Finite Difference Method

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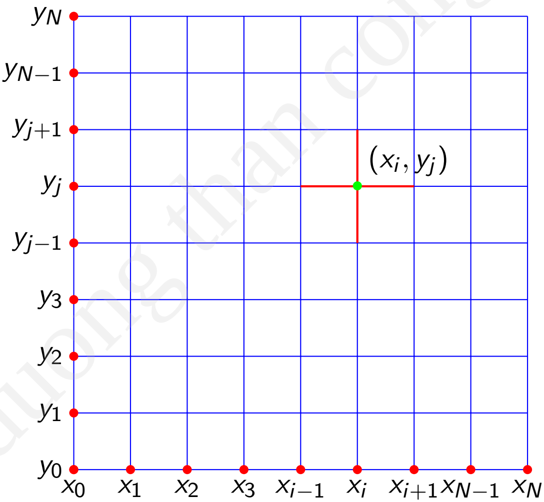
Poisson's equation

We will use the finite difference method on two dimensions to discretize the following equation with subset $\Omega \subset \mathbb{R}^2$ and $f \in L^2(\Omega)$:

$$\begin{cases} -\Delta u & = f(x,y) \quad \text{in } \Omega \\ u(x,y) & = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1}$$

Mesh



Mesh

For simplicity, we consider the diffusion equation on $\Omega = (0,1) \times (0,1)$. On interval $[0,1]$, we make two uniform partition $(x_i)_{i \in \overline{0,N}}$, $(y_j)_{j \in \overline{0,N}}$, such that

$$x_i = ih \quad \text{for all } i \in 0, \cdots, N$$

$$y_j = jh \quad \text{for all } j \in 0, \cdots, N$$

We would like to find the value of the function u at points (x_i, x_j) for all $i, j = 0, \cdots N$, it means that

$$u_{i,j} \simeq u(x_i, x_j)$$

Scheme

From the first equation of (1), we have

$$-\frac{\partial^2 u}{\partial x^2}(x_i, y_j) - \frac{\partial^2 u}{\partial y^2}(x_i, y_j) = f(x_i, y_j) \quad \forall i, j = 1, \cdots, N-1$$

Using the approximation of the second order derivative respect x, we have

$$-\frac{\partial^2 u}{\partial x^2}(x_i, y_j) = \frac{-u_{i-1,j} + 2u_{i,j} - u_{i+1,j}}{h^2}$$

It is similar, there holds

$$-\frac{\partial^2 u}{\partial y^2}(x_i, y_j) = \frac{-u_{i,j-1} + 2u_{i,j} - u_{i,j+1}}{h^2}$$

Scheme

Then, we get the scheme for finite voulume dicretization:

$$\frac{-u_{i,j-1} - u_{i-1,j} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = f_{i,j}, \qquad (2)$$

and

$$u_{0,j} = u_{N,j} = u_{i,0} = u_{i,N} = 0, \qquad \forall i, j = 0, \cdots, N$$

- If $j = 1$, and $i = 1$ then $u_{i,j-1} = u_{i-1,j} = 0$, the equation in (2) becomes

$$\frac{4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = f_{i,j}.$$

- If $j = 1$,and $i = N - 1$ then $u_{i,j-1} = u_{i+1,j} = 0$, the equation in (2) becomes

$$\frac{-u_{i-1,j} + 4u_{i,j} - u_{i,j+1}}{h^2} = f_{i,j}.$$

Scheme

- If $j = 1$ and $i \notin \{1, N-1\}$ then $u_{i,j-1} = 0$, the equation in (2) becomes

$$\frac{-u_{i-1,j} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = f_{i,j}.$$

- If $j = N-1$ and $i = 1$ then $u_{i,j+1} = u_{i-1,j} = 0$, the equation in (2) becomes

$$\frac{-u_{i,j-1} + 4u_{i,j} - u_{i+1,j}}{h^2} = f_{i,j}.$$

- If $j = N-1$ and $i = N-1$ then $u_{i,j+1} = u_{i+1,j} = 0$, the equation in (2) becomes

$$\frac{-u_{i,j-1} - u_{i-1,j} + 4u_{i,j}}{h^2} = f_{i,j}.$$

Scheme

- If $j = N-1$ and $i \notin \{1, N-1\}$ then $u_{i,j+1} = 0$, the equation in (2) becomes

$$\frac{-u_{i,j-1} - u_{i-1,j} + 4u_{i,j} - u_{i+1,j}}{h^2} = f_{i,j},$$

- $j \notin \{1, N-1\}$ and $i = 1$ then $u_{i-1,j} = 0$, the equation in (2) becomes

$$\frac{-u_{i,j-1} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = f_{i,j}$$

Scheme

- $j \notin \{1, N-1\}$ and $i = N-1$ then $u_{i+1,j} = 0$, the equation in (2) becomes

$$\frac{-u_{i,j-1} - u_{i-1,j} + 4u_{i,j} - u_{i,j+1}}{h^2} = f_{i,j}$$

- $j \notin \{1, N-1\}$ and $i \notin \{1, N-1\}$ then $u_{i+1,j} = 0$, the equation in (2) becomes

$$\frac{-u_{i,j-1} - u_{i-1,j} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = f_{i,j}.$$

Linear System

We get a linear system

$$Au = F, \qquad A \in \mathbb{R}^{(N-1)^2} \times \mathbb{R}^{(N-1)^2}$$

$$u = [u_{1,1}, \cdots, u_{N-1,1}, u_{1,2}, \cdots, u_{N-1,2}, \cdots, u_{1,N-1}, \cdots, u_{N-1,N-1}]^T$$
$$F = [f_{1,1}, \cdots, f_{N-1,1}, f_{1,2}, \cdots, f_{N-1,2}, \cdots, f_{1,N-1}, \cdots, f_{N-1,N-1}]^T$$

$$A = \frac{1}{h^2}\begin{bmatrix} B & -I & & & & \\ -I & B & -I & & & \\ & -I & B & -I & & \\ & & & \cdots & & \\ & & & -I & B & -I \\ & & & & -I & B \end{bmatrix}$$

Linear System

where

$$B = \begin{bmatrix} 4 & -1 & & & & & \\ -1 & 4 & -1 & & & & \\ & -1 & 4 & -1 & & & \\ & & & \cdots & & & \\ & & & -1 & 4 & -1 \\ & & & & -1 & 4 \end{bmatrix}$$

and

$$I = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \cdots & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

Other types of boundary condition

- Dirichlet-Neumann boundary condition

$$\frac{\partial u}{\partial y}(0, y) = u(x, 0) = u(1, y) = u(x, 1)$$

We can use the forward difference or the second order approximation of derivative respect y at $(0, y)$ or the central differnce (see 1D, changing only the matrix A).

- In-homogeneuos Dirichet boundary condition

$$u(x, y) = g(x, y) \text{ on } \partial\Omega$$

with this condition, changing only the vector F.

Other types of boundary condition

- In-homogeneous Dirichlet-Neumann boundary condtion

$$
\begin{aligned}
u(x,0) &= g_1(x), \\
u(1,y) &= g_2(y), \\
u(x,1) &= g_3(x), \\
\frac{\partial u}{\partial y}(0,y) &= g_1(y).
\end{aligned}
$$

With this condition, it make vector F and matrix A change.

Numerical experiments

We will demonstrate the convergence and convergent rate of the scheme for 4 previous boundary condtion.

Stability

Definition

The numerical solution itself should remain uniformly bounded

Now, we prove the stability of the scheme for Dirichlet boundary condition with $\Omega =]0,1[^2$. Firstly, we define discrete L^2_h-norm

$$\|u\|^2_{2,h} = \sum_{i,j} u^2_{i,j} h^2$$

Multiplying (2) by $u_{i,j}$ then sum over $i,j = 1, \cdots, N-1$, we get

$$\sum_{i,j=1}^{N-1} \frac{(u_{i,j} - u_{i-1,j})u_{i,j}}{h^2} + \frac{(u_{i,j} - u_{i+1,j})u_{i,j}}{h^2}$$
$$+ \sum_{i,j=1}^{N-1} \frac{(u_{i,j} - u_{i,j-1})u_{i,j}}{h^2} + \frac{(u_{i,j} - u_{i,j+1})u_{i,j}}{h^2} = \sum_{i,j=1}^{N-1} f_{i,j} u_{i,j}$$

Stability

We can change the index in the sum, we have

$$\sum_{i,j=1}^{N-1} \frac{(u_{i,j} - u_{i-1,j})u_{i,j}}{h^2} + \sum_{i=2,j=1}^{N,N-1} \frac{(u_{i-1,j} - u_{i,j})u_{i-1,j}}{h^2}$$

$$+ \sum_{i,j=1}^{N-1} \frac{(u_{i,j} - u_{i,j-1})u_{i,j}}{h^2} + \sum_{i=1,j=2}^{N-1,N} \frac{(u_{i,j-1} - u_{i,j})u_{i,j-1}}{h^2} = \sum_{i,j=1} f_{i,j}u_{i,j}$$

Sine $u_{0,j} = u_{N,j} = u_{i,0} = u_{i,N}$, then

$$\sum_{i,j=1}^{N,N-1} \frac{(u_{i,j} - u_{i-1,j})^2}{h^2} + \sum_{i,j=1}^{N-1,N} \frac{(u_{i,j} - u_{i,j-1})^2}{h^2} = \sum_{i,j=1}^{N-1} f_{i,j}u_{i,j}$$

Stability

We can write again

$$\sum_{i,j=1}^{N,N-1}(D_{x-}u)_{i,j}^2+\sum_{i,j=1}^{N-1,N}(D_{y-}u)_{i,j}^2=\sum_{i,j=1}^{N-1}f_{i,j}u_{i,j}, \qquad (3)$$

where

$$(D_{x-}u)_{i,j}=\frac{u_{i,j}-u_{i-1,j}}{h} \qquad (D_{y-}u)_{i,j}=\frac{u_{i,j}-u_{i,j-1}}{h}$$

Let's define the discrete H_h^1-norm

$$|||u|||_{1,h}^2=\sum_{i,j=1}^{N,N-1}(D_{x-}u)_{i,j}^2h^2+\sum_{i,j=1}^{N-1,N}(D_{y-}u)_{i,j}^2h^2$$

Stability

Applying Holder inequality, there hold

$$h^2 \sum_{i,j=1}^{N-1} f_{i,j} u_{i,j} \leq \left(\sum_{i,j=1}^{N-1} h^2 f_{i,j}^2 \right)^{1/2} \left(\sum_{i,j=1}^{N-1} h^2 u_{i,j}^2 \right)^{1/2}$$
$$= \|f\|_{2,h} \|u\|_{2,h}$$

From (3), we get

$$|||u|||_{1,h}^2 \leq \|f\|_{2,h} \|u\|_{2,h} \quad (4)$$

Stability

Lemma

There exists a constant positive C_Ω such that

$$\|u\|_{2,h} \le C_\Omega |||u|||_{1,h}$$

Proof: Since $u_{0,j} = 0$ then

$$u_{i,j} = \sum_{i'=1}^{i} (u_{i',j} - u_{i'-1,j}) = \sum_{i'=1}^{i} \frac{u_{i',j} - u_{i'-1,j}}{h}.h$$

$$= \sum_{i'=1}^{i} (D_{x-}u)_{i',j}.h$$

Stability

Thus

$$u_{i,j}^2 \le \sum_{i'=1}^{i} 1 \sum_{i'=1}^{i} (D_{x-}u)_{i',j}^2 h^2 \le N \sum_{i'=1}^{N-1} (D_{x-}u)_{i',j}^2 h^2$$

So

$$\|u\|_h^2 = \sum_{j=1}^{N-1} \sum_{i=1}^{N-1} h^2 u_{i,j}^2 \le \sum_{j=1}^{N-1} N^2 h^2 \sum_{i'=1}^{N-1} (D_{x-}u)_{i',j}^2 h^2$$

$$= N^2 h^2 \sum_{i',j=1}^{N-1} (D_{x-}u)_{i',j}^2 h^2 \le |||u|||_{1,h}^2$$

We have completed the proof of the lemma. Using the lemma and (4), we get

$$|||u|||_{1,h} \le \|f\|_{1,h}$$

Consistency

Let L be the differential operator, $\widehat{u}$ be a exact solution of the following equation:

$$L\widehat{u}(x, y) = f(x, y), \text{ for all } x \in \Omega$$

Let L_h be the discrete differential operator of L, and u be the discrete solution, we have

$$L_h u(x_i, y_j) = f(x_i, y_j) \text{ for all } i, j \in [1, N-1]$$

Consistency (Cont.)

Definition

A finite differential scheme is said to be consistent with the partial differential equation it present, if for any smooth solution u, the truncation error of the scheme:

$$\tau_{i,j} = L_h\widehat{u}(x_i, y_j) - f(x_i, y_j) \text{ for all } i, j \in [1, N-1]$$

tends uniformly forward to zero when h tends to zero, that mean that

$$\lim_{h\to 0} \|\tau\|_{\infty,h} = 0$$

Consistency (Cont.)

Lemma

Suppose $\widehat{u} \in C^4(\Omega)$. Then, the numerical scheme in (2) is cosistent and second-order accuracy for the norm $\|\cdot\|_\infty$

Proof: We write again the definition L, L_h operators of our case:

$$L(\widehat{u})(x_i, y_j) = -\frac{\partial^2 \widehat{u}}{\partial x^2}(x_i, y_j) - \frac{\partial^2 \widehat{u}}{\partial y^2}(x_i, y_j)$$

$$L_h(\widehat{u})(x_i, y_j) = -\frac{\widehat{u}(x_{i-1}, y_j) + \widehat{u}(x_{i+1}, y_j) - 4\widehat{u}(x_i, y_j) + \widehat{u}(x_i, y_{j-1}) + \widehat{u}(x_i, y_{j+1})}{h^2}$$

By using the fact that

$$L(\widehat{u})(x_i, y_j) = -\frac{\partial^2 \widehat{u}}{\partial x^2}(x_i, y_j) - \frac{\partial^2 \widehat{u}}{\partial y^2}(x_i, y_j) = f(x_i, y_j)$$

Consistency (Cont.)

We have

$$\tau_{i,j} = L_h(\widehat{u})(x_i, y_j) - f(x_i, y_j)$$
$$= L_h(\widehat{u})(x_i, y_j) - L(\widehat{u})(x_i, y_j)$$

Using the defintion of L and L_h, there holds

$$\tau_{i,j} = -\frac{\widehat{u}(x_{i-1}, y_j) - 2\widehat{u}(x_i, y_j) + \widehat{u}(x_{i+1}, y_j)}{h^2} + \frac{\partial^2 \widehat{u}}{\partial x^2}(x_i, y_j)$$
$$-\frac{\widehat{u}(x_i, y_{j-1}) - 2\widehat{u}(x_i, y_j) + \widehat{u}(x_i, y_{j+1})}{h^2} + \frac{\partial^2 \widehat{u}}{\partial y^2}(x_i, y_j)$$

Using the Taylor series expansion respect x, there exists $\eta_i \in [x_{i-1}, x_{i+1}]$ such that

$$-\frac{\widehat{u}(x_{i-1}, y_j) - 2\widehat{u}(x_i, y_j) + \widehat{u}(x_{i+1}, y_j)}{h^2} + \frac{\partial^2 \widehat{u}}{\partial x^2}(x_i, y_j) = \frac{-h^2}{12}\frac{\partial^4 \widehat{u}}{\partial x^4}(\eta_i, y_j)$$

Consistency (Cont.)

It is similar, there exists $\zeta_j \in [y_{j-1}, y_{j+1}]$ such that

$$-\frac{\widehat{u}(x_i, y_{j-1}) - 2\widehat{u}(x_i, y_j) + \widehat{u}(x_i, y_{j+1})}{h^2} + \frac{\partial^2 \widehat{u}}{\partial y^2}(x_i, y_j) = \frac{-h^2}{12}\frac{\partial^4 \widehat{u}}{\partial y^4}(x_i, \zeta_j)$$

Adding two previous equations, we get

$$\tau_{i,j} = -\frac{h^2}{12}\frac{\partial^4 \widehat{u}}{\partial x^4}(\eta_i, y_j) - \frac{h^2}{12}\frac{\partial^4 \widehat{u}}{\partial y^4}(x_i, \zeta_j)$$

Thus,

$$\|\tau\|_{2,h} \le \|\tau\|_{\infty,h} \le \frac{h^2}{12}\left(\|\frac{\partial^4 \widehat{u}}{\partial x^4}\|_\infty + \|\frac{\partial^4 \widehat{u}}{\partial y^4}\|_\infty\right)$$

Convergence

Lemma

Let u be the exact solution and u_h be the discrete solution, there holds

$$\lim_{h\to 0} |||\widehat{u} - u|||_{1,h} = 0.$$

Proof: We have

$$\tau_{i,j} = L_h(\widehat{u})(x_i, y_j) - f(x_i, y_j) = L_h(\widehat{u})(x_i, y_j) - L_h(u)(x_i, y_j)$$
$$= L_h(\widehat{u} - u)(x_i, y_j)$$

Using the proof of stability, we have

$$|||\widehat{u} - u|||_{1,h} \le \|\tau\|_{2,h} \le \frac{h^2}{12}\left(\|\frac{\partial^4 \widehat{u}}{\partial x^4}\|_\infty + \|\frac{\partial^4 \widehat{u}}{\partial y^4}\|_\infty\right)$$