 Rayleigh-Ritz method

Rayleigh-Ritz method is a method for finding the approximate solution based on the variational form of the problem.

In this method, we construct a series of trial approximating functions that satisfy the boundary conditions but not the differential equations. For the elasticity displacement formulation, this concept would express the displacements in the form

$$u = u_0 + \sum_{j=1}^{N} a_j u_j \;\; ; \;\; v = v_0 + \sum_{j=1}^{N} b_j v_j \;\; ; \;\; w = w_0 + \sum_{j=1}^{N} c_j w_j$$

where the functions u_0, v_0, and w_0 are chosen to satisfy any non-homogeneous boundary conditions and u_j, v_j, w_j satisfy the corresponding homogeneous boundary conditions. Note that these forms are not required to satisfy the traction boundary conditions. Normally, these trial functions are chosen from some combination of elementary functions such as polynomial, trigonometric, or hyperbolic forms.

The unknown constant coefficients a_j, b_j, c_j are to be determined so as to minimize the potential energy of the problem, thus approximately satisfying the variational formulation of the problem.

6.8 Rayleigh-Ritz method

Using this type of approximation, the total potential energy will thus be a function of these unknown coefficients

$$\Pi = \Pi\left(a_j, b_j, c_j\right) \quad (6.7.2)$$

and the minimizing condition can be expressed as a series of expressions

$$\frac{\partial \Pi}{\partial a_j} = 0, \quad \frac{\partial \Pi}{\partial b_j} = 0, \quad \frac{\partial \Pi}{\partial c_j} = 0 \qquad (6.7.3)$$

This set forms a system of 3N algebraic equations that can be solved to obtain the parameters a_j, b_j, c_j. Under suitable conditions on the choice of trial functions (completeness property), the approximation will be improved as the number of included terms is increased.

A big disadvantage of this method is the selection of the approximating functions. There exists no systematic procedure of constructing them. The selection process becomes more difficult when the domain is geometrically complex and/or boundary conditions are complicated.

6.8 Rayleigh-Ritz method

Example 6.2: Rayleigh-Ritz solution of a simply supported Euler-Bernoulli Beam

Consider a simply supported Euler-Bernoulli beam od length l carrying a uniform loading q_0. This one-dimensional problem has displacement boundary conditions

$$w = 0 \text{ at } x = 0, l \qquad (6.7.4)$$

and tractions or moment conditions

$$EI\frac{d^2w}{dx^2} = 0 \text{ at } x = 0, l \qquad (6.7.5)$$

The Ritz approximation for this problem is of form

$$w = w_0 + \sum_{j=1}^{N} c_j w_j \qquad (6.7.6)$$

With no nonhomogeneous boundary conditions, $w_0 = 0$. For the example, we choose a polynomial form for the trial solution. An appropriate choice that satisfies the homogeneous conditions (6.7.4) is $w_j = x^j(l - x)$. Note this form dose not satisfy the traction conditions (6.7.5). Using the previously developed relation for the potential energy (6.6.12), we get

6.8 Rayleigh-Ritz method

Example 6.2: Rayleigh-Ritz solution of a simply supported Euler-Bernoulli Beam

$$\Pi = \int_0^l \left[\frac{EI}{2}\left(\frac{d^2 w}{dx^2}\right)^2 - q_0 w \right] dx \qquad (6.7.7)$$

$$= \int_0^l \left[\frac{EI}{2}\left(\sum_{j=1}^{N} c_j [j(j-1) l x^{j-2} - j(j+1) x^{j-1}]\right)^2 - q_0 \sum_{j=1}^{N} c_j x^j (l-x) \right] dx$$

Retaining only a two-term approximation ($N = 2$), the coefficients are found to be

$$c_1 = \frac{q_0 l^2}{24EI}, c_2 = 0$$

And this gives the following approximate solution:

$$w = \frac{q_0 l^2}{24EI} x(l-x) \qquad (6.7.8)$$

6.8 Rayleigh-Ritz method

Example 6.2: Rayleigh-Ritz solution of a simply supported Euler-Bernoulli Beam

Note that the approximate solution predicts a parabolic displacement distribution, while the exact solution to this problem is given by the cubic relation

$$w = \frac{q_0 l^2}{24EI}(l^3 + x^3 - 2lx^2) \quad (6.7.9)$$

Actually, for this special case, the exact solution can be obtained from b Ritz scheme by including polynomials of degree three.

See you next week