ĐẠI HỌC TÔN ĐỨC THẮNG
TON DUC THANG UNIVERSITY

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Home of Computational Scientists

SOLID MECHANICS

Chapter 7: Two-dimensional Formulation

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7.1 Review of two-dimensional formulation

- Three-dimensional elasticity problems are very difficult to solve. Thus, most solutions are developed for reduced problems that typically include axisymmetric or two-dimensionality. We will first develop governing equations for two-dimensional problems, and will explore four different theories:

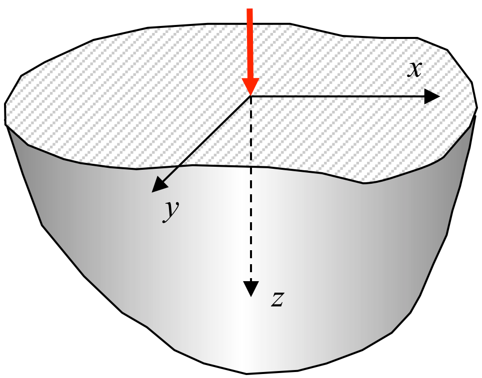
- **Plane Strain**
- **Plane Stress**
- **Generalized Plane Stress**
- **Anti-Plane Strain**

- Since all real elastic structures are three-dimensional, theories set forth here will be approximate models. The nature and accuracy of the approximation will depend on problem and loading geometry

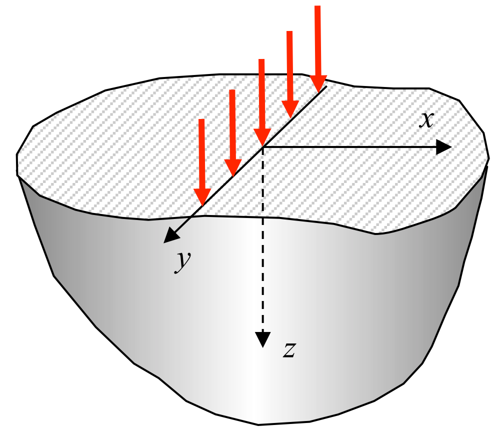
- The basic theories of *plane strain* and *plane stress* represent the fundamental plane problem in elasticity. While these two theories apply to significantly different types of two-dimensional bodies, their formulations yield very similar field equations.

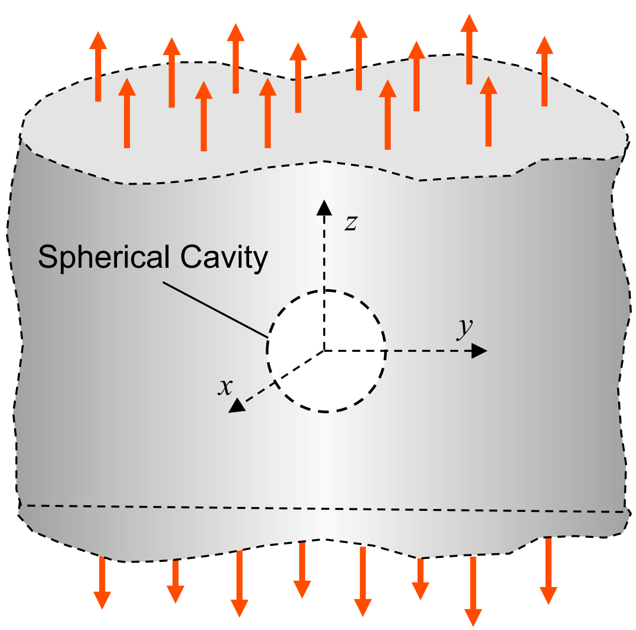
7.1 Review of two-dimensional formulation

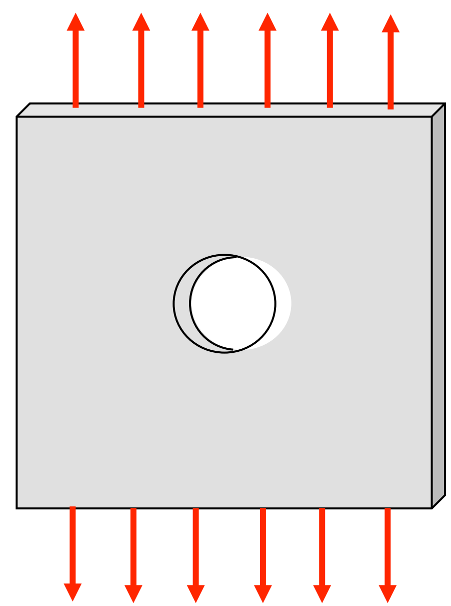
Three-Dimensional



Two-Dimensional



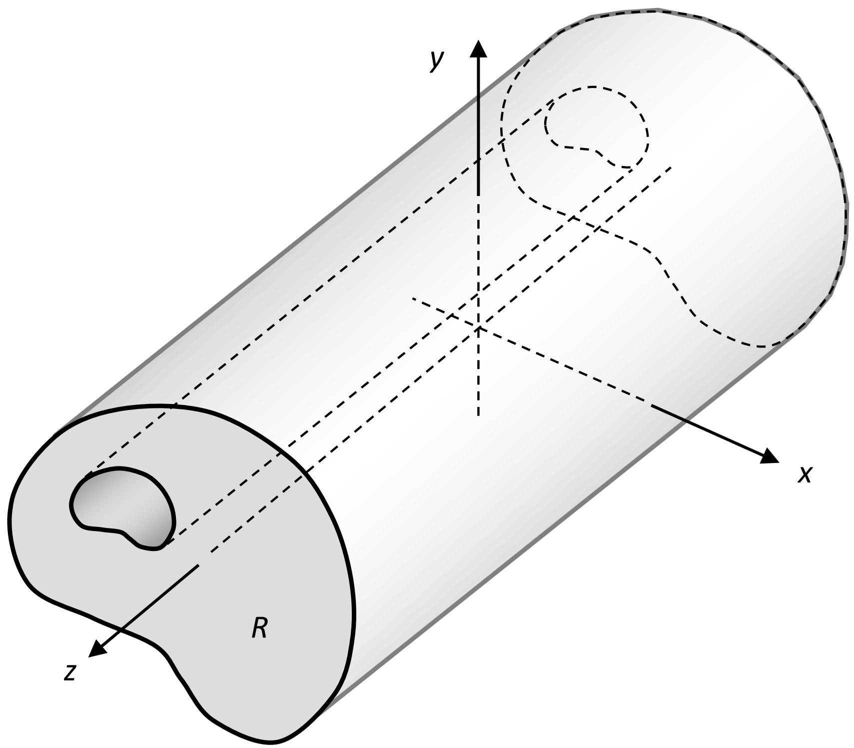




7.2 Plane strain

- Consider an infinitely long cylindrical (prismatic) body as shown in Figure. If the body forces and tractions on lateral boundaries are independent of the *z*-coordinate and have no *z*-component, then the deformation field can be taken in the reduced form

$$u = u(x, y)\ ,\ v = v(x, y)\ ,\ w = 0$$



7.2 Plane strain

Plane Strain Field Equations

Note that: Although $e_z = 0$, the normal stress σ_z will not in general vanish.

Strains $$e_x = \frac{\partial u}{\partial x},\ e_y = \frac{\partial v}{\partial y},\ e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),\ e_z = e_{xz} = e_{yz} = 0$$

Stresses $$\begin{cases} \sigma_x = \lambda(e_x + e_y) + 2\mu e_x\ ,\ \sigma_y = \lambda(e_x + e_y) + 2\mu e_y \\ \sigma_z = \lambda(e_x + e_y) = \nu(\sigma_x + \sigma_y) \\ \tau_{xy} = 2\mu e_{xy}\ ,\ \tau_{xz} = \tau_{yz} = 0 \end{cases}$$

Equilibrium Equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + F_y = 0$$

Strain Compatibility

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y}$$

Navier Equations

$$\mu\nabla^2 u + (\lambda + \mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_x = 0$$

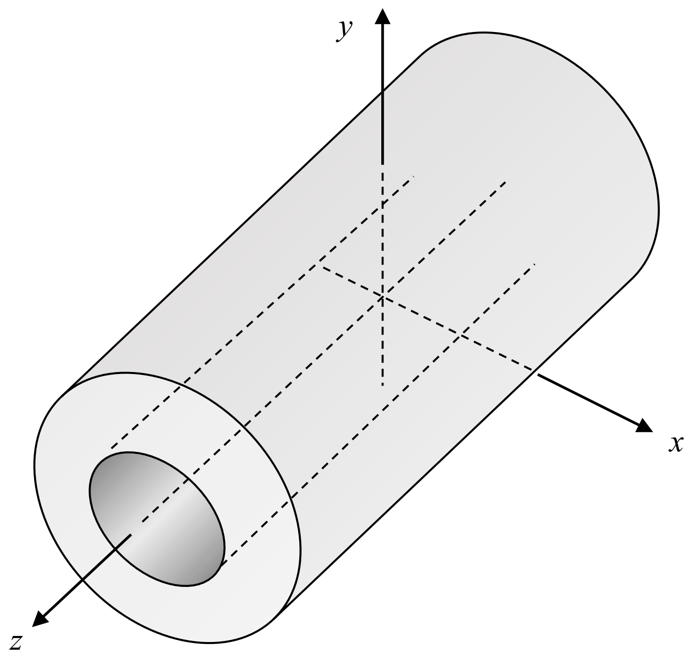
$$\mu\nabla^2 v + (\lambda + \mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_y = 0$$

Beltrami-Michell Equation

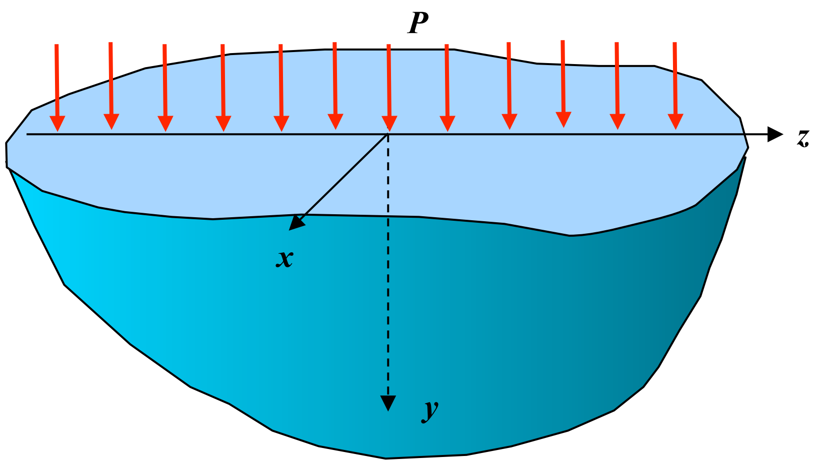
$$\nabla^2(\sigma_x + \sigma_y) = -\frac{1}{1-\nu}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)$$

7.2 Plane strain

Examples of Plane Strain Problems



Long Cylinders
Under Uniform Loading

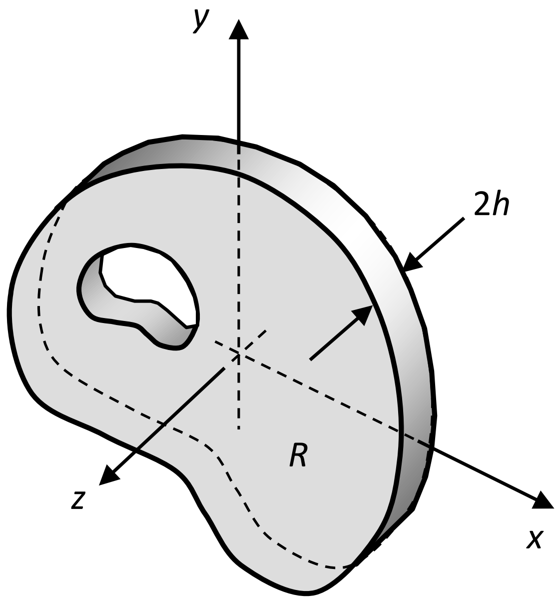


Semi-Infinite Regions
Under Uniform
Loadings

7.3 Plane stress

Consider the domain bounded two stress free planes $z = \pm h$, where h is small in comparison to other dimensions in the problem. Since the region is thin in the z-direction, there can be little variation in the stress components σ_z , τ_{xz} , τ_{yz} through the thickness, and thus they will be approximately zero throughout the entire domain. Finally since the region is thin in the z-direction it can be argued that the other non-zero stresses will have little variation with z. Under these assumptions, the stress field can be taken as

$$\sigma_x = \sigma_x(x, y)$$
$$\sigma_y = \sigma_y(x, y)$$
$$\tau_{xy} = \tau_{xy}(x, y)$$
$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$



7.3 Plane stress

Plane Stress Field Equations

Note that: Although $\sigma_z = 0$, the normal strain e_z will not in general vanish.

Strains (Hooke law)

$$e_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)\ ,\ e_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

$$e_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) = -\frac{\nu}{1-\nu}(e_x + e_y)$$

$$e_{xy} = \frac{1+\nu}{E}\tau_{xy}\ ,\ e_{xz} = e_{yz} = 0$$

Strain Displacement Relations

$$e_x = \frac{\partial u}{\partial x}\ ,\ e_y = \frac{\partial v}{\partial y}\ ,\ e_z = \frac{\partial w}{\partial z}\ ,\ e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0\ ,\ e_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0$$

Equilibrium Equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + F_y = 0$$

Strain Compatibility

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = 2\frac{\partial^2 e_{xy}}{\partial x \partial y}$$

Navier Equations

$$\mu\nabla^2 u + \frac{E}{2(1-\nu)}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_x = 0$$

$$\mu\nabla^2 v + \frac{E}{2(1-\nu)}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_y = 0$$

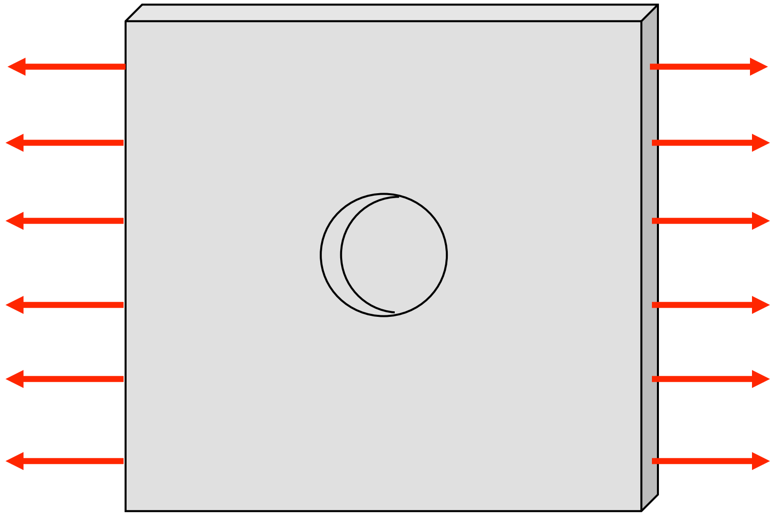
Beltrami-Michell Equation

$$\nabla^2(\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)$$

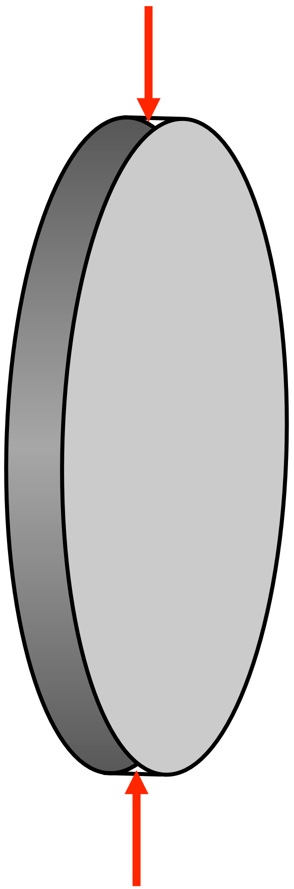
Note that: Navier equations and Beltrami-Michell equations are similar but not identical to the corresponding plane strain relation.

7.3 Plane stress

Examples of Plane Stress Problems

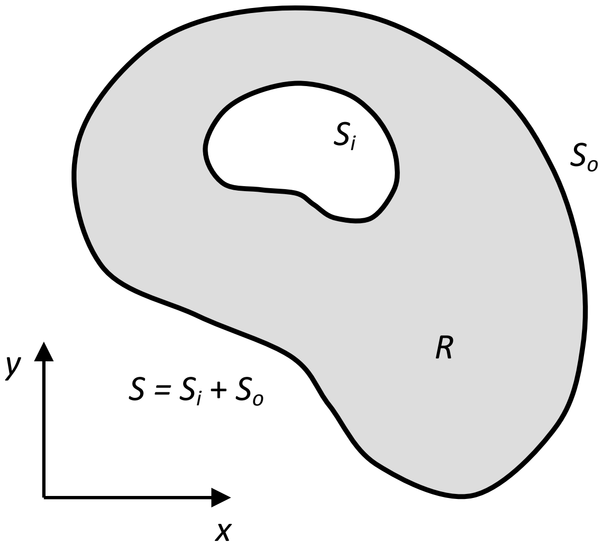


Thin Plate With Central Hole



Circular Plate Under Edge Loadings

7.3 Plane stress



Displacement Boundary Conditions

$$u = u_b(x, y)\ ,\ v = v_b(x, y)\ \text{on}\ \ S$$

Stress/Traction Boundary Conditions

$$T_x^n = T_x^{(b)}(x, y) = \sigma_x^{(b)} n_x + \tau_{xy}^{(b)} n_y$$
$$T_y^n = T_y^{(b)}(x, y) = \tau_{xy}^{(b)} n_x + \sigma_y^{(b)} n_y \quad \text{on}\ \ S$$

7.3 Plane stress

Correspondence Between Plane Formulations

Plane Strain

$$\mu\nabla^2 u+(\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_x=0$$

$$\mu\nabla^2 v+(\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_y=0$$

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+F_x=0$$

$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+F_y=0$$

$$\nabla^2(\sigma_x+\sigma_y)=-\frac{1}{1-\nu}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right)$$

Plane Stress

$$\mu\nabla^2 u+\frac{E}{2(1-\nu)}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_x=0$$

$$\mu\nabla^2 v+\frac{E}{2(1-\nu)}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+F_y=0$$

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+F_x=0$$

$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+F_y=0$$

$$\nabla^2(\sigma_x+\sigma_y)=-(1+\nu)\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right)$$

7.3 Plane stress

Transformation Between Plane Strain and Plane Stress

Plane strain and plane stress field equations had identical equilibrium equations and boundary conditions. Navier's equations and compatibility relations were similar but not identical with differences occurring only in particular coefficients involving just elastic constants. So perhaps a simple change in elastic moduli would bring one set of relations into an exact match with the corresponding result from the other plane theory. This in fact can be done using results in the following table.

	E	ν
Plane Stress to Plane Strain	$\frac{E}{1-\nu^2}$	$\frac{\nu}{1-\nu}$
Plane Strain to Plane Stress	$\frac{E(1+2\nu)}{(1+\nu)^2}$	$\frac{\nu}{1+\nu}$

Therefore the solution to one plane problem also yields the solution to the other plane problem through this simple transformation scheme.

7.4 Generalized plane stress

The plane stress formulation produced some inconsistencies in particular out-of-plane behavior and resulted in some three-dimensional effects where in-plane displacements were functions of z. We avoided these issues by simply neglecting some of the troublesome equations thereby producing an approximate elasticity formulation. In order to avoid this unpleasant situation, an alternate approach called *Generalized Plane Stress* can be constructed based on *averaging* the field quantities through the thickness of the domain.

Using the averaging operator defined by $\bar{\phi}(x,y) = \frac{1}{2h}\int_{-h}^{h} \phi(x,y,z)dz$

all plane stress equations are satisfied *exactly* by the averaged stress, strain and displacements variables; thereby eliminating the inconsistencies found in the original plane stress formulation. However, this gain in rigor does not generally contribute much to applications .

7.5 Anti-plane strain

An additional plane theory of elasticity called *Anti-Plane Strain* involves a formulation based on the existence of only out-of-plane deformation starting with an assumed displacement field

$$u = v = 0\ ,\ w = w(x, y)$$

Strains

$$e_x = e_y = e_z = e_{xy} = 0$$

$$e_{xz} = \frac{1}{2}\frac{\partial w}{\partial x}\ ,\ e_{yz} = \frac{1}{2}\frac{\partial w}{\partial y}$$

Stresses

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$$

$$\tau_{xz} = 2\mu e_{xz}\ ,\ \tau_{yz} = 2\mu e_{yz}$$

Equilibrium Equations

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + F_z = 0$$

$$F_x = F_y = 0$$

Navier's Equation

$$\mu \nabla^2 w + F_z = 0$$

This theory is sometimes used in geomechanic applications to model deformations of portions of the earth's interior.

7.6 Airy stress function

Numerous solutions to plane strain and plane stress problems can be determined using an *Airy Stress Function* technique. The method will reduce the general formulation to a single governing equation in terms of a single unknown. The resulting equation is then solvable by several methods of applied mathematics, and thus many analytical solutions to problems of interest can be found.

The method is started by reviewing the equilibrium equations for the plane problems. We retain the body forces but assume that they are derivable from a potential function V such that

$$F_x = -\frac{\partial V}{\partial x}\ ,\ F_y = -\frac{\partial V}{\partial y}$$

This assumption is not very restrictive because many body forces found in applications (e.g. gravity loading) fall into this category. Under this form, the plane equilibrium equations can be written as

$$\frac{\partial(\sigma_x - V)}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \ ;\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial(\sigma_y - V)}{\partial y} = 0$$

7.6 Airy stress function

These equations will be identically satisfied by choosing a representation

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} + V \quad ; \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} + V \quad ; \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

where $\phi = \phi(x,y)$ is an arbitrary form called the Airy stress function

In the case of zero body forces, then we have

$$\sigma_x = \frac{\partial^2 \varphi}{\partial y^2}, \ \sigma_y = \frac{\partial^2 \varphi}{\partial x^2}, \ \tau_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y}$$

It is easily shown that this form satisfies equilibrium (zero body force case) and substituting it into the compatibility equations gives

$$\frac{\partial^4 \varphi}{\partial x^4} + 2\frac{\partial^4 \varphi}{\partial x^2 \partial y^2} + \frac{\partial^4 \varphi}{\partial y^4} = \nabla^4 \varphi = 0$$

This relation is called the *biharmonic equation* and its solutions are known as *biharmonic functions*.

7.6 Airy stress function

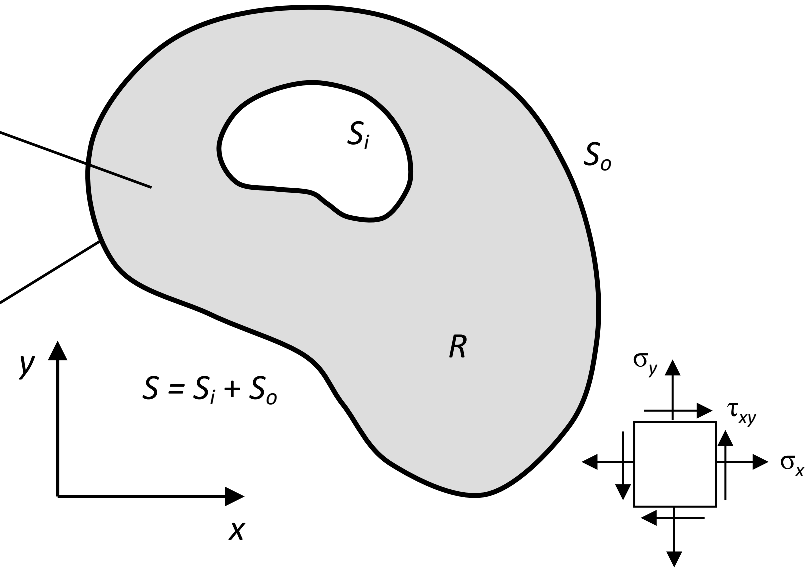
Airy Stress Function Formulation

The plane problem of elasticity can be reduced to a single equation in terms of the Airy stress function. This function is to be determined in the two-dimensional region *R* bounded by the boundary *S* as shown in the figure. Appropriate boundary conditions over *S* are necessary to complete a solution. Traction boundary conditions would involve the specification of second derivatives of the stress function; however, this condition can be reduced to specification of first order derivatives.

$$\frac{\partial^4 \varphi}{\partial x^4} + 2\frac{\partial^4 \varphi}{\partial x^2 \partial y^2} + \frac{\partial^4 \varphi}{\partial y^4} = \nabla^4 \varphi = 0$$

$$T_x^{(n)} = \sigma_x n_x + \tau_{xy} n_y = \frac{\partial^2 \varphi}{\partial y^2} n_x - \frac{\partial^2 \varphi}{\partial x \partial y} n_y$$

$$T_y^{(n)} = \tau_{xy} n_x + \sigma_y n_y = -\frac{\partial^2 \varphi}{\partial x \partial y} n_x + \frac{\partial^2 \varphi}{\partial x^2} n_y$$



7.7 Polar coordinate formulation

7.7 Polar coordinate formulation

Plane Elasticity Problem

Strain-Displacement

$$\begin{cases} e_r = \dfrac{\partial u_r}{\partial r} \\ e_\theta = \dfrac{1}{r}\left(u_r + \dfrac{\partial u_\theta}{\partial \theta} \right) \\ e_{r\theta} = \dfrac{1}{2}\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r} \right) \end{cases}$$

Hooke's Law

Plane strain

$$\sigma_r = \lambda(e_r + e_\theta) + 2\mu e_r$$
$$\sigma_\theta = \lambda(e_r + e_\theta) + 2\mu e_\theta$$
$$\sigma_z = \lambda(e_r + e_\theta) = \nu(\sigma_r + \sigma_\theta)$$
$$\tau_{r\theta} = 2\mu e_{r\theta} \ , \ \tau_{\theta z} = \tau_{rz} = 0$$

Plane stress

$$e_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta)$$
$$e_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$
$$e_z = -\frac{\nu}{E}(\sigma_r + \sigma_\theta) = -\frac{\nu}{1-\nu}(e_r + e_\theta)$$
$$e_{r\theta} = \frac{1+\nu}{E}\tau_{r\theta} \ , \ e_{\theta z} = e_{rz} = 0$$

7.7 Polar coordinate formulation

Navier's Equations

Plane strain

$$\begin{cases} \mu\nabla^2 u_r + (\lambda+\mu)\dfrac{\partial}{\partial r}\left(\dfrac{\partial u_r}{\partial r}+\dfrac{u_r}{r}+\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}\right)+F_r=0 \\ \mu\nabla^2 u_\theta + (\lambda+\mu)\dfrac{1}{r}\dfrac{\partial}{\partial \theta}\left(\dfrac{\partial u_r}{\partial r}+\dfrac{u_r}{r}+\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}\right)+F_\theta=0 \end{cases}$$

Plane stress

$$\begin{cases} \mu\nabla^2 u_r + \dfrac{E}{2(1-\nu)}\dfrac{\partial}{\partial r}\left(\dfrac{\partial u_r}{\partial r}+\dfrac{u_r}{r}+\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}\right)+F_r=0 \\ \mu\nabla^2 u_\theta + \dfrac{E}{2(1-\nu)}\dfrac{1}{r}\dfrac{\partial}{\partial \theta}\left(\dfrac{\partial u_r}{\partial r}+\dfrac{u_r}{r}+\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}\right)+F_\theta=0 \end{cases}$$

Equilibrium Equations

$$\begin{cases} \dfrac{\partial \sigma_r}{\partial r}+\dfrac{1}{r}\dfrac{\partial \tau_{r\theta}}{\partial \theta}+\dfrac{(\sigma_r-\sigma_\theta)}{r}+F_r=0 \\ \dfrac{\partial \tau_{r\theta}}{\partial r}+\dfrac{1}{r}\dfrac{\partial \sigma_\theta}{\partial \theta}+\dfrac{2\tau_{r\theta}}{r}+F_\theta=0 \end{cases}$$

Compatibility Equations

$$\nabla^2(\sigma_r+\sigma_\theta)=-\frac{1}{1-\nu}\left(\frac{\partial F_r}{\partial r}+\frac{F_r}{r}+\frac{1}{r}\frac{\partial F_\theta}{\partial \theta}\right)$$

7.7 Polar coordinate formulation

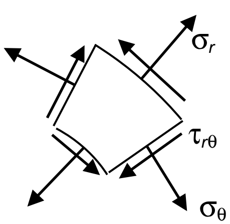
Airy Stress Function Approach $\varphi = \varphi(r,\theta)$

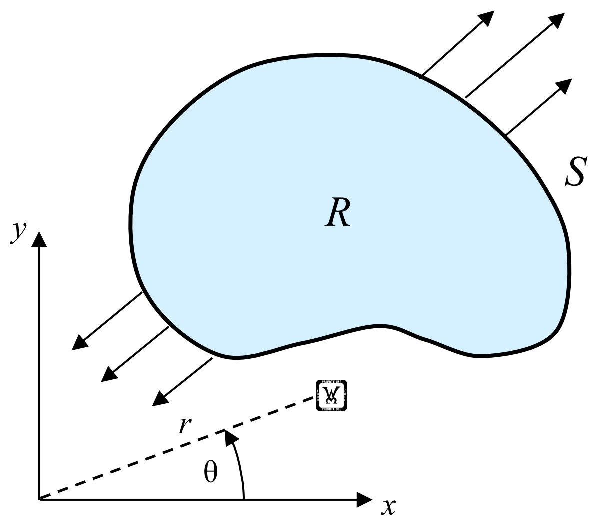
Airy Representation

$$\begin{cases} \sigma_r = \dfrac{1}{r}\dfrac{\partial \varphi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \varphi}{\partial \theta^2} \\ \sigma_\theta = \dfrac{\partial^2 \varphi}{\partial r^2} \\ \tau_{r\theta} = -\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial \varphi}{\partial \theta}\right) \end{cases}$$

Biharmonic Governing Equation

$$\nabla^4 \varphi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\varphi = 0$$





Traction Boundary Conditions

$T_r = f_r(r,\theta)$, $T_\theta = f_\theta(r,\theta)$

See you next week