ĐẠI HỌC TÔN ĐỨC THẮNG
TON DUC THANG UNIVERSITY

ĐH TÔN ĐỨC THẮNG

INCOS
Home of Computational Scientists

SOLID MECHANICS

Chapter 8: Two-dimensional problem solution (Part 2)

Lecturer: Assoc. Prof. Nguyen Thoi Trung
Assistant: Dang Trung Hau

TDT University - 2015

8.4 Polar Coordinate Formulation

8.5 General Solutions in Polar Coordinates

8.6 Example Polar Coordinate Solutions

8.4 Polar Coordinate Formulation

Airy Stress Function Approach $\varphi = \varphi(r,\theta)$

Airy Representation

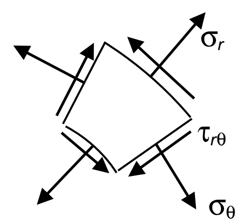
$$\begin{cases} \sigma_r = \dfrac{1}{r}\dfrac{\partial \varphi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \varphi}{\partial \theta^2} \\ \sigma_\theta = \dfrac{\partial^2 \varphi}{\partial r^2} \\ \tau_{r\theta} = -\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial \varphi}{\partial \theta}\right) \end{cases}$$

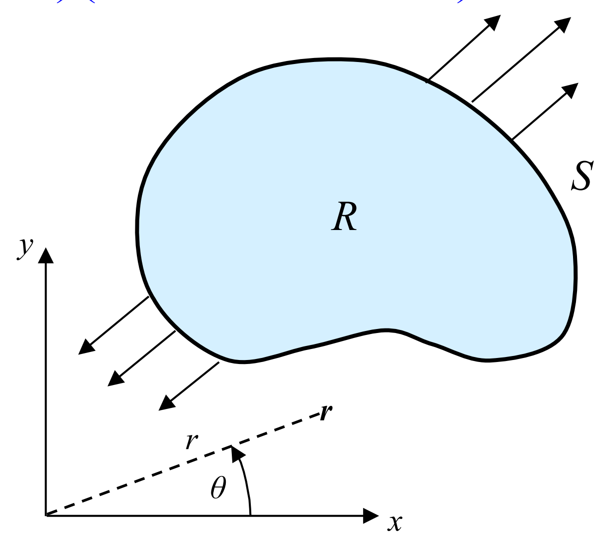
Biharmonic Governing Equation

$$\nabla^4 \varphi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\varphi = 0$$

Traction Boundary Conditions

$$T_r = f_r(r,\theta) \ , \ T_\theta = f_\theta(r,\theta)$$





8.4 Polar Coordinate Formulation

Plane Elasticity Problem

Strain-Displacement

$$\begin{cases} e_r = \dfrac{\partial u_r}{\partial r} \\ e_\theta = \dfrac{1}{r}\left(u_r + \dfrac{\partial u_\theta}{\partial \theta}\right) \\ e_{r\theta} = \dfrac{1}{2}\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r}\right) \end{cases}$$

Plane strain

$$\sigma_r = \lambda(e_r + e_\theta) + 2\mu e_r$$
$$\sigma_\theta = \lambda(e_r + e_\theta) + 2\mu e_\theta$$
$$\sigma_z = \lambda(e_r + e_\theta) = \nu(\sigma_r + \sigma_\theta)$$
$$\tau_{r\theta} = 2\mu e_{r\theta} \ , \ \tau_{\theta z} = \tau_{rz} = 0$$

Plane stress

$$e_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) \ , \ e_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$
$$e_z = -\frac{\nu}{E}(\sigma_r + \sigma_\theta) = -\frac{\nu}{1-\nu}(e_r + e_\theta)$$
$$e_{r\theta} = \frac{1+\nu}{E}\tau_{r\theta} \ , \ e_{\theta z} = e_{rz} = 0$$

Hooke's Law

8.4 Polar Coordinate Formulation

8.5 General Solutions in Polar Coordinates

8.6 Example Polar Coordinate Solutions

8.5 General Solutions in Polar Coordinates

8.3.1 General Michell Solution

$$\varphi(r,\theta) = f(r)e^{b\theta} \Rightarrow \nabla^4\varphi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi = 0$$

$$f'''' + \frac{2}{r}f''' - \frac{1-2b^2}{r^2}f'' + \frac{1-2b^2}{r^3}f' + \frac{b^2(4+b^2)}{r^4}f = 0$$

Solving the equation gives the general Michell solution (restricted to the periodic case)

$$\left.\begin{aligned}
\varphi &= a_0 + a_1\log r + a_2 r^2 + a_3 r^2\log r \\
&\quad + (a_4 + a_5\log r + a_6 r^2 + a_7 r^2\log r)\theta \\
&\quad + \left(a_{11}r + a_{12}r\log r + \frac{a_{13}}{r} + a_{14}r^3 + a_{15}r\theta + a_{16}r\theta\log r\right)\cos\theta \\
&\quad + \left(b_{11}r + b_{12}r\log r + \frac{b_{13}}{r} + b_{14}r^3 + b_{15}r\theta + b_{16}r\theta\log r\right)\sin\theta \\
&\quad + \sum_{n=2}^{\infty}(a_{n1}r^n + a_{n2}r^{2+n} + a_{n3}r^{-n} + a_{n4}r^{2-n})\cos n\theta \\
&\quad + \sum_{n=2}^{\infty}(b_{n1}r^n + b_{n2}r^{2+n} + b_{n3}r^{-n} + b_{n4}r^{2-n})\sin n\theta
\end{aligned}\right\}$$

We will use various terms from this general solution to solve several plane problems in polar coordinates

8.5 General Solutions in Polar Coordinates

8.3.2 Axisymmetric Solutions

Stress Function Approach

$$\varphi = a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r$$

$$\sigma_r = 2a_3 \log r + \frac{a_1}{r^2} + a_3 + 2a_2$$

$$\sigma_\theta = 2a_3 \log r - \frac{a_1}{r^2} + 3a_3 + 2a_2$$

$$\tau_{r\theta} = 0$$

Displacements - Plane Stress Case

$$u_r = \frac{1}{E}\left[-\frac{(1+\nu)}{r}a_1 + 2(1-\nu)a_3 r \log r - (1+\nu)a_3 r + 2a_2(1-\nu)r\right] + \underline{A\sin\theta + B\cos\theta}$$

$$u_\theta = \frac{4r\theta}{E}a_3 + \underline{A\cos\theta - B\sin\theta + Cr}$$

Underlined terms represent rigid-body motion

Navier Equation Approach

$\boldsymbol{u} = u_r(r)\boldsymbol{e}_r$

(Plane Stress or Plane Strain)

$$\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{1}{r^2}u_r = 0$$

$$u_r = C_1 r + C_2 \frac{1}{r}$$

Gives Stress Forms

$$\sigma_r = \frac{A}{r^2} + B,\ \sigma_\theta = -\frac{A}{r^2} + B,\ \tau_{r\theta} = 0$$

- a_3 term leads to multivalued behavior, and is not found following the displacement formulation approach

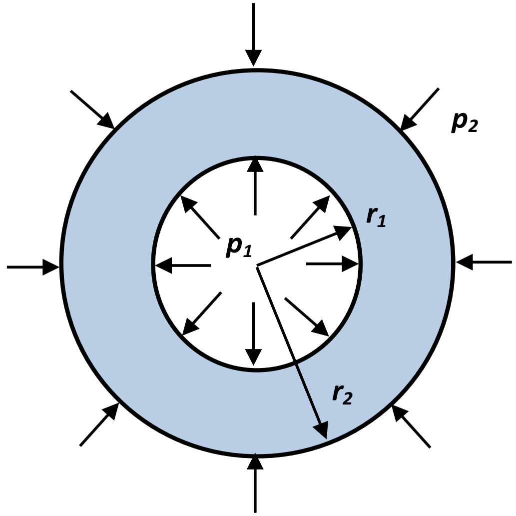
8.4 Polar Coordinate Formulation

8.5 General Solutions in Polar Coordinates

8.6 Example Polar Coordinate Solutions

8.6 Example Polar Coordinate Solutions

Example 8.6 Thick-Walled Cylinder Under Uniform Boundary Pressure



General Axisymmetric Stress Solution

$$\sigma_r = \frac{A}{r^2} + B$$

$$\sigma_\theta = -\frac{A}{r^2} + B$$

Boundary Conditions

$$\sigma_r(r_1) = -p_1 \ , \ \sigma_r(r_2) = -p_2$$

$$\Rightarrow \quad A = \frac{r_1^2 r_2^2 (p_2 - p_1)}{r_2^2 - r_1^2}$$

$$B = \frac{r_1^2 p_1 - r_2^2 p_2}{r_2^2 - r_1^2}$$

$$\sigma_r = \frac{r_1^2 r_2^2 (p_2 - p_1)}{r_2^2 - r_1^2}\frac{1}{r^2} + \frac{r_1^2 p_1 - r_2^2 p_2}{r_2^2 - r_1^2}$$

$$\sigma_\theta = -\frac{r_1^2 r_2^2 (p_2 - p_1)}{r_2^2 - r_1^2}\frac{1}{r^2} + \frac{r_1^2 p_1 - r_2^2 p_2}{r_2^2 - r_1^2}$$

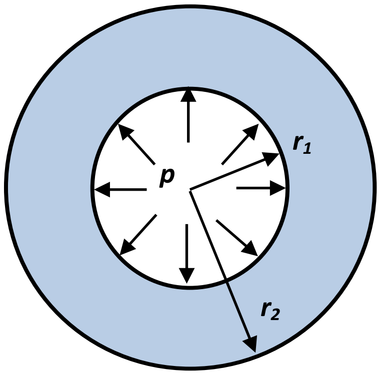
Using Strain Displacement Relations and Hooke’s Law for plane strain gives the radial displacement

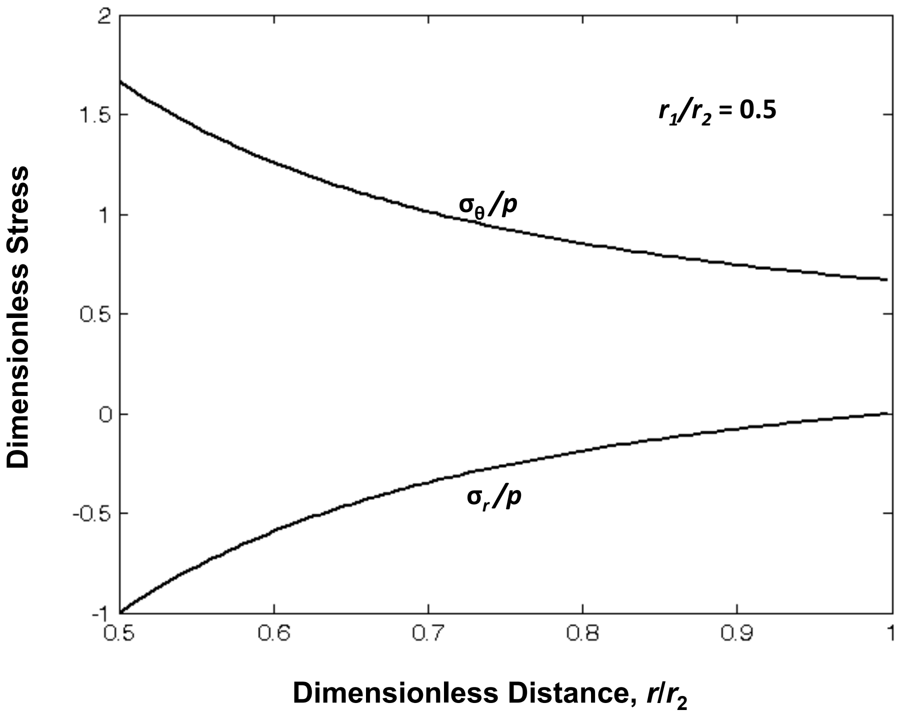
$$\Rightarrow \quad u_r = \frac{1+\nu}{E} r[(1-2\nu)B - \frac{A}{r^2}]$$

$$= \frac{1+\nu}{E}\left[-\frac{r_1^2 r_2^2 (p_2 - p_1)}{r_2^2 - r_1^2}\frac{1}{r} + (1-2\nu)\frac{r_1^2 p_1 - r_2^2 p_2}{r_2^2 - r_1^2} r\right]$$

8.6 Example Polar Coordinate Solutions

Example 8.6 Cylinder Problem Results Internal Pressure Only





For the case of only internal pressure (p_2 = 0 and p_1 = p) with r_1/r_2 = 0.5. Radial stress decays from $-p$ to zero. Hoop stress is positive with a maximum value at the inner radius:

$$(\sigma_\theta)_{\max} = (r_1^2 + r_2^2)/(r_2^2 - r_1^2)\,p = (5/3)\,p$$

Thin-Walled Tube Case:

$$t = r_2 - r_1 << 1 \qquad r_o = (r_1 + r_2)/2 \quad \Rightarrow \quad \sigma_\theta \approx \frac{pr_o}{t}$$

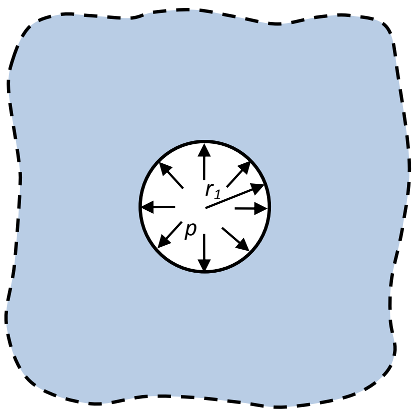
Matches with Strength of Materials Theory

8.6 Example Polar Coordinate Solutions

Special Cases of Example 8-6

Pressurized Hole in an Infinite Medium

$$p_2 = 0 \text{ and } r_2 \to \infty$$

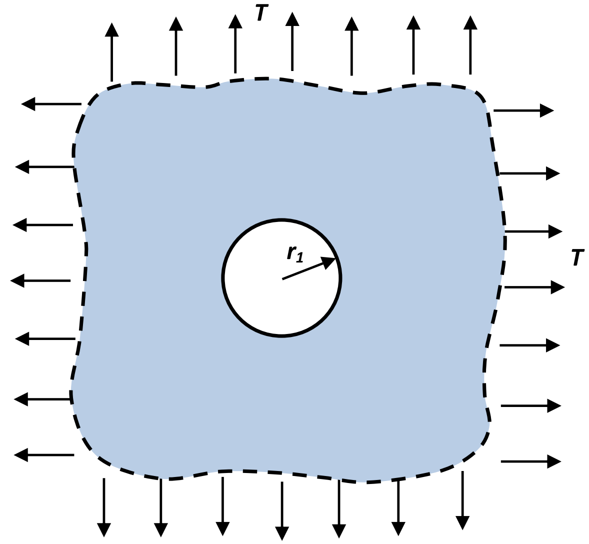


$$\sigma_r = -p_1 \frac{r_1^2}{r^2}, \ \sigma_\theta = p_1 \frac{r_1^2}{r^2}, \ \sigma_z = 0$$

$$u_r = \frac{1+\nu}{E} \frac{p_1 r_1^2}{r}$$

Stress Free Hole in an Infinite Medium Under Equal Biaxial Loading at Infinity

$$p_1 = 0, \ p_2 = -T, \ r_2 \to \infty$$

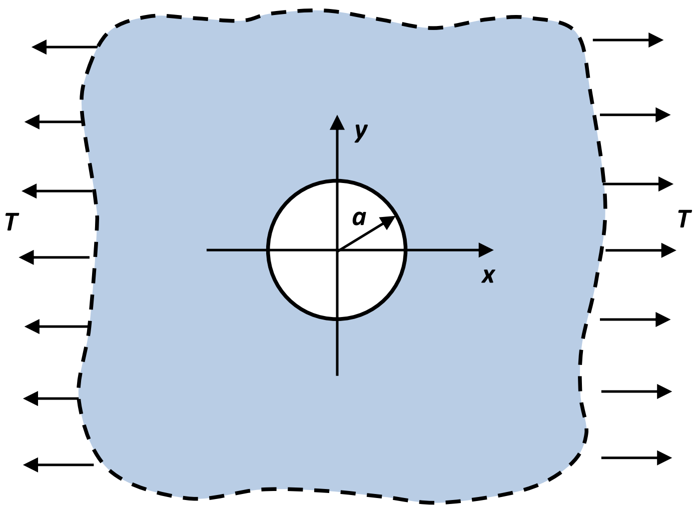


$$\sigma_r = T\left(1 - \frac{r_1^2}{r^2}\right), \ \sigma_\theta = T\left(1 + \frac{r_1^2}{r^2}\right)$$

$$\sigma_{\max} = (\sigma_\theta)_{\max} = \sigma_\theta(r_1) = 2T$$

8.6 Example Polar Coordinate Solutions

Example 8.7 Infinite Medium with a Stress Free Hole Under Uniform Far Field Loading



Boundary Conditions

$$\sigma_r(a,\theta) = \tau_{r\theta}(a,\theta) = 0$$

$$\sigma_r(\infty,\theta) = \frac{T}{2}(1+\cos 2\theta)$$

$$\sigma_\theta(\infty,\theta) = \frac{T}{2}(1-\cos 2\theta)$$

$$\tau_{r\theta}(\infty,\theta) = -\frac{T}{2}\sin 2\theta$$

Note: Far-field condition derived from the law in Exercise 3.3

Try Stress Function

$$\varphi = a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r$$
$$+ (a_{21} r^2 + a_{22} r^4 + a_{23} r^{-2} + a_{24})\cos 2\theta$$

$$\sigma_r = a_3(1+2\log r) + 2a_2 + \frac{a_1}{r^2} - (2a_{21} + \frac{6a_{23}}{r^4} + \frac{4a_{24}}{r^2})\cos 2\theta$$

$$\sigma_\theta = a_3(3+2\log r) + 2a_2 - \frac{a_1}{r^2} + (2a_{21} + 12a_{22}r^4 + \frac{6a_{23}}{r^4})\cos 2\theta$$

$$\tau_{r\theta} = (2a_{21} + 6a_{22}r^2 - \frac{6a_{23}}{r^4} - \frac{2a_{24}}{r^2})\sin 2\theta$$

For finite stresses at infinity => $a_3 = a_{22} = 0$

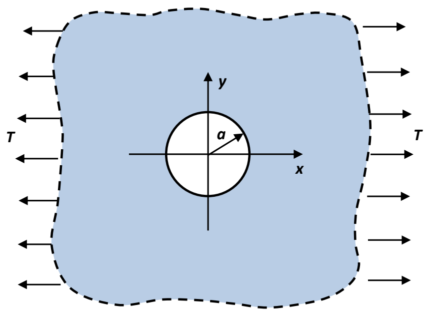
$$\sigma_r = \frac{T}{2}\left(1-\frac{a^2}{r^2}\right) + \frac{T}{2}\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta = \frac{T}{2}\left(1+\frac{a^2}{r^2}\right) - \frac{T}{2}\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = -\frac{T}{2}\left(1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right)\sin 2\theta$$

8.6 Example Polar Coordinate Solutions

Example 8.7 Stress Results

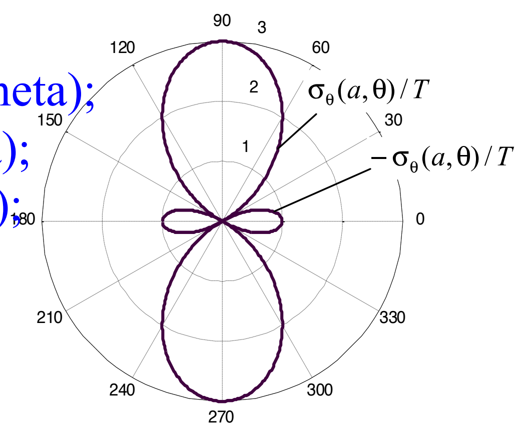


$$\sigma_r = \frac{T}{2}\left(1-\frac{a^2}{r^2}\right)+\frac{T}{2}\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos 2\theta$$

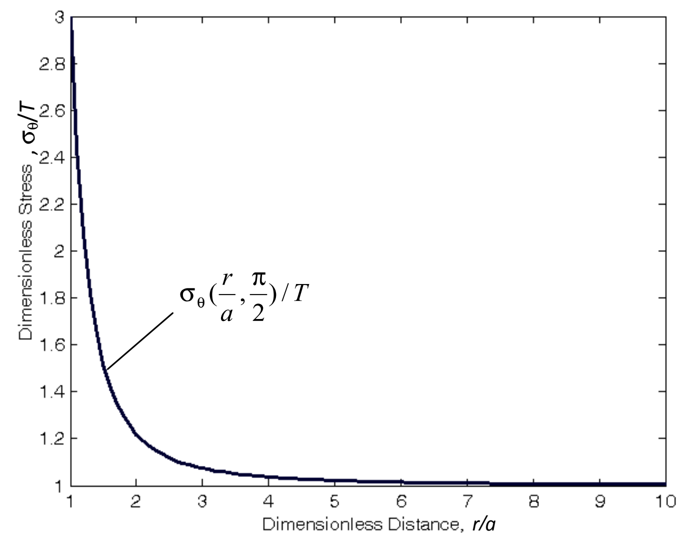
$$\sigma_\theta = \frac{T}{2}\left(1+\frac{a^2}{r^2}\right)-\frac{T}{2}\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = -\frac{T}{2}\left(1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right)\sin 2\theta$$

```
theta=0:pi/24:2*pi;
sigma=1-2*cos(2*theta);
X=sigma.*cos(theta);
Y=sigma.*sin(theta);
figure
plot(X,Y,'-r')
axis equal
```



$$\sigma_{\max} = \sigma_\theta(a, \pm\pi/2) = 3T$$



$$\sigma_\theta(a,\theta) = T(1-2\cos 2\theta)$$

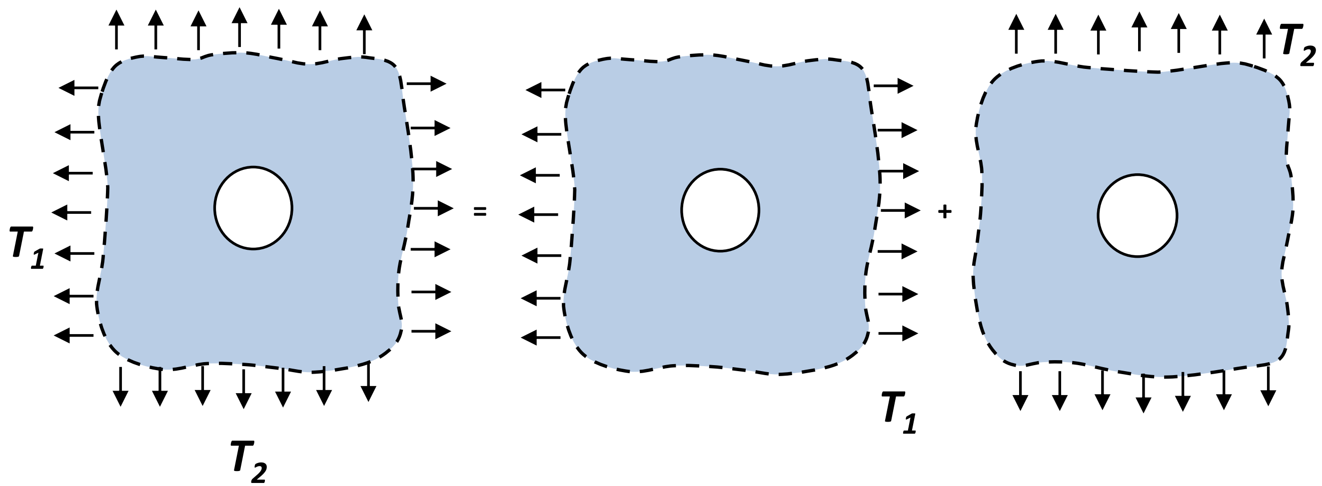
$$\sigma_\theta(a,0) = -T,\ \sigma_\theta(a,15^o) = (1-\sqrt{3})T,$$

$$\sigma_\theta(a,30^o) = 0,\ \sigma_\theta(a,45^o) = T,$$

$$\sigma_\theta(a,60^o) = 2T,\ \sigma_\theta(a,90^o) = 3T$$

8.6 Example Polar Coordinate Solutions

Superposition of Example 8.: Biaxial Loading Cases



Equal Biaxial Tension Case

$T_1 = T_2 = T$

$$\sigma_r = T\left(1 - \frac{r_1^2}{r^2}\right), \quad \sigma_\theta = T\left(1 + \frac{r_1^2}{r^2}\right)$$

$$\sigma_{\max} = (\sigma_\theta)_{\max} = \sigma_\theta(r_1) = 2T$$

Tension/Compression Case

$T_1 = T$, $T_2 = -T$

$$\sigma_r = T\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta$$

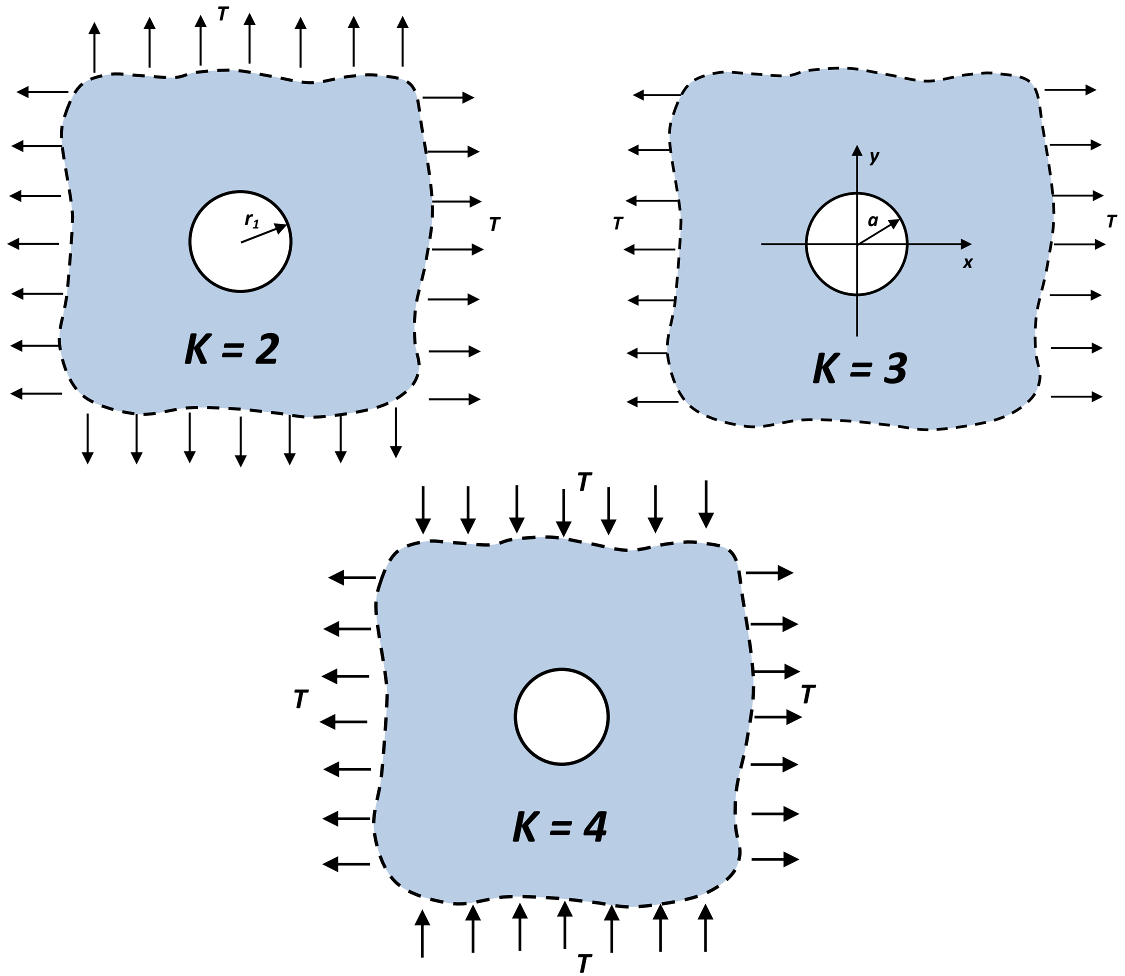
$$\sigma_\theta = -T\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = -T\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta$$

$$\sigma_\theta(a,0) = \sigma_\theta(a,\pi) = -4T \ , \ \sigma_\theta(a,\pi/2) = \sigma_\theta(a,3\pi/2) = 4T$$

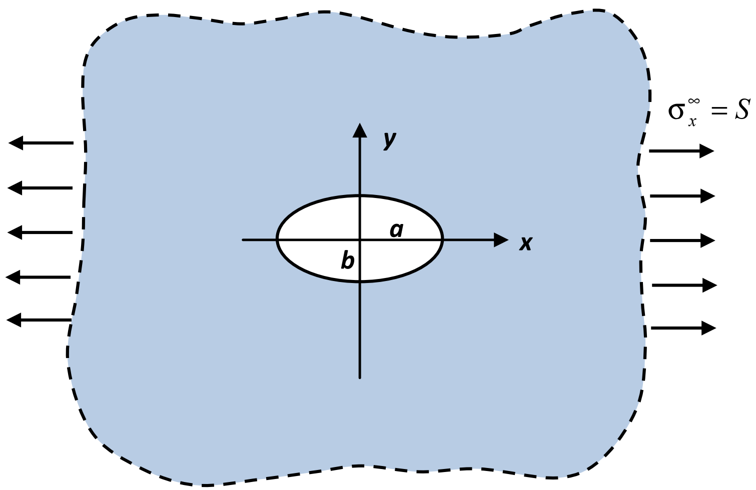
8.6 Example Polar Coordinate Solutions

Review Stress Concentration Factors Around Stress Free Holes



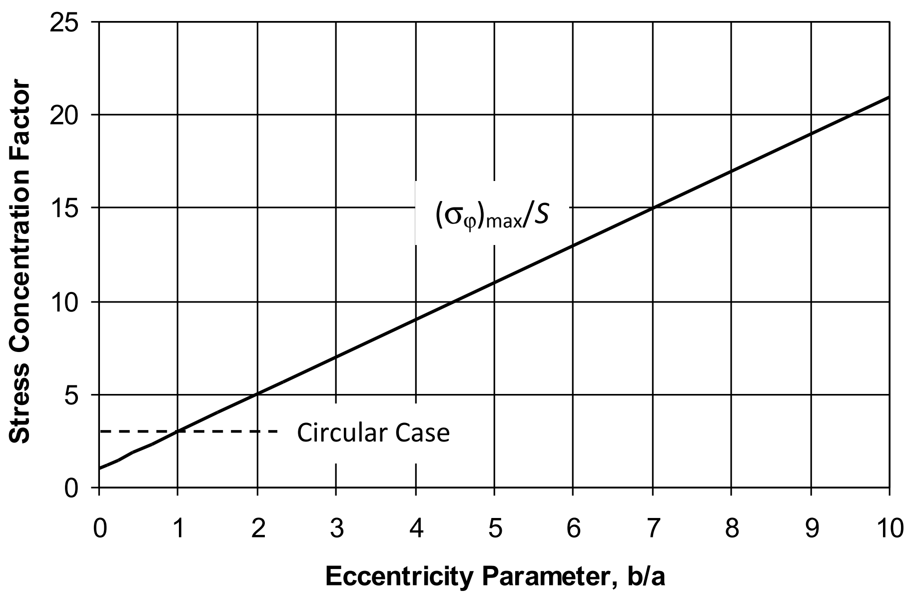
8.6 Example Polar Coordinate Solutions

Stress Concentration Around - Stress Free Elliptical Hole – Chapter 10



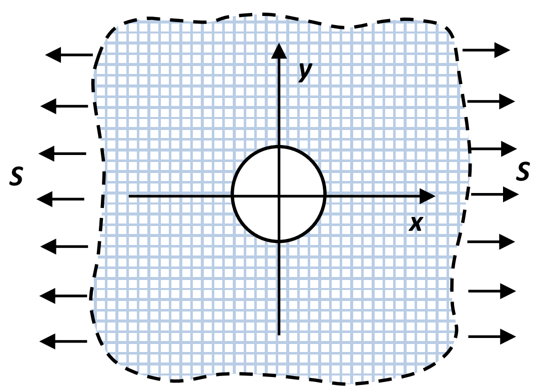
Maximum Stress Field

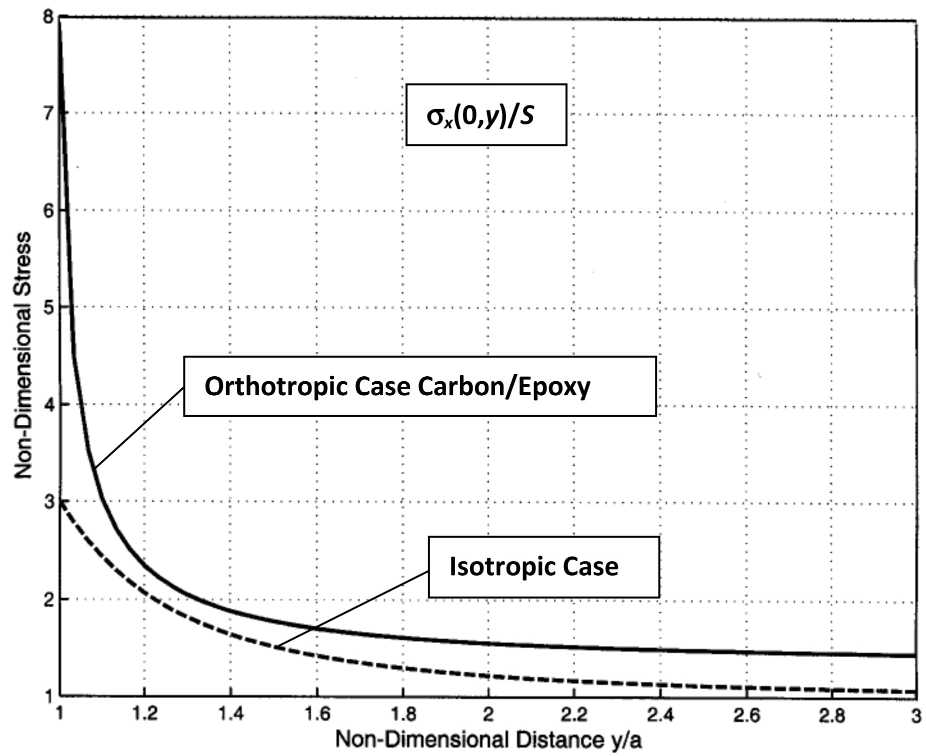
$$(\sigma_\phi)_{max} = S\left(1+2\frac{b}{a}\right)$$



8.6 Example Polar Coordinate Solutions

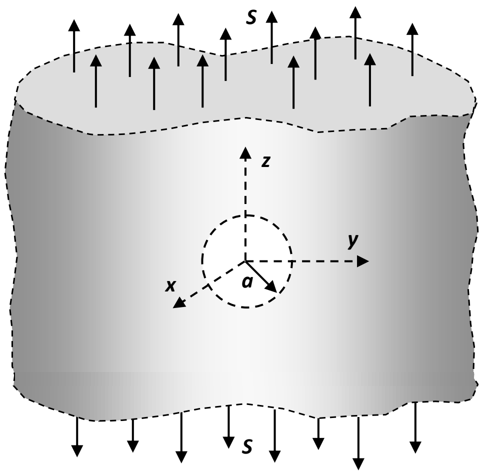
Stress Concentration Around Stress Free Hole in Orthotropic Material – Chapter 11





8.6 Example Polar Coordinate Solutions

Three Dimensional Stress Concentration Problem – Chapter 13

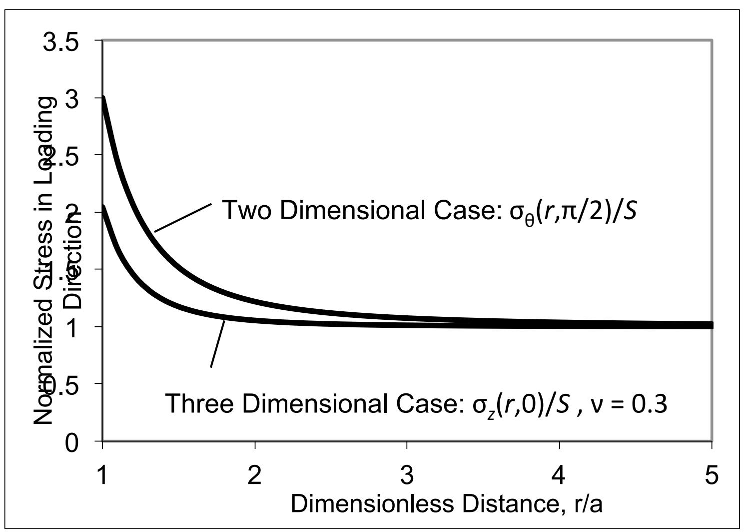


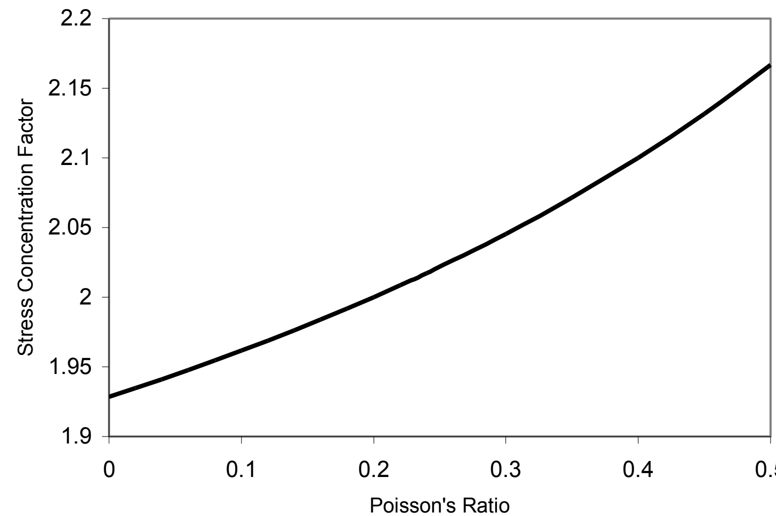
Normal Stress on the *x,y*-plane (*z* = 0)

$$\sigma_z(r,0) = S\left(1 + \frac{4-5\nu}{2(7-5\nu)}\frac{a^3}{r^3} + \frac{9}{2(7-5\nu)}\frac{a^5}{r^5}\right)$$

Stress Field

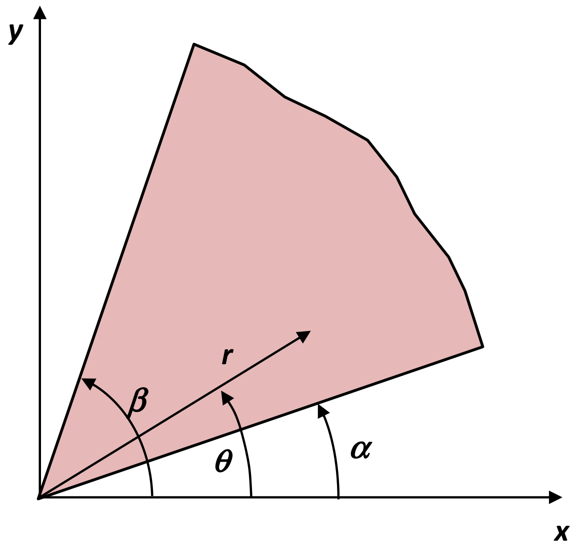
$$\sigma_z(a,0) = (\sigma_z)_{\max} = \frac{27-15\nu}{2(7-5\nu)}S \qquad \nu = 0.3 \Rightarrow \frac{(\sigma_z)_{\max}}{S} = 2.04$$





8.6 Example Polar Coordinate Solutions

Wedge Domain Problems



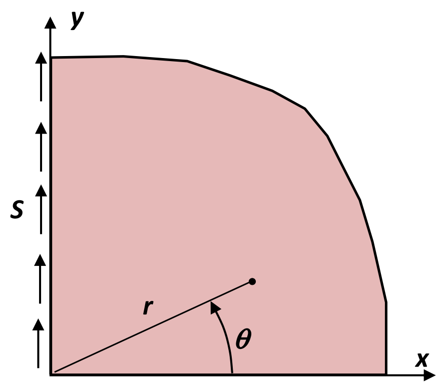
Use general stress function solution to include terms that are bounded at origin and give uniform stresses on the boundaries

$$\varphi = r^2(a_2 + a_6\theta + a_{21}\cos 2\theta + b_{21}\sin 2\theta)$$

$$\sigma_r = 2a_2 + 2a_6\theta - 2a_{21}\cos 2\theta - 2b_{21}\sin 2\theta$$
$$\sigma_\theta = 2a_2 + 2a_6\theta + 2a_{21}\cos 2\theta + 2b_{21}\sin 2\theta$$
$$\tau_{r\theta} = -a_6 - 2b_{21}\cos 2\theta + 2a_{21}\sin 2\theta$$

Quarter Plane Example ($\alpha = 0$ and $\beta = \pi/2$)

$$\sigma_\theta(r, \pi/2) = 0$$
$$\tau_{r\theta}(r, \pi/2) = S$$

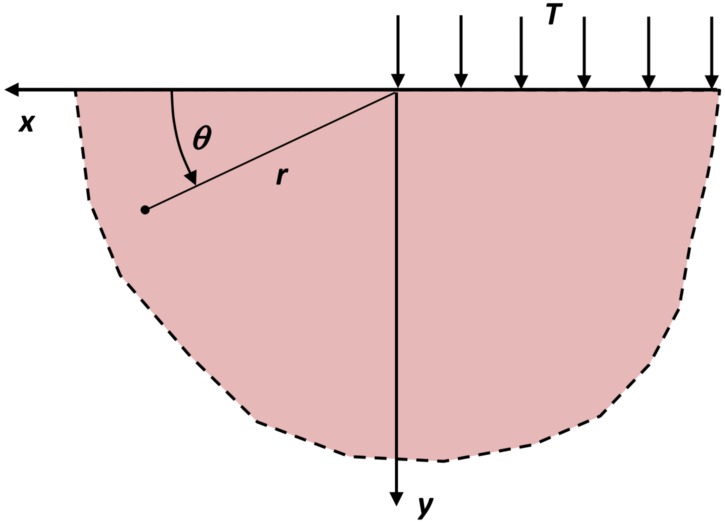


$$\sigma_\theta(r,0) = \tau_{r\theta}(r,0) = 0$$

$$\sigma_r = \frac{S}{2}\left(\frac{\pi}{2} - 2\theta + \frac{\pi}{2}\cos 2\theta - \sin 2\theta\right)$$
$$\sigma_\theta = \frac{S}{2}\left(\frac{\pi}{2} - 2\theta - \frac{\pi}{2}\cos 2\theta + \sin 2\theta\right)$$
$$\tau_{r\theta} = \frac{S}{2}\left(1 - \cos 2\theta - \frac{\pi}{2}\sin 2\theta\right)$$

8.6 Example Polar Coordinate Solutions

Example 6: Half-Space Examples Uniform Normal Stress Over x ≤ 0



Boundary Conditions

$$\sigma_\theta(r,0)=\tau_{r\theta}(r,0)=0$$
$$\tau_{r\theta}(r,\pi)=0,\ \sigma_\theta(r,\pi)=-T$$

Try Airy Stress Function

$$\boxed{\varphi=a_6 r^2\theta+b_{21}r^2\sin 2\theta}$$

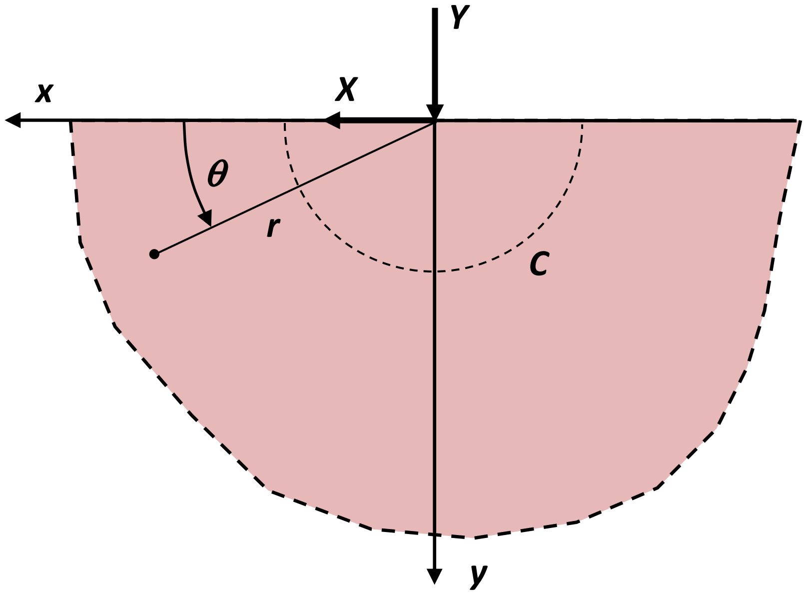
$$\sigma_\theta=2a_6\theta+2b_{21}\sin 2\theta$$
$$\tau_{r\theta}=-a_6-2b_{21}\cos 2\theta$$

Use BC's To Determine Stress Solution

$$\sigma_r=-\frac{T}{2\pi}(\sin 2\theta+2\theta)$$
$$\sigma_\theta=\frac{T}{2\pi}(\sin 2\theta-2\theta)$$
$$\tau_{r\theta}=\frac{T}{2\pi}(1-\cos 2\theta)$$

8.6 Example Polar Coordinate Solutions

Example 7: Half-Space Under Concentrated Surface Force System (Flamant Problem)



Boundary Conditions

$$\sigma_\theta(r,0)=\tau_{r\theta}(r,0)=0$$

$$\tau_{r\theta}(r,\pi)=0,\ \sigma_\theta(r,\pi)=0$$

$$\int_C \textbf{Forces} = -(X\mathbf{e}_1+Y\mathbf{e}_2)$$

Try Airy Stress Function

$$\boxed{\begin{aligned}\varphi &= (a_{12} r\log r + a_{15} r\theta)\cos\theta \\ &\quad + (b_{12} r\log r + b_{15} r\theta)\sin\theta\end{aligned}}$$

The tractions on any semicircular arc C enclosing the origin must balance the applied concentrated loadings. Because the area of such an arc is proportional to the radius r, the stresses must be of order 1/r to allow such an equilibrium statement to hold on any radius. The appropriate terms in the general Michell solution (8.3.6) that will give stresses of order 1/r are specified by

Use BC's To Determine Stress Solution

$$\sigma_r = -\frac{2}{\pi r}[X\cos\theta + Y\sin\theta]$$

$$\sigma_\theta = \tau_{r\theta} = 0$$

8.6 Example Polar Coordinate Solutions

Example 8: Flamant Solution Stress Results Normal Force Case

$$\sigma_r = -\frac{2Y}{\pi r}\sin\theta$$

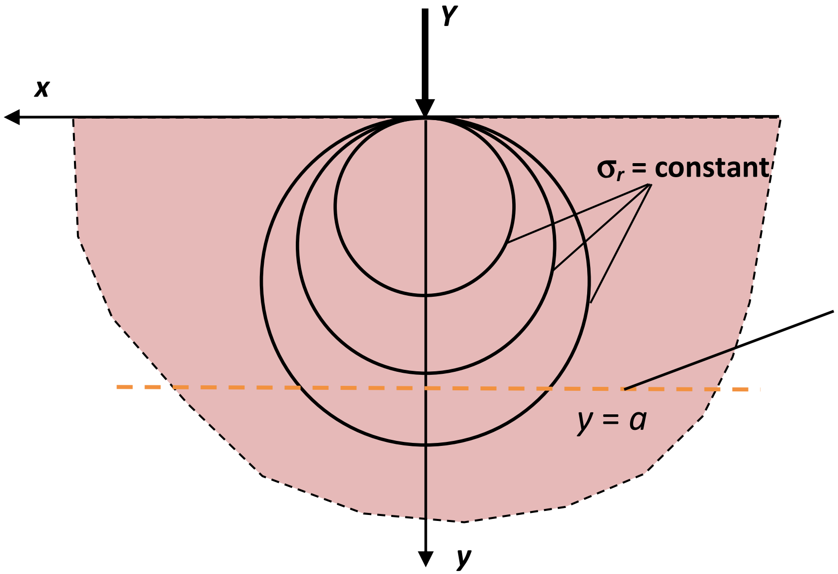
$$\sigma_\theta = \tau_{r\theta} = 0$$

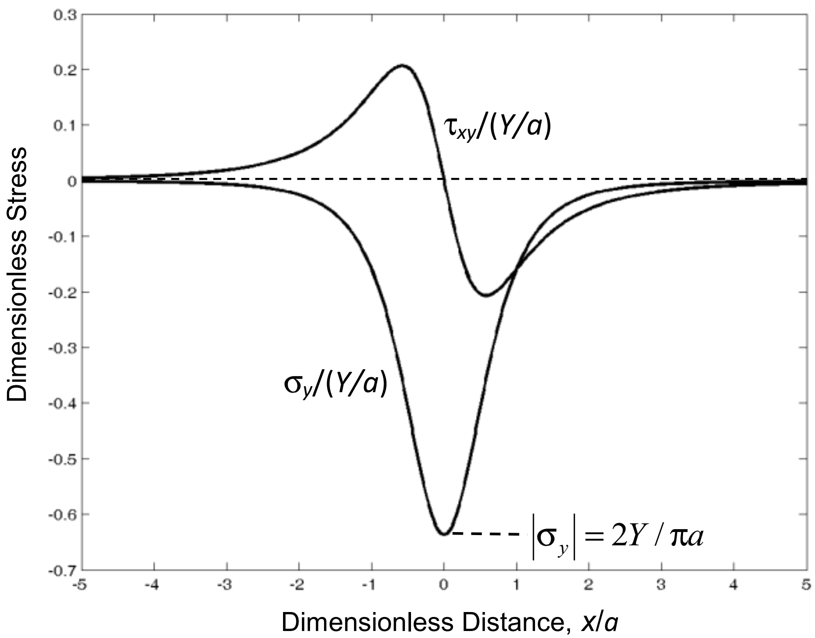
or in Cartesian components

$$\sigma_x = \sigma_r \cos^2\theta = -\frac{2Yx^2 y}{\pi(x^2+y^2)^2}$$

$$\sigma_y = \sigma_r \sin^2\theta = -\frac{2Yy^3}{\pi(x^2+y^2)^2}$$

$$\tau_{xy} = \sigma_r \sin\theta\cos\theta = -\frac{2Yxy^2}{\pi(x^2+y^2)^2}$$





8.6 Example Polar Coordinate Solutions

Example 8: Flamant Solution Stress Results Normal Force Case

$$\varepsilon_r = \frac{\partial u_r}{\partial r} = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) = -\frac{2Y}{\pi E r}\sin\theta$$

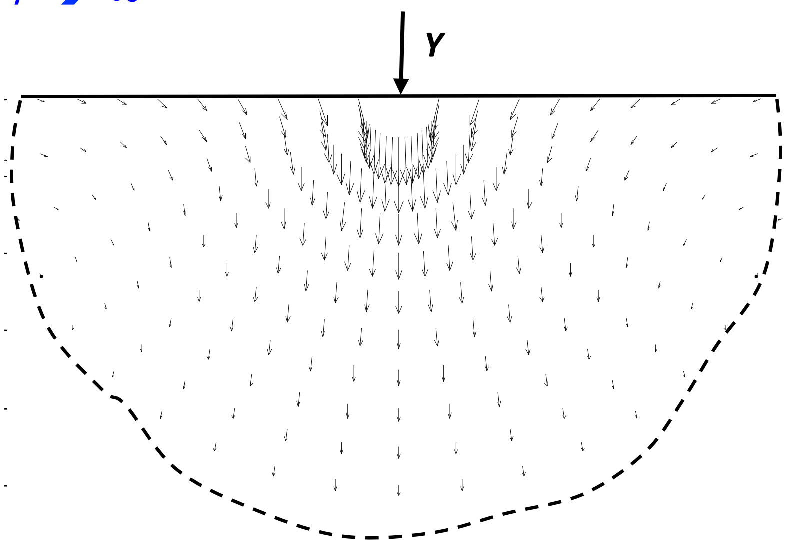
$$\varepsilon_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) = \frac{2\nu Y}{\pi E r}\sin\theta$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} = \frac{1}{\mu}\tau_{r\theta} = 0$$

$$u_r = \frac{Y}{\pi E}[(1-\nu)(\theta - \frac{\pi}{2})\cos\theta - 2\log r \sin\theta]$$

$$u_\theta = \frac{Y}{\pi E}[-(1-\nu)(\theta - \frac{\pi}{2})\sin\theta - 2\log r \cos\theta - (1+\nu)\cos\theta]$$

Note unpleasant feature of 2-D model that displacements become unbounded as $r \rightarrow \infty$



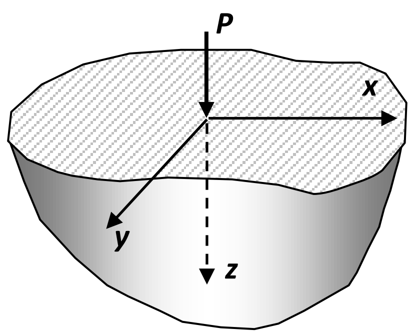
<u>On Free Surface $y = 0$</u>

$$u_r(r,0) = u_r(r,\pi) = -\frac{Y}{2E}(1-\nu)$$

$$u_\theta(r,0) = -u_\theta(r,\pi) = -\frac{Y}{\pi E}[(1+\nu) + 2\log r]$$

8.6 Example Polar Coordinate Solutions

Comparison of Flamant Results with 3-D Theory-Boussinesq's Problem



Cartesian Solution

$$u=\frac{Px}{4\pi\mu R}\left(\frac{z}{R^2}-\frac{1-2\nu}{R+z}\right),\ v=\frac{Py}{4\pi\mu R}\left(\frac{z}{R^2}-\frac{1-2\nu}{R+z}\right),\ w=\frac{P}{4\pi\mu R}\left(2(1-\nu)+\frac{z^2}{R^2}\right)$$

$$\sigma_x=-\frac{P}{2\pi R^2}\left[\frac{3x^2z}{R^3}-(1-2\nu)\left(\frac{z}{R}-\frac{R}{R+z}+\frac{x^2(2R+z)}{R(R+z)^2}\right)\right]$$

$$\sigma_y=-\frac{P}{2\pi R^2}\left[\frac{3y^2z}{R^3}-(1-2\nu)\left(\frac{z}{R}-\frac{R}{R+z}+\frac{y^2(2R+z)}{R(R+z)^2}\right)\right]$$

$$\sigma_z=-\frac{3Pz^3}{2\pi R^5},\ \tau_{xy}=-\frac{P}{2\pi R^2}\left[\frac{3xyz}{R^3}-\frac{(1-2\nu)(2R+z)xy}{R(R+z)^2}\right]$$

$$\tau_{yz}=-\frac{3Pyz^2}{2\pi R^5},\ \tau_{xz}=-\frac{3Pxz^2}{2\pi R^5}$$

Free Surface Displacements

$$u_z(R,0)=\frac{P(1-\nu)}{2\pi\mu R}$$

Corresponding 2-D Results

$$u_\theta(r,0)=-\frac{P}{\pi E}\left[(1+\nu)+2\log r\right]$$

3-D Solution eliminates the unbounded far-field behavior

Cylindrical Solution

$$u_r=\frac{P}{4\pi\mu R}\left[\frac{rz}{R^2}-\frac{(1-2\nu)r}{R+z}\right]$$

$$u_z=\frac{P}{4\pi\mu R}\left[2(1-\nu)+\frac{z^2}{R^2}\right]$$

$$u_\theta=0$$

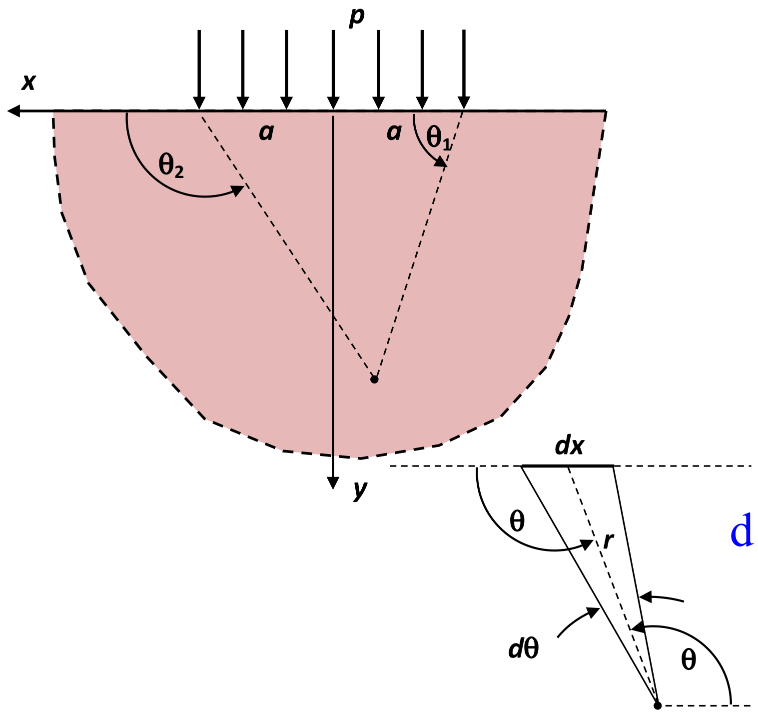
$$\sigma_r=\frac{P}{2\pi R^2}\left[-\frac{3r^2z}{R^3}+\frac{(1-2\nu)R}{R+z}\right]$$

$$\sigma_\theta=\frac{(1-2\nu)P}{2\pi R^2}\left[\frac{z}{R}-\frac{R}{R+z}\right]$$

$$\sigma_z=-\frac{3Pz^3}{2\pi R^5},\ \tau_{rz}=-\frac{3Prz^2}{2\pi R^5}$$

8.6 Example Polar Coordinate Solutions

Example 9: Half-Space Under Uniform Normal Loading $a \le x \le - a$



$$\sigma_x = \sigma_r \cos^2\theta = -\frac{2Y}{\pi r}\sin\theta\cos^2\theta$$

$$\sigma_y = \sigma_r \sin^2\theta = -\frac{2Y}{\pi r}\sin^3\theta$$

$$\tau_{xy} = \sigma_r \sin\theta\cos\theta = -\frac{2Y}{\pi r}\sin^2\theta\cos\theta$$

$$\mathrm{d}Y = p\mathrm{d}x = pr\mathrm{d}\theta / \sin\theta$$

$$\mathrm{d}\sigma_x = -\frac{2p}{\pi}\cos^2\theta\,\mathrm{d}\theta$$

$$\mathrm{d}\sigma_y = -\frac{2p}{\pi}\sin^2\theta\,\mathrm{d}\theta$$

$$\mathrm{d}\tau_{xy} = -\frac{2p}{\pi}\sin\theta\cos\theta\,\mathrm{d}\theta$$

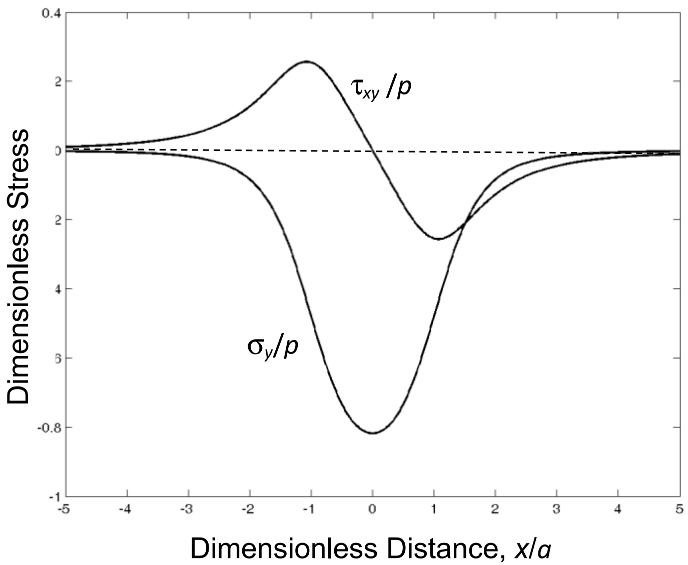
$$\sigma_x = -\frac{2p}{\pi}\int_{\theta_1}^{\theta_2}\cos^2\theta\, d\theta = -\frac{p}{2\pi}[2(\theta_2-\theta_1)+(\sin 2\theta_2 - \sin 2\theta_1)]$$

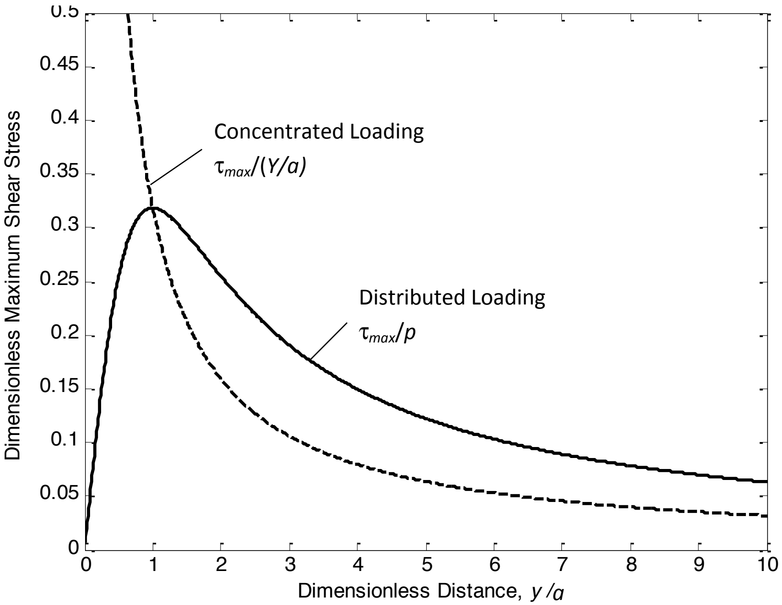
$$\sigma_y = -\frac{2p}{\pi}\int_{\theta_1}^{\theta_2}\sin^2\theta\, d\theta = -\frac{p}{2\pi}[2(\theta_2-\theta_1)-(\sin 2\theta_2 - \sin 2\theta_1)]$$

$$\tau_{xy} = -\frac{2p}{\pi}\int_{\theta_1}^{\theta_2}\sin\theta\cos\theta\, d\theta = \frac{p}{2\pi}[\cos 2\theta_2 - \cos 2\theta_1]$$

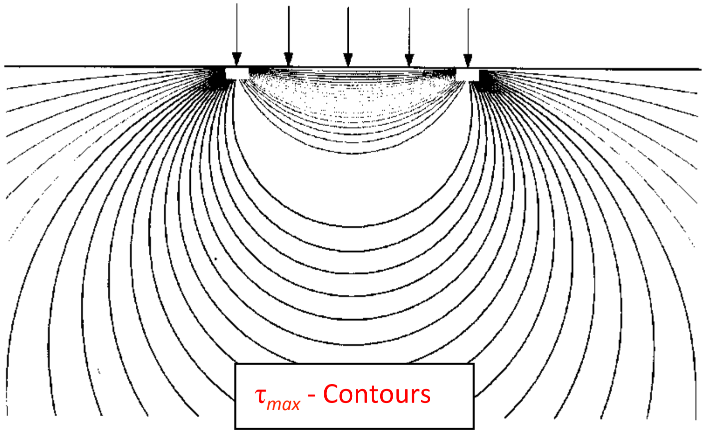
8.6 Example Polar Coordinate Solutions

Example 9: Half-Space Under Uniform Normal Loading $a \leq x \leq - a$





Along y-axis below the loading, $\tau_{xy} = 0$, and the x- and y-axes are principal at these points and the maximum shear stress is given by $\tau_{max} = \frac{1}{2}|\sigma_x - \sigma_y|$



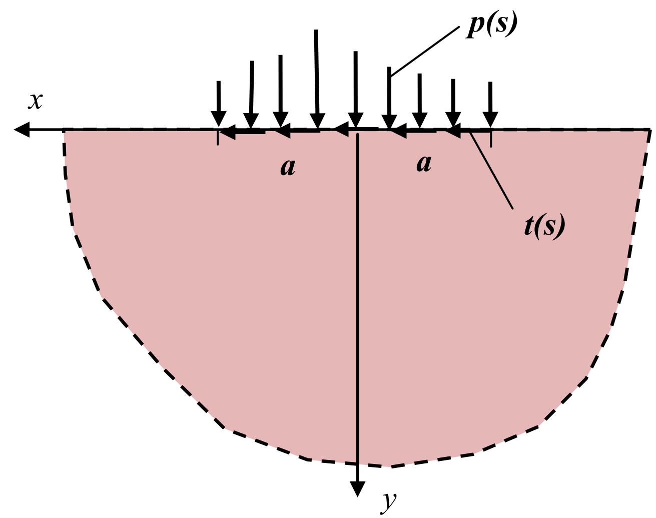
A plot of this maximum shear stress versus depth below the surface is shown in Figure 8-27.

8.6 Example Polar Coordinate Solutions

Example 9: Half-Space Under Uniform Normal Loading $a \le x \le - a$

Generalized Superposition Method

Half-Space Loading Problems



$$\sigma_x = -\frac{2y}{\pi}\int_{-a}^{a}\frac{p(s)(x-s)^2}{[(x-s)^2+y^2]^2}ds - \frac{2}{\pi}\int_{-a}^{a}\frac{t(s)(x-s)^3}{[(x-s)^2+y^2]^2}ds$$

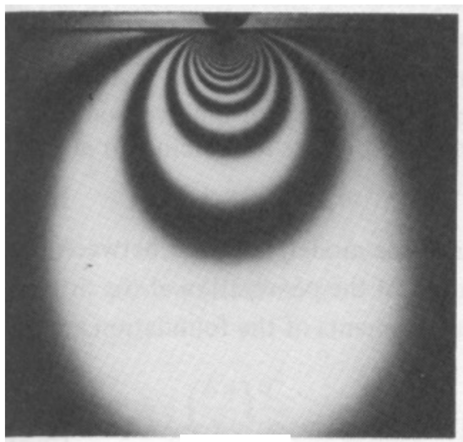
$$\sigma_y = -\frac{2y^3}{\pi}\int_{-a}^{a}\frac{p(s)}{[(x-s)^2+y^2]^2}ds - \frac{2y^2}{\pi}\int_{-a}^{a}\frac{t(s)(x-s)}{[(x-s)^2+y^2]^2}ds$$

$$\tau_{xy} = -\frac{2y^2}{\pi}\int_{-a}^{a}\frac{p(s)(x-s)^2}{[(x-s)^2+y^2]^2}s - \frac{2y}{\pi}\int_{-a}^{a}\frac{t(s)(x-s)^2}{[(x-s)^2+y^2]^2}ds$$

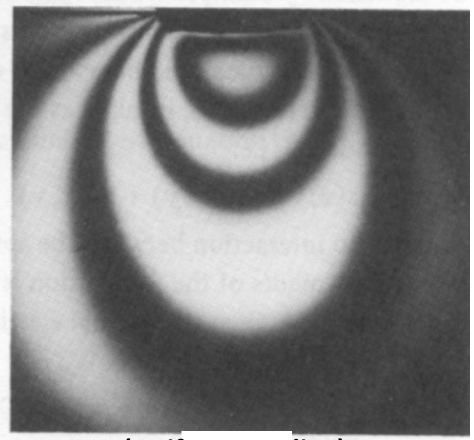
8.6 Example Polar Coordinate Solutions

Example 9: Half-Space Under Uniform Normal Loading $a \leq x \leq - a$

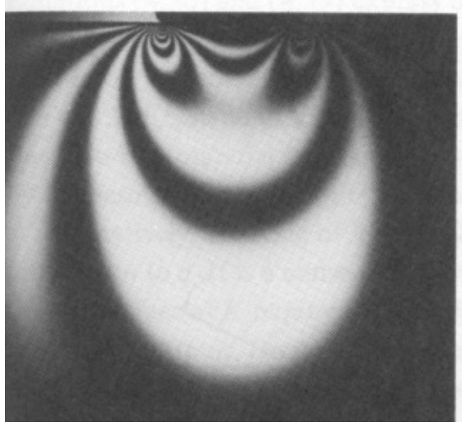
Photoelastic Contact Stress Fields



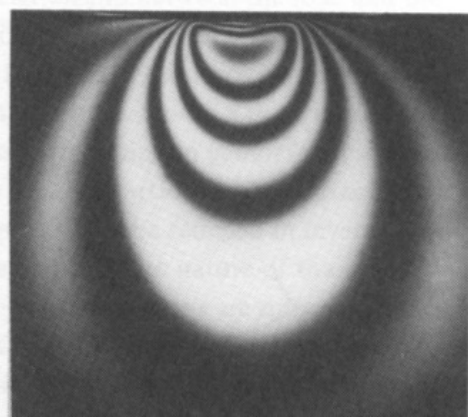
(Point Loading)



(Uniform Loading)



(Flat Punch Loading)

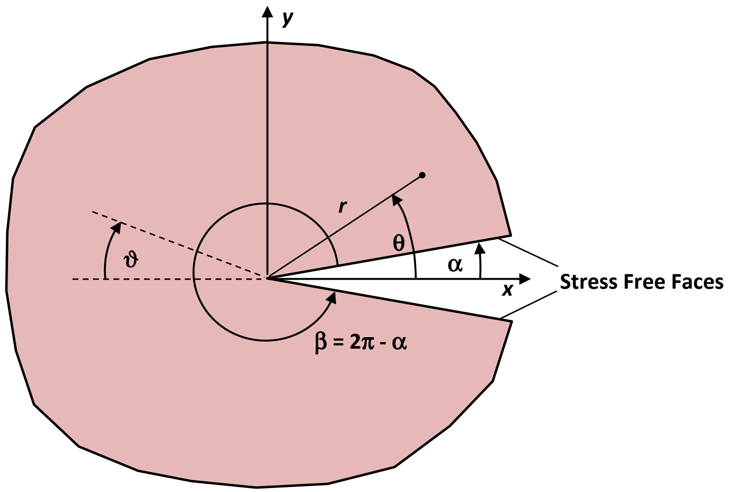


(Cylinder Contact Loading)

8.6 Example Polar Coordinate Solutions

Example 10: Notch/Crack Problems

- Consider the wedge problem for the case where angle α is small and β is 2π-α.
- We pursue the case where $\alpha \approx 0$, and the notch becomes a crack.
- The boundary surfaces of the notch are taken to be stress free, and thus the problem involves only far-field loadings.
- Start with Michell solution, we try the stress Function in generalized form:



$$\varphi = r^{\lambda}[A\sin\lambda\theta + B\cos\lambda\theta + C\sin(\lambda-2)\theta + D\cos(\lambda-2)\theta]$$ where λ is allowed a non-integer

$$\sigma_\theta = \lambda(\lambda-1)r^{\lambda-2}[A\sin\lambda\theta + B\cos\lambda\theta + C\sin(\lambda-2)\theta + D\cos(\lambda-2)\theta]$$
$$\tau_{r\theta} = -(\lambda-1)r^{\lambda-2}[A\lambda\cos\lambda\theta - B\lambda\sin\lambda\theta + C(\lambda-2)\cos(\lambda-2)\theta - D(\lambda-2)\sin(\lambda-2)\theta]$$

Boundary Conditions: $\sigma_\theta(r,0) = \tau_{r\theta}(r,0) = \sigma_\theta(r,2\pi) = \tau_{r\theta}(r,2\pi) = 0 \Rightarrow$

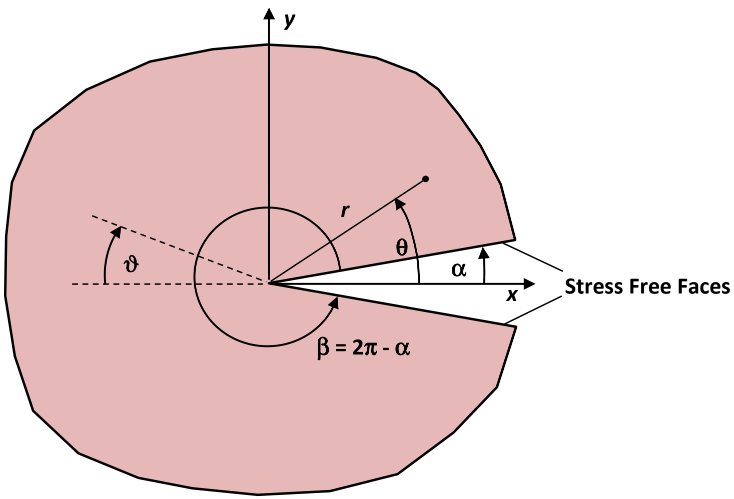
$$\sin 2\pi(\lambda-1) = 0 \Rightarrow \lambda = \frac{n}{2} + 1\,,\ n = 0,1,2,\cdots$$

At Crack Tip r ➔ 0: Stress $= O(r^{\lambda-2})$, Displacement $= O(r^{\lambda-1})$

Finite Displacements and Singular Stresses at Crack Tip ➔ $1 < \lambda < 2$ ➔ $\lambda = 3/2$

8.6 Example Polar Coordinate Solutions

Example 10: Notch/Crack Problems



$$\sigma_r = -\frac{3}{4}\frac{1}{\sqrt{r}}\left[A(\sin\frac{3}{2}\theta + 5\sin\frac{\theta}{2}) + B(\cos\frac{3}{2}\theta + \frac{5}{3}\cos\frac{\theta}{2})\right]$$

$$\sigma_\theta = \frac{3}{4}\frac{1}{\sqrt{r}}\left[A(\sin\frac{3}{2}\theta - 3\sin\frac{\theta}{2}) + B(\cos\frac{3}{2}\theta - \cos\frac{\theta}{2})\right]$$

$$\tau_{r\theta} = -\frac{3}{4}\frac{1}{\sqrt{r}}\left[A(\cos\frac{3}{2}\theta - \cos\frac{\theta}{2}) - B(\sin\frac{3}{2}\theta - \frac{1}{3}\sin\frac{\theta}{2})\right]$$

Transform to υ Variable

$$\sigma_r = -\frac{3}{2}\frac{A}{\sqrt{r}}\cos\frac{\vartheta}{2}(3-\cos\vartheta) - \frac{B}{2\sqrt{r}}\sin\frac{\vartheta}{2}(1-3\cos\vartheta)$$

$$\sigma_\theta = -\frac{3}{2}\frac{A}{\sqrt{r}}\cos\frac{\vartheta}{2}(1+\cos\vartheta) + \frac{3B}{2\sqrt{r}}\sin\frac{\vartheta}{2}(1+\cos\vartheta)$$

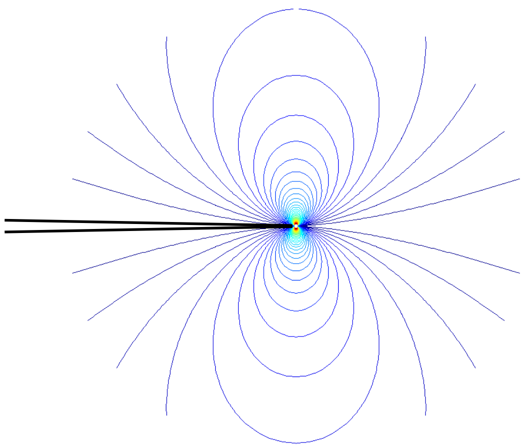
$$\tau_{r\theta} = -\frac{3}{2}\frac{A}{\sqrt{r}}\sin\frac{\vartheta}{2}(1+\cos\vartheta) + \frac{B}{2\sqrt{r}}\cos\frac{\vartheta}{2}(1-3\cos\vartheta)$$

- Note special singular behavior of stress field $O(1/\sqrt{r})$
- *A* and *B* coefficients are related to *stress intensity factors* and are useful in fracture mechanics theory
- *A* terms give symmetric stress fields – Opening or Mode I behavior
- *B* terms give anti-symmetric stress fields – Shearing or Mode II behavior

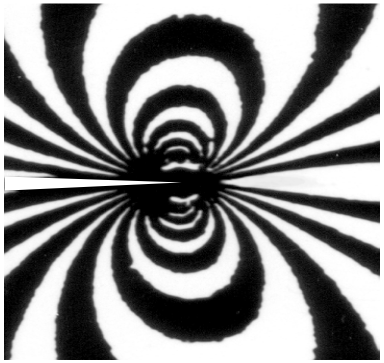
8.6 Example Polar Coordinate Solutions

Example 10: Notch/Crack Problems

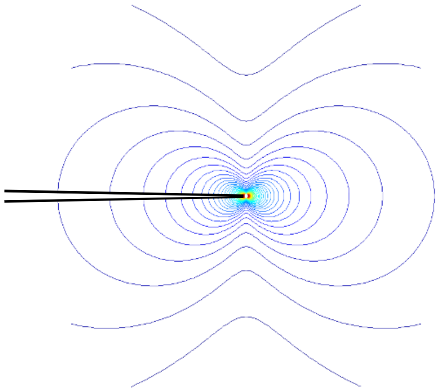
Crack Problem Results - Contours of Maximum Shear Stress



Mode I (Maximum shear stress contours)



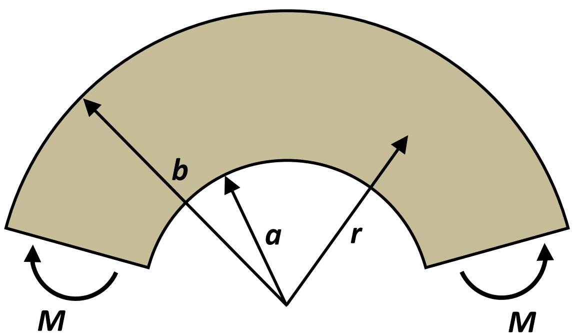
Experimental Photoelastic Isochromatics Courtesy of URI Dynamic Photomechanics Laboratory

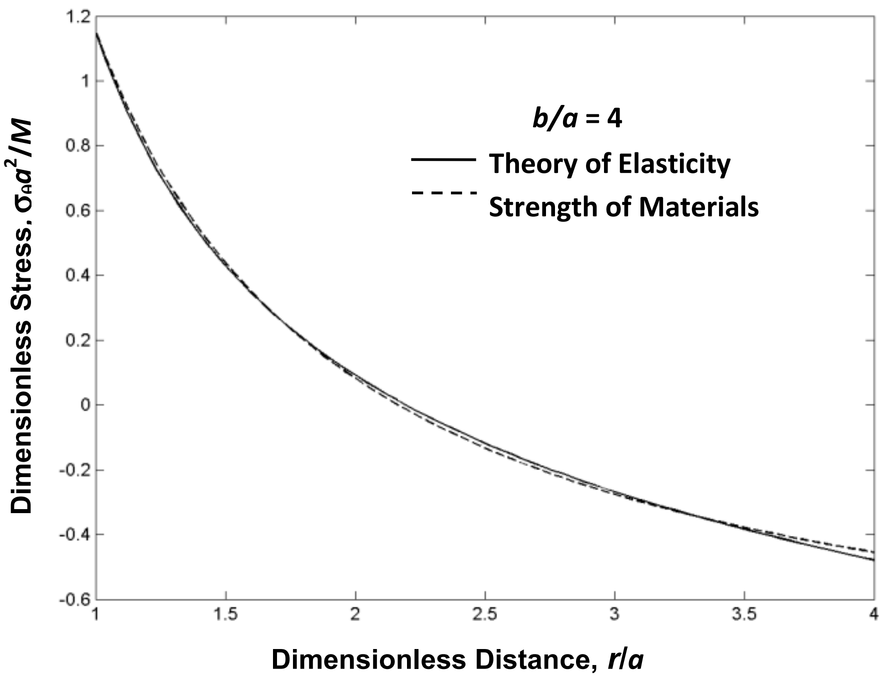


Mode II (Maximum shear stress contours)

8.6 Example Polar Coordinate Solutions

Example 11: Curved Beam Under End Moments





$$\sigma_r(a) = \sigma_r(b) = 0$$

$$\tau_{r\theta}(a) = \tau_{r\theta}(b) = 0$$

$$\int_a^b \sigma_\theta dr = 0$$

$$\int_a^b \sigma_\theta r dr = -M$$

$$\varphi = a_0 + a_1 \log r + a_2 r^2 + a_3 r^2 \log r \Rightarrow$$

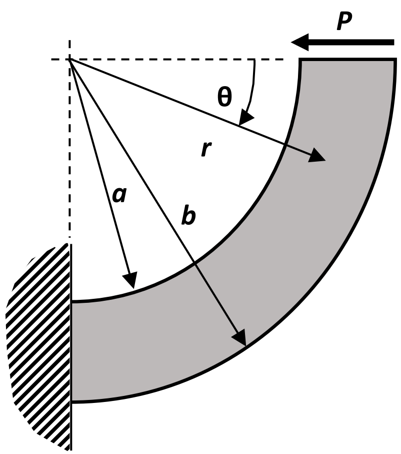
$$\sigma_r = -\frac{4M}{N}\left[\frac{a^2b^2}{r^2}\log\left(\frac{b}{a}\right) + b^2\log\left(\frac{r}{b}\right) + a^2\log\left(\frac{a}{r}\right)\right]$$

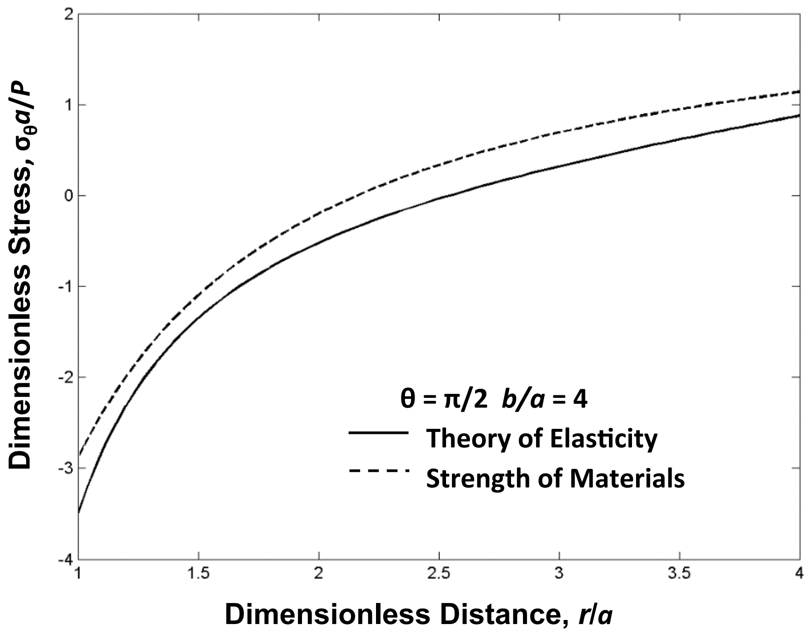
$$\sigma_\theta = -\frac{4M}{N}\left[-\frac{a^2b^2}{r^2}\log\left(\frac{b}{a}\right) + b^2\log\left(\frac{r}{b}\right) + a^2\log\left(\frac{a}{r}\right) + b^2 - a^2\right]$$

$$\tau_{r\theta} = 0$$

8.6 Example Polar Coordinate Solutions

Example 12: Curved Cantilever Under End Loading





$$\sigma_r(a,\theta)=\sigma_r(b,\theta)=0$$

$$\tau_{r\theta}(a,\theta)=\tau_{r\theta}(b,\theta)=0$$

$$\int_a^b \tau_{r\theta}(r,0)dr=P$$

$$\int_a^b \sigma_\theta(r,0)dr=\int_a^b \sigma_\theta(r,0)rdr=0$$

$$\int_a^b \sigma_\theta(r,\pi/2)dr=-P$$

$$\int_a^b \sigma_\theta(r,\pi/2)rdr=P(a+b)/2$$

$$\int_a^b \tau_{r\theta}(r,\pi/2)dr=0$$

$$\varphi=(Ar^3+\frac{B}{r}+Cr+Dr\log r)\sin\theta$$

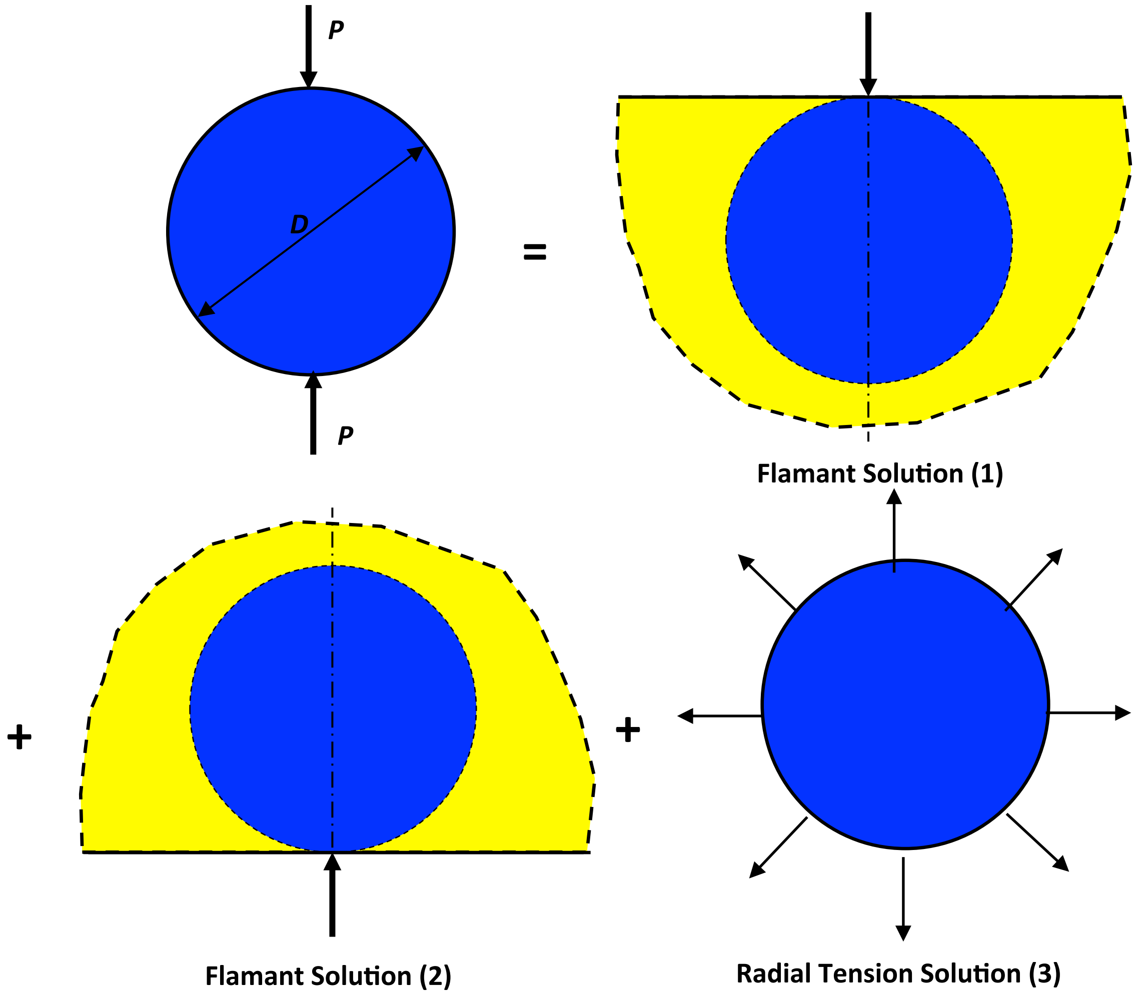
$$\sigma_r=\frac{P}{N}(r+\frac{a^2b^2}{r^3}-\frac{a^2+b^2}{r})\sin\theta$$

$$\sigma_\theta=\frac{P}{N}(3r-\frac{a^2b^2}{r^3}-\frac{a^2+b^2}{r})\sin\theta$$

$$\tau_{r\theta}=-\frac{P}{N}(r+\frac{a^2b^2}{r^3}-\frac{a^2+b^2}{r})\cos\theta$$

8.6 Example Polar Coordinate Solutions

Example 13: Disk Under Diametrical Compression



8.6 Example Polar Coordinate Solutions

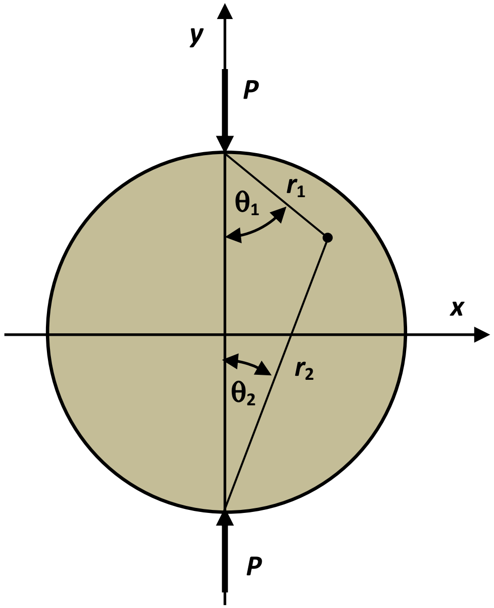
Example 13: Disk Under Diametrical Compression

Superposition of Stresses

$$\sigma_x^{(1)} = -\frac{2P}{\pi r_1}\cos\theta_1 \sin^2\theta_1$$

$$\sigma_y^{(1)} = -\frac{2P}{\pi r_1}\cos^3\theta_1$$

$$\tau_{xy}^{(1)} = -\frac{2P}{\pi r_1}\cos^2\theta_1 \sin\theta_1$$



$$\sigma_x^{(2)} = -\frac{2P}{\pi r_2}\cos\theta_2 \sin^2\theta_2$$

$$\sigma_y^{(2)} = -\frac{2P}{\pi r_2}\cos^3\theta_2$$

$$\tau_{xy}^{(2)} = -\frac{2P}{\pi r_2}\cos^2\theta_2 \sin\theta_2$$

$$\sigma_x^{(3)} = \sigma_y^{(3)} = \frac{2P}{\pi D}\ ,\ \tau_{xy}^{(3)} = 0$$

$$\Rightarrow$$

$$\sigma_x = -\frac{2P}{\pi}\left[\frac{(R-y)x^2}{r_1^4} + \frac{(R+y)x^2}{r_2^4} - \frac{1}{D}\right]$$

$$\sigma_y = -\frac{2P}{\pi}\left[\frac{(R-y)^3}{r_1^4} + \frac{(R+y)^3}{r_2^4} - \frac{1}{D}\right]$$

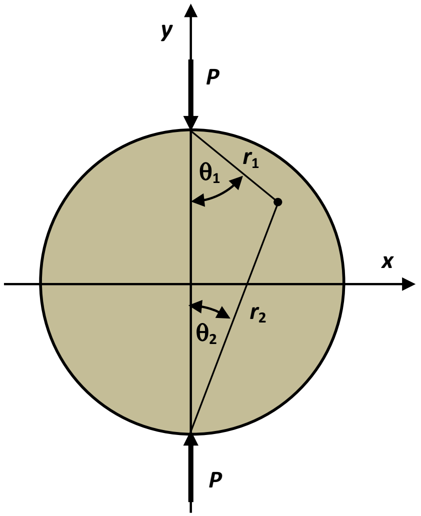
$$\tau_{xy} = \frac{2P}{\pi}\left[\frac{(R-y)^2 x}{r_1^4} - \frac{(R+y)^2 x}{r_2^4}\right]$$

$$r_{1,2} = \sqrt{x^2 + (R \mp y)^2}$$

8.6 Example Polar Coordinate Solutions

Example 13: Disk Under Diametrical Compression

Superposition of Stresses



x-axis ($y = 0$)

$$\sigma_x(x,0) = \frac{2P}{\pi D}\left[\frac{D^2 - 4x^2}{D^2 + 4x^2}\right]^2$$

$$\sigma_y(x,0) = -\frac{2P}{\pi D}\left[\frac{4D^4}{(D^2 + 4x^2)^2} - 1\right]$$

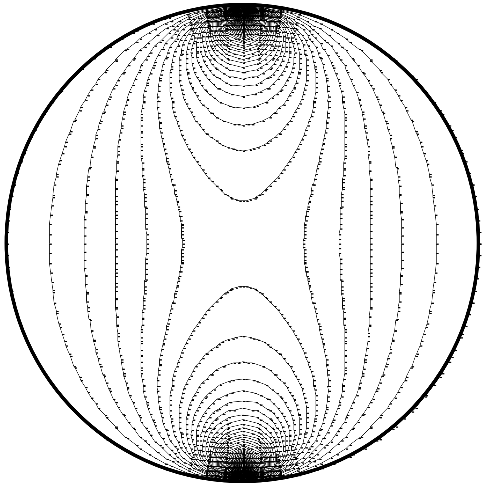
$$\tau_{xy}(x,0) = 0$$

y-axis ($x = 0$)

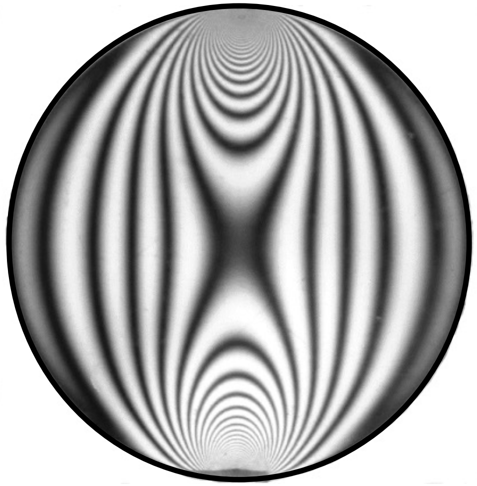
$$\sigma_x(0,y) = \frac{2P}{\pi D} = \text{Constant}$$

$$\sigma_y(0,y) = -\frac{2P}{\pi}\left[\frac{2}{D-2y} + \frac{2}{D+2y} - \frac{1}{D}\right]$$

$$\tau_{xy}(0,y) = 0$$



(Theoretical τ_{max} Contours)



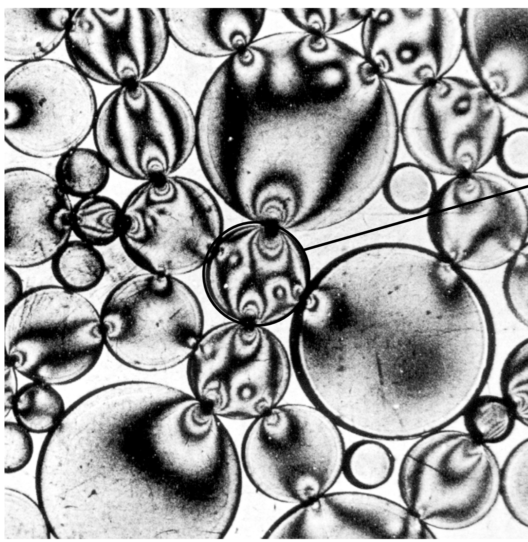
(Photoelastic Contours)
(Courtesy of URI Dynamic Photomechanics Lab)

8.6 Example Polar Coordinate Solutions

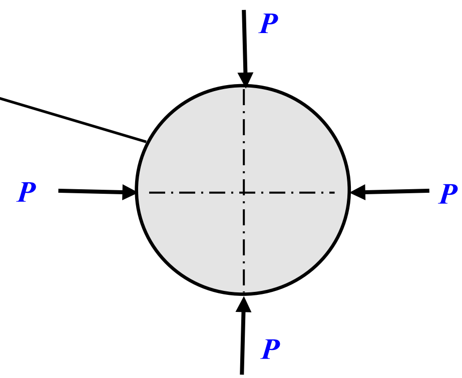
Example 13: Disk Under Diametrical Compression

Applications to Granular Media Modeling

Contact Load Transfer Between Idealized Grains



(Courtesy of URI Dynamic Photomechanics Lab)



Four-Contact Grain

Thank you for your attention!