Lecture 1

$L^p(\Omega)$ SPACES

$L^p(\Omega)$ SPACES

Theorem (Lebesgue measure) There exists a positive measure m defined on a σ- algebra $\mathfrak{M}$ in $\mathbb{R}^n$, with the following properties:

(a) $m((a_1, b_1)\times\ldots\times(a_k, b_n)) = (a_1 - b_1)\times\ldots\times(a_k - b_n)$

(b) $\mathfrak{M}$ contains all open sets and closed sets in $\mathbb{R}^n$; more precisely, $E \in \mathfrak{M}$ if and only if there are a sequence of closed sets $\{A_k\}$ and a sequence of open subsets $\{B_k\}$ in $\mathbb{R}^n$ such that

$$\bigcup_{k=1}^{\infty} A_k \subset E \subset \bigcap_{k=1}^{\infty} B_k \quad \text{and} \quad m\Big(\bigcap_{k=1}^{\infty} B_k \setminus \bigcup_{k=1}^{\infty} A_k\Big) = 0$$

(c) m is translation-invariant, i.e., $m(E + x) = m(E)$ for every E in $\mathfrak{M}$ and every x in $\mathbb{R}^n$.

(d) If E is in $\mathfrak{M}$ and c is a positive real number then

$$m\,(cE) = c^n m(E)\,,$$

where $cE = \{cx : x \in E\}$.

The members of $\mathfrak{M}$ are called the Lebesgue measurable (or simply "measurable") sets in $\mathbb{R}^n$ and m is called the Lebesgue measure (or simply "measurable") on $\mathbb{R}^n$.

Let f be a real function on a measurable subset A of $\mathbb{R}^n$. We say f is a measurable function on A *if and only if* $f^{-1}((c,\infty)) \in \mathfrak{M}$ for every real number c.

Definition. A real function s is said to be a simple function if there are k measurable subsets $A_1, \ldots, A_k$ and k real numbers $c_1, \ldots, c_k$ such that

$$s = \sum_{i=1}^{k} c_i \chi_{A_i} \quad ,$$

where

$$\chi_{A_i}(x) = \begin{cases} 1 & \forall x \in A_i \ , \\ 0 & \forall x \in \mathbb{R}^n \setminus A_i . \end{cases}$$

Lecture 2

SOBOLEV SPACES

SOBOLEV SPACES

Definition. Let f be a real function on an open subset D of $\mathbb{R}^n$, $x = (x_1, \ldots, x_n) \in D$ and $I \in \{1, \ldots, n\}$. We define

$$\frac{\partial f}{\partial x_i}(x) = \lim_{t \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + t, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)}{t}$$

provided the limit exists., and $\frac{\partial f}{\partial x_i}(x)$ is called the partial derivative of f at x with respect to the variable x_i.

If $\frac{\partial f}{\partial x_i}(x)$ exits for any i in $\{1, \ldots, n\}$, we say f is differentiable at x and has derivative

$$Df(x) = \nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \cdots, \frac{\partial f}{\partial x_n}(x)\right)$$

Definition. Let f be a real function on an open subset D of $\mathbb{R}^n$. We say :

- f is differentiable on D if $\nabla f(x)$ exists for any x in D,
- f is of class $C^1(D)$ if f is differentiable on D and ∇f is a continuous from D into $\mathbb{R}^n$.
- f is of class $C_c^1(D)$ if f is of class $C^1(D)$ and $f(x) = 0$ for any x in $D \setminus K_f$, where K_f is a compact set contained in D.
- f is of class $C^1(\bar{D})$ if f is of class $C^1(D_f)$, where D_f is a open set containing D.

Definition. Let f be a real differentiable function on an open subset D of $\mathbb{R}^n$ and $x \in D$. Put $g_j = \frac{\partial f}{\partial x_j}$, then g_j is a real function on D for any j in $\{1, \ldots, n\}$. Let i be in $\{1, \ldots, n\}$. We say :

- f has the second-order partial derivative $\frac{\partial^2 f}{\partial x_i \partial x_j}(x)$ at x if g_j has the partial derivative $\frac{\partial g_j}{\partial x_i}(x)$ at x.
- f has the second-order partial derivative at x if $\frac{\partial^2 f}{\partial x_i \partial x_j}(x)$ exists for any i, j in $\{1, \ldots, n\}$. In this case the second-order derivative $D^2 f(x)$ of f at x is the $n \times n$- *matrix* $\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right]_{i,j=1,2,\ldots,n}$

Definition. Let f be a real function on an open subset D of $\mathbb{R}^n$. We say :

- f is differentiable 2-times on D if $D^2 f(x)$ exists for any x in D,
- f is of class $C^2(D)$ if f is differentiable 2-times on D and $D^2 f$ is a continuous from D into $\mathbb{R}^{n\times n}$.
- f is of class $C_c^2(D)$ if f is of class $C^2(D)$ and $f(x) = 0$ for any x in $D \setminus K_f$, where K_f is a compact set contained in D.
- f is of class $C^2(\bar{D})$ if f is of class $C^2(D_f)$, where D_f is a open set containing D.

Similarly we can define the classes $C^r(D)$, $C_c^r(D)$ and $C^r(\bar{D})$ for any integer $r > 2$. We put

$$C^\infty(D) = \bigcap_{r=1}^{\infty} C^r(D),$$

$$C_C^\infty(D) = \bigcap_{r=1}^{\infty} C_c^r(D),$$

$$C^\infty(\bar{D}) = \bigcap_{r=1}^{\infty} C^r(\bar{D}).$$

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Theorem. Let D be an open subset of $\mathbb{R}^n$, $p \in [1,\infty)$ and f be in $L^p(D)$. Assume

$$\int_D fg dx = 0 \qquad \forall g \in C_c^\infty(D).$$

Then $f = 0$ a.e. on D.

Theorem. Let D be an open subset of $\mathbb{R}^n$ with smooth boundary ∂D, $i \in \{1, \ldots, n\}$ and f be in $C^1(\bar{D})$. Then

$$(i)\ \int_D f\frac{\partial g}{\partial x_i}dx = \int_{\partial D} fgds - \int_D \frac{\partial f}{\partial x_i}gdx \qquad \forall g \in C^1(\bar{D}),$$

$$(ii)\ \int_D f\frac{\partial g}{\partial x_i}dx = -\int_D \frac{\partial f}{\partial x_i}gdx \qquad \forall g \in C_c^1(D),$$

where ds is the measure on the boundary ∂D .

Put

$$\|f\|_{1,p} = \{\int_D (|f|^p + \|\nabla f\|^p)dx]^{1/p} \qquad \forall f \in C^1(\bar{D}),$$

$$\|f\|_{2,p} = \{\int_D (|f|^p + \|\nabla f\|^p + \|D^2 f\|^p)dx]^{1/p} \quad \forall f \in C^2(\bar{D}),$$

$$\|f\|_{k,p} = \{\int_D (|f|^p + \sum_{r=1}^{k}\|D^r f\|^p)dx]^{1/p} \qquad \forall f \in C^k(\bar{D}).$$

We see that $(C_c^k(D), \|.\|_{1,p})$ and $(C^k(\bar{D}), \|.\|_{1,p})$ are normed linear spaces. We denote by $W_0^{k,p}(D)$ and $W^{k,p}(D)$ their completions respectively. These Banach spaces are called Sobolev spaces.

We see that

- $W_0^{k,p}(D) \subset W^{k,p}(D) \qquad \forall k \geqslant 1,$
- $W^{k,p}(D) \subset W^{k-1,p}(D) \subset L^p(D) \quad \forall k > 1.$

Let $p \in [1,\infty)$ and $u \in W^{1,p}(D)$. There is a Cauchy sequence $\{u_m\}$ in $(C^1(\bar{D}), \|.\|_{1,p})$ such that $\{u_m\}$ "converges" to u in following sense : $\{u_m\}$ converges to u in $L^p(D)$, $\{\frac{\partial u_m}{\partial x_i}\}$ is a Cauchy sequence in $L^p(D)$ for any $i \in \{1,\ldots,n\}$

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Let $p \in [1,\infty)$ and $u \in W^{1,p}(D)$. There is a Cauchy sequence $\{u_m\}$ in $(C^1(\bar{D}), \|.\|_{1,p})$ such that $\{u_m\}$ "converges" to u in following sense : $\{u_m\}$ converges to u in $L^p(D)$, $\{\frac{\partial u_m}{\partial x_i}\}$ is a Cauchy sequence in $L^p(D)$ for any $i \in \{1, \ldots , n\}$.

We can choose $\{u_m\}$ and $v_1, \ldots, v_n$ in $L^p(D)$ such that

$$\lim_{m\to\infty} \| \frac{\partial u_m}{\partial x_i} - v_i \|_p = 0 \qquad \forall i \in \{1,\ldots,n\},$$

$$u(x) = \lim_{m\to\infty} u_m(x) \qquad \text{a.e. on } D,$$

$$v_i(x) = \lim_{m\to\infty} \frac{\partial u_m}{\partial x_i}(x) \qquad \text{a.e. on } D, \forall i \in \{1,\ldots,n\}.$$

$$\int_D u_m \frac{\partial \varphi}{\partial x_i} dx = -\int_D \frac{\partial u_m}{\partial x_i} \varphi dx \qquad \forall \varphi \in C_\infty^1(D), m \in \mathbb{N} \quad (1)$$

$$\left| \int_D u_n \frac{\partial \varphi}{\partial x_i} dx - \int_D u \frac{\partial \varphi}{\partial x_i} dx \right| = \left| \int_D (u_m - u) \frac{\partial \varphi}{\partial x_i} dx \right| \leqslant \int_D \left| (u_m - u) \frac{\partial \varphi}{\partial x_i} \right| dx$$

$$\leqslant \{\int_D | u_m - u |^p \, dx\}^{1/p} \{\int_D | \frac{\partial \varphi}{\partial x_i} |^{p/(p-1)} \, dx\}^{(p-1)/p} \to 0 \ \ as \ m \to \infty \quad (2)$$

$$\left| \int_D \frac{\partial u_m}{\partial x_i} \varphi dx - \int_D v_i \varphi dx \right| = \left| \int_D (\frac{\partial u_m}{\partial x_i} - v_i) \varphi dx \right| \leqslant \int_D \left| (\frac{\partial u_m}{\partial x_i} - v_i) \varphi \right| dx$$

$$\leqslant \{\int_D | \frac{\partial u_m}{\partial x_i} - v_i |^p \, dx\}^{1/p} \{\int_D |\varphi|^{p/(p-1)} \, dx\}^{(p-1)/p} \to 0 \ \ as \ m \to \infty \quad (3).$$

$$(1),(2),(3) \Rightarrow \int_D u \frac{\partial \varphi}{\partial x_i} dx = -\int_D v_i \varphi dx \qquad \forall \varphi \in C_\infty^1(D), i \in \{1,\ldots n\}$$

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$$(1),(2),(3) \Rightarrow \int_D u \frac{\partial \varphi}{\partial x_i} dx = -\int_D v_i \varphi dx \qquad \forall \varphi \in C_\infty^1(D), i \in \{1,\ldots n\}$$

We say v_i is the generalized partial derivative of u with respect to x_i and denote it by $\frac{\partial u}{\partial x_i}$.

Thus, let u be in $W^{1,p}(D)$, then u has its generalized partial derivatives $\frac{\partial u}{\partial x_i} \in L^p(D)$ such that

$$\int_D u \frac{\partial \varphi}{\partial x_i} dx = -\int_D \frac{\partial u}{\partial x_i} \varphi dx \qquad \forall \varphi \in C_\infty^1(D), i \in \{1,\ldots n\}.$$

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Thus, let u be in $W^{1,p}(D)$, then u has its generalized partial derivatives $\frac{\partial u}{\partial x_i} \in L^p(D)$ such that

$$\int_D u\frac{\partial \varphi}{\partial x_i}dx = -\int_D \frac{\partial u}{\partial x_i}\varphi dx \qquad \forall \varphi \in C_c^1(D), i \in \{1,\ldots n\}.$$

Let η be in $W_0^{1,p}(D)$. We can choose a sequence $\{\varphi_m\}$ in $C_c^1(D)$, which converges to η in $W_0^{1,p}(D)$. Arguing as in (1),(2) and (3), we get

$$\int_D u\frac{\partial \eta}{\partial x_i}dx = -\int_D \frac{\partial u}{\partial x_i}\eta dx \qquad \forall \eta \in W_0^{1,p}(D), i \in \{1,\ldots n\}.$$

Let D = (-1 , 1) and $u(x) = |x|$ for any x in D. Put

$$u_m(x) = \sqrt{x^2 + m^{-1}} \qquad \forall\ x \in D, m \in \{1,2,\ldots\}.$$

We have

- $|u_m(x)| \leqslant \sqrt{2}$ *and* $\lim_{m\to\infty} u_m(x) = \sqrt{x^2} = u(x) \quad \forall x \in D,$
- $|u_m'(x)| = \left|\frac{x}{\sqrt{x^2+m^{-1}}}\right| \leqslant 1 \qquad \forall x \in D\setminus\{0\},$
- $\lim_{m\to\infty} u_m'(x) = \frac{x}{\sqrt{x^2}} = sign\ x \qquad \forall x \in D\setminus\{0\}.$

By the Lebesgue dominated convergence theorem, u is in $W^{1,2}(D)$ and its generalized derivative is $u'(x)$ = sign x.

Let D = (-1 , 1). Put

$$u(x) = \begin{cases} 1 & \forall x \in (-1,0], \\ 0 & \forall x \in (0,1). \end{cases}$$

We see that $u \in L^2(D)$.

Now assume there is $v \in L^2(D)$ such that

$$\int_D u\varphi' dx = -\int_D v\varphi dx \qquad \forall \varphi \in C_c^1(D) \qquad (1)$$

We have

$$\int_D u\varphi' dx = \int_{-1}^0 \varphi' dx = \varphi(0) - \varphi(-1) = \varphi(0) \qquad \forall \varphi \in C_c^1(D) \quad (2),$$

Now assume there is $v \in L^2(D)$ such that

$$\int_D u\varphi' dx = -\int_D v\varphi dx \qquad \forall \varphi \in C_c^1(D) \qquad (1)$$

We have

$$\int_D u\varphi' dx = \int_{-1}^0 \varphi' dx = \varphi(0) - \varphi(-1) = \varphi(0) \qquad \forall \varphi \in C_c^1(D) \quad (2),$$

By (1) and (2), we see that

$$\int_D v\varphi dx = 0 \qquad \forall \varphi \in C_c^1(D\setminus\{0\}),$$

which implies $v = 0$ a.e. on $D \setminus \{0\}$. Thus $v = 0$ a.e. on D or

$$\int_D v\varphi dx = 0 \qquad \forall \varphi \in C_c^1(D) \qquad (3)$$

By (2) and (3), φ (0) = 0 for any $\varphi \in C_c^1(D)$!

Therefore $W^{1,2}(D) \subset L^2(D)$, but $W^{1,2}(D) \neq L^2(D)$.

The following properties of generalized derivatives are proved in Chapter 7 of the book " D. Gilbarg and N. Trudinger, Elliptic partial differential equations of second order".

Theorem. Let D be an open subset of $\mathbb{R}^n$, p and q be in $(1,\infty)$ such that $p^{-1}+q^{-1} = 1$. Let $u \in W^{1,p}(D)$ and $v \in W^{1,q}(D)$. Then uv belongs to $u \in W^{1,1}(D)$ and

$$\frac{\partial(uv)}{\partial x_i} = \frac{\partial u}{\partial x_i} v + u \frac{\partial v}{\partial x_i} \qquad \forall\ i \in \{1,\dots,n\}.$$

Theorem. Let $a_1 < a_2 < \dots < a_k$ be k real numbers, D be an open subset of $\mathbb{R}^n$. Put $B = \{a_1, a_2, \dots, a_k\}$. Let f be a real function on $\mathbb{R}$ of class $C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus B)$ such that f' is discontinuous at every point of B, and $f' \in L^\infty(\mathbb{R} \setminus B)$. Let $u \in W^{1,p}(D)$ with $p \in [1, \infty)$. Then $v = f \circ u$ belongs to $W^{1,p}(D)$ and

$$\frac{\partial v}{\partial x_i}(x) = \begin{cases} f'(u(x)) \dfrac{\partial u}{\partial x_i}(x) & \text{if } u(x) \in \mathbb{R} \setminus B, \\ 0 & \text{if } u(x) \in B. \end{cases}$$

Theorem. Let D be an open subset of $\mathbb{R}^n$ and $u \in W^{1,p}(D)$ with $p \in [1, \infty)$. Put $u^+ = \max\{0, u\}$ and $u^- = \max\{0, -u\}$. Then u^+ , u^- and $|u|$ belong to $W^{1,p}(D)$ and

$$\frac{\partial u^+}{\partial x_i}(x) = \begin{cases} \dfrac{\partial u}{\partial x_i}(x) & \text{if } u(x) > 0, \\ 0 & \text{if } u(x) \leqslant 0. \end{cases}$$

$$\frac{\partial u^-}{\partial x_i}(x) = \begin{cases} \dfrac{\partial u}{\partial x_i}(x) & \text{if } u(x) < 0, \\ 0 & \text{if } u(x) \geqslant 0. \end{cases}$$

$$\frac{\partial |u|}{\partial x_i}(x) = \begin{cases} \dfrac{\partial u}{\partial x_i}(x) & \text{if } u(x) > 0, \\ 0 & \text{if } u(x) = 0, \\ -\dfrac{\partial u}{\partial x_i}(x) & \text{if } u(x) < 0. \end{cases}$$

We see that

- $W_0^{k,p}(D) \subset W^{k,p}(D) \qquad \forall k \geqslant 1,$
- $W^{k,p}(D) \subset W^{k-1,p}(D) \subset L^p(D) \quad \forall k > 1,$
- $W_0^{1,p}(D) \subset W^{1,p}(D) \subset L^p(D).$

Theorem (Sobolev imbedding). Let D be an open subset with smooth boundary in $\mathbb{R}^n$, and $u \in W^{1,p}(D)$ with $p \in [1,\infty)$. Then

(i) u is in $L^q(D)$ where $q = \dfrac{np}{n-p}$ if $p < n$,

(ii) *u is of class* $C^r(\bar{D})$ if $0 \leqslant r < 1 - n^{-1}p$.

Theorem (Sobolev imbedding). Let D be an open subset with smooth boundary in $\mathbb{R}^n$, and $u \in W^{k,p}(D)$ with $p \in [1,\infty)$. Then

(i) u is in $L^q(D)$ where $q = \frac{np}{n-kp}$ if $kp < n$,

(ii) *u is of class* $C^r(\bar{D})$ if $0 \leqslant r < k - n^{-1}p$.

The proof of this theorem is in the book of Adams.

Theorem (Sobolev imbedding). Let D be an open subset with smooth boundary in $\mathbb{R}^n$, and $u \in W^{k,p}(D)$ with $p \in [1,\infty)$. Then u is in $L^q(D)$ if $q \in [p, \frac{np}{n-kp}]$ and $kp < n$.

Theorem (Poincare inequality). Let D be a bounded open subset with smooth boundary in $\mathbb{R}^n$, n be a positive integer, $p \in [1,\infty)$ such that $p < n$.

Then for any $q \in [1, \frac{np}{n-p}]$ there is a positive real number C such that

$$\| u \|_q \leqslant C \| \nabla u \|_p \qquad \forall u \in W_0^{1,p}(D).$$

Theorem (Sobolev imbedding). Let D be a bounded open subset with smooth boundary in $\mathbb{R}^n$, and $u \in W^{k,p}(D)$ with $p \in [1,\infty)$. Then u is in $L^q(D)$ if $q \in [1, \frac{np}{n-kp}]$ and $kp < n$.

Theorem (Sobolev inequality). Let D be a bounded open subset with smooth boundary in $\mathbb{R}^n$, *n and k* be positive integers and $p \in [1,\infty)$ such that $kp < n$.

Then for any $q \in [1, \frac{np}{n-kp}]$ there is a positive real number C such that

$$\|u\|_q \leqslant C \|u\|_{k,p} \qquad \forall\, u \in W^{k,p}(D).$$

Theorem . Let D be a bounded open subset with smooth boundary in $\mathbb{R}^n$, n be a positive integer, $p \in [1,\infty)$ such that $p < n$. Put

$$|||\, u \,|||_{1,p} = \{\int_D \| \nabla u \|^p \, dx\}^{1/p} \qquad \forall u \in W_0^{1,p}(D).$$

Then there are a positive real number c such that

$$c \| u \|_{1,p} \leqslant |||\, u \,|||_{1,p} \leqslant \| u \|_{1,p} \qquad \forall u \in W_0^{1,p}(D).$$

Theorem. $(W_0^{1,2}(D), |||\,.\,|||)$ is a Hilbert space with the following inner product

$$< u, v > = \int_D \nabla u \nabla v dx \qquad \forall u, v \in W_0^{1,2}(D).$$

Theorem. $W^{1,2}(D)$ is a Hilbert space with the following inner product

$$< u,v > \; = \int_D (uv + \nabla u \nabla v)dx \qquad \forall u,v \in W_0^{1,2}(D).$$

Theorem (Sobolev imbedding). Let D be a bounded open subset with smooth boundary in ☼, and $u \in W^{1,p}(D)$ with $p \in (1,\infty)$. Then u is in $L^q(D)$ for any $q \in [1,\infty)$.

Theorem (Sobolev inequality). Let D be a bounded open subset with smooth boundary in ☼, and $p \in (1,\infty)$. Then for any $q \in [1,\infty)$, there is a positive real number C such that

$$\|u\|_q \leqslant C\,\|u\|_{1,p} \qquad \forall\, u \in W^{1,p}(D).$$

Theorem(Rellich-Kondrachov). Let D be a bounded open subset with smooth boundary in ☼^{n}, k be positive integer, and $p \in [1,\infty)$ such that $kp < n$. Let $q \in [1, \frac{np}{n-kp})$ and put

$$T(u) = u \qquad \forall\, u \in W^{k,p}(D).$$

Then T is a bounded linear mapping from $W^{k,p}(D)$ into $L^q(D)$, and the closure $T(A)$ in $L^q(D)$ is compact in $L^q(D)$ for any bounded subset A in $W^{k,p}(D)$.

Theorem(Rellich-Kondrachov). Let D be a bounded open subset with smooth boundary in ☼, $p \in (1,\infty)$ and $q \in [1,\infty)$. Put

$$T(u) = u \qquad \forall\, u \in W^{1,p}(D).$$

Then T is a bounded linear mapping from $W^{1,p}(D)$ into $L^q(D)$, and the closure $T(A)$ in $L^q(D)$ is compact in $L^q(D)$ for any bounded subset A in $W^{1,p}(D)$.

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Theorem. Let D be a bounded open subset with smooth boundary in ☼^{n}, $p \in (1,\infty)$, and T be a linear mapping from $W^{1,p}(D)$ into ☼. Then T is continuous on $W^{1,p}(D)$ if and only if there are $g, g_1, \ldots, g_n$ in $L^{p/(p-1)}(D)$ such that

$$T(u) \; = \int_D [ug + \frac{\partial u}{\partial x_1} g_1 + \cdots + \frac{\partial u}{\partial x_n} g_n]dx \qquad \forall u \in W^{1,p}(D).$$

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Theorem. Let D be a bounded open subset with smooth boundary in $\mathbb{R}^n$, and T be a linear mapping from $W_0^{1,2}(D)$ into $\mathbb{R}$. Then T is continuous on $W_0^{1,2}(D)$ if and only if there is g in $W_0^{1,2}(D)$ such that

$$T(u) = \int_D \left[\frac{\partial u}{\partial x_1}\frac{\partial g}{\partial x_1} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial g}{\partial x_n}\right]dx \qquad \forall u \in W_0^{1,2}(D).$$

Definition. Let D be a bounded open subset with smooth boundary in $\mathbb{R}^n$, $p \in (1,\infty)$, v in $W^{1,p}(D)$ and $\{v_m\}$ be a sequence in $W^{1,p}(D)$. Then we say $\{v_m\}$ weakly converges to v in $W^{1,p}(D)$ if $\{T(v_m)\}$ converges to $T(u)$ for any bounded linear mapping T from $W^{1,p}(D)$ into $\mathbb{R}$.

Theorem. Let D be a bounded open subset with smooth boundary in $\mathbb{R}^n$, $p \in (1,\infty)$, and $\{u_m\}$ be a bounded sequence in $W^{1,p}(D)$. Then there are u in $W^{1,p}(D)$ and a subsequence $\{u_{m_k}\}$ such that $\{u_{m_k}\}$ weakly converges to u

Lecture 3

Variational calculus

1

Variational calculus

Definition. Let f be a mapping from an open subset U of a normed space $(E,\|.\|_E)$ into another normed space $(G,\|.\|_G)$, and $x \in U$. We say f has the directional derivative at x if and only if there is a bounded linear mapping T from E into G such that

$$T(h) = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t} \qquad \forall\, h \in E.$$

In this case, we call T the directional derivative at x of f and denote it by $Df(x)$.

If $Df(x)$ exits for any x in U, we say f is directional differentiable on U.

Denote by $L(E,G)$ the set of all bounded linear mappings from $(E,\|.\|_E)$ into $(G,\|.\|_G)$, then $L(E,G)$ is a normed space with the following norm

$$\|T\| = \sup_{\|h\|_E \leqslant 1} \|T(h)\|_G \qquad \forall h \in E.$$

Let f be a directionally differentiable mapping from an open subset U of a normed space $(E,\|.\|E)$ into another normed space $(G,\|.\|G)$. We say f is of class $C^1(U)$ if and only if Df is a continuous mapping from U into $(L(E,G),\|.\|)$

If Df is of class $C^1(U)$, then we say f is of class $C^2(U)$ and has the second order derivative $D^2 f(x) = D(Df)(x)$ for any x in U .

Let Ω be a bounded open subset of $\mathbb{R}^n$. Put

$$f(u) = \int_\Omega |\nabla u|^2 dx \qquad \forall u \in W^{1,2}(\Omega).$$

Then f is of class $C^1(\Omega)$.

Let Ω be a bounded open subset of $\mathbb{R}^n$ and g be a real function of class C^2 on $\Omega\times\mathbb{R}$ such that there are a positive real number c and a real function v in $L^{(2n+1)/2n}(\Omega)$ such that

$$|g(x,s)| + |\frac{\partial g}{\partial s}(x,s)| \leqslant cv(x) \qquad \forall (x,s) \in \Omega \times \mathbb{R}^n.$$

Put

$$f(u) = \int_\Omega g(x,u(x))dx \qquad \forall u \in W^{1,2}(\Omega).$$

Then f is directionally differentiable on $W^{1,2}(\Omega)$.

Theorem. Let f be a mapping from an open subset U of a normed space $(E, \|.\|_E)$ into $\mathbb{R}$ and $x \in U$ such that

(i) f has the directional derivative at x ,

(ii) $f(x) \leqslant f(y)$ for any $y \in U$.

Then

$$Df(x)h = 0 \qquad \forall\ h \in E . \qquad (1)$$

Therefore if we can find x as in the foregoing theorem, we can solve the equation (1).

Definition. Let D be an open subset with smooth boundary in $\mathbb{R}^n$, and f be a real function on a subset M of $W^{k,p}(D)$ with $k \in \{0,1,2, \ldots\}$, $p \in (1,\infty)$. Then we say f is weakly lower semi-continuous on M if and only if for any sequence $\{u_m\}$ weakly converging to u in M , we have

$$f(u) \leqslant \liminf_{m\to\infty} f(u_m)$$

Let D be an open subset with smooth boundary in $\mathbb{R}^3$. Put

$$f(u) = \int_D u^6(x)dx \qquad \forall\ u \in W^{1,2}(D).$$

Then f is weakly lower semi-continuous on $W^{1,2}(D)$.

Let D be an open subset with smooth boundary in $\mathbb{R}^n$, and F be a real function on $D\times\mathbb{R}\times\mathbb{R}^n$ such that $F(x,u(x), \nabla u(x))$ is integrable on D for any u in $W^{1,p}(D)$. Assume

(i) $F(x,s,.)$ is convex on $\mathbb{R}^n$ for every $(x,s) \in D\times\mathbb{R}$,

(ii) There is an integrable function g on D such that

$$g(x) \leqslant F(x,s,z) \qquad \forall\ (x,s,z) \in D\times\mathbb{R}^n.$$

Put

$$f(u) = \int_D F(x,u(x),\nabla u(x)dx \qquad \forall\ u \in W^{1,p}(D).$$

Then f is weakly lower semi-continuous on $W^{1,p}(D)$.

Definition. Let D be an open subset with smooth boundary in $\mathbb{R}^n$, and M be a subset of $W^{k,p}(D)$ with $k \in \{0,1,2, \ldots\}$, $p \in (1,\infty)$. Then we say M is weakly closed in $W^{k,p}(D)$ if and only if for any sequence $\{u_m\}$ in M such that $\{u_m\}$ weakly converging to u in $W^{k,p}(D)$, we have $u \in M$.

Let $D = (0,2\pi)$. Put

$$S = \{u \in L^2(D) : \|u\|_2 = 1\} \quad \text{and}$$

$$B = \{u \in L^2(D) : \|u\|_2 \leqslant 1\}.$$

Then S and B are closed in $L^2(D)$, B is weakly closed in $L^2(D)$, and S is not weakly closed in $L^2(D)$.

Theorem. Let D be an open subset with smooth boundary in ☼^{n}, and M be a closed convex subset of $W^{k,p}(D)$ with $k \in \{0,1,2, . . .\}$, $p \in (1,\infty)$. Then M is weakly closed in $W^{k,p}(D)$.

Theorem. Let D be an open subset with smooth boundary in ☼^{n}, and M be a weakly closed subset of $W^{k,p}(D)$ with $k \in \{0,1,2, . . .\}$, $p \in (1,\infty)$. Let f be a real weakly lower semi-continuous function on M. Assume : $\{u_m\}$ is bounded in $W^{k,p}(D)$ if it is a sequence in M and $\{f(u_m)\}$ is bounded in ☼ . Then there is u in M such that

$$f(u) \leqslant f(v) \qquad \forall\, v \in M.$$

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Let k be a nonnegative function in $L^{n/2}(D)$, Then there is a u in $W_0^{1,2}(D)$ such that

$$\int_D [\nabla u \nabla v + kuv + v \sin u^2]dx \;=\; 0 \qquad \forall v \in W_0^{1,2}(D)$$

This u is called a weak solution $W_0^{1,2}(D)$ to the following equation

$$-\Delta u + ku + \cos u^2 \;=\; 0$$

10

Theorem.(Lagrange multiplier) Let f and g be real functions of class C^1 from an open subset U of a Banach space E, and $r \in$ ☼. Let $x_0 \in M = \{ x \in U : g(x) = r\}$ such that $Dg(x_0) \neq 0$ and $f(x_0) \leqslant f(x)$ for any x in M. Then there is a real number c such that

$$Df(x_0) \;=\; cDg(x_0)$$

Using this theorem we can find weak solution u to the following eigenvalue problem

$$\Delta u \;=\; \lambda k(x,u)$$

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Lecture 4

TOPOLOGICAL DEGREE

TOPOLOGICAL DEGREE

Definition. Let T be a continuous mapping from a subset A of a normed space $(E,\|.\|_E)$ into E. We say T is a compact mapping on A if and only if the closure of $T(A)$ in E is compact.

In this case, put

$$f(x) = x - T(x) \qquad \forall\ x \in A\ .$$

Then f is called a compact vector field on A.

Let T and S be compact mappings on a subset A of a normed space $(E,\|.\|E)$. Then $T + S$ also is compact on A.

Let D be an open bounded subset with smooth boundary in $\mathbb{R}^3$ and g be in $L^3(D)$. Put

$$\int_D \nabla(S(u)).\nabla v dx \ \equiv\ < S(u), v >$$

$$= \int_D g(x)v(x)dx \qquad \forall\ u \in W_0^{1,2}(D).$$

Then S is a compact mapping on every bounded subset A of $W^{1,2}(D)$.

Let D be an open bonded subset with smooth boundary in $\mathbb{R}^3$. Put

$$\int_D \nabla(T(u)).\nabla v dx \ \equiv\ < T(u), v >$$

$$= -\int_D u^3(x)v(x)dx \qquad \forall\ u \in W_0^{1,2}(D)$$

Then T is a compact mapping on every bounded subset A of $W^{1,2}(D)$.

Put $f(w) = w - S(w) - T(w)$ for any w in $W_0^{1,2}(D)$. Let u be in $W_0^{1,2}(D)$ such that $f(u) = 0$. Then u is a weak solution in $W_0^{1,2}(D)$ to the following equation

$$-\Delta u + \tfrac{1}{4}u^4 = g$$

5

Theorem. Let U be và open subset in a Banach space E with closure $\bar{U}$ and boundary ∂U , and f be a compact vector field on $\bar{U}$. Then $f(\partial U)$ is closed in E.

Theorem. Let U be và open subset in a Banach space E with closure $\bar{U}$ and boundary ∂U , and f be a compact vector field on $\bar{U}$. Then there is a continuous mapping deg(f , U , .) from $E \setminus f(\partial U)$ into $\mathbb{Z}$ having the following properties :

(D1) If $a \in E \setminus f(\partial U)$ and deg(f , U , a) ≠ 0. Then there is x in U such that $f(x) = a$.

(D1) If $a \in E \setminus f(\partial U)$ and deg(f , U , a) ≠ 0. Then there is x in U such that $f(x) = a$.

(D2) deg(Id , U , a) = 1 if $a \in U$ and deg(Id , U , a) = 0 if $a \in E \setminus \bar{U}$.

(D2) If there are a compact mapping H from $[0,1]\times\bar{U}$ into E and $a \in E \setminus H([0,1]\times\partial U)$. Then

deg(f_1 , U , a) = deg(f_0 , U , a)

where $f_i(x) = x - H(i,x)$ for any (i,x) in $\{0,1\}\times\bar{U}$.

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Let f be a compact vector field on a closed $B'(0,r)$ in a Hilbert space H such that

$$< f(x) , x > \; > 0 \qquad \forall\, x ,\ \|x\| = r .$$

Then there is u in $B(0,r)$ such that $f(u) = 0$.

Let D be an open bounded subset with smooth boundary in $\mathbb{R}^3$ and g be in $L^3(D)$. Then there is a weak solution in $W_0^{1,2}(D)$ to the following equation

$$-\Delta u + \tfrac{1}{4}u^4 = g$$

°

Definition . Let E be a measurable subset and s be a simple function such that

$$s = \sum_{i=1}^{k} c_i \chi_{A_i} \quad .$$

We define the integral of s on E as follows

$$\int_E s\,dx = \sum_{i=1}^{k} c_i m(E \cap A_i)$$

Definition . Let E be a measurable subset and f be a positive measurable function on E . Put $F(f)$ is the set of all nonnegative simple function $s \leqslant f$. Then the integral of f on E is defined as follows

$$\int_E f\,dx = \sup_{s \in F(f)} \int_E s\,dx \quad ,$$

Definition . Let E be a measurable subset and f be a measurable function on E . We say f is integrable on E if and only if

$$\int_E |f|\,dx < \infty \quad .$$

In this case we put

$$\int_E f\,dx = \int_E f^+ dx - \int_E f^- dx \quad ,$$

where $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$.

We have following results (see the proofs in the book " Real and complex analysis" of W. Rudin)

Theorem (Lebesgue's Monotone Convergence theorem)

Let $\{f_m\}$ be a sequence of measurable functions on E, and suppose that

(a) $0 \leqslant f_1(x) \leqslant f_2(x) \leqslant \ldots \leqslant f_m(x) \leqslant \ldots$ for every $x \in E$,

(b) $f_m(x) \to f(x)$ as $n \to \infty$, for every $x \in E$.

Then f is measurable on E, and

$$\int_E f\,dx = \lim_{m\to\infty} \int_E f_m dx$$

Fatou's Lemma: If $f_m : E \to [0, \infty)$ is measurable, for each positive integer m, then

$$\int_E (\liminf_{m\to\infty} f_m) dx \leqslant \liminf_{m\to\infty} \int_E f_m dx \quad .$$

Lebesgue's Dominated Convergence Theorem

Suppose $\{f_m\}$ is a sequence of real measurable functions on E such that there is a real function f and an integrable real function g on E having the following propreties

$$f(x) = \lim_{m\to\infty} f_m(x) \qquad \forall\ x \in E,$$

$$|f_m(x)| \leqslant g(x) \qquad \forall\ x \in E, m = 1, 2, \ldots.$$

Then f is integrable on E ,

$$\lim_{m\to\infty} \int_E |f_m - f|\,dx = 0 \quad \text{and}$$

$$\int_E f\,dx = \lim_{m\to\infty} \int_E f_m dx$$

Let E be a measurable subset of $\mathbb{R}^n$ with $m(E) > 0$. Denote by $\mathfrak{M}(E)$ the set of all real measurable functions on E. If f and g are in $\mathfrak{M}(E)$ and if $m(\{x : f(x) \neq g(x)\}) = 0$, we say that $f = g$ a.e. (almost everywhere) on E, and we may write $f \sim g$. This is easily seen to be an equivalence relation. The transitivity ($f \sim g$ and $g \sim h$ implies $f \sim h$) is a consequence of the fact that the union of two sets of measure 0 has measure 0.

Note that if $f \sim g$ and $u \sim v$, then

- $f + u \sim g + v$,
- $f . u \sim g . v$,
- $c\,u \sim c\,v$ for any real number c .

Let f be in $\mathfrak{M}(E)$, we put

$$\tilde{f} = \{g \in \mathfrak{M}(E) : g \sim f\}$$

We see that $\tilde{f}$ is an equivalent class of $\mathfrak{M}(E)$ with respect to relation $\sim$. The set of these equivalent classes is a vector space with the following operations :

$$\tilde{f} + \tilde{g} = \widetilde{f + g} \qquad \forall\ f, g \in \mathfrak{M}(E),$$

$$\alpha \tilde{f} = \widetilde{\alpha f} \qquad \forall\ f \in \mathfrak{M}(E), \alpha \in \mathbb{R},$$

$$\tilde{f} . \tilde{g} = \widetilde{f . g} \qquad \forall\ f, g \in \mathfrak{M}(E),$$

$$|\tilde{f}| = \widetilde{|f|} \qquad \forall\ f \in \mathfrak{M}(E).$$

Denote by $M(E)$ be this vector space. An element of $M(E)$ is a class of functions.

We can consider every element of $M(E)$ as a real function on E, which belongs to it. We say:

- $\tilde{f}$ is continuous if there is a continuous map g in $\tilde{f}$,
- $\tilde{f}$ is bounded if there is a bounded map g in $\tilde{f}$,
- $\tilde{f}$ is differentiable if there is a differentiable map g in $\tilde{f}$.

Hereafter we consider every element u of $M(E)$ as a real function f on E and apply the differential and integral calculus to f in order to get estimations about u.

For example, if we can prove that $|f(x)| \leqslant 5$ for any x in E, then we say $|u| \leqslant 5$ for almost everywhere on E, that is : for any g in the class u there is a subset A_g of E such that $m(A_g) = 0$ and $|g(x)| \leqslant 5$ for any x in $E \setminus A_g$.

Let A be a measurable subset of E with $m(A) > 0$, then we can define the restriction $u|_A$ in usual way. But $u|_A$ is nonsense if $m(A) = 0$.

Let u be in $M(E)$. If there is an integrable function f in the class u, we say u is integrable on E and put

$$\int_A u\,dx = \int_A f\,dx \qquad \forall \text{ measurable subset } A \text{ of } E.$$

This notation is well-defined, because

$$\int_A f\,dx = \int_A g\,dx \qquad \forall \text{ measurable subset } A \text{ of } E, f, g \in M(E).$$

$$(m(\{x \in E : f(x) \neq g(x)\}) = 0)$$

Let p be in the interval $[1, \infty)$ and E be a measurable subset of ☼^{n} with $m(E) > 0$, and u be in $M(E)$. We say

- $u \in L^p(E)$ if $|u|^p$ is integrable on E,
- $u \in L^\infty(E)$ if there is a real number K such that $|u| \leqslant K$ almost everywhere on E.

We put

$$\| u \|_p = \{\int_E |u|^p\}^{1/p} \qquad \forall\ u \in L^p(E),\ 1 \leqslant p < \infty,$$

$$\| u \|_\infty = \inf\{K > 0 : |u| \leqslant K \text{ a.e. on } E\} \qquad \forall\ u \in L^\infty(E).$$

We have following properties of $L^p(E)$ (see the proofs in the book " Real and complex analysis" of W. Rudin)

Theorem . $(L^r(E), \|.\|_r)$ is a Banach space for any r in $[1,\infty]$.

Theorem (Holder) Let p and q be in $(1, \infty)$, f be in $L^p(E)$ and g be in $L^q(E)$ such that $p^{-1} + q^{-1} = 1$. Then

$$\left| \int_E fg\,dx \right| \leqslant \| f \|_p \| g \|_q$$

Theorem. Let p be in $(1,\infty)$ and T be a continuous linear mapping from $L^p(E)$ into ☼. Then there exists a unique g in $L^q(E)$, $p^{-1} + q^{-1} = 1$ such that $\|T\| = \|g\|_q$ and

$$T(f) = \int_E fg\,dx \qquad \forall\ f \in L^p(E).$$

Theorem. $L^2(E)$ is a Hilbert space with respect to following inner-product

$$< u,v > = \int_E uv\,dx \qquad \forall\ u,v \in L^2(E).$$

Definition. Let D an open subset of ☼^{n} and f be a continuous real function on D. We say f is of class $C_c(D)$ if and only if there is a compact subset K of ☼^{n} such that $K \subset D$ and $f(x) = 0$ for any x in $D \setminus K$.

Theorem. Let D an open subset of ☼^{n}, $p \in [1,\infty)$ and u be in $L^p(D)$. Then there is a sequence $\{u_m\}$ in $C_c(D)$ such that

$$\lim_{m\to\infty} \| u - u_m \|_p = 0.$$