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INCOS
Home of Computational Scientists

SOLID MECHANICS

Chapter 8: Two-dimensional problem solution (Part 1)

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8.1 Two-dimensional problem solution

Using the Airy Stress Function approach, it was shown that the plane elasticity formulation with zero body forces reduces to a single governing biharmonic equation. In Cartesian coordinates it is given by

$$\frac{\partial^4 \varphi}{\partial x^4} + 2\frac{\partial^4 \varphi}{\partial x^2 \partial y^2} + \frac{\partial^4 \varphi}{\partial y^4} = \nabla^4 \varphi = 0$$

and the stresses are related to the stress function by

$$\sigma_x = \frac{\partial^2 \varphi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \varphi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y}$$

We now explore solutions to several specific problems in both Cartesian and Polar coordinate systems

8.2 Cartesian Coordinate Solutions Using Polynomials

The biharmonic equation

$$\frac{\partial^4\varphi}{\partial x^4}+2\frac{\partial^4\varphi}{\partial x^2\partial y^2}+\frac{\partial^4\varphi}{\partial y^4}=\nabla^4\varphi=0$$

In Cartesian coordinates we choose Airy stress function solution of polynomial form

$$\varphi(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}x^m y^n$$

where A_{mn} are constant coefficients to be determined. This method produces polynomial stress distributions, and thus would not satisfy general boundary conditions. However, we can modify such boundary conditions using Saint-Venant's principle and replace a non-polynomial condition with a statically equivalent loading. This formulation is most useful for problems with rectangular domains, and is commonly based on the inverse solution concept where we assume a polynomial solution form and then try to find what problem it will solve.

8.2 Cartesian Coordinate Solutions Using Polynomials

$$\varphi(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{mn}x^m y^n$$

$$\frac{\partial^4\varphi}{\partial x^4}+2\frac{\partial^4\varphi}{\partial x^2\partial y^2}+\frac{\partial^4\varphi}{\partial y^4}=\nabla^4\varphi=0$$

Noted that the three lowest order terms with $m + n \leq 1$ do not contribute to the stresses and will therefore be dropped. It should be noted that second order terms will produce a constant stress field, third-order terms will give a linear distribution of stress, and so on for higher-order polynomials.

Terms with $m + n \leq 3$ will automatically satisfy the biharmonic equation for any choice of constants A_{mn}. However, for higher order terms, constants A_{mn} will have to be related in order to have the polynomial satisfy the biharmonic equation. For example, the 4th-order polynomial terms $A_{40}x^4+A_{22}x^2y^2+A_{04}y^4$ will not satisfy the biharmonic equation unless $3A_{40}+A_{22}+3A_{04}=0$. This condition specifies one constant in terms of the other two, thus leaving two constants to be determined by the boundary conditions.

8.2 Cartesian Coordinate Solutions Using Polynomials

Considering the general case, substituting the series into the governing biharmonic equation yields

$$\sum_{m=4}^{\infty}\sum_{n=0}^{\infty} m(m-1)(m-2)(m-3)A_{mn}x^{m-4}y^{n} + 2\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} m(m-1)n(n-1)A_{mn}x^{m-2}y^{n-2}$$
$$+\sum_{m=0}^{\infty}\sum_{n=4}^{\infty} n(n-1)(n-2)(n-3)A_{mn}x^{m}y^{n-4} = 0$$

Collecting like powers of x and y, the preceding equation may be written as

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\Big[(m+2)(m+1)m(m-1)A_{m+2,n-2} + 2m(m-1)n(n-1)A_{mn} +$$
$$+(n+2)(n+1)n(n-1)A_{m-2,n+2}\Big]x^{m-2}y^{n-2} = 0$$

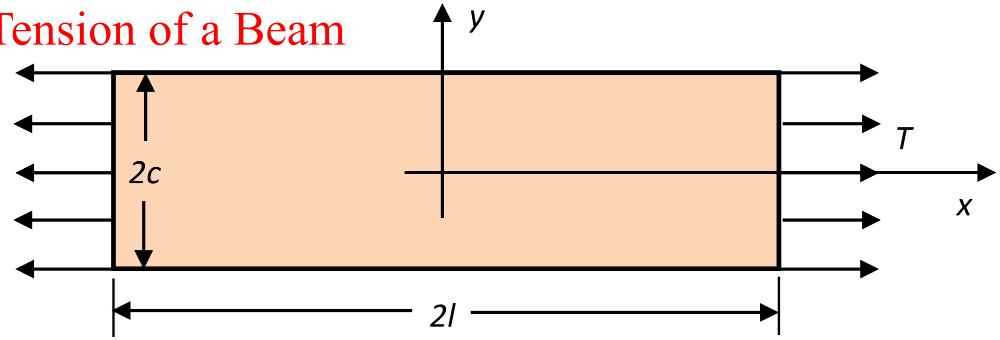
Because this relation must be satisfied for all values of x and y, the coefficient in brackets must vanish, giving the result

$$(m+2)(m+1)m(m-1)A_{m+2,n-2} + 2m(m-1)n(n-1)A_{mn} + (n+2)(n+1)n(n-1)A_{m-2,n+2} = 0$$

For each m, n pair, this equation is the general relation that must be satisfied to ensure that the polynomial grouping is biharmonic.

8.2 Cartesian Coordinate Solutions Using Polynomials

Example 8.1 Uniaxial Tension of a Beam



Stress Field

Boundary Conditions: $\begin{cases} \sigma_x(\pm l, y) = T, \ \sigma_y(x, \pm c) = 0 \\ \tau_{xy}(\pm l, y) = \tau_{xy}(x, \pm c) = 0 \end{cases}$

Since the boundary conditions specify constant stresses on all boundaries, try a second-order stress function of the form

$$\varphi = A_{02} y^2 \Rightarrow \boxed{\sigma_x = 2A_{02} \ , \sigma_y = \tau_{xy} = 0}$$

The first boundary condition implies that $A_{02} = T/2$, and all other boundary conditions are identically satisfied. Therefore the stress field solution is given by

$$\sigma_x = T \ , \ \sigma_y = \tau_{xy} = 0$$

Displacement Field (Plane Stress)

$$\frac{\partial u}{\partial x} = e_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = \frac{T}{E}$$

$$\frac{\partial v}{\partial y} = e_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = -\nu\frac{T}{E}$$

$$\Rightarrow \boxed{u = \frac{T}{E}x + f(y), \ v = -\nu\frac{T}{E}y + g(x)}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2e_{xy} = \frac{\tau_{xy}}{\mu} = 0 \Rightarrow f'(y) + g'(x) = 0$$

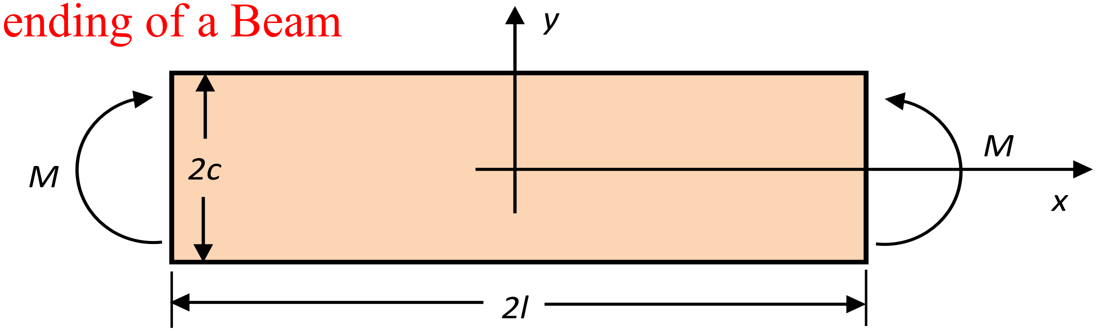
$$f(y) = -\omega_o y + u_o$$
$$g(x) = \omega_o x + v_o$$
... *Rigid-Body Motion*

"Fixity conditions" needed to determine RBM terms

$$u(0,0) = v(0,0) = u(0,c) = 0 \Rightarrow f(y) = g(x) = 0$$

8.2 Cartesian Coordinate Solutions Using Polynomials

Example 8.2 Pure Bending of a Beam



Stress Field

Boundary Conditions:

$$\sigma_y(x,\pm c)=0\ ,\ \tau_{xy}(x,\pm c)=\tau_{xy}(\pm l,y)=0$$

$$\int_{-c}^{c}\sigma_x(\pm l,y)dy=0\ ,\ \int_{-c}^{c}\sigma_x(\pm l,y)ydy=-M$$

Expecting a linear bending stress distribution, try 2nd- stress function of the form

$$\varphi=A_{03}y^3 \Rightarrow \sigma_x=6A_{03}y\ ,\ \sigma_y=\tau_{xy}=0$$

Moment boundary condition implies that A_{03} = $-M/4c^3$, and all other boundary conditions are identically satisfied. Thus the stress field is

$$\sigma_x=-\frac{3M}{2c^3}y\ ,\ \sigma_y=\tau_{xy}=0$$

Displacement Field (Plane Stress)

$$\frac{\partial u}{\partial x}=-\frac{3M}{2Ec^3}y \Rightarrow u=-\frac{3M}{2Ec^3}xy+f(y)$$

$$\frac{\partial v}{\partial y}=\nu\frac{3M}{2Ec^3}y \Rightarrow v=\frac{3M\nu}{4Ec^3}y^2+g(x)$$

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0 \Rightarrow -\frac{3M}{2Ec^3}x+f'(y)+g'(x)=0$$

$$\Rightarrow \begin{cases} f(y)=-\omega_0 y+u_0 \\ g(x)=\dfrac{3M}{4Ec^3}x^2+\omega_0 x+v_0 \end{cases}$$

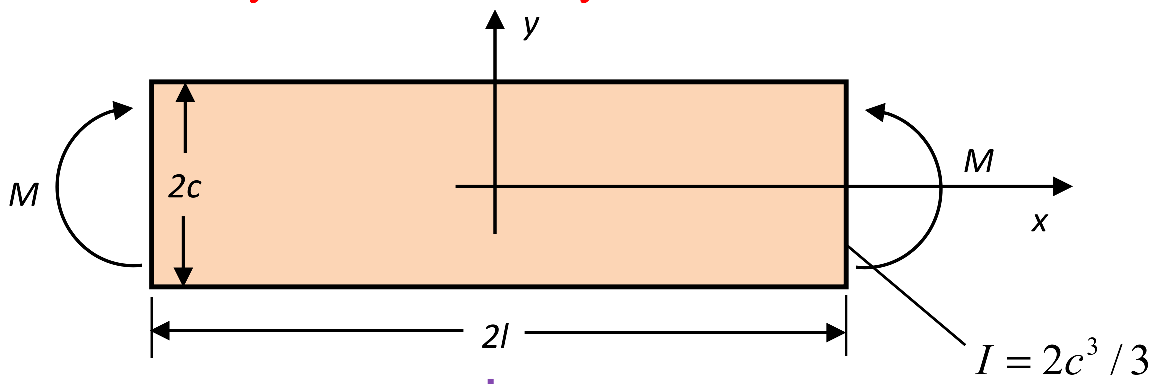
"Fixity conditions" to determine RBM terms:

$$v(\pm l,0)=0 \quad \text{and} \quad u(-l,0)=0$$

$$\Rightarrow u_0=\omega_0=0\ ,\ v_0=-3Ml^2/16Ec^3$$

8.2 Cartesian Coordinate Solutions Using Polynomials

Solution Comparison of Elasticity with Elementary Mechanics of Materials



Elasticity Solution

$$\sigma_x = -\frac{M}{I}y \ , \ \sigma_y = \tau_{xy} = 0$$

$$u = -\frac{Mxy}{EI} \ , \ v = \frac{M}{8EI}[4\nu y^2 + 4x^2 - l^2]$$

Mechanics of Materials Solution

Uses Euler-Bernoulli beam theory to find bending stress and deflection of beam centerline

$$\sigma_x = -\frac{M}{I}y \ , \ \sigma_y = \tau_{xy} = 0$$

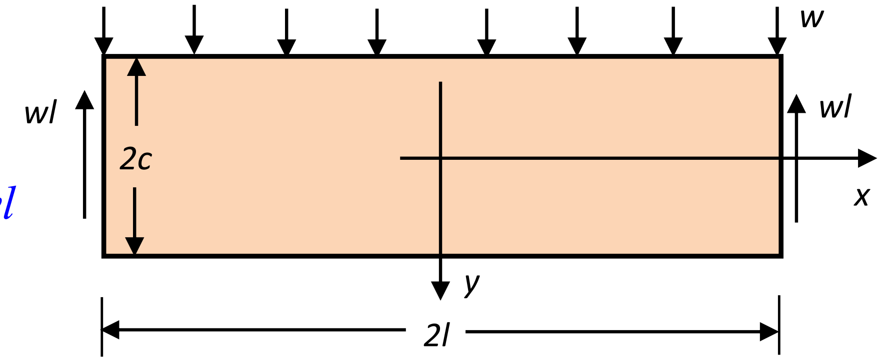
$$v = v(x,0) = \frac{M}{8EI}[4x^2 - l^2]$$

Two solutions are identical, with the exception of the *x*-displacements

8.2 Cartesian Coordinate Solutions Using Polynomials

Example 8.3 Bending of a Beam by Uniform Transverse Loading

Stress Field



Boundary Conditions:

$$\begin{cases} \tau_{xy}(x,\pm c)=0; & \int_{-c}^{c}\tau_{xy}(\pm l,y)\,\mathrm{d}y=\mp wl \\ \sigma_y(x,c)=0; & \int_{-c}^{c}\sigma_x(\pm l,y)\,y\,\mathrm{d}y=0 \\ \sigma_y(x,-c)=-w; & \int_{-c}^{c}\sigma_x(\pm l,y)\,\mathrm{d}y=0 \end{cases}$$

$$\varphi = A_{20}x^2 + A_{21}x^2y + A_{03}y^3 + A_{23}x^2y^3 - \frac{A_{23}}{5}y^5$$

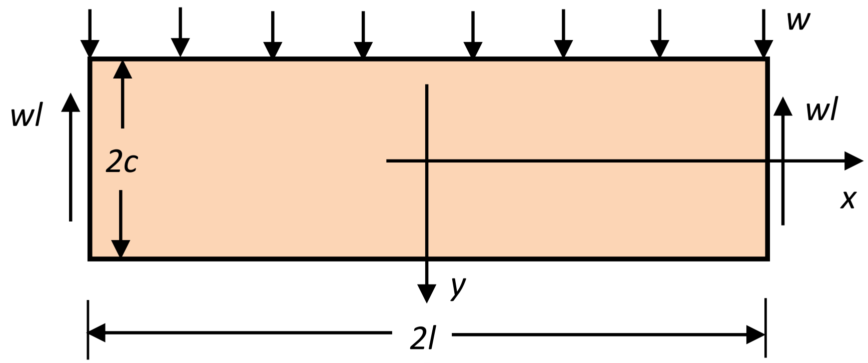
$$\begin{cases} \sigma_x = 6A_{03}y + 6A_{23}(x^2y - \frac{2}{3}y^3) \\ \sigma_y = 2A_{20} + 2A_{21}y + 2A_{23}y^3 \\ \tau_{xy} = -2A_{21}x - 6A_{23}xy^2 \end{cases}$$

BC's ⇒

$$\begin{cases} \sigma_x = \frac{3w}{4c}\left(\frac{l^2}{c^2}-\frac{2}{5}\right)y - \frac{3w}{4c^3}(x^2y-\frac{2}{3}y^3) \\ \sigma_y = -\frac{w}{2}+\frac{3w}{4c}y-\frac{w}{4c^3}y^3 \\ \tau_{xy} = -\frac{3w}{4c}x+\frac{3w}{4c^3}xy^2 \end{cases}$$

8.2 Cartesian Coordinate Solutions Using Polynomials

Example 8.3 Bending of a Beam by Uniform Transverse Loading



Elasticity Solution

$$\sigma_x = \frac{w}{2I}(l^2 - x^2)y + \frac{w}{I}\left(\frac{y^3}{3} - \frac{c^2 y}{5}\right)$$

$$\sigma_y = -\frac{w}{2I}\left(\frac{y^3}{3} - c^2 y + \frac{2}{3}c^3\right)$$

$$\tau_{xy} = -\frac{w}{2I}x(c^2 - y^2)$$

Mechanics of Materials Solution

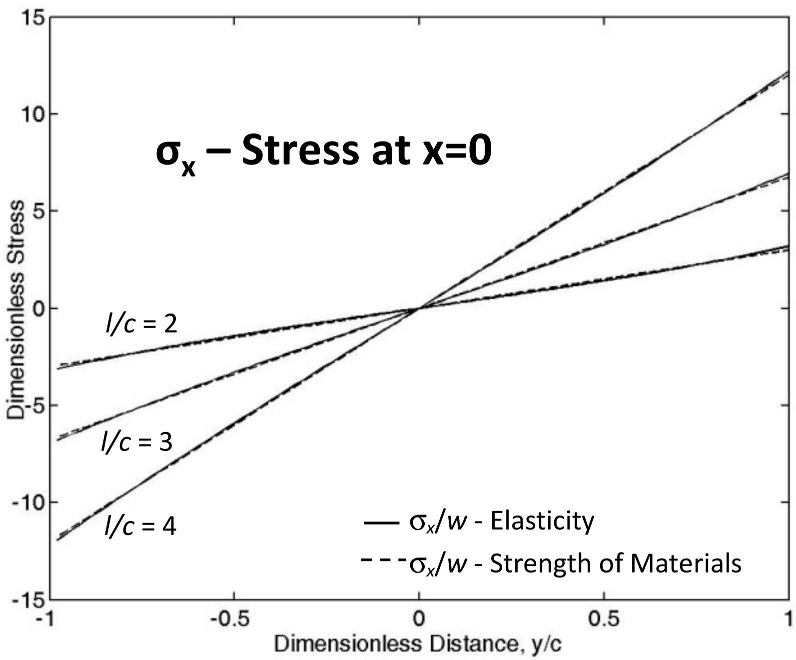
$$\sigma_x = \frac{My}{I} = \frac{w}{2I}(l^2 - x^2)y$$

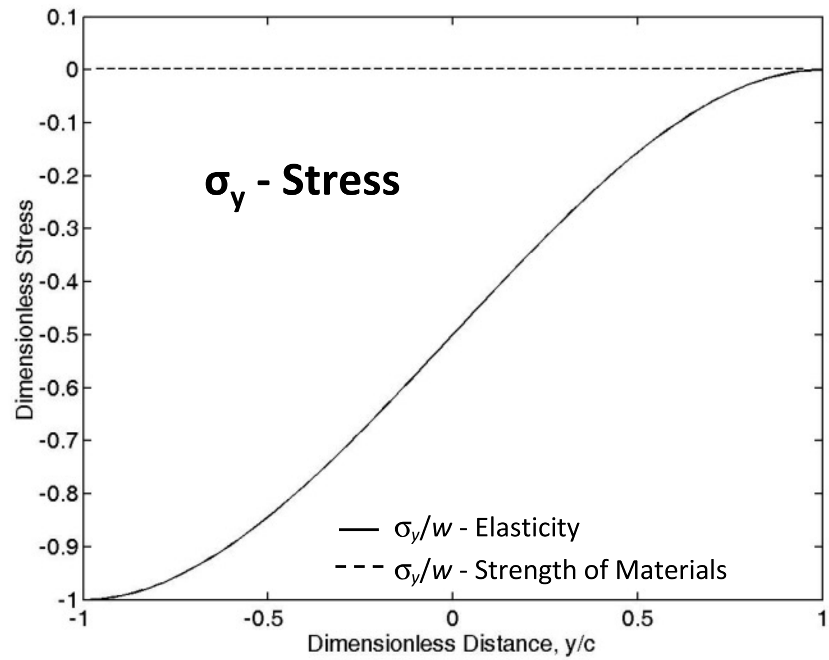
$$\sigma_y = 0$$

$$\tau_{xy} = \frac{VQ}{It} = -\frac{w}{2I}x(c^2 - y^2)$$

Shear stresses are identical, while normal stresses are not

8.2 Cartesian Coordinate Solutions Using Polynomials



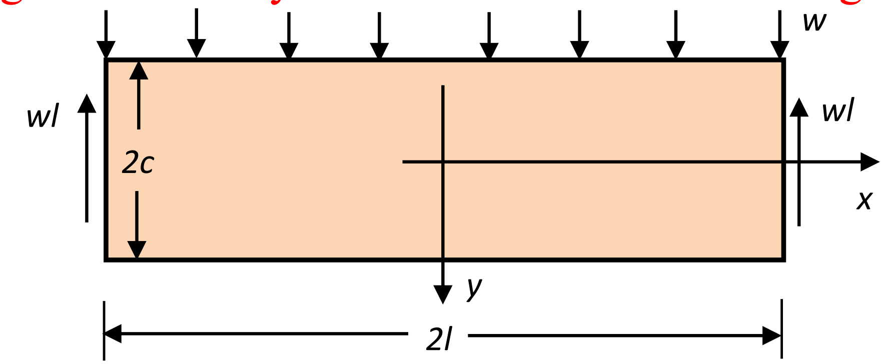


Maximum differences between the two theories exist at top and bottom of beam, and actual difference in stress values is $w/5$. For most beam problems where $l >> c$, the bending stresses will be much greater than w, and thus the differences between elasticity and strength of materials will be relatively small.

Maximum difference between the two theories is w and this occurs at the top of the beam. Again this difference will be negligibly small for most beam problems where $l >> c$. These results are generally true for beam problems with other transverse loadings.

8.2 Cartesian Coordinate Solutions Using Polynomials

Example 8.3 Bending of a Beam by Uniform Transverse Loading



Displacement Field (Plane Stress)

$$\begin{cases} u = \frac{w}{2EI}[(l^2x - \frac{x^3}{3})y + x(\frac{2y^3}{3} - \frac{2c^2y}{5}) + \nu x(\frac{y^3}{3} - c^2y + \frac{2c^3}{3})] + f(y) \\ v = -\frac{w}{2EI}[(\frac{y^4}{12} - \frac{c^2y^2}{2} + \frac{2c^3y}{3}) + \nu(l^2 - x^2)\frac{y^2}{2} + \nu(\frac{y^4}{6} - \frac{c^2y^2}{5})] + g(x) \end{cases}$$

$$f(y) = \omega_0 y + u_0 \ , \quad g(x) = \frac{w}{24EI}x^4 - \frac{w}{4EI}[l^2 - (\frac{8}{5} + \nu)c^2]x^2 - \omega_0 x + v_0$$

Choosing Fixity Conditions $\boxed{u(0, y) = v(\pm l, 0) = 0}$

$$\Rightarrow \quad u_0 = \omega_0 = 0, v_0 = \frac{5wl^4}{24EI}\left[1 + \frac{12}{5}\left(\frac{4}{5} + \frac{\nu}{2}\right)\frac{c^2}{l^2}\right]$$

8.2 Cartesian Coordinate Solutions Using Polynomials

Example 8.3 Bending of a Beam by Uniform Transverse Loading

Displacement Field (Plane Stress)

$$u = \frac{w}{2EI}\left[\left(l^2 x - \frac{x^3}{3}\right)y + x\left(\frac{2y^3}{3} - \frac{2c^2 y}{5}\right) + \nu x\left(\frac{y^3}{3} - c^2 y + \frac{2c^3}{3}\right)\right]$$

$$v = -\frac{w}{2EI}\left\{\begin{array}{l}\frac{y^4}{12} - \frac{c^2 y^2}{2} + \frac{2c^3 y}{3} + \nu\left[\left(l^2 - x^2\right)\frac{y^2}{2} + \frac{y^4}{6} - \frac{c^2 y^2}{5}\right] \\ -\frac{x^4}{12} + \left[\frac{l^2}{2} + \left(\frac{4}{5} + \frac{\nu}{2}\right)c^2\right]x^2\end{array}\right\} + \frac{5wl^4}{24EI}\left[1 + \frac{12}{5}\left(\frac{4}{5} + \frac{\nu}{2}\right)\frac{c^2}{l^2}\right]$$

⇨ $$v(0,0) = v_{\max} = \frac{5wl^4}{24EI}\left[1 + \frac{12}{5}\left(\frac{4}{5} + \frac{\nu}{2}\right)\frac{c^2}{l^2}\right]$$

Strength of Materials: $v_{\max} = \frac{5wl^4}{24EI}$ Good match for beams where $l >> c$

8.3 Cartesian Coordinate Solutions Using Fourier Methods

8.3 Cartesian Coordinate Solutions Using Fourier Methods

A more general solution scheme for the biharmonic equation may be found using *Fourier methods*. Such techniques generally use *separation of variables* along with *Fourier series* or *Fourier integrals*.

$$\varphi(x,y)=X(x)Y(y) \Rightarrow \frac{\partial^4\varphi}{\partial x^4}+2\frac{\partial^4\varphi}{\partial x^2\partial y^2}+\frac{\partial^4\varphi}{\partial y^4}=0$$

Choosing $X=e^{\alpha x}$, $Y=e^{\beta y}$ $\Rightarrow$ $\alpha=\pm i\beta$

$$\begin{aligned}\phi &= \sin\beta x\left[(A+C\beta y)\sinh\beta y+(B+D\beta y)\cosh\beta y\right]\\ &+\cos\beta x\left[(A'+C'\beta y)\sinh\beta y+(B'+D'\beta y)\cosh\beta y\right]\\ &+\sin\alpha y\left[(E+G\alpha x)\sinh\alpha x+(F+H\alpha x)\cosh\alpha x\right]\\ &+\cos\alpha y\left[(E'+G'\alpha x)\sinh\alpha x+(F'+H'\alpha x)\cosh\alpha x\right]\\ &+\phi_{\alpha=0}+\phi_{\beta=0}\end{aligned}$$

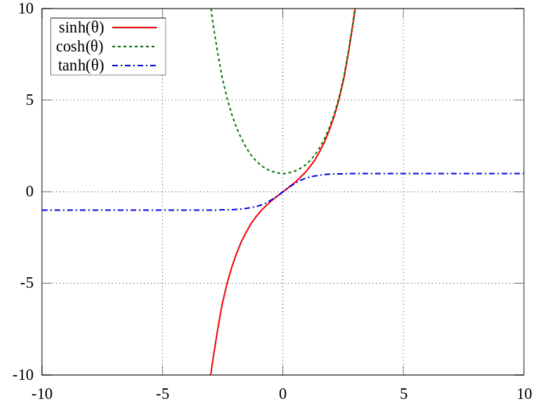
(zero root solutions)

where
$$\begin{cases}\phi_{\beta=0}=C_0+C_1x+C_2x^2+C_3x^3\\ \phi_{\alpha=0}=C_4y+C_5y^2+C_6y^3+C_7xy+C_8x^2y+C_9xy^2\end{cases}$$

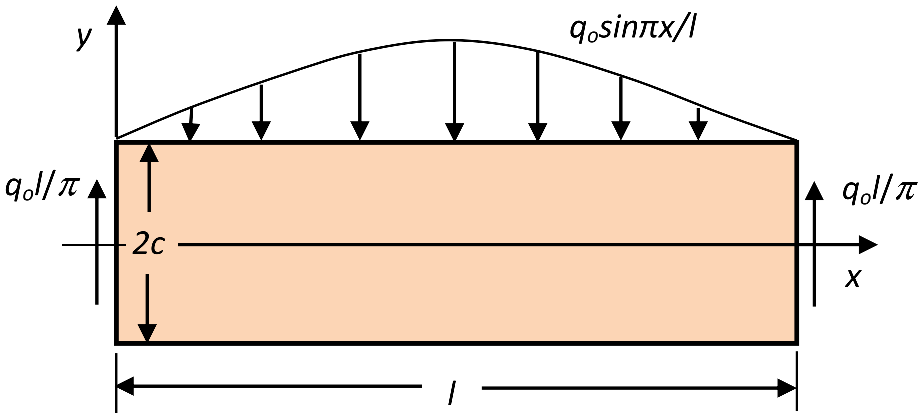
The general solution includes the superposition of the general roots plus the zero root cases

$$\sinh(x)=\frac{e^x-e^{-x}}{2}=-i\sin(ix)$$

$$\cosh(x)=\frac{e^x+e^{-x}}{2}=\cos ix$$



8.3 Cartesian Coordinate Solutions Using Fourier Methods



Example 8.4 Beam with Sinusoidal Loading

Stress Field

Boundary Conditions:

$$\varphi = \sin\beta x\left[(A + C\beta y)\sinh\beta y + (B + D\beta y)\cosh\beta y\right]$$

$$\sigma_x(0, y) = \sigma_x(l, y) = 0 \qquad (1)$$

$$\tau_{xy}(x, \pm c) = 0 \qquad (2)$$

$$\sigma_y(x, -c) = 0 \qquad (3)$$

$$\sigma_y(x, c) = -q_o \sin(\pi x / l) \qquad (4)$$

$$\int_{-c}^{c} \tau_{xy}(0, y)dy = -q_o l / \pi \qquad (5)$$

$$\int_{-c}^{c} \tau_{xy}(l, y)dy = q_o l / \pi \qquad (6)$$

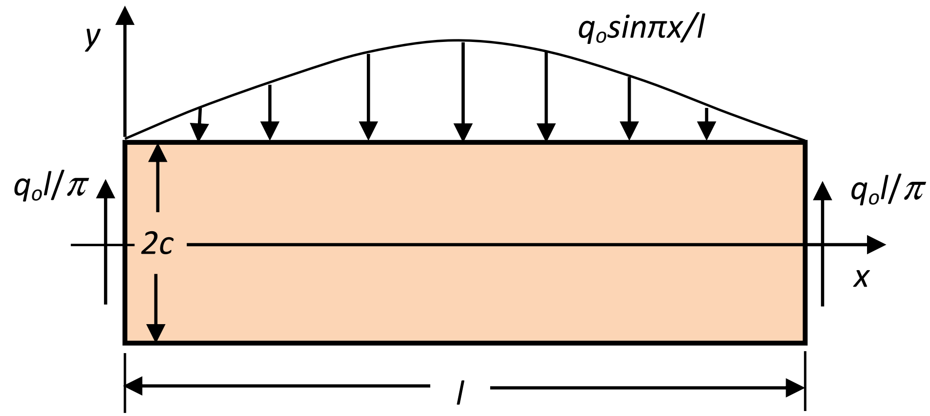
$$\sigma_x = \beta^2 \sin\beta x\left[A\sinh\beta y + C(\beta y\sinh\beta y + 2\cosh\beta y) + B\cosh\beta y + D(\beta y\cosh\beta y + 2\sinh\beta y)\right]$$

$$\sigma_y = -\beta^2 \sin\beta x\left[(A + C\beta y)\sinh\beta y + (B + D\beta y)\cosh\beta y\right]$$

$$\tau_{xy} = -\beta^2 \cos\beta x\left[A\cosh\beta y + C(\beta y\cosh\beta y + 2\sinh\beta y) + B\sinh\beta y + D(\beta y\sinh\beta y + 2\cosh\beta y)\right]$$

8.3 Cartesian Coordinate Solutions Using Fourier Methods

Example 8.4 Beam with Sinusoidal Loading

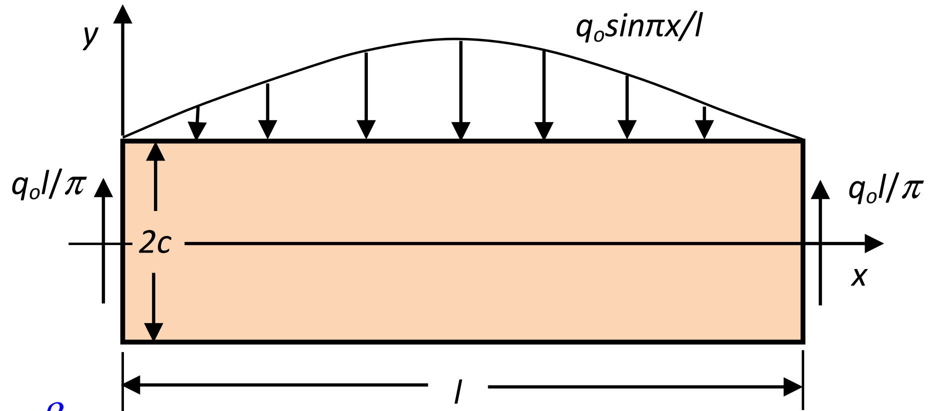


Condition (2) gives
$$\begin{cases} A = -D\left(\beta c \tanh \beta c + 1\right) \\ B = -C\left(\beta c \coth \beta c + 1\right) \end{cases} \qquad \beta = \frac{\pi}{l}$$

Condition (3) gives
$$C = \frac{-q_o \sinh \frac{\pi c}{l}}{2\frac{\pi^2}{l^2}\left[\frac{\pi c}{l} + \sinh \frac{\pi c}{l} \cosh \frac{\pi c}{l}\right]} \qquad D = \frac{-q_o \sinh \frac{\pi c}{l}}{2\frac{\pi^2}{l^2}\left[\frac{\pi c}{l} - \sinh \frac{\pi c}{l} \cosh \frac{\pi c}{l}\right]}$$

Condition (1) and condition (5,6) will be satisfied

8.3 Cartesian Coordinate Solutions Using Fourier Methods



Example 8.4 Beam with Sinusoidal Loading

Displacement Field

$$u=-\frac{\beta}{E}\cos\beta x\{A(1+\nu)\sinh\beta y+B(1+\nu)\cosh\beta y$$
$$+C[(1+\nu)\beta y\sinh\beta y+2\cosh\beta y]$$
$$+D[(1+\nu)\beta y\cosh\beta y+2\sinh\beta y]\}-\omega_o y+u_o$$

$$v=-\frac{\beta}{E}\sin\beta x\{A(1+\nu)\cosh\beta y+B(1+\nu)\sinh\beta y$$
$$+C[(1+\nu)\beta y\cosh\beta y-(1+\nu)\sinh\beta y]$$
$$+D[(1+\nu)\beta y\sinh\beta y-(1+\nu)\cosh\beta y]\}+\omega_o y+v_o$$

$$u(0,0)=v(0,0)=v(l,0)=0 \Rightarrow \omega_0=v_0=0\,,\; u_0=\frac{\beta}{E}[B(1+\nu)+2C]$$

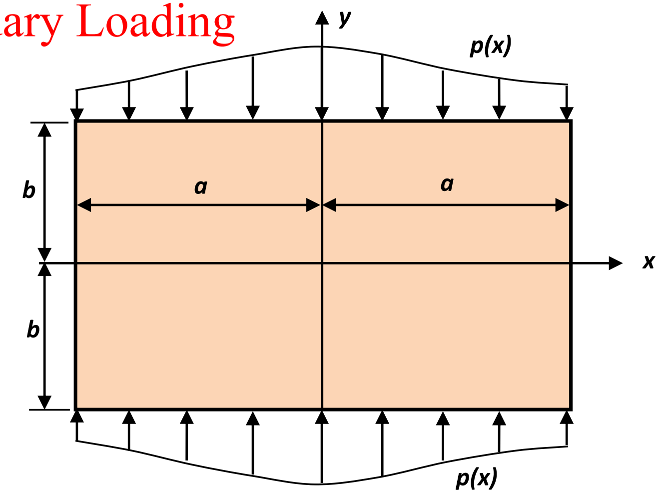
$$v(x,0)=\frac{D\beta}{E}\sin\beta x[2+(1+\nu)\beta c\tanh\beta c]$$

For the case $l>>c$ $\Rightarrow$ $D\approx-\frac{3q_0 l^5}{4c^3\pi^5}$ $\Rightarrow$ $v(x,0)=-\frac{3q_0 l^4}{2c^3\pi^4 E}\sin\frac{\pi x}{l}\left[1+\frac{1+\nu}{2}\frac{\pi c}{l}\tanh\frac{\pi c}{l}\right]$

Strength of Materials $\quad v(x,0)=-\frac{3q_0 l^4}{2c^3\pi^4 E}\sin\frac{\pi x}{l}$

8.3 Cartesian Coordinate Solutions Using Fourier Methods

Example 8.5 Rectangular Domain with Arbitrary Boundary Loading



Must use series representation for Airy stress function to handle general boundary loading.

$$\beta_n = \frac{n\pi}{l}$$

$$\varphi = \sum_{n=1}^{\infty} \cos\beta_n x[B_n \cosh\beta_n y + C_n \beta_n y \sinh\beta_n y] + \sum_{m=1}^{\infty} \cos\alpha_m y[F_m \cosh\alpha_m x + G_m \alpha_m x \sinh\alpha_m x] + C_0 x^2$$

⇓

$$\sigma_x = \sum_{n=1}^{\infty} \beta_n^2 \cos\beta_n x\left[B_n \cosh\beta_n y + C_n\left(\beta_n y \sinh\beta_n y + 2\cosh\beta_n y\right)\right] - \sum_{m=1}^{\infty} \alpha_m^2 \cos\alpha_m y[F_m \cosh\alpha_m x + G_m \alpha_m x \sinh\alpha_m x]$$

$$\sigma_y = -\sum_{n=1}^{\infty} \beta_n^2 \cos\beta_n x[B_n \cosh\beta_n y + C_n \beta_n y \sinh\beta_n y] + 2C_0 + \sum_{m=1}^{\infty} \alpha_m^2 \cos\alpha_m y\left[F_m \cosh\alpha_m x + G_m\left(\alpha_m x \sinh\alpha_m x + 2\cosh\alpha_m x\right)\right]$$

$$\tau_{xy} = \sum_{n=1}^{\infty} \beta_n^2 \sin\beta_n x\left[B_n \sinh\beta_n y + C_n\left(\beta_n y \cosh\beta_n y + \sinh\beta_n y\right)\right] + \sum_{m=1}^{\infty} \alpha_m^2 \sin\alpha_m y\left[F_m \sinh\alpha_m x + G_m\left(\alpha_m x \cosh\alpha_m x + \sinh\alpha_m x\right)\right]$$

Boundary Conditions

$$\sigma_x(\pm a, y) = 0$$
$$\tau_{xy}(\pm a, y) = 0$$
$$\tau_{xy}(x, \pm b) = 0$$
$$\sigma_y(x, \pm b) = -p(x)$$

Use Fourier series theory to handle general boundary conditions, and this generates a doubly infinite set of equations to solve for unknown constants in stress function form. See text for details

See you next week