CTT310: Digital Image Processing

Filtering in the Frequency Domain

Dr. Nguyen Ngoc Thao
Department of Computer Science, FIT, HCMUS

Outline

- The basics of filtering in the frequency domain
- Image enhancement using frequency domain filters
  - Image smoothing
  - Image sharpening
  - Selective filtering

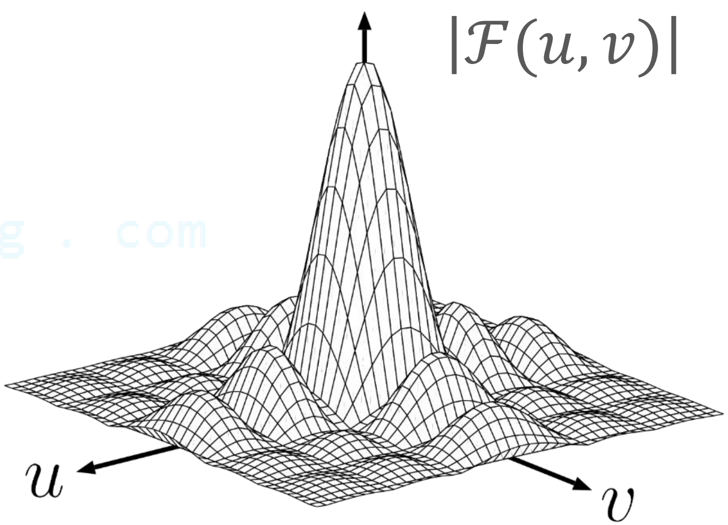
Section 4.1

THE BASICS OF FILTERING IN THE FREQUENCY DOMAIN

cuu duong than cong . com

Frequency Domain: Additional characteristics

- It is intuitively to associate frequencies in the Fourier Transform with patterns of intensity variations in an image
- The slowest varying frequency component $(u = v = 0)$ is proportional to the average intensity of an image
- Low frequencies $\rightarrow$ slowly varying intensity components, higher frequencies $\rightarrow$ faster intensity changes
  - E.g., (wall and floor) vs. (edges and other abrupt changes in intensity)

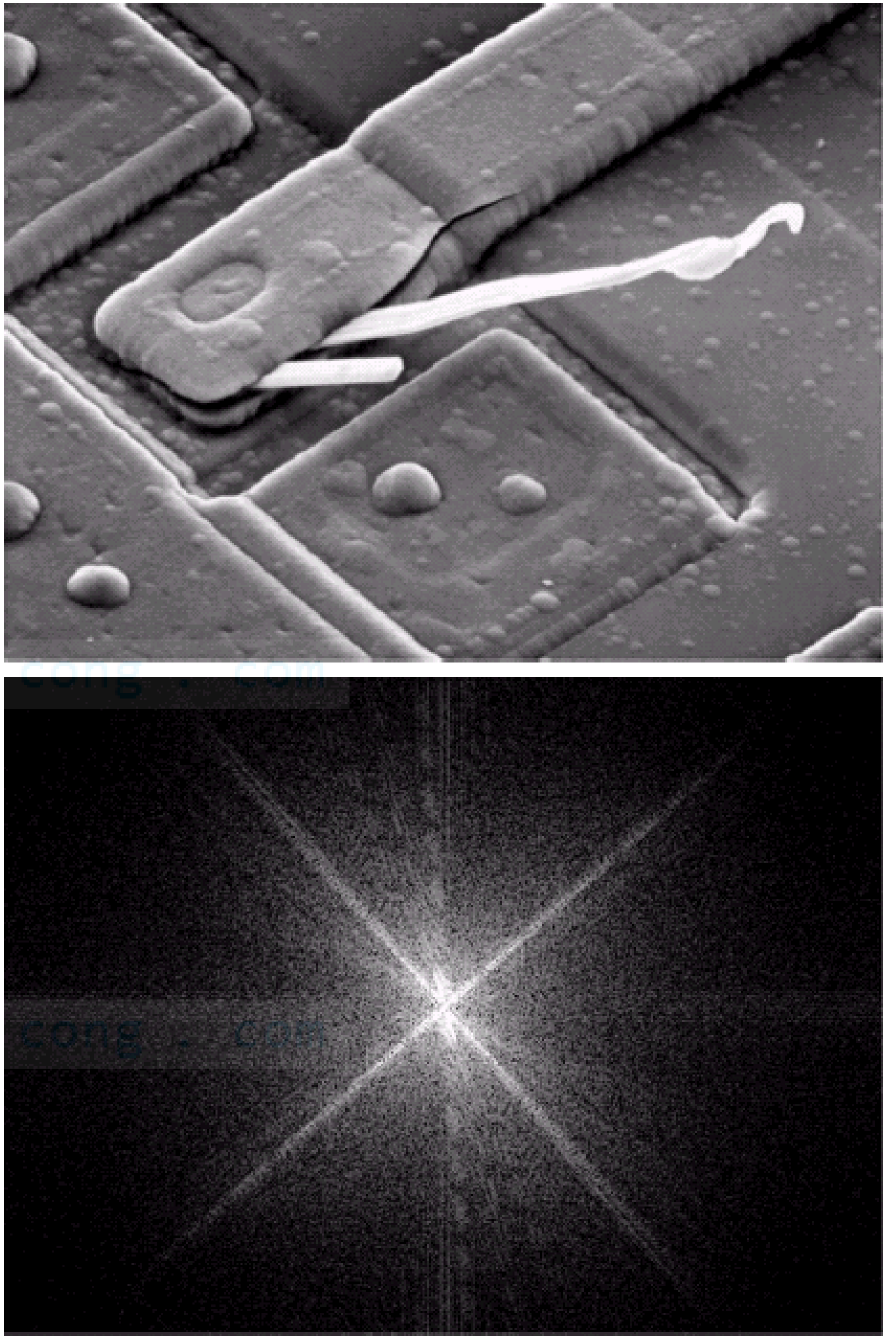


Frequency Domain: Additional characteristics

- The transform magnitude (spectrum) determines the amplitudes of the sinusoids that combine to form the resulting image
  - The information of intensities in the image $\rightarrow$ gross characteristics of the image from which the spectrum was generated.
- Meanwhile, the phase is a measure of displacement of the various sinusoids with respect to their origin
  - It carries much of the information about where discernable objects are located in the image
  - Visual analysis of the phase component generally is not very useful

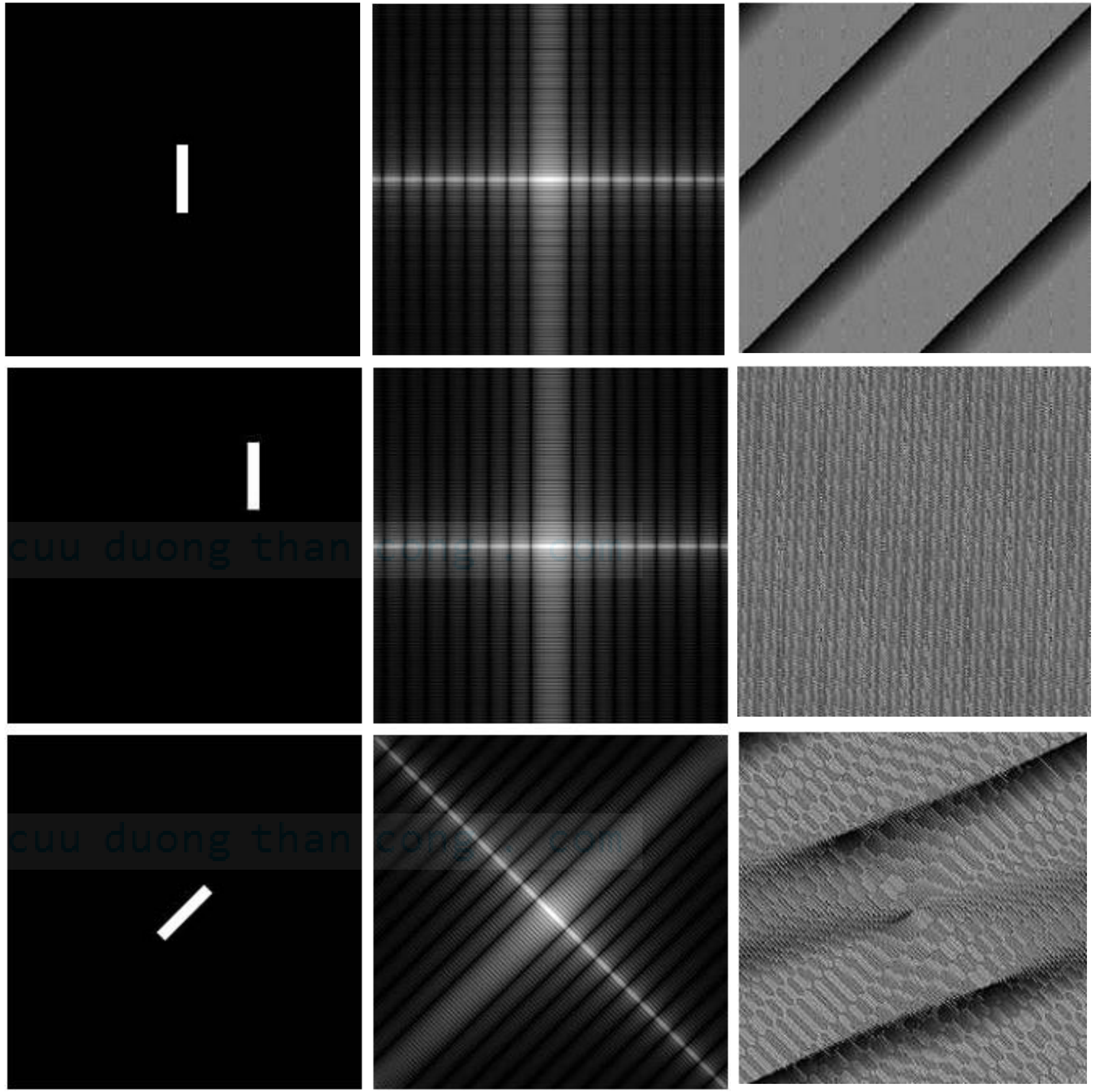
a b

(a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)



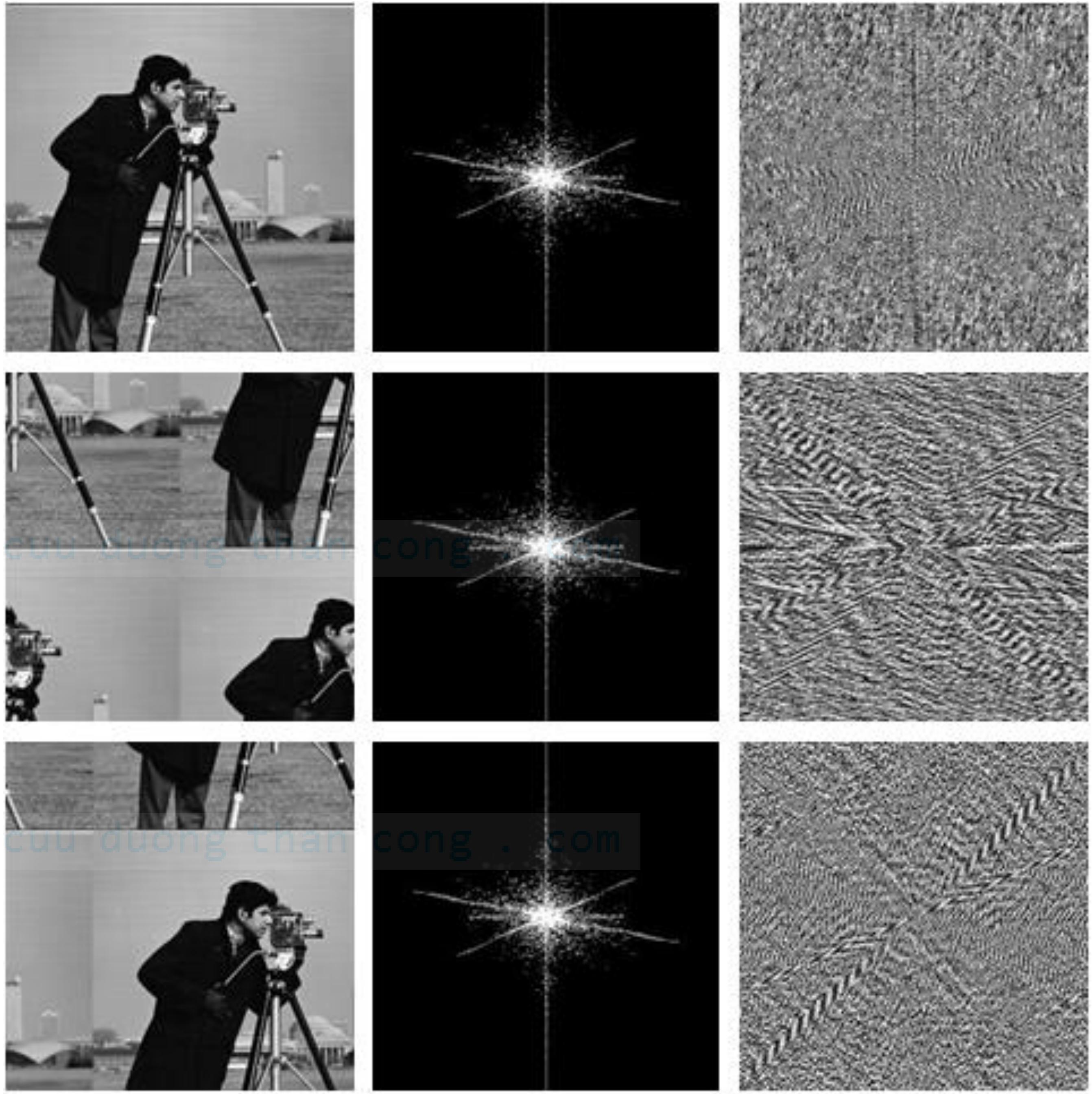
a	b	c
d	e	f
g	h	i

(a) Original image. (b)-(c) The corresponding spectrum and phase angle of (a)
(d) Translated image of (a). (e)-(f) The corresponding spectrum and phase angle of (d)
(g) Rotated image of (a). (h)-(i) The corresponding spectrum and phase angle of (g)



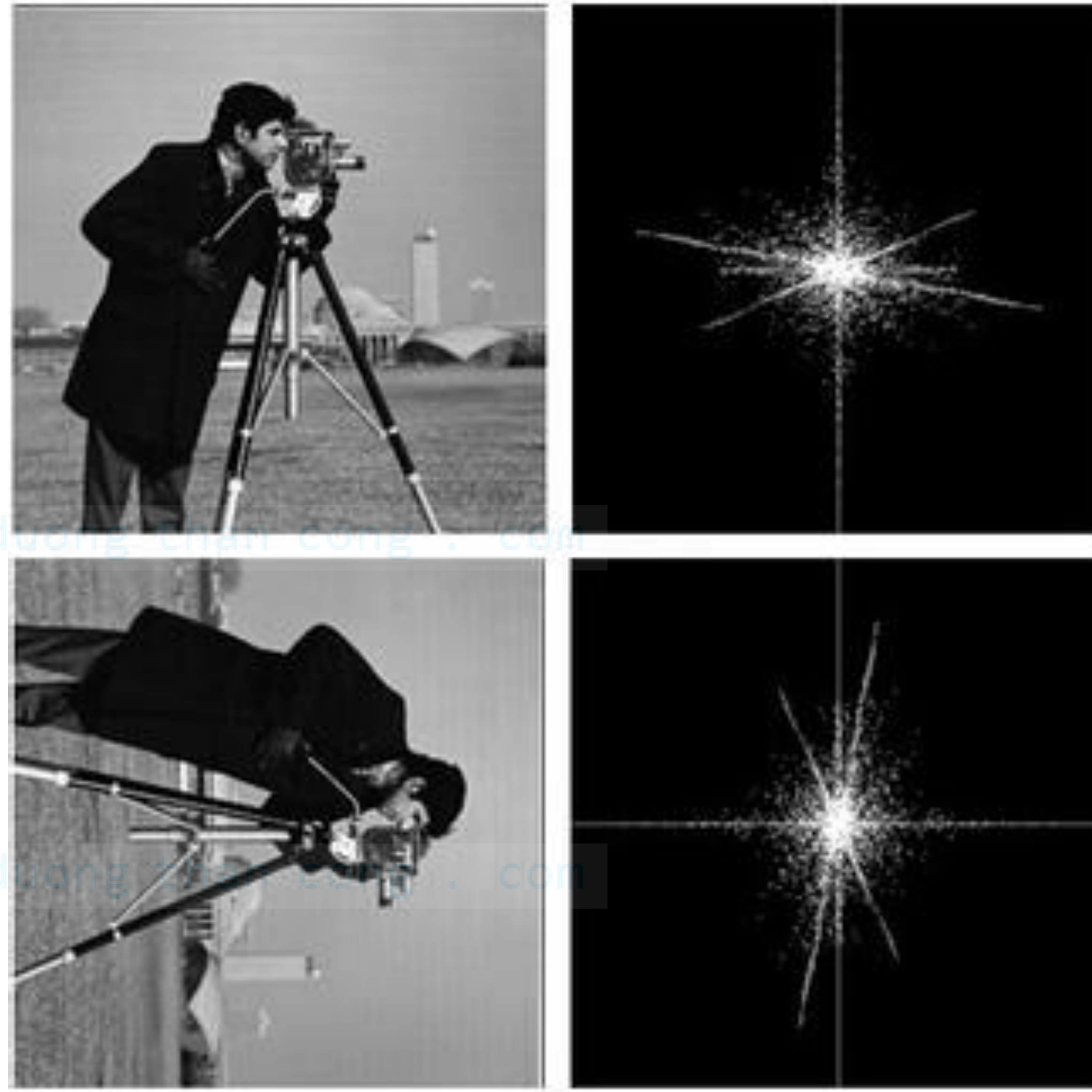
a	b	c
d	e	f
g	h	i

(a) Original image. (b)-(c) The corresponding spectrum and phase angle of (a)
(d) The image (a) is translated 128 rows and 128 columns. (e)-(f) The corresponding spectrum and phase angle of (d)
(g) The image (a) is translated 64 rows and 64 columns. (h)-(i) The corresponding spectrum and phase angle of (g)



a	b
c	d

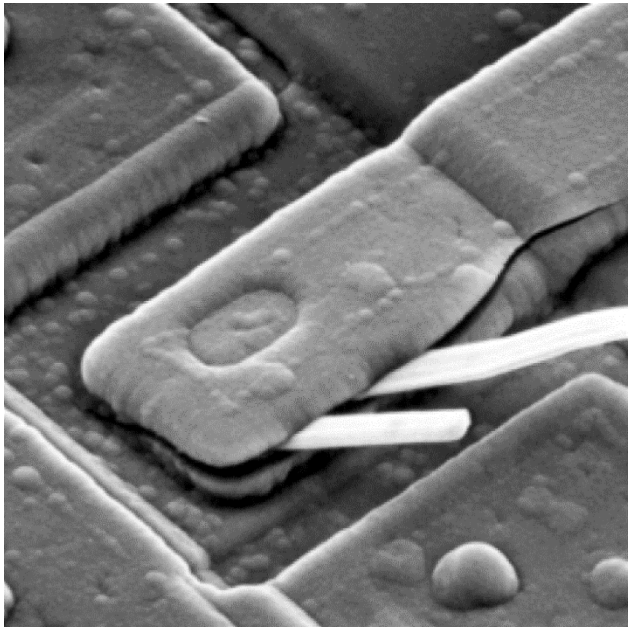
(a) The original image. (b) Fourier spectrum of (a). (c) Image resulting from rotating (a) 90^0. (d) Fourier spectrum of (c).

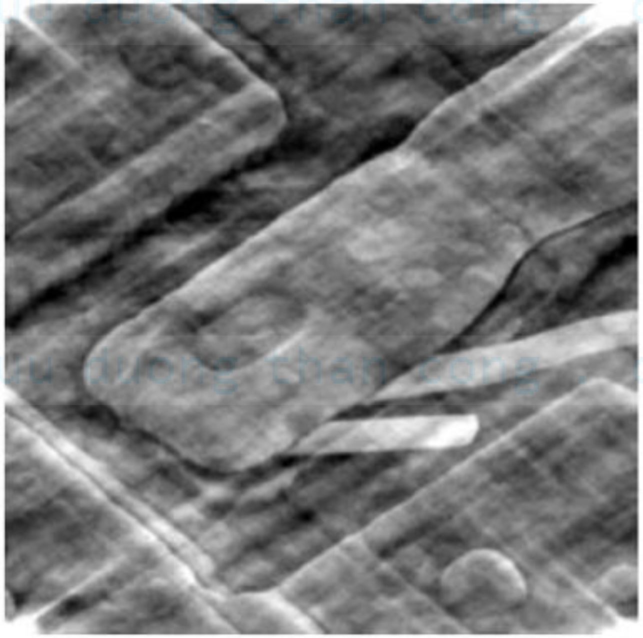


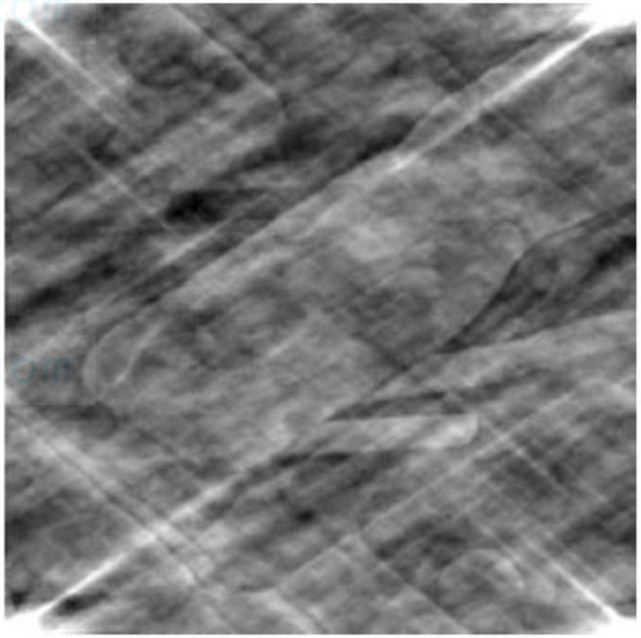
a

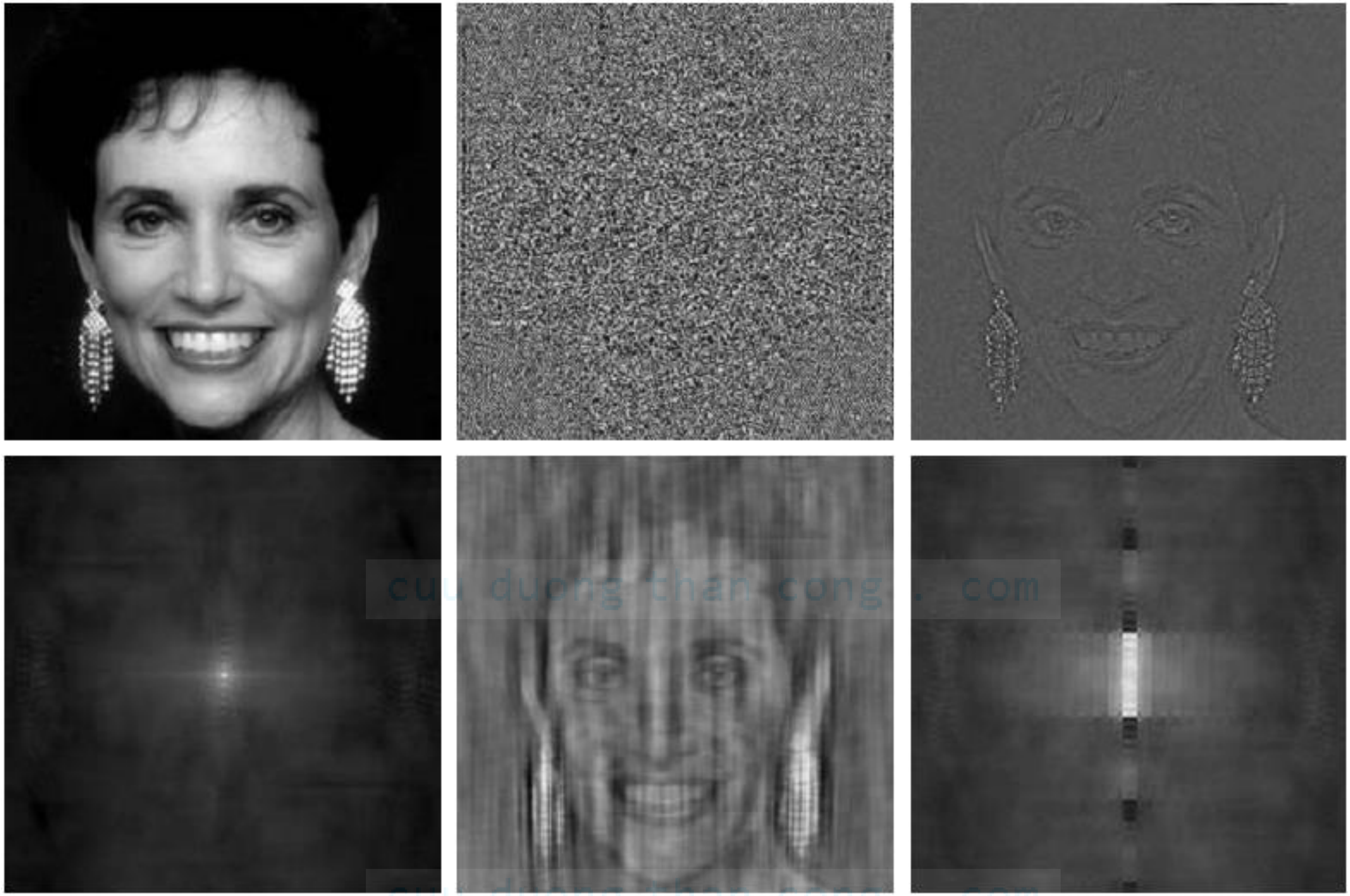
b c

(a) The original image. (b)-(c) Image resulting from multiplying phase angle with 0.5 and 0.25 then performing IDFT, respectively. The spectrum is unchanged in two cases.









a	b	c
d	e	f

(a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle (a) in slide 7. (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.

Frequency domain filtering fundamentals

- The filtering process in the frequency domain is based on the convolution property of the Fourier Transform

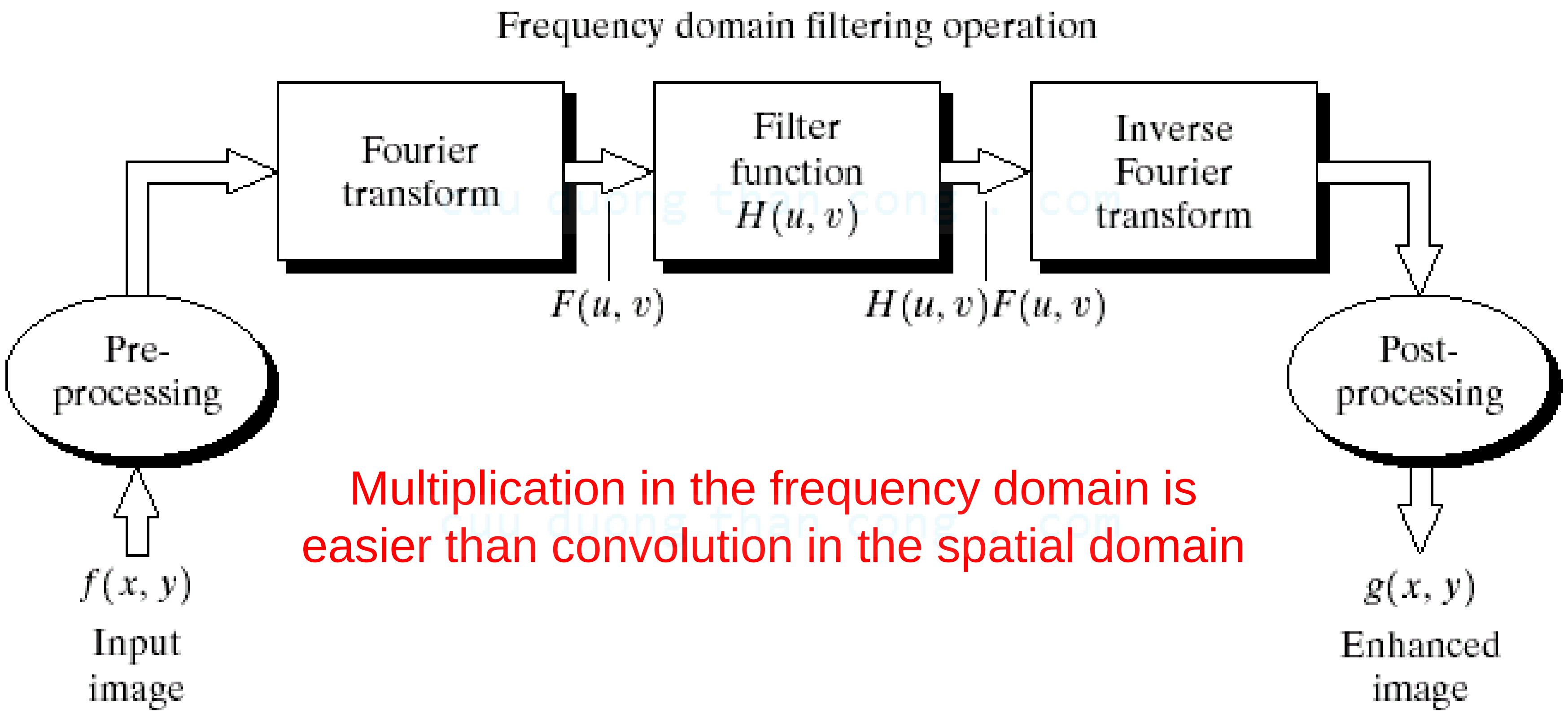
$$g(x,y) = f(x,y) \star h(x,y)$$

$$\Updownarrow$$

$$\mathcal{F}(u,v)\mathcal{H}(u,v) = G(u,v)$$

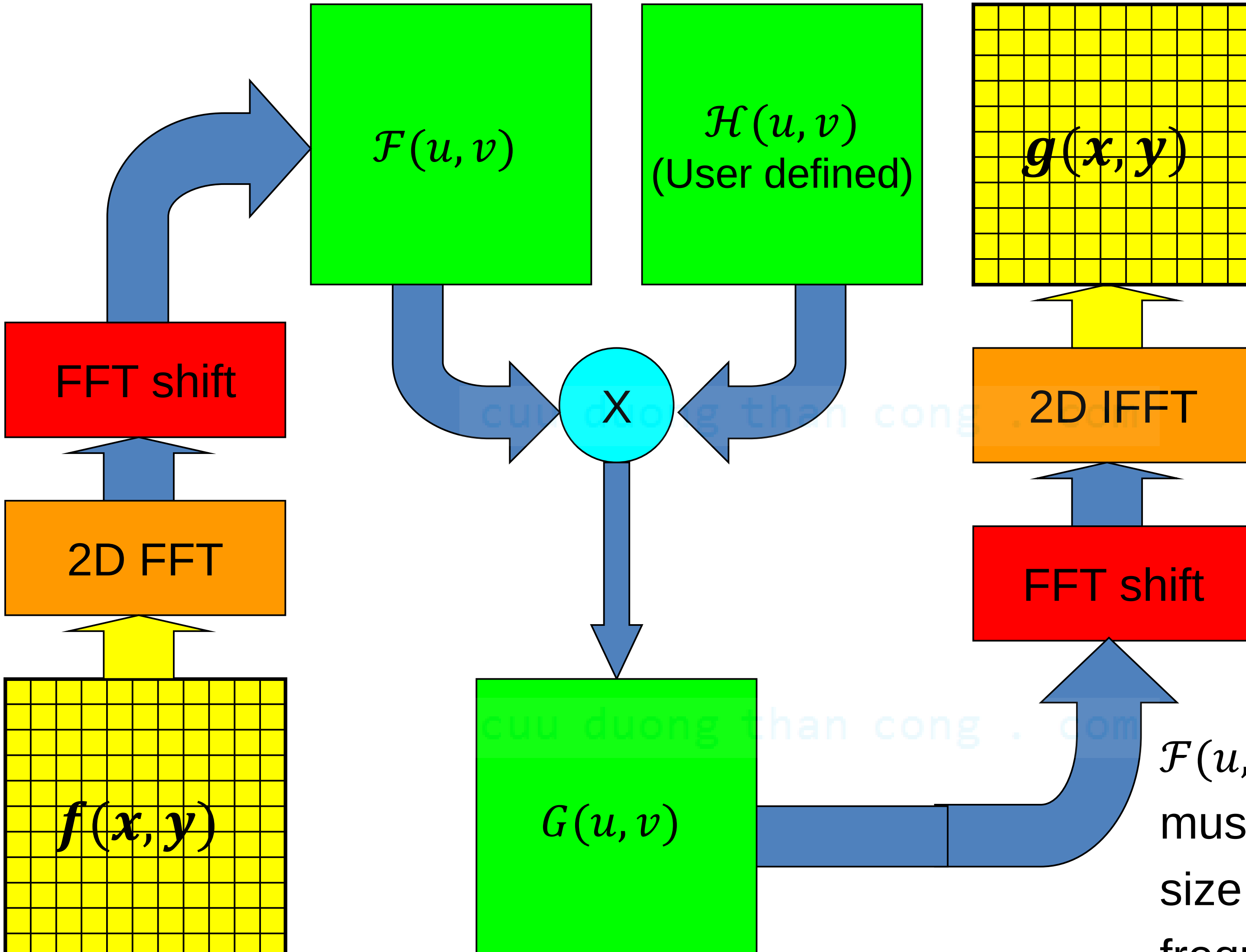
Frequency domain filtering fundamentals

- Modify the Fourier Transform to achieve a specific objective
- Compute the inverse DFT to get back to the image domain



Multiplication in the frequency domain is easier than convolution in the spatial domain

Filtering in the frequency domain with FFT shift



$\mathcal{F}(u,v)$ and $\mathcal{H}(u,v)$ must have the same size and have the zero frequency at the center

Multiplication in the frequency domain

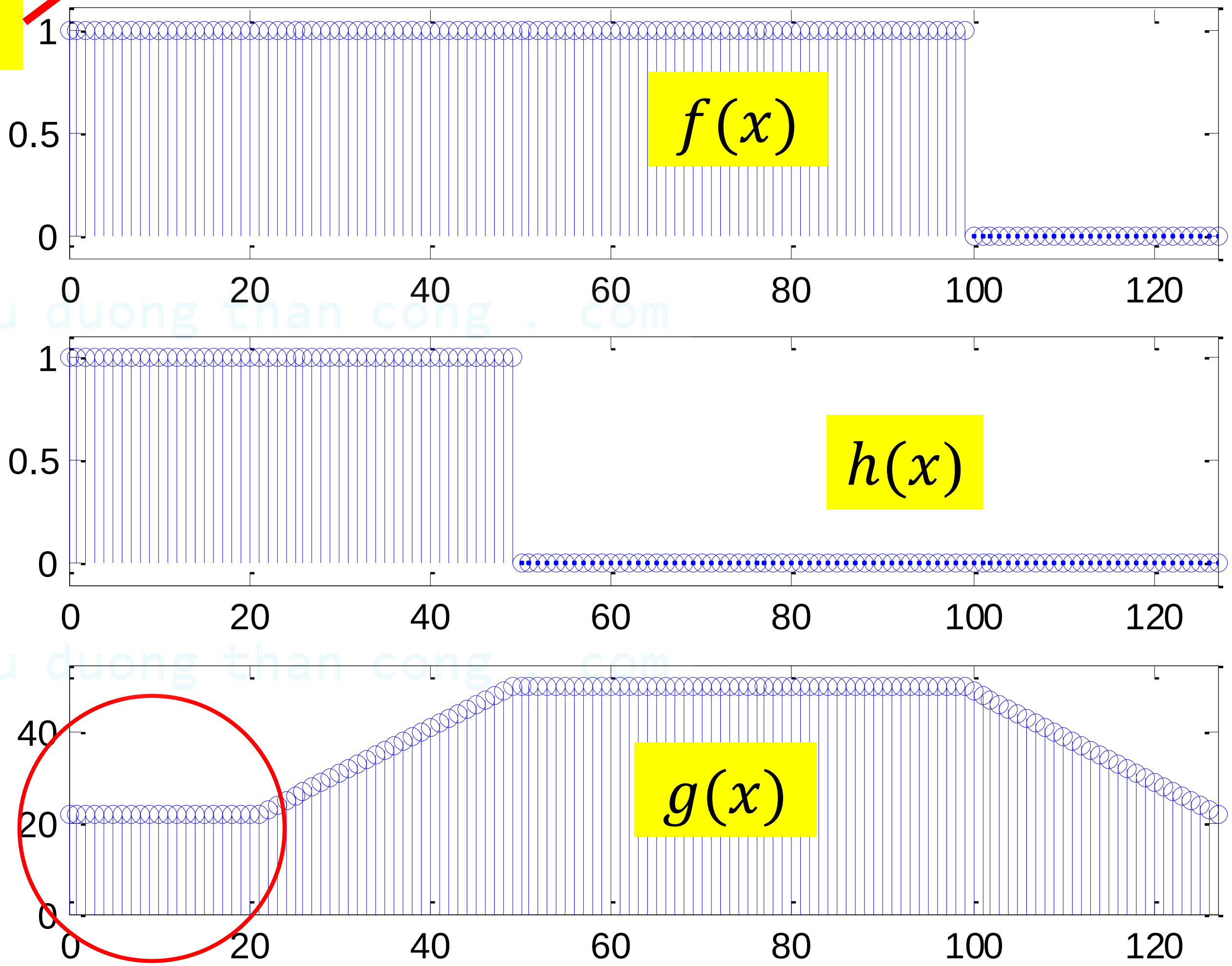
$f(x)$ → DFT → $\mathcal{F}(u)$

$h(x)$ → DFT → $\mathcal{H}(u)$

$G(u) = \mathcal{F}(u)\mathcal{H}(u)$ → IDFT → $g(x)$

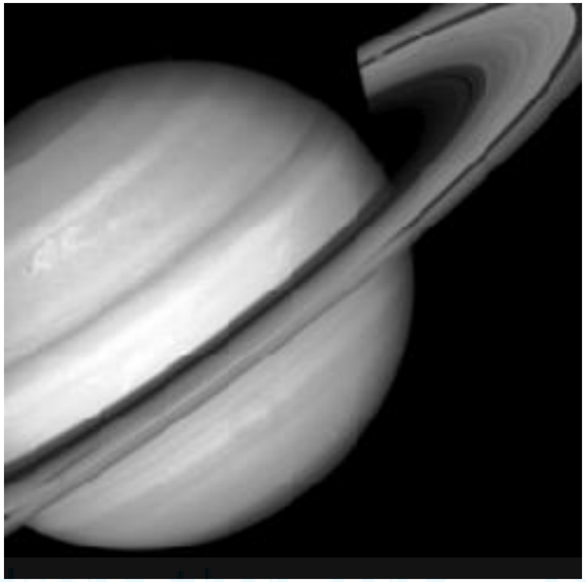
Multiplication of DFTs of 2 signals is equivalent to perform circular convolution in the spatial domain.

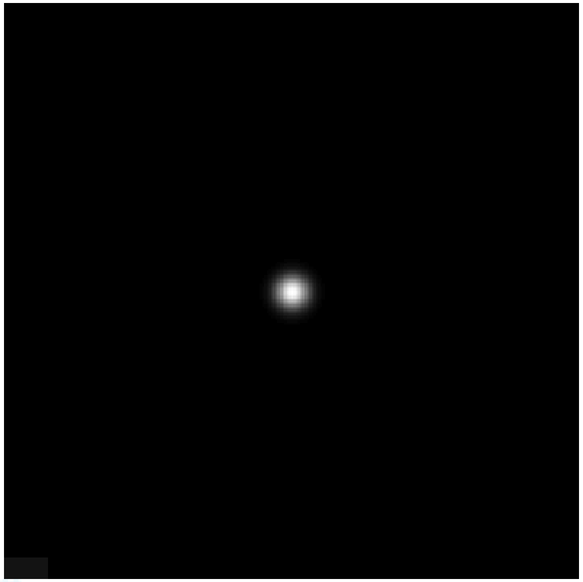
"Wrap around" effect

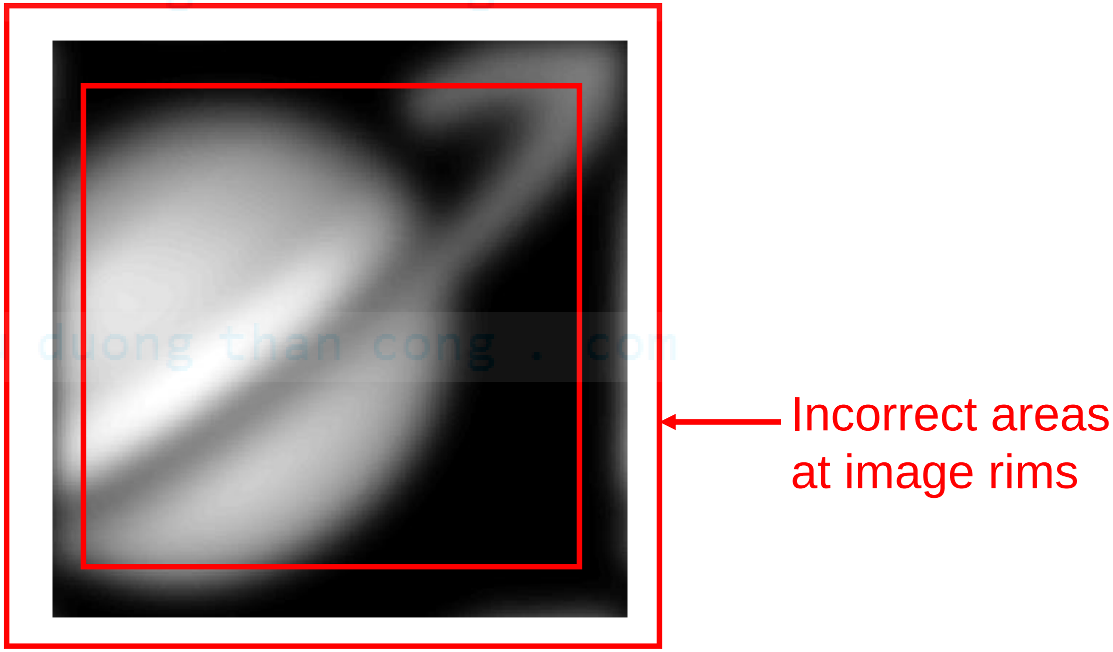


a b
c

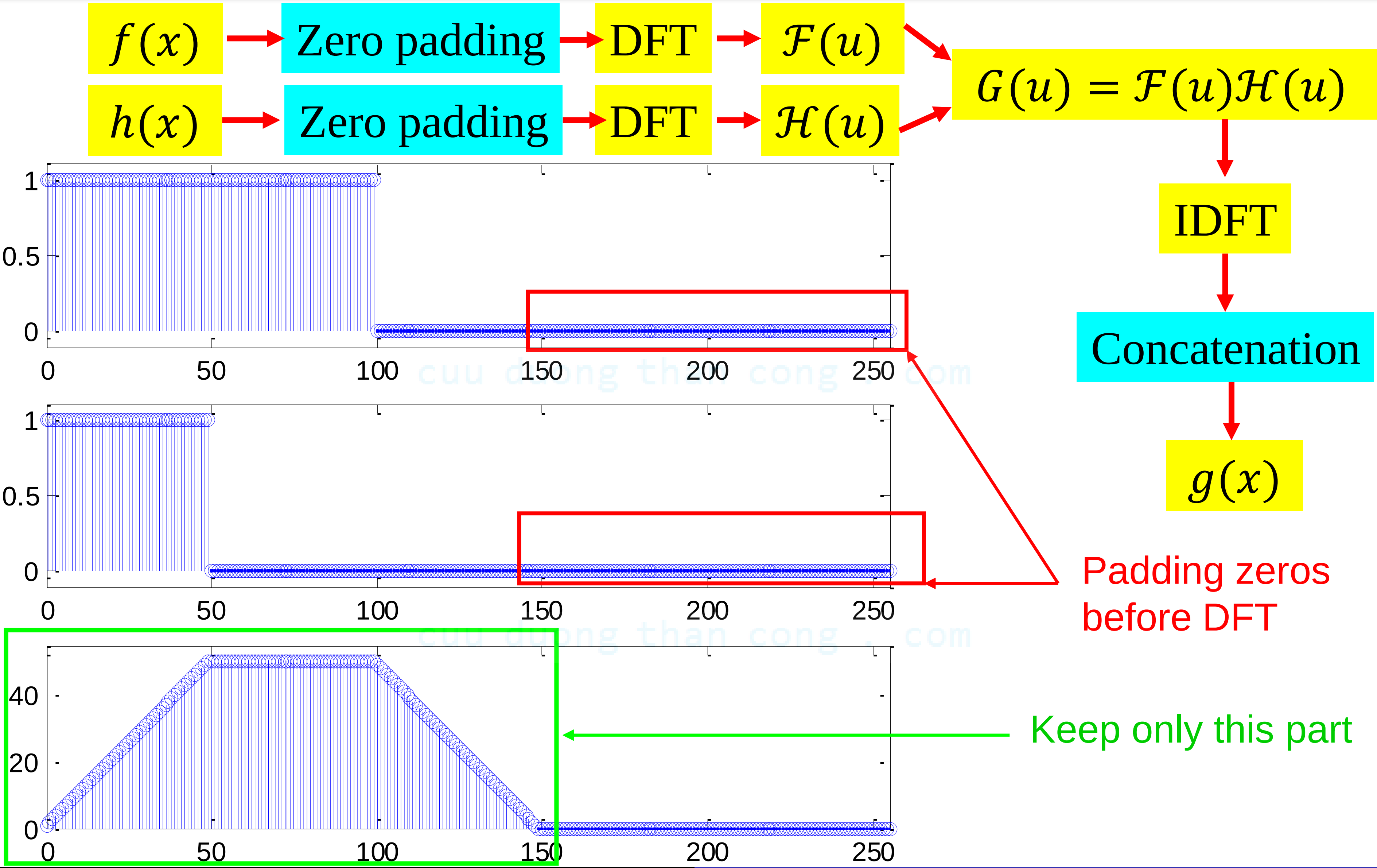
(a) Original image. (b) Gaussian lowpass $H(u, v)$ with $D_0 = 5$. (c) Filtered image (obtained using circular convolution)





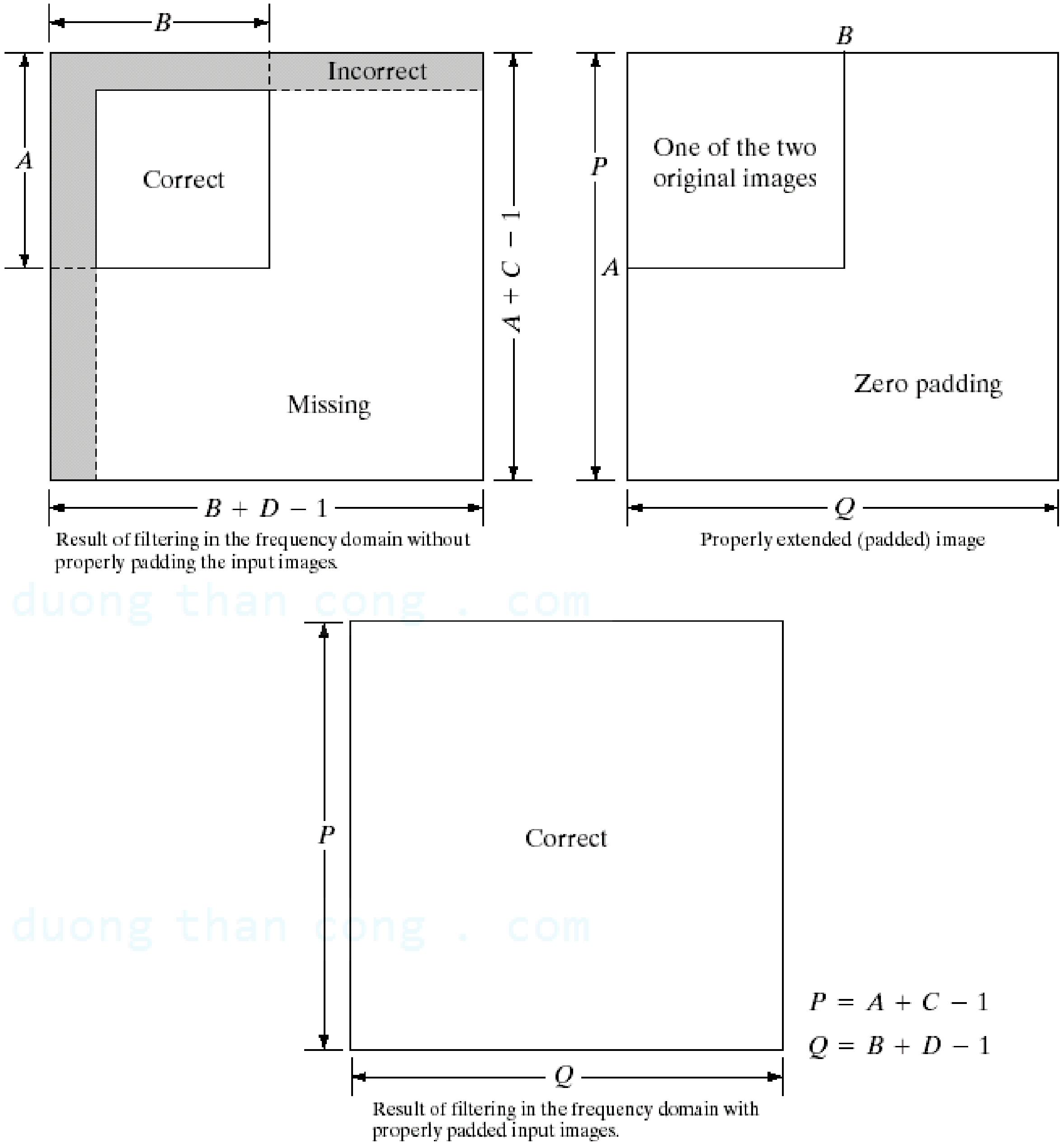


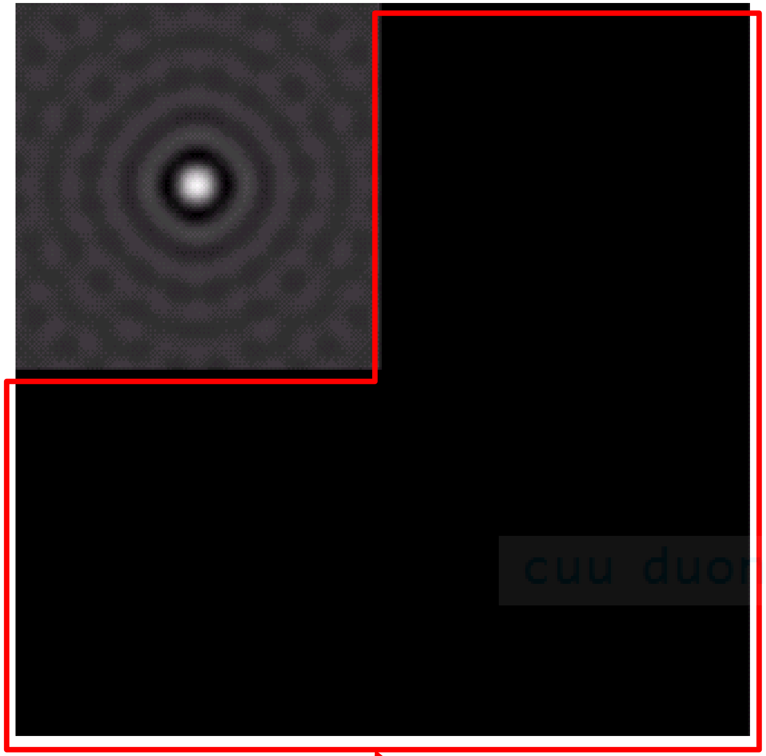
Multiplication in the frequency domain



a b
c

Illustration of the need for function padding. (a) Result of performing 2-D convolution without padding. (b) Proper function padding. (c) Correct convolution result.





Filtered image

Zero padding area in the spatial domain of the mask image (the ideal lowpass filter)

Only this area is kept

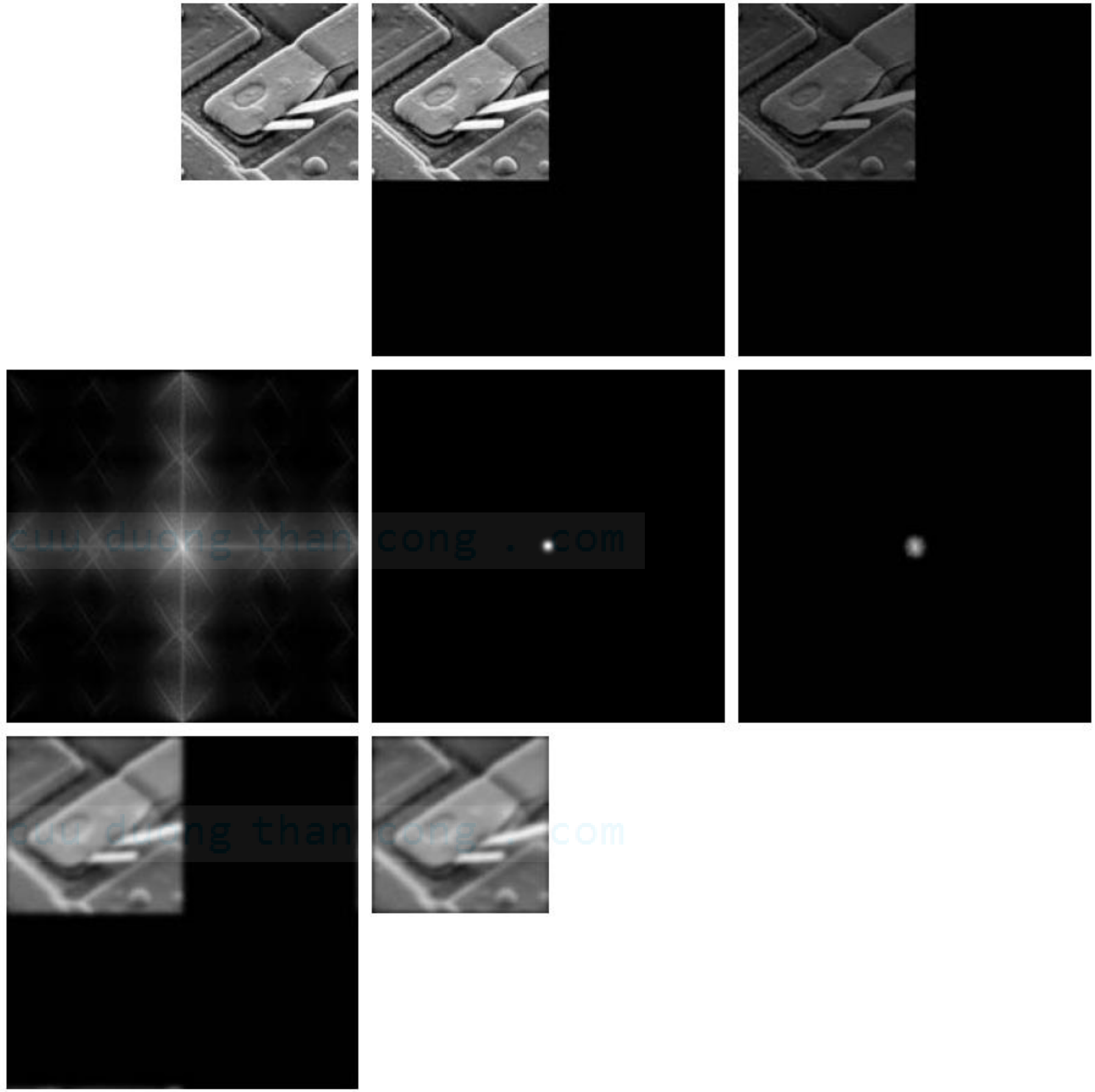
Steps for Filtering in the frequency domain

1. Given an input image $f(x, y)$ of size $M \times N$, obtain the padding parameters P and Q from Eqs. (4.6-31) and (4.6-32). Typically, we select $P = 2M$ and $Q = 2N$.
2. Form a padded image, $f_p(x, y)$, of size $P \times Q$ by appending the necessary number of zeros to $f(x, y)$.
3. Multiply $f_p(x, y)$ by $(-1)^{x+y}$ to center its transform.
4. Compute the DFT, $F(u, v)$, of the image from step 3.
5. Generate a real, symmetric filter function, $H(u, v)$, of size $P \times Q$ with center at coordinates $(P/2, Q/2)$.† Form the product $G(u, v) = H(u, v)F(u, v)$ using array multiplication; that is, $G(i, k) = H(i, k)F(i, k)$.
6. Obtain the processed image:

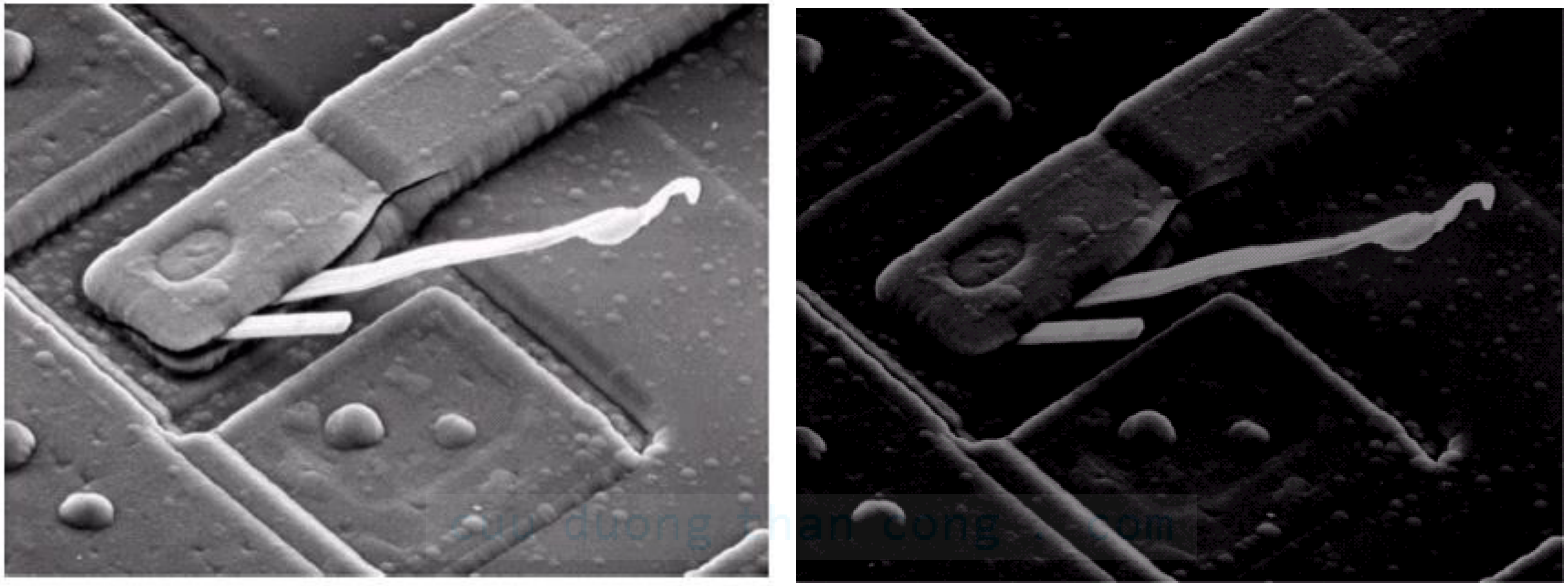
$$g_p(x, y) = \left\{\text{real}\left[\Im^{-1}[G(u, v)]\right]\right\}(-1)^{x+y}$$

   where the real part is selected in order to ignore parasitic complex components resulting from computational inaccuracies, and the subscript p indicates that we are dealing with padded arrays.
7. Obtain the final processed result, $g(x, y)$, by extracting the $M \times N$ region from the top, left quadrant of $g_p(x, y)$.

a	b	c
d	e	f
g	h	

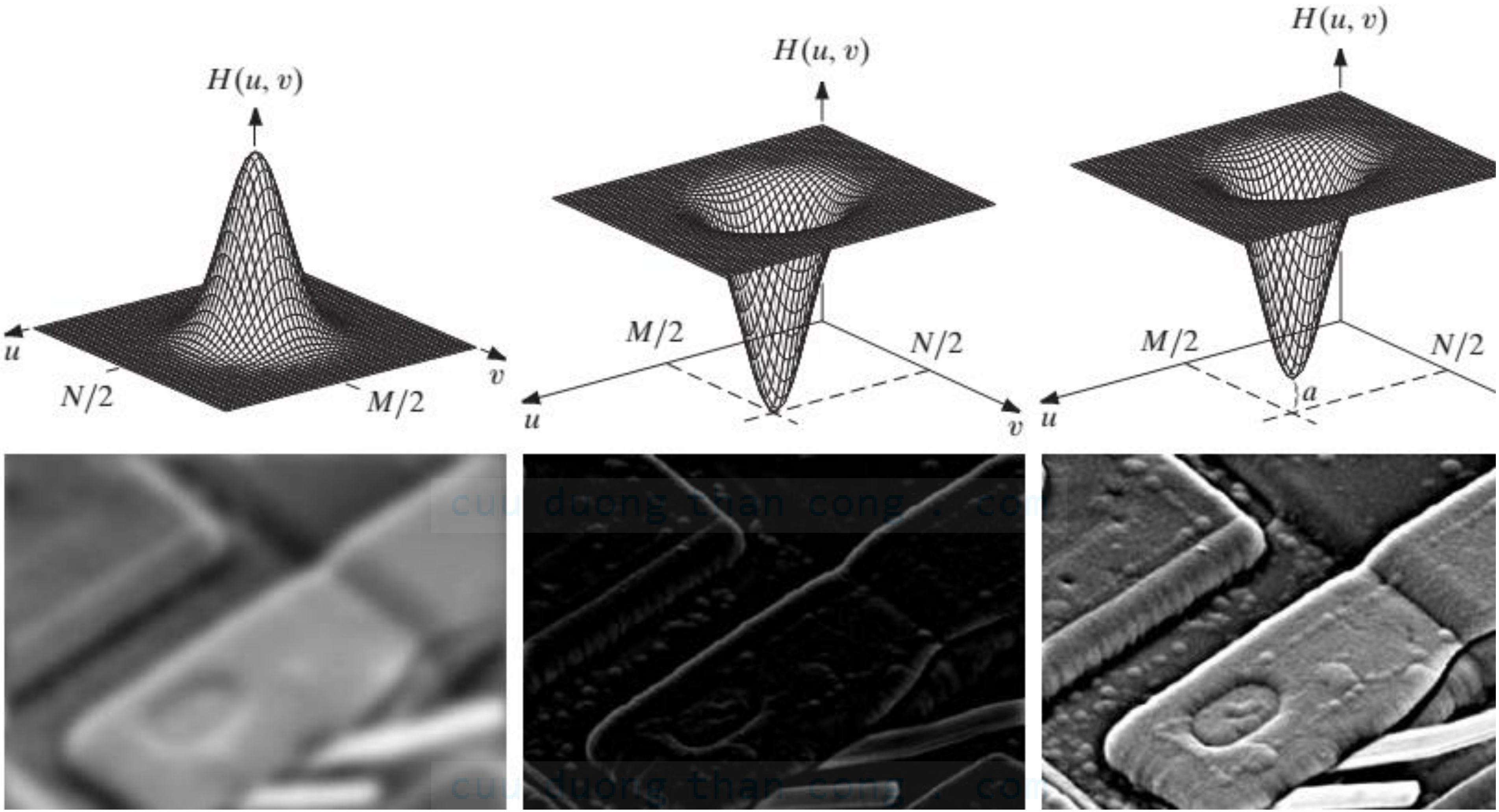


(a) An $M \times N$ image, f. (b) Padded image, f_p, of size $P \times Q$. (c) Result of multiplying f_p by $(-1)^{x+y}$. (d) Spectrum of F_p. (e) Centered Gaussian lowpass filter, H, of size $P \times Q$. (f) Spectrum of the product HF_p. (g) g_p, the product of $(-1)^{x+y}$ and the real part of the IDFT of HF_p, (h) Final result, g, obtained by cropping the first M rows and N columns of g_p



a b (a) Original image. (b) Result of filtering (a) by setting to 0 the term (M 2, N 2) in the Fourier transform.

- In this example, $\mathcal{F}(0,0)$ is set to zero $\rightarrow$ the zero frequency component is removed
  - Zero frequency = average intensity of an image



a	b	c
d	e	f

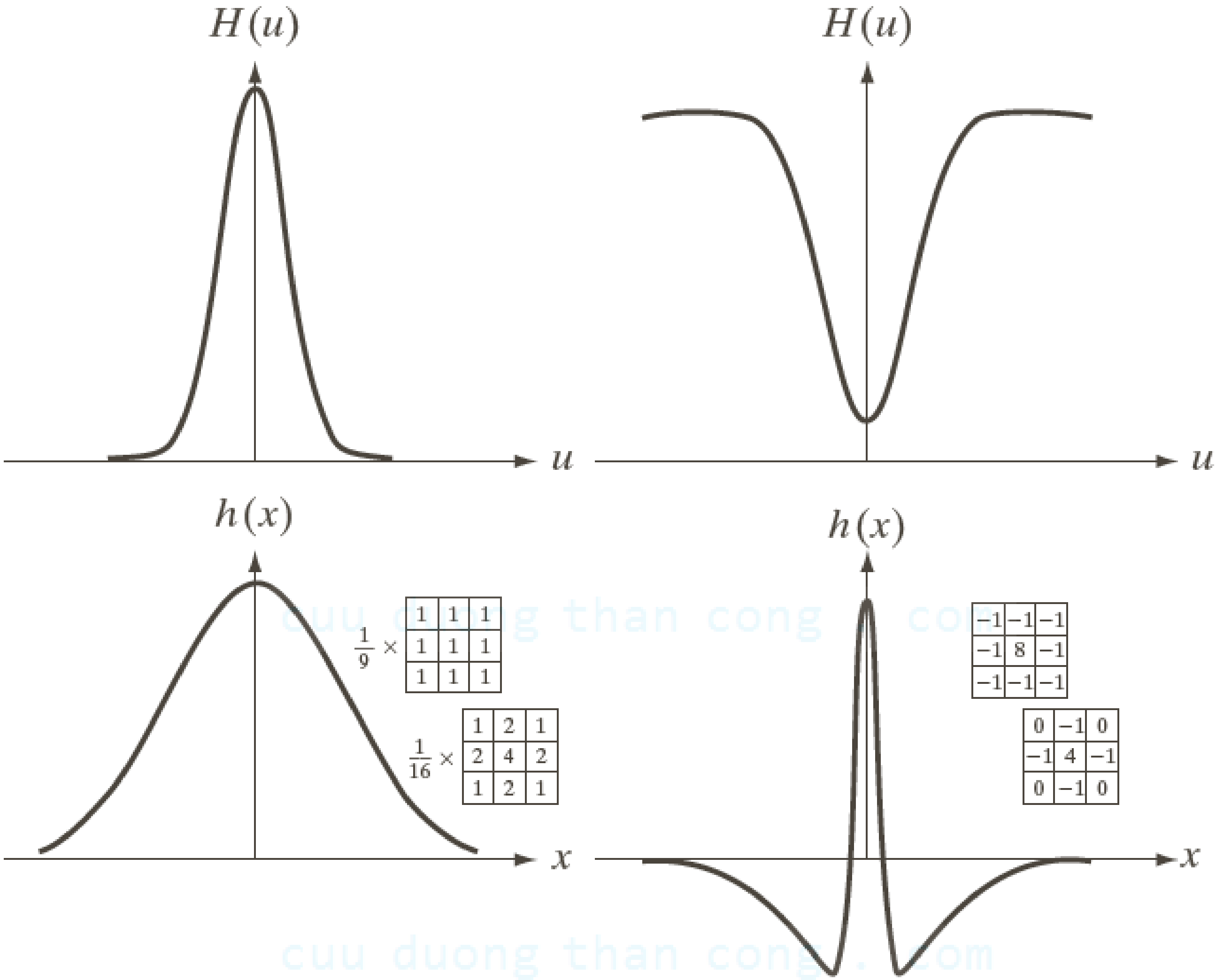
Top row: frequency domain filters – (a) lowpass filter, (b) highpass filfer, (c) Set a = 0.85 to modify (b) (the height of the filter itself is 1). Bottom row: corresponding filtered images. Compare (f) with the original image

Spatial filter and their Fourier Transforms

- The spatial filter and the frequency domain filter are related to each other through **the convolution theorem**

$$\boldsymbol{h(x,y) \Leftrightarrow H(u,v)}$$

- The narrower the frequency domain filter, the more it will attenuate the high frequencies to increase blurring.
- A larger mask in the spatial domain must be used to yield the same effect

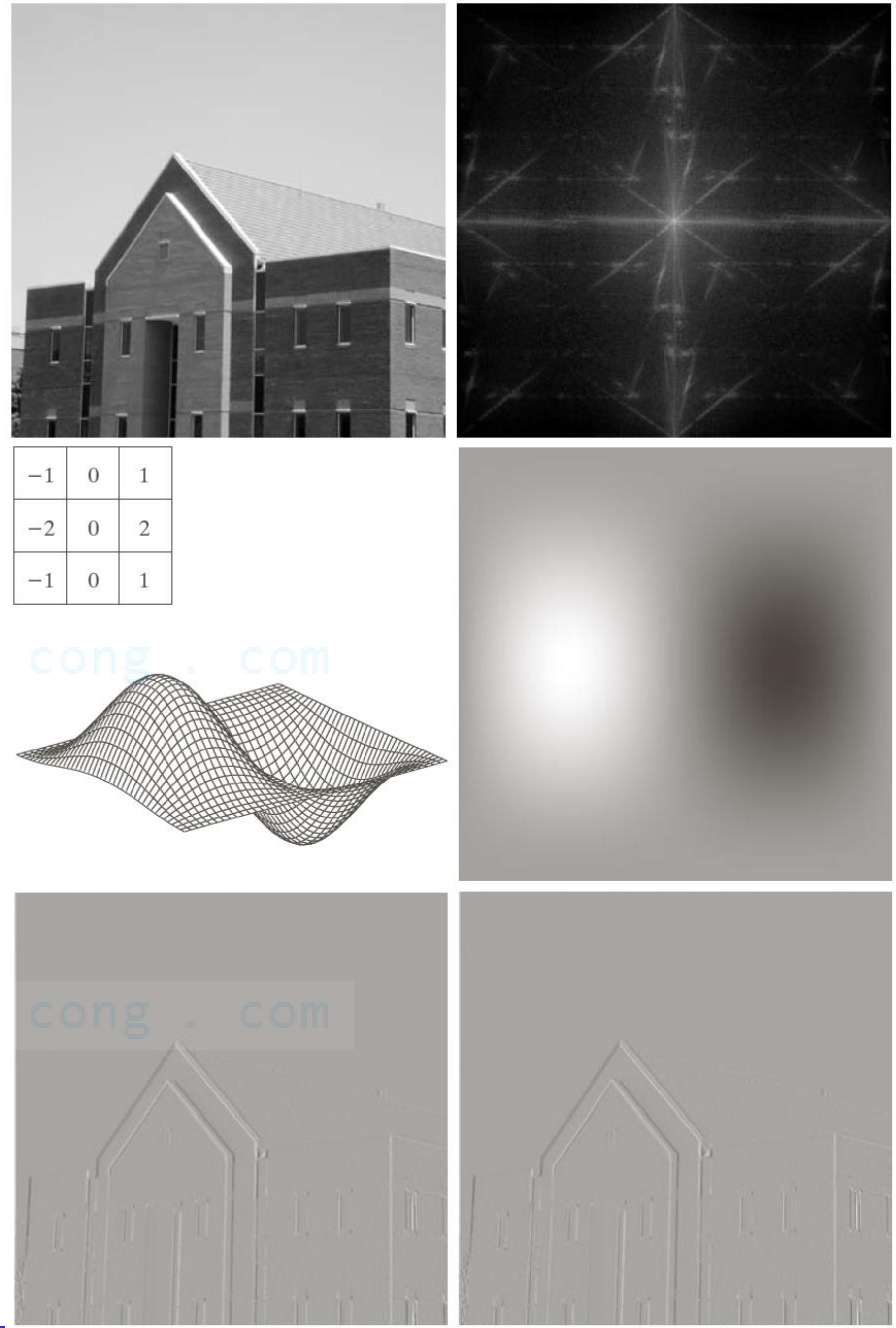


a	c
b	d

(a) A 1-D Gaussian lowpass filter in the frequency domain. (b) Spatial lowpass filter corresponding to (a). (c) Gaussian highpass filter in the frequency domain. (d) Spatial highpass filter corresponding to (c).

a	b
c	d
e	f

(a) Original image. (b) Spectrum of (a). (c) A spatial mask and perspective plot of its corresponding frequency domain filter. (d) Frequency domain filter shown as an image. (e) Result of filtering (a) in the frequency domain with the filter in (d). (f) Result of filtering the same image with the spatial filter in (c). The results are identical.



−1	0	1
−2	0	2
−1	0	1

Section 4.2

IMAGE SMOOTHING USING FREQUENCY DOMAIN FILTERS

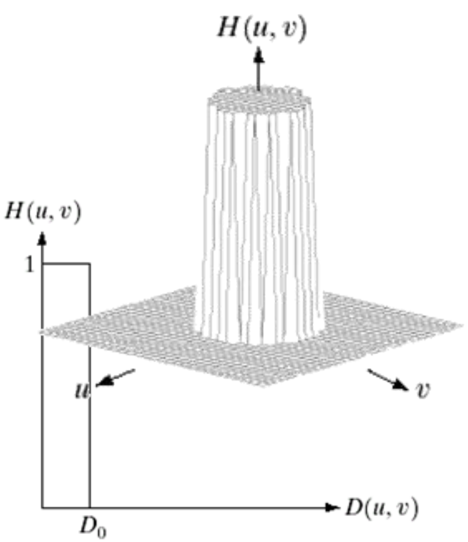
The frequency domain filter $H(u,v)$

- $H(u,v)$ is a discrete function of size $\mathrm{P} \times Q$ (after padding 0s)
  - I.e. the discrete frequency variables are in the range $u = 0, 1, 2, \ldots, P-1$, $v = 0, 1, 2, \ldots, Q-1$
- $H(u,v)$ are zero-phase-shift and radially symmetric
  - The real and imaginary parts are treated equally → the phase is not affected.
  - The zero frequency is at the center of the frequency rectangle
- Filtering with $H(u,v)$ to achieve smoothing and sharpening effects follows the aforementioned steps (see slide 20)

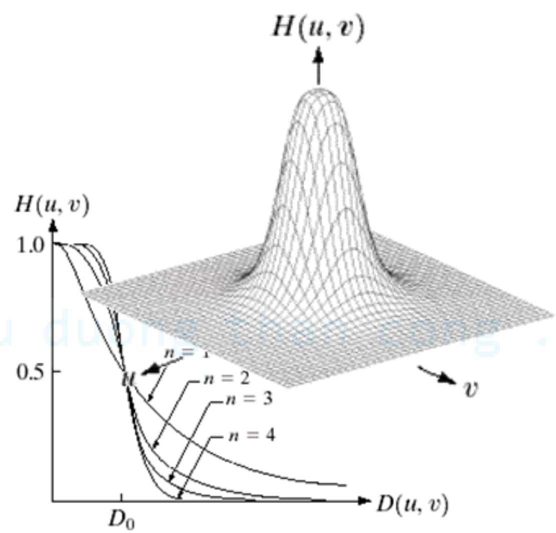
Lowpass filtering

- Smoothing (blurring) is achieved in the frequency domain by high-frequency attenuation, called lowpass filtering
  - Edges and other sharp intensity transitions (e.g. noise) contribute significantly to the high-frequency content of the Fourier Transform

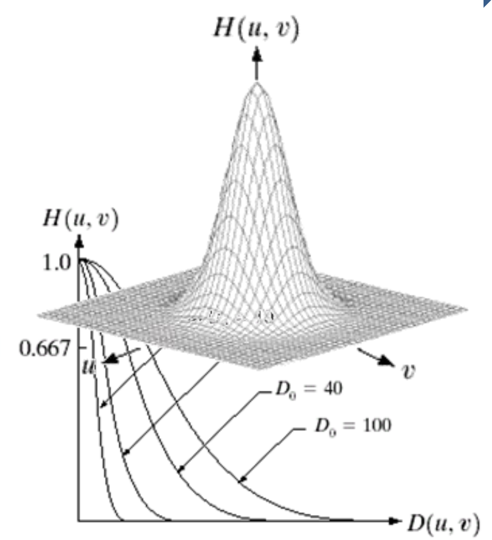
Very sharp — TRANSITION — Very smooth



Ideal lowpass filter



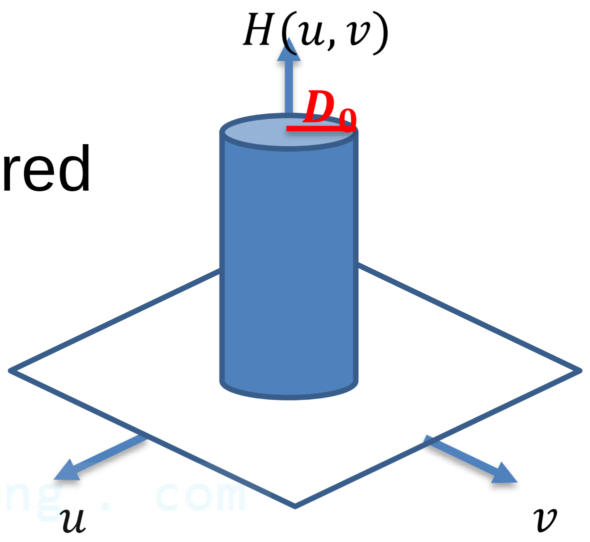
Butterworth lowpass filter



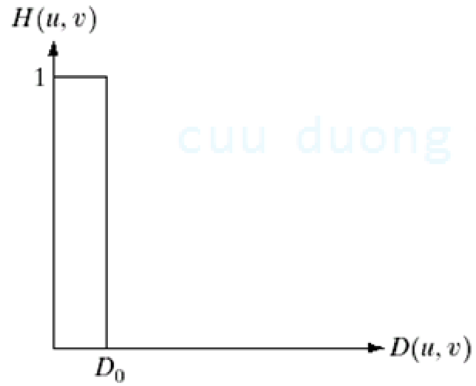
Gaussian lowpass filter

Basic notations: Cut-off frequency D_0

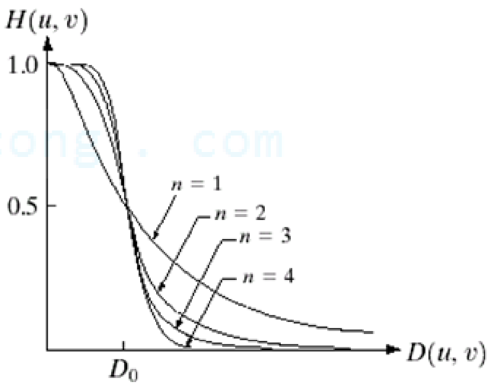
- Positive constant
- Define a circle of radius D_0 centered at the origin



- The point of transition between $H(u,v) = 1$ and $H(u,v) = 0$
  - Sharp transition (ILPF) or gradual transition (BLPF and GLBF)



Ideal lowpass filter



Butterworth lowpass filter

Basic notations: Distance $D(u, v)$

- $D(u, v)$ is the distance between a point (u, v) in the frequency domain and the center of the frequency rectangle

$$D(u,v) = \left[\left(u - \frac{P}{2}\right)^2 + \left(v - \frac{Q}{2}\right)^2\right]^{1/2}$$

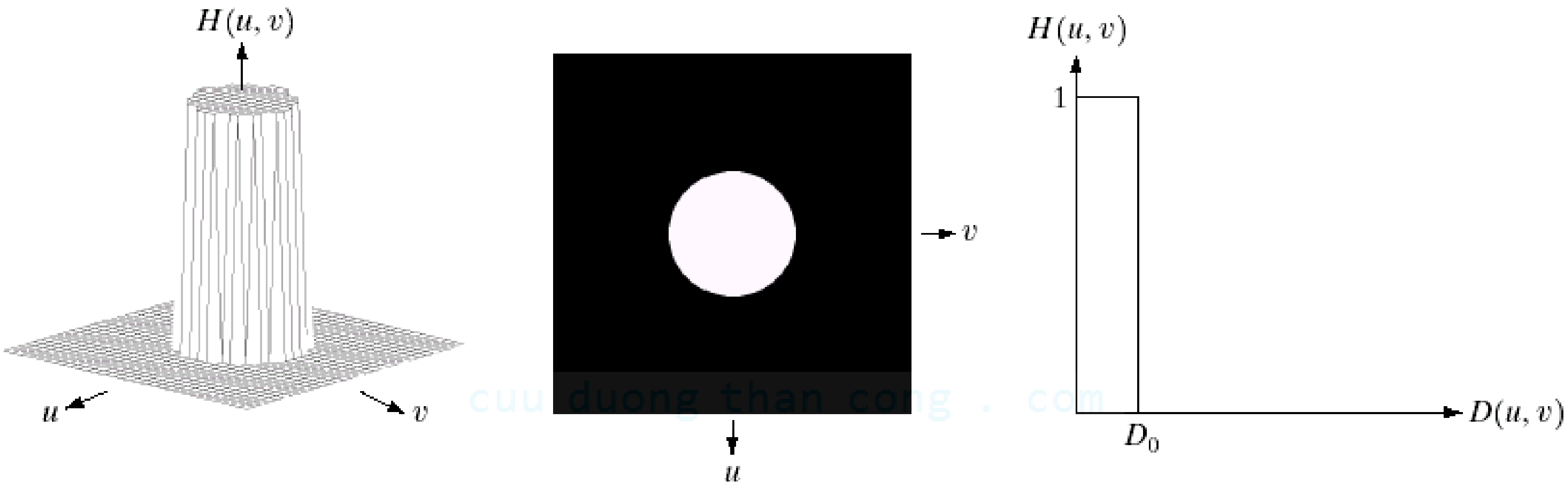
  - P and Q are the padded sizes

Ideal lowpass filter (ILPF)

- Pass without attenuation all frequencies within a circle of radius D_0 from the origin and “cuts off” all frequencies outside this circle

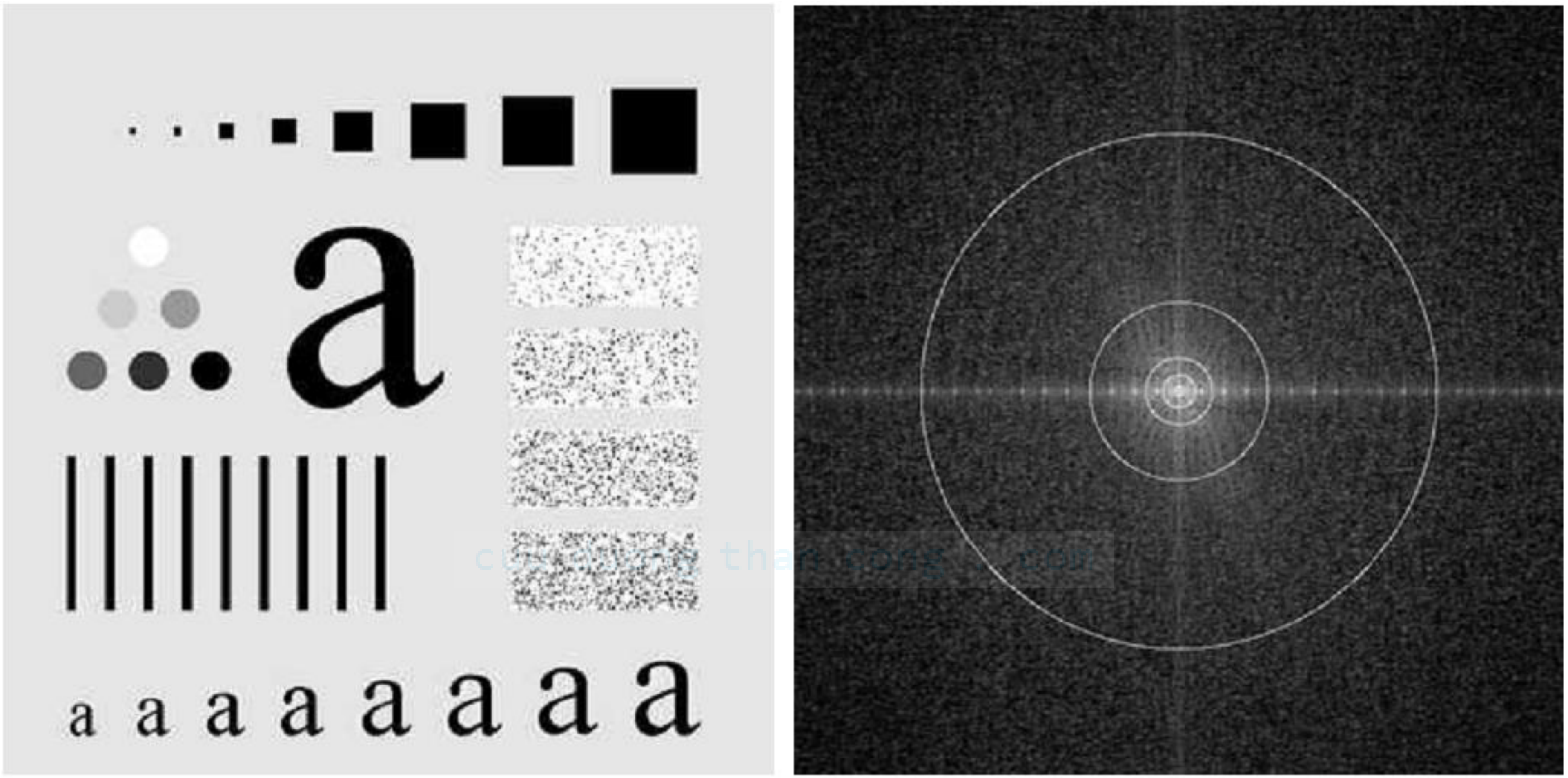
$$H(u,v) = \begin{cases} 1 & if\ D(u,v) \le D_0 \\ 0 & if\ D(u,v) > D_0 \end{cases}$$

- The sharp cutoff frequencies of an ILPF can only be simulated in a computer, it cannot be realized with electronic components

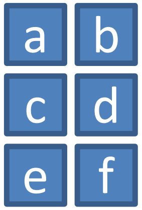


a b c

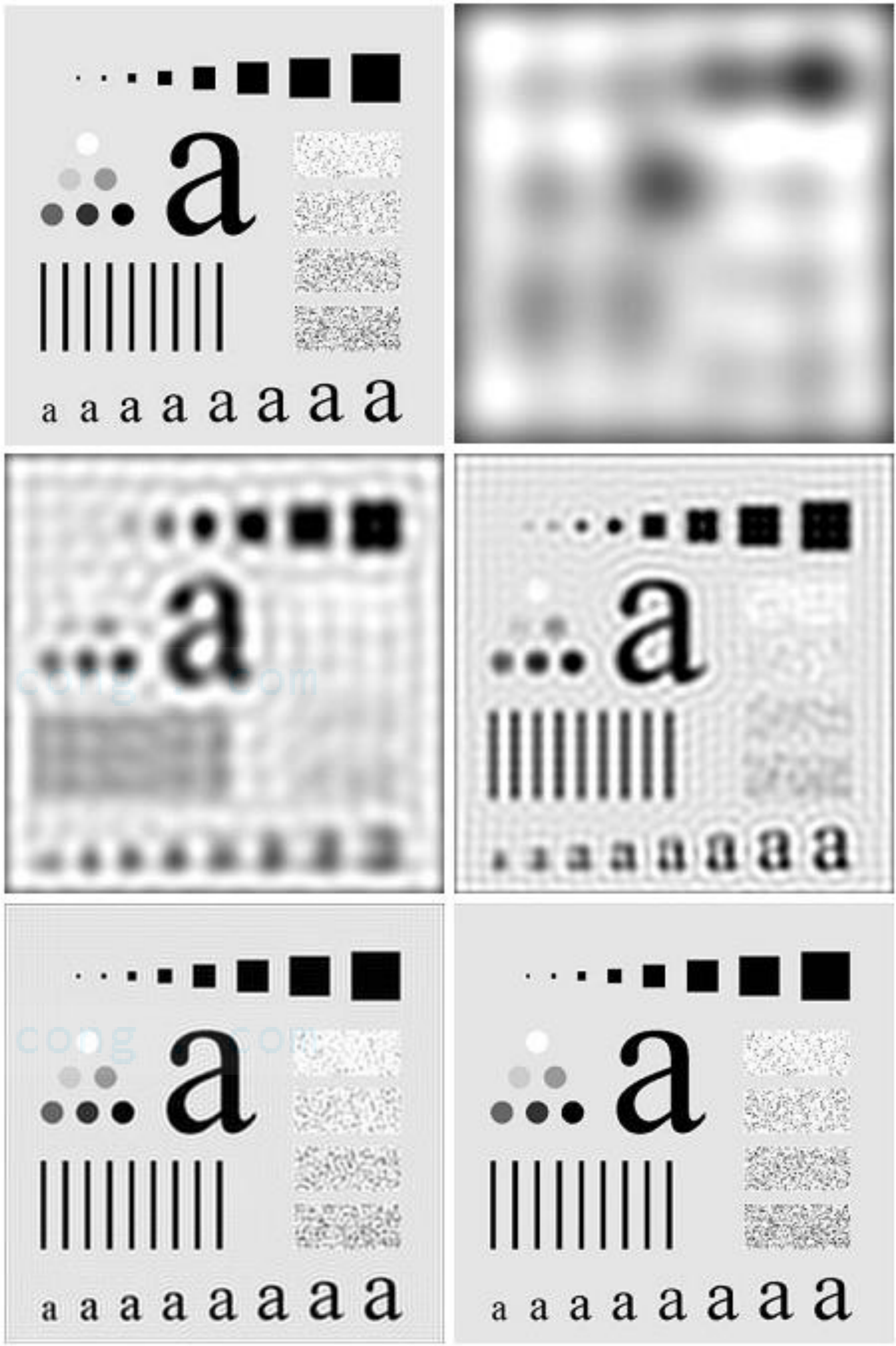
(a) Perspective plot of an ideal lowpass filter function. (b) Filter displayed as an image. (c) Filter radial cross section.



a b (a) Test pattern of size pixels, and (b) its Fourier spectrum. The superimposed circles have radii equal to 10, 30, 60, 160, and 460 with respect to the full-size spectrum image. These radii enclose 87.0, 93.1, 95.7, 97.8, and 99.2% of the padded image power, respectively.



(a) Original image. (b)–(f) Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460. The power removed by these filters was 13, 6.9, 4.3, 2.2, and 0.8% of the total, respectively.

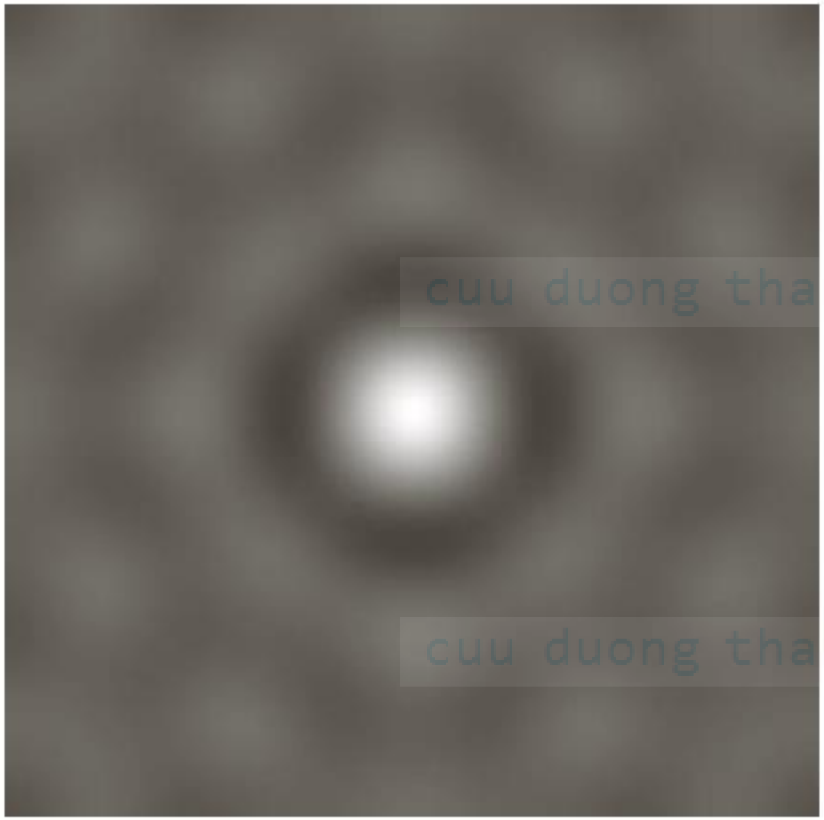


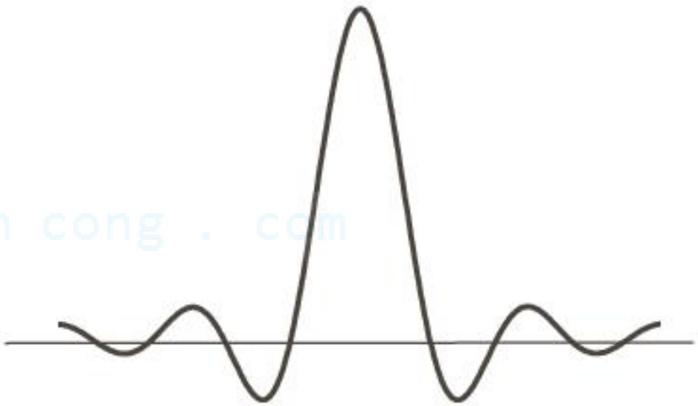
Ideal lowpass filter: Ringing effect

- A cross section of the ILPF in the frequency domain looks like a box filter $\Rightarrow$ a cross section of the corresponding spatial filter has the shape of a sinc function
- Imagine each pixel in the image being a discrete impulse whose strength is proportional to the intensity of the image at that location
- Convolving a sinc with an impulse copies the sinc at the location of the impulse
  - The center lobe of the sinc is the principal cause of blurring, while
  - The outer, smaller lobes are mainly responsible for ringing

a b

(a) Representation in the spatial domain of an ILPF of radius 5 and size 1000 × 1000. (b) Intensity profile of a horizontal line passing through the center of the image.

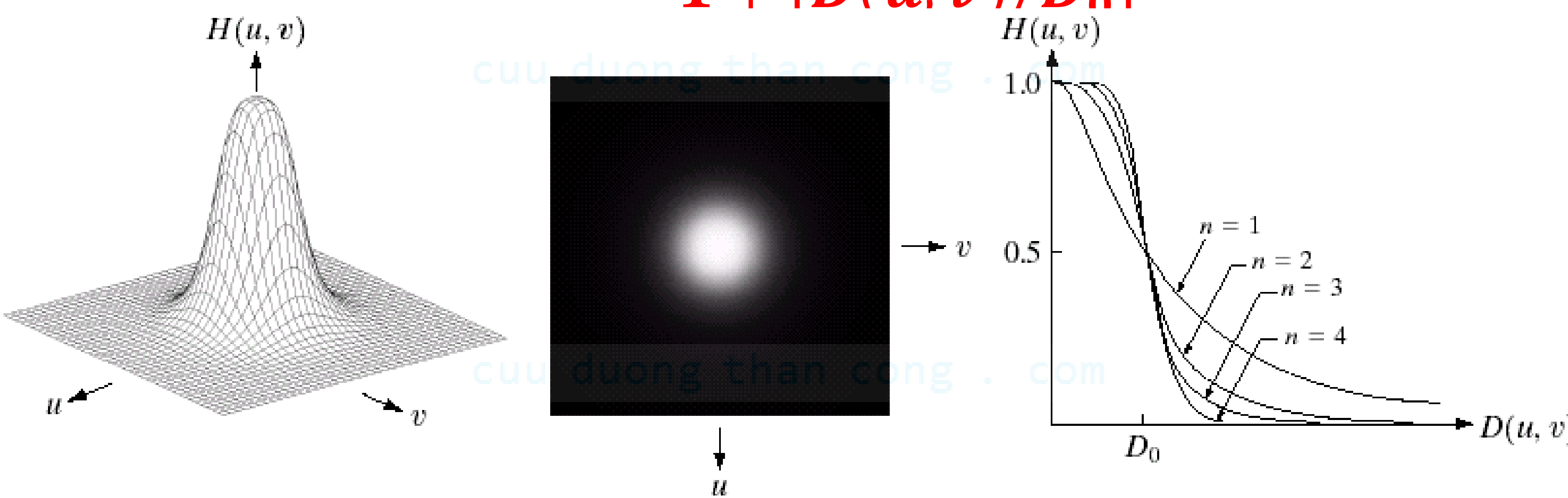




Butterworth lowpass filter (BLPF)

- The transfer function of a **Butterworth lowpass filter** (BLPF) of order n is defined as

$$H(u,v) = \frac{1}{1 + [D(u,v)/D_0]^{2n}}$$

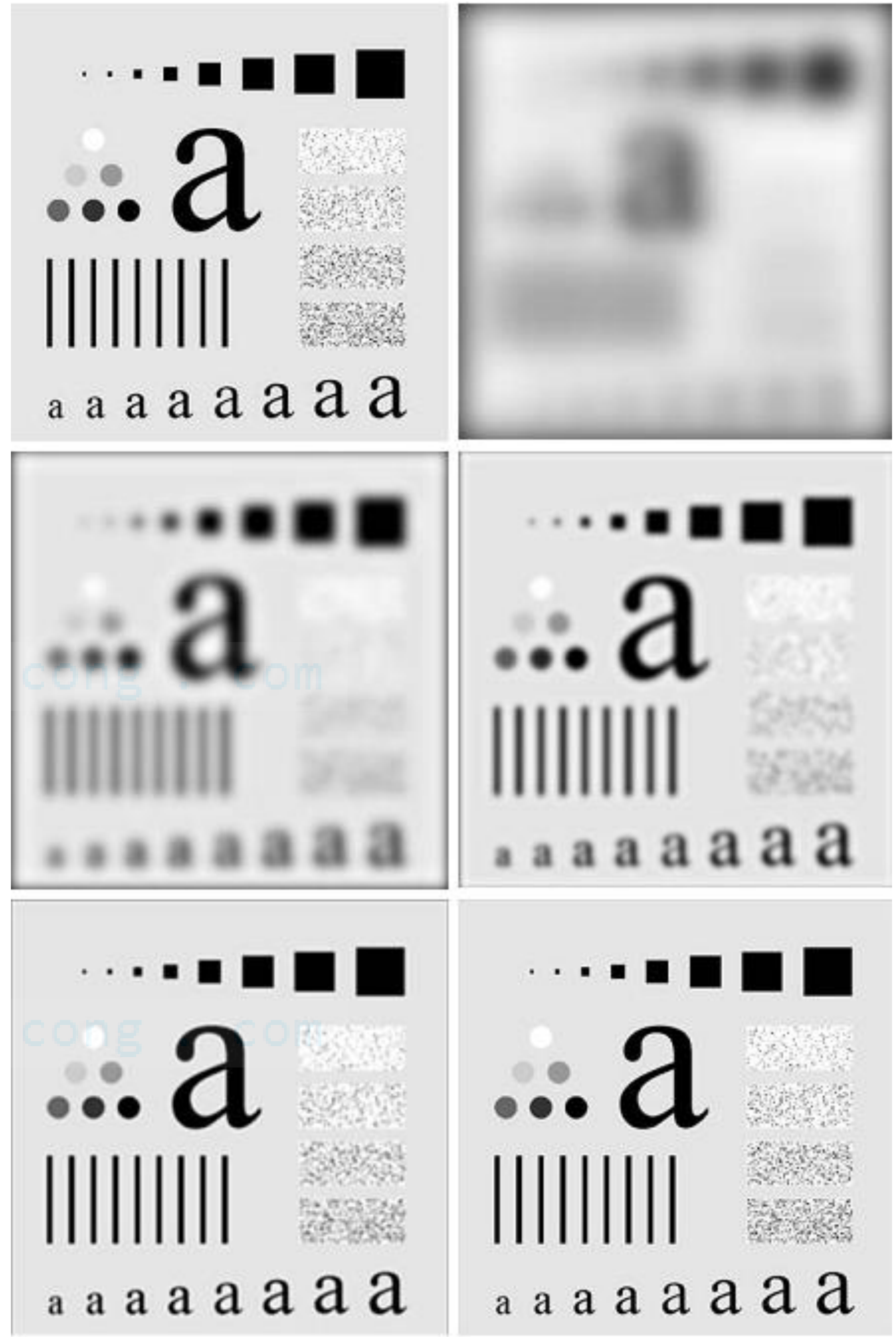


a b c

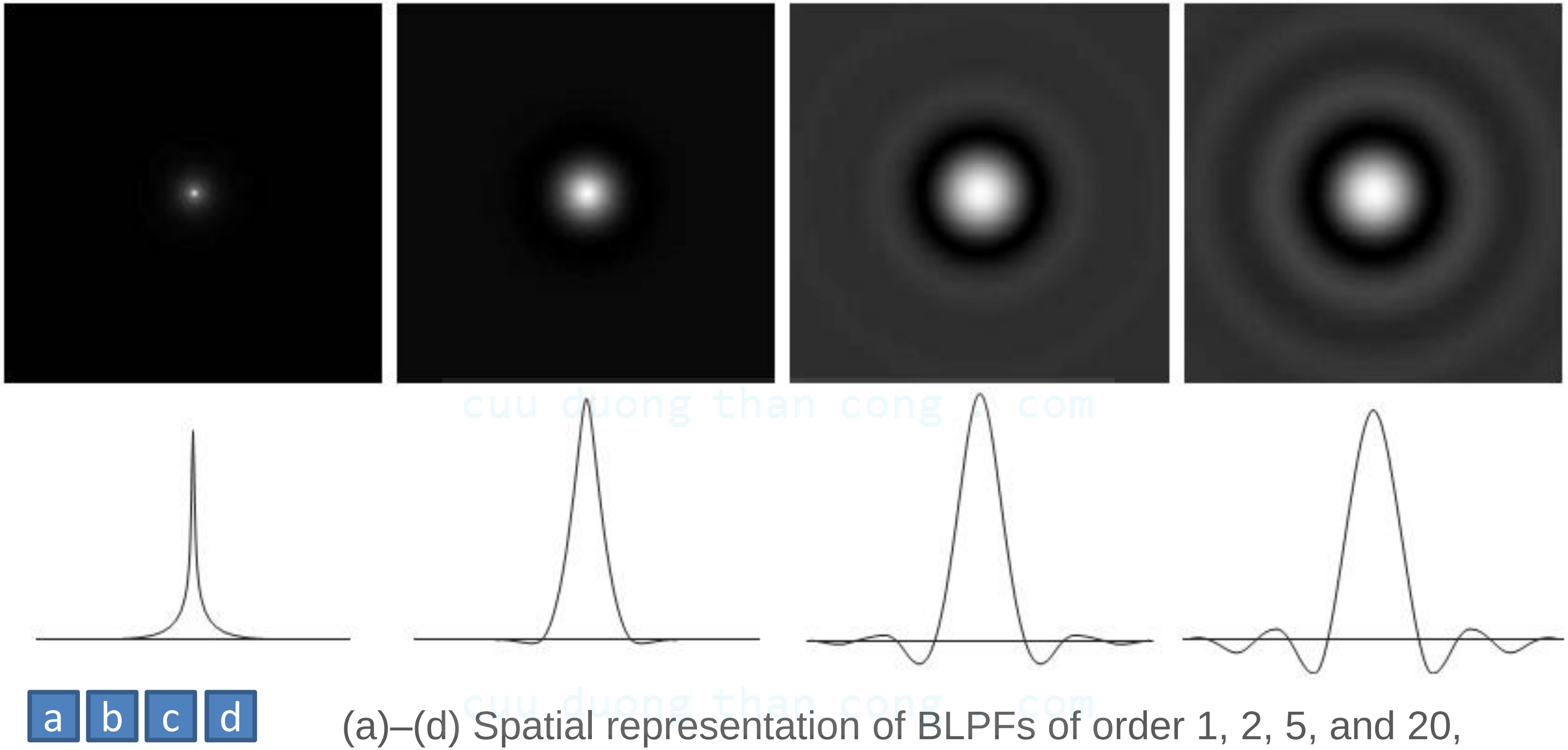
(a) Perspective plot of a Butterworth lowpass filter function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

a	b
c	d
e	f

(a) Original image. (b)–(f) Results of filtering using BLPFs of order 2, with cutoff frequencies at the radii values 10, 30, 60, 160, and 460



The "ring effect" of BLPFs



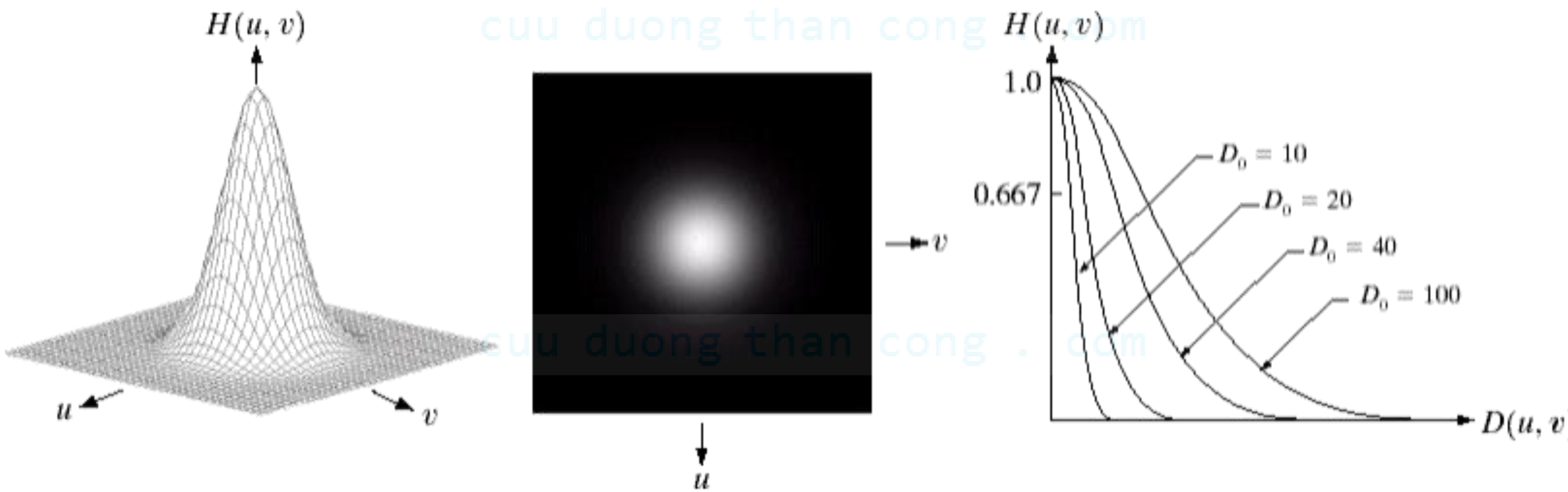
(a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding intensity profiles through the center of the filters (the size in all cases is 1000×1000 and the cutoff frequency is 5). Observe how ringing increases as a function of filter order

Gaussian lowpass filter (GLPF)

- The transfer function of a **Gaussian lowpass filter** (GLPF) is defined as

$$H(u,v) = e^{-\frac{D(u,v)^2}{2D_0^2}}$$

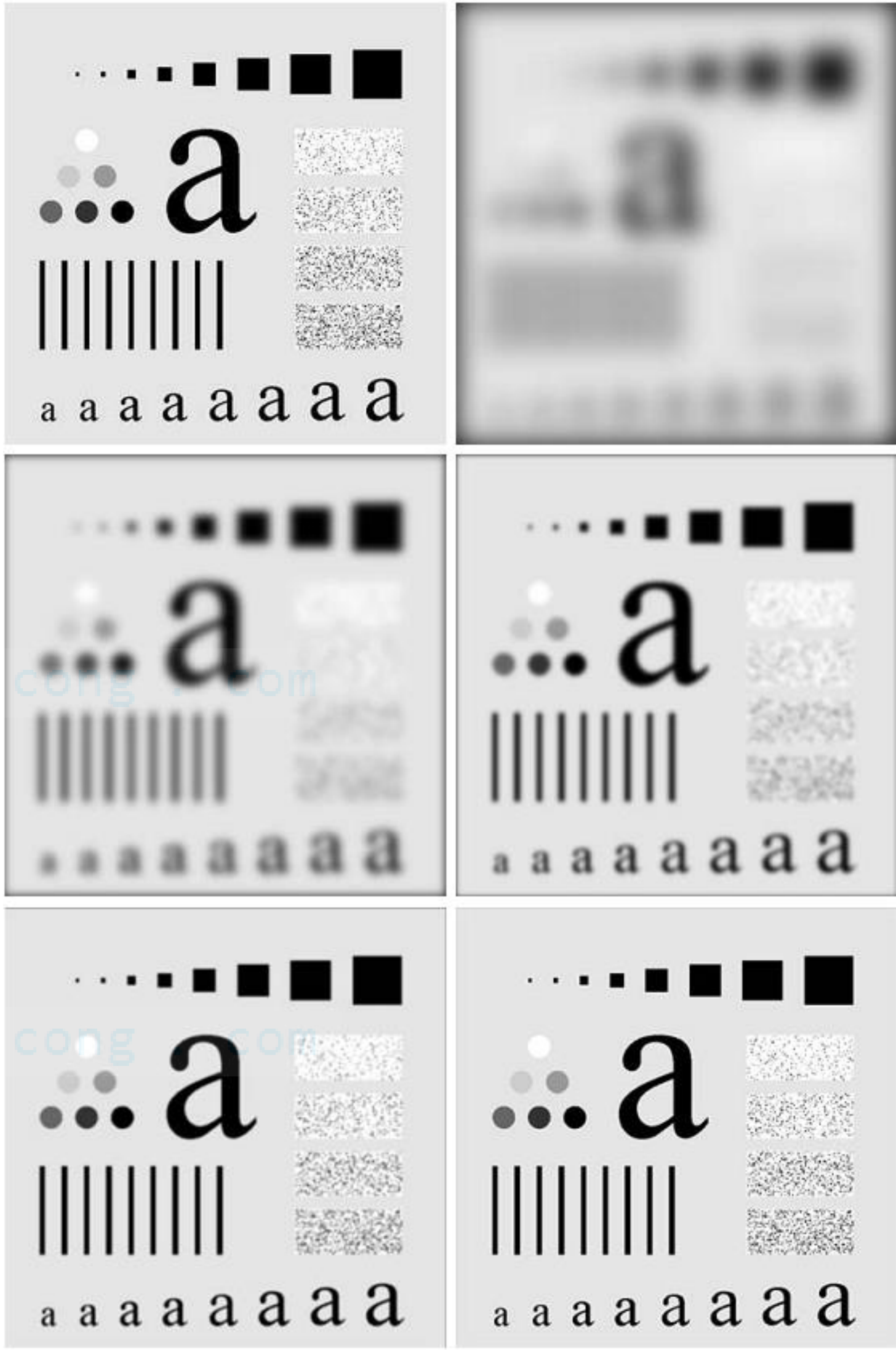
  - The IFT of the GLPF is also a spatial Gaussian filter without ringing



a b c (a) Perspective plot of a Gaussian lowpass filter function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

a	b
c	d
e	f

(a) Original image. (b)–(f) Results of filtering using GLPFs with cutoff frequencies set at the radii values 10, 30, 60, 160, and 460



A summary of lowpass filters

Name of the filter	Transfer function
Ideal lowpass filter (ILPF)	$H(u,v) = \begin{cases} 1 & if\ D(u,v) \leq D_0 \\ 0 & if\ D(u,v) > D_0 \end{cases}$
Butterworth lowpass filter (BLPF)	$H(u,v) = \frac{1}{1 + [D(u,v)/D_0]^{2n}}$
Gaussian lowpass filter (GLPF)	$H(u,v) = e^{-D(u,v)^2/2{D_0}^2}$

*D_0: cut-off frequency, n: the order of Butterworth LPF

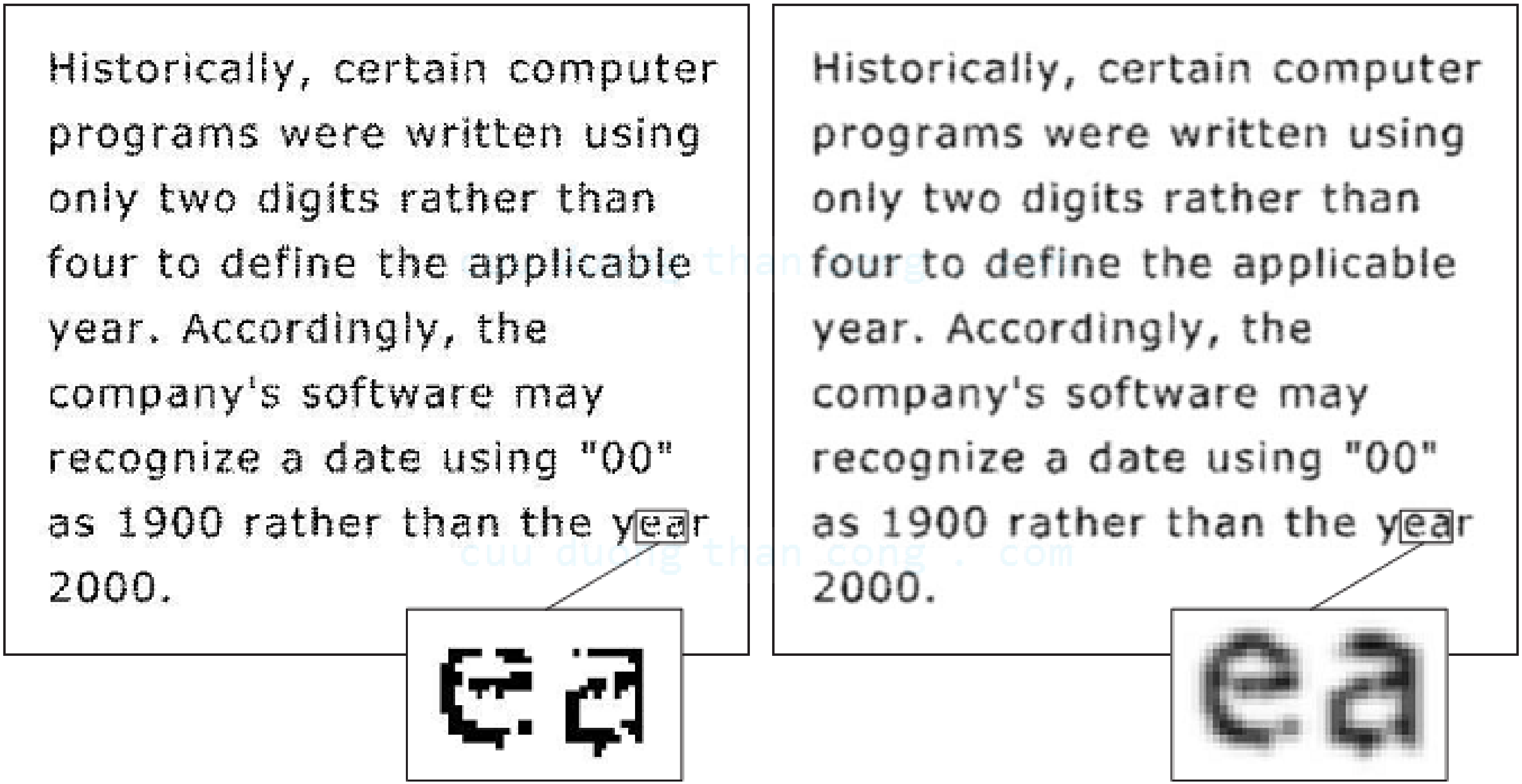
Application of lowpass filtering

- Machine perception: character recognition
- Printing and publishing industry
- Satellite and aerial images processing

- Similar results can be obtained using the lowpass spatial filtering techniques

a b

(a) Sample text of low resolution (note broken characters in magnified view) (444×508 pixels). (b) Result of filtering with a GLPF with $D_0 = 80$ (broken character segments were joined)

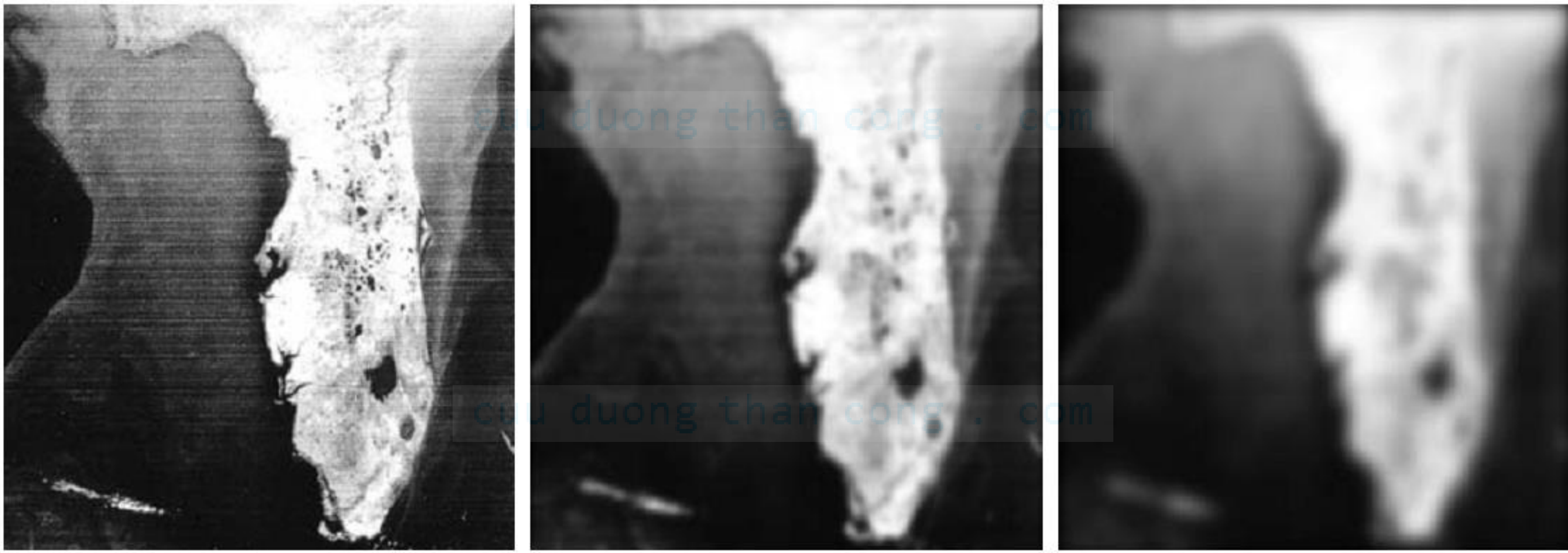


a b c

(a) Original image (784×732 pixels). (b) Result of filtering using a GLPF with $D_0 = 100$. (c) Result of filtering using a GLPF with $D_0 = 80$. Note the reduction in fine skin lines in the magnified sections in (b) and (c)

a b c

(a) Image showing prominent horizontal scan lines. (b) Result of filtering using a GLPF with $D_0 = 50$. (c) Result of using a GLPF with $D_0 = 20$. (Original image courtesy of NOAA.)



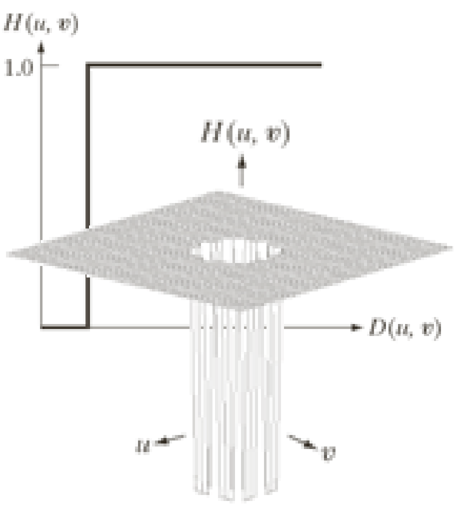
Section 4.3

IMAGE SHARPENING USING FREQUENCY DOMAIN FILTERS

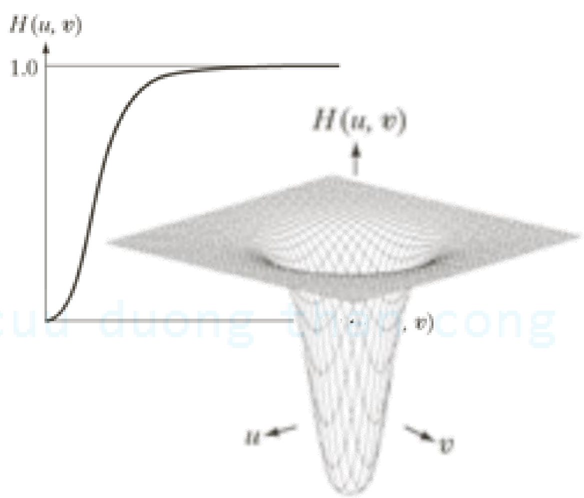
Highpass filtering

- Sharpening is achieved in the frequency domain by low-frequency attenuation, called highpass filtering.
  - The high-frequency information in the FT is left unchanged.

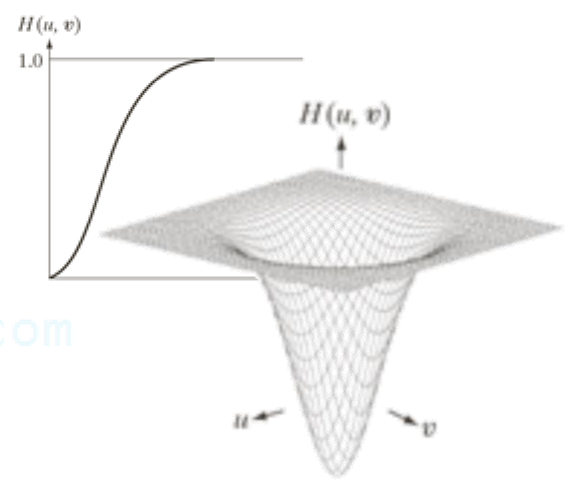
Very sharp — TRANSITION — Very smooth



Ideal highpass filter



Butterworth highpass filter



Gaussian highpass filter

Highpass filters vs. Lowpass filters

- A highpass filter is obtained from a given lowpass filter using the equation

$$\boldsymbol{H_{HP}(u,v) = 1 - H_{LP}(u,v)}$$

  - $H_{LP}(u,v)$: the transfer function of the lowpass filter

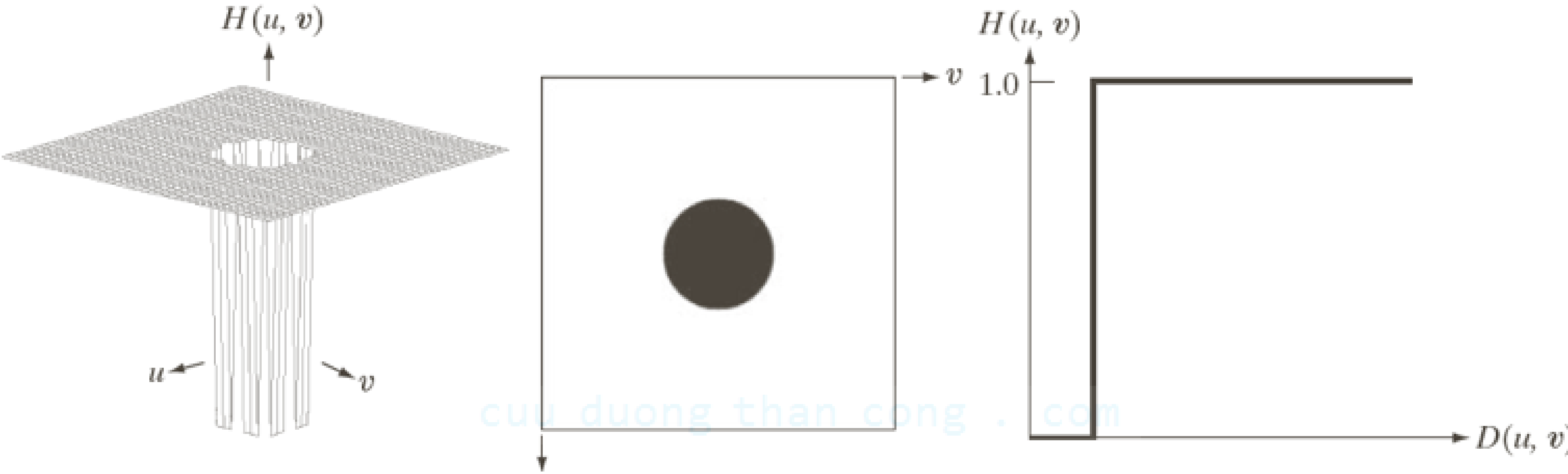
- When the lowpass filter attenuates frequencies, the highpass filter passes them, and vice versa

Ideal highpass filter (IHPF)

- Set to zero all the frequencies inside a circle of radius D_0 while passing, without attenuation, all frequencies outside the circle

$$H(u,v) = \begin{cases} 0 & if\ D(u,v) \leq D_0 \\ 1 & if\ D(u,v) > D_0 \end{cases}$$

- Similar to ILPF, the IHPF is not physically realizable either

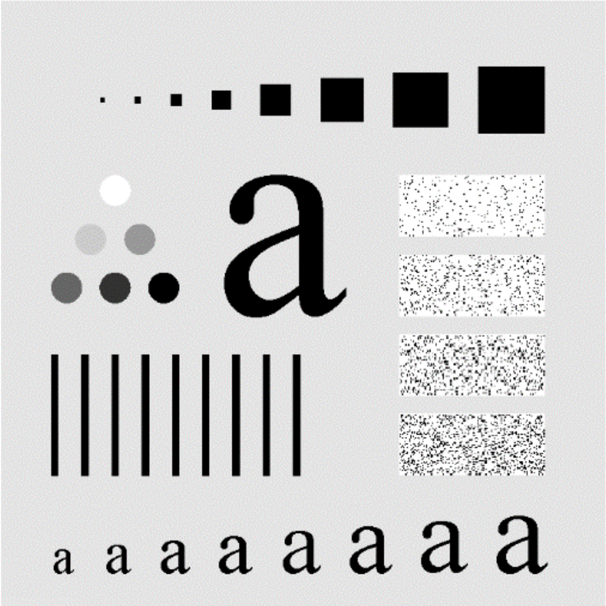


a b c

(a) Perspective plot of an ideal highpass filter function. (b) Filter displayed as an image. (c) Filter radial cross section.

a
b c d

(a) Original image. (b)-(d) Results of highpass filtering (a) using an IHPF with $D_0 = 30, 60$ and 160

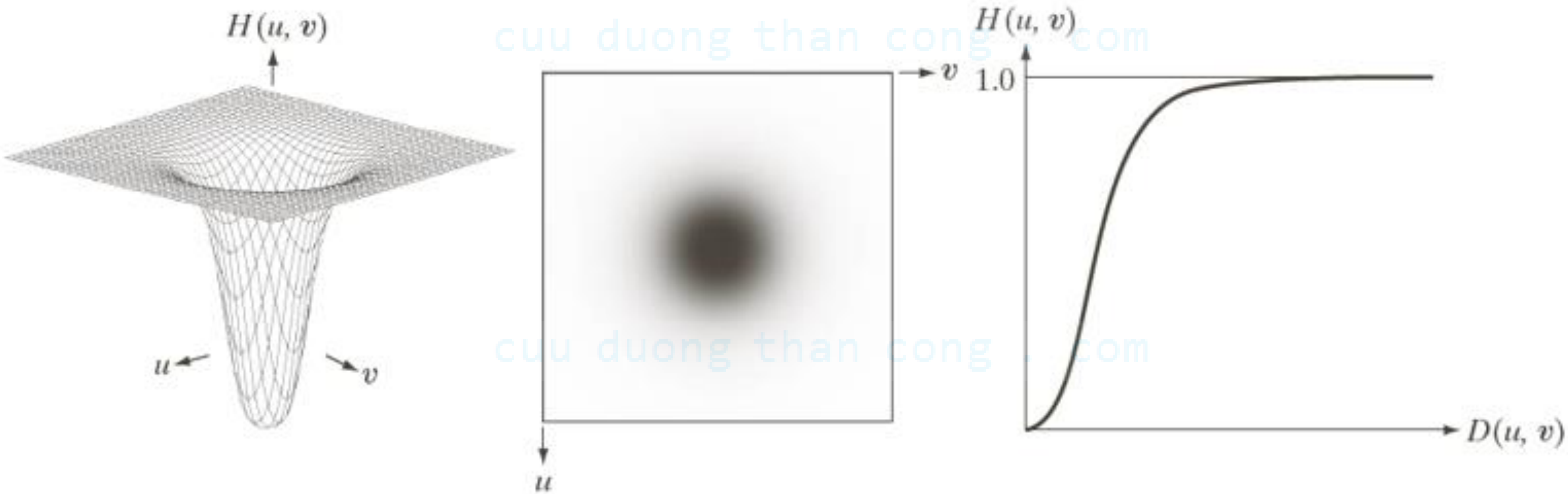


cuu duong than cong . com

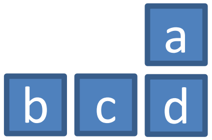
Butterworth highpass filter (BHPF)

- A 2-D **Butterworth highpass filter** (BHPF) of order n and cutoff frequency D_0 is defined as

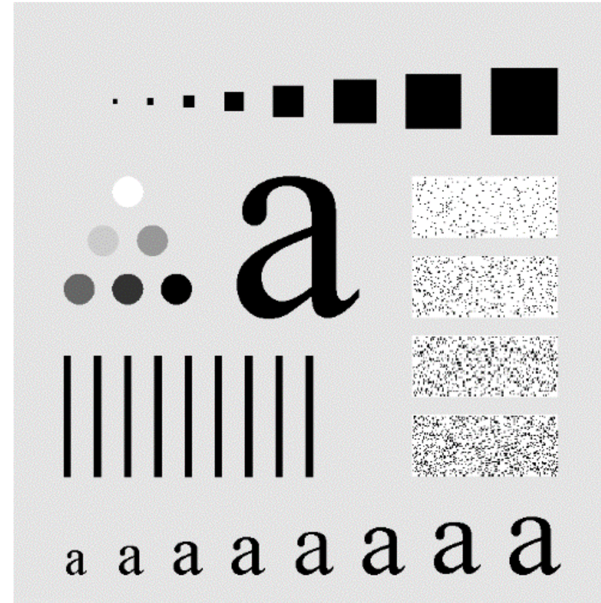
$$H(u,v) = \frac{1}{1 + [D_0/D(u,v)]^{2n}}$$

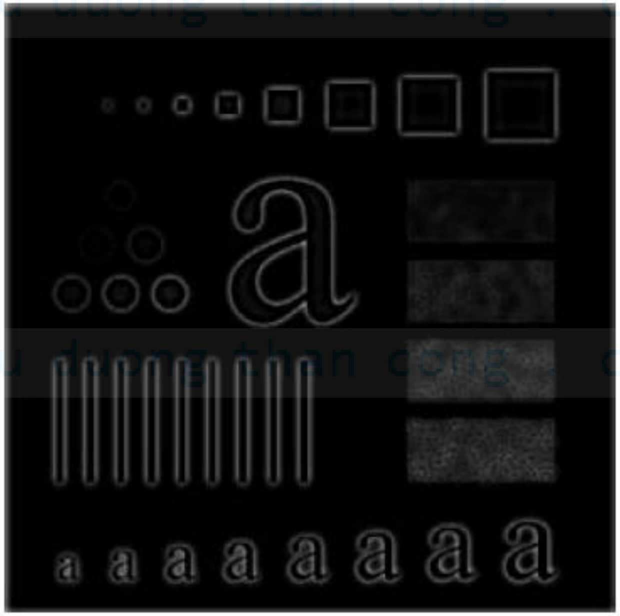


a b c (a) Perspective plot of a Butterworth highpass filter function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.



(a) Original image. (b)-(d) Results of highpass filtering (a) using a BHPF of order 2 with $D_0 = 30, 60$ and 160

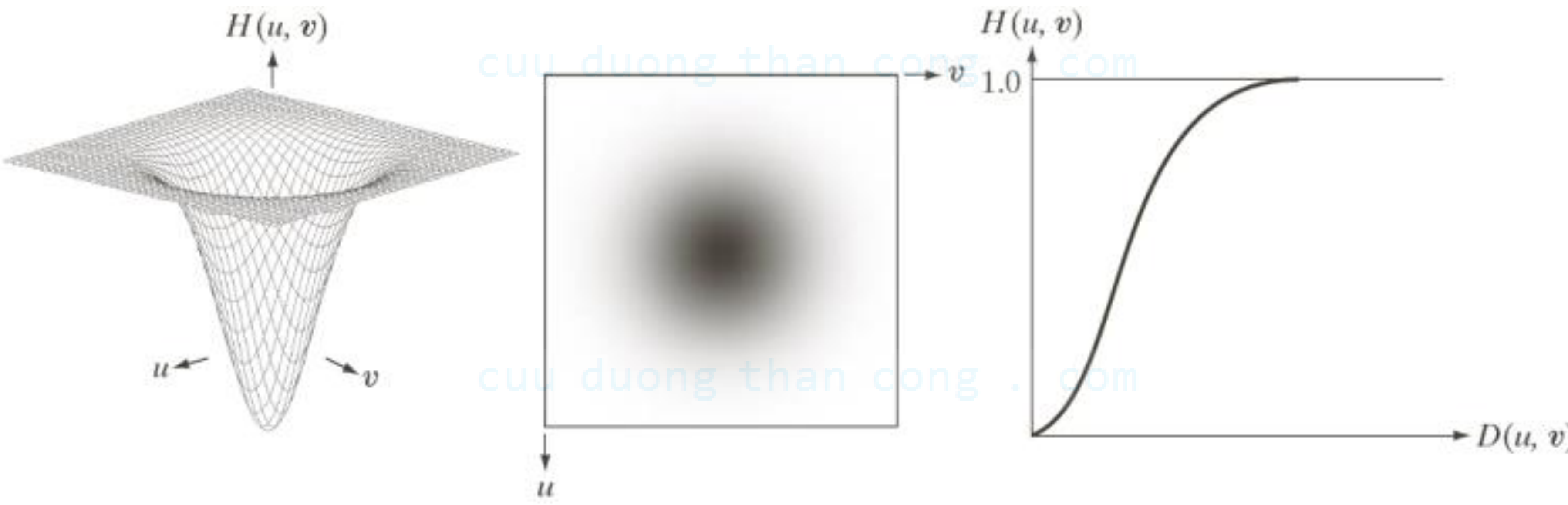




Gaussian highpass filter (GHPF)

- A 2-D **Gaussian highpass filter** (GHPF) with the cutoff frequency D_0 is defined as

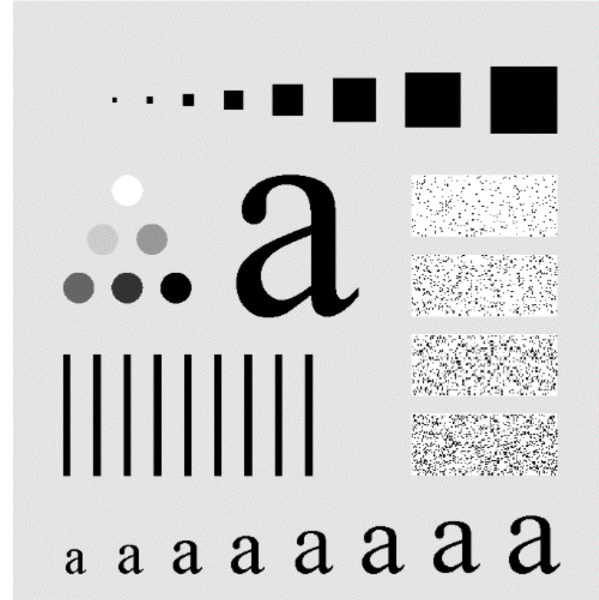
$$H(u,v) = 1 - e^{-\frac{D(u,v)^2}{2D_0^2}}$$



a b c (a) Perspective plot of a Gaussian highpass filter function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

a
b c d

(a) Original image. (b)-(d) Results of highpass filtering (a) using a GHPF with $D_0 = 30, 60$ and 160

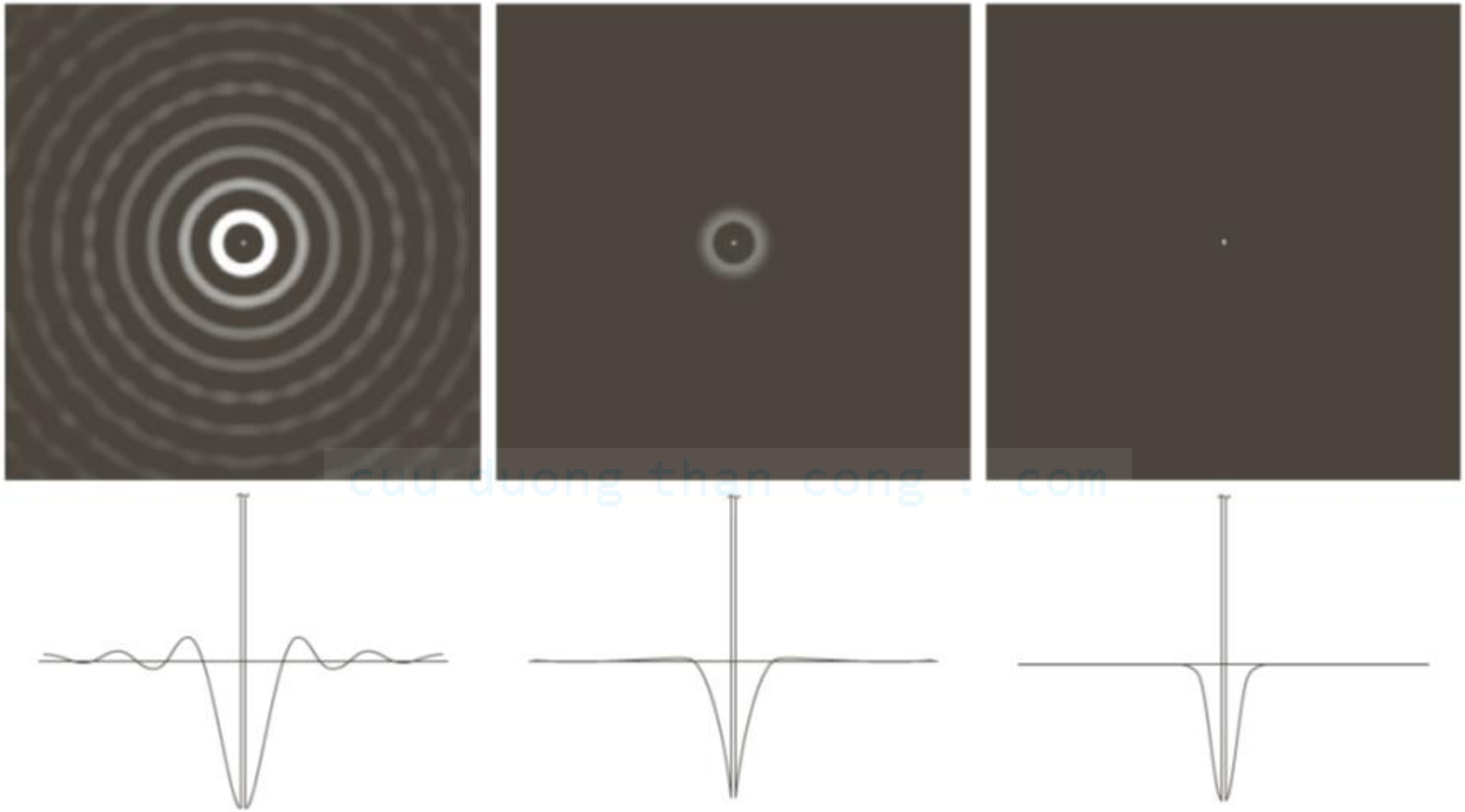


A summary of highpass filters

Name of the filter	Transfer function
Ideal highpass filter (IHPF)	$H(u,v) = \begin{cases} 0 & if\ D(u,v) \leq D_0 \\ 1 & if\ D(u,v) > D_0 \end{cases}$
Butterworth highpass filter (BHPF)	$H(u,v) = \dfrac{1}{1 + [D_0/D(u,v)]^{2n}}$
Gaussian highpass filter (GHPF)	$H(u,v) = 1 - e^{-D(u,v)^2/2{D_0}^2}$

*D_0: cut-off frequency, n: the order of Butterworth HPF

The “ring effect” of HLPFs



a b c Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.

Application of highpass filtering

- The enhancement of print ridges and reduction of smudges
  - The ridges contain high frequencies → unchanged with a highpass filter. The background and smudges correspond to slowly varying intensities → filtered out.

a b c (a) Thumb print. (b) Result of highpass filtering (a). (c) Result of thresholding (b).

The Laplacian in the frequency domain

- The Laplacian is implemented in the frequency domain by

$$\boldsymbol{H(u,v) = -4\pi^2 D^2(u,v)}$$

- Then, the Laplacian image is obtained as

$$\nabla^2 f(x,y) = \mathcal{F}^{-1}\{H(u,v)F(u,v)\}$$

- Finally, enhancement is achieved using the equation

$$\boldsymbol{g(x,y) = f(x,y) + c\nabla^2 f(x,y)}$$

  - $c = -1$ because $H(u,v)$ is negative
  - DFT scaling factors that can be several orders of magnitude larger than the maximum value of f
    - Bring f to the range [0, 1] (before computing its DFT) and $\boldsymbol{\nabla^2 f(x,y)}$ to the range [-1, 1] by diving its values with the maximum value

a b c

(a) Original, blurry image. (b) Image enhanced using the Laplacian in the frequency domain. Compare with the result of spatial Laplacian filtering in Lecture 3.

Unsharp masking and highboost filtering

1. Blur the original image $f(x, y)$ to produce a blurred image

$$f_{LP}(x,y) = \mathcal{F}^{-1}\{H_{lp}(u,v)F(u,v)\}$$

2. The mask is defined as

$$g_{mask}(x,y) = f(x,y) - f_{LP}(x,y)$$

3. Then,

$$g(x,y) = f(x,y) + k * g_{mask}(x,y)$$

- *Unsharp masking* when $k = 1$ or *highboost filtering* when $k > 1$

Unsharp masking and highboost filtering

- $g(x, y)$ can be represented entirely in terms of frequency domain computations

$$g(x,y) = \mathcal{F}^{-1}\{[1 + k * (1 - H_{LP}(u,v)]F(u,v)\}$$
$$= \mathcal{F}^{-1}\{\underbrace{[1 + k * H_{HP}(u,v)]}_{\text{High-frequency-emphasis filter}}F(u,v)\}$$

High-frequency-emphasis filter

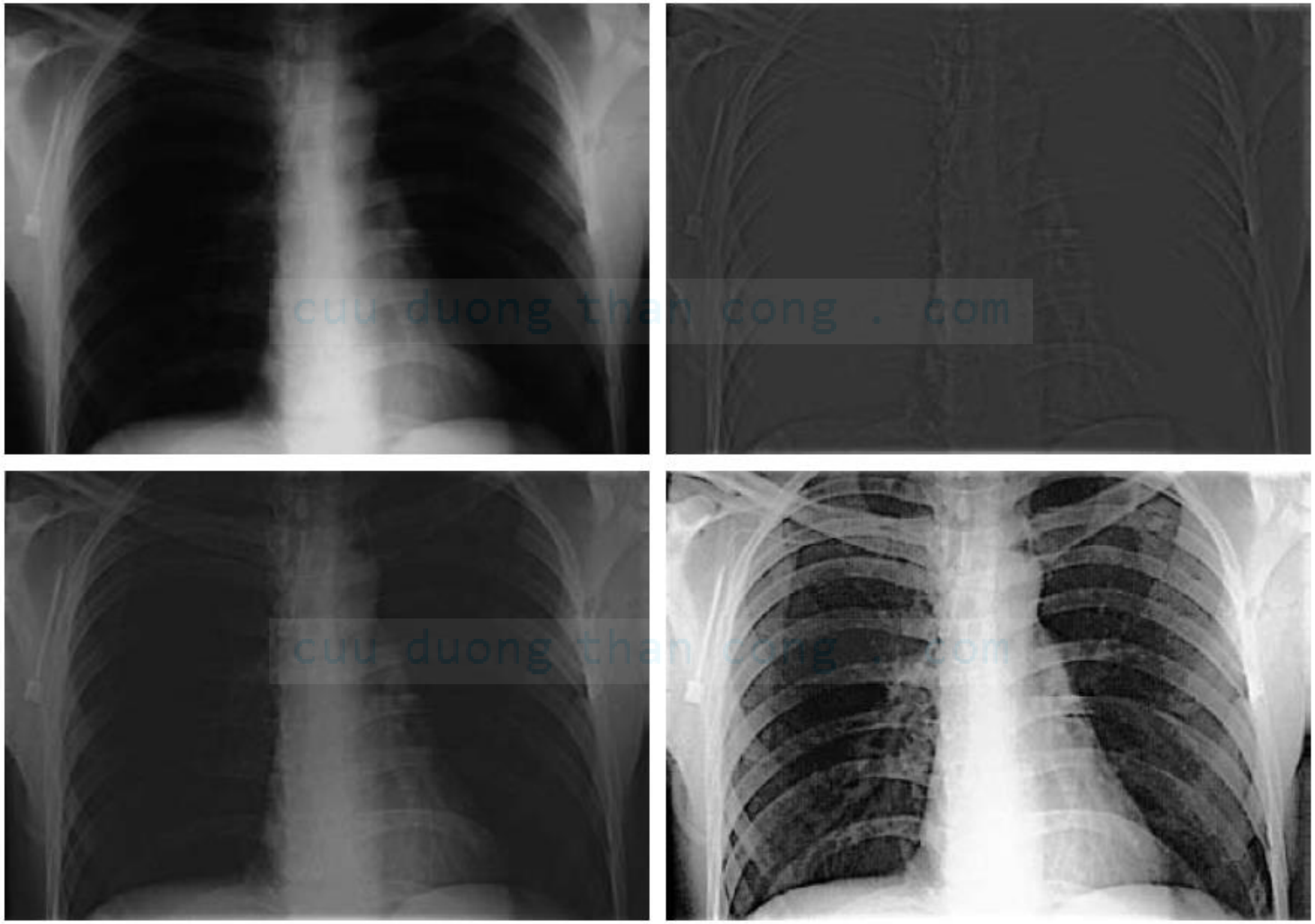
- A slightly more general formulation of the high-frequency-emphasis filtering is

$$\boldsymbol{g(x,y) = \mathcal{F}^{-1}\{[k_1 + k_2 * H_{HP}(u,v)]F(u,v)\}}$$

  - where $k_1 \geq 0$ gives controls of the offset from the origin and $k_2 \geq 0$ controls the contribution of high frequencies

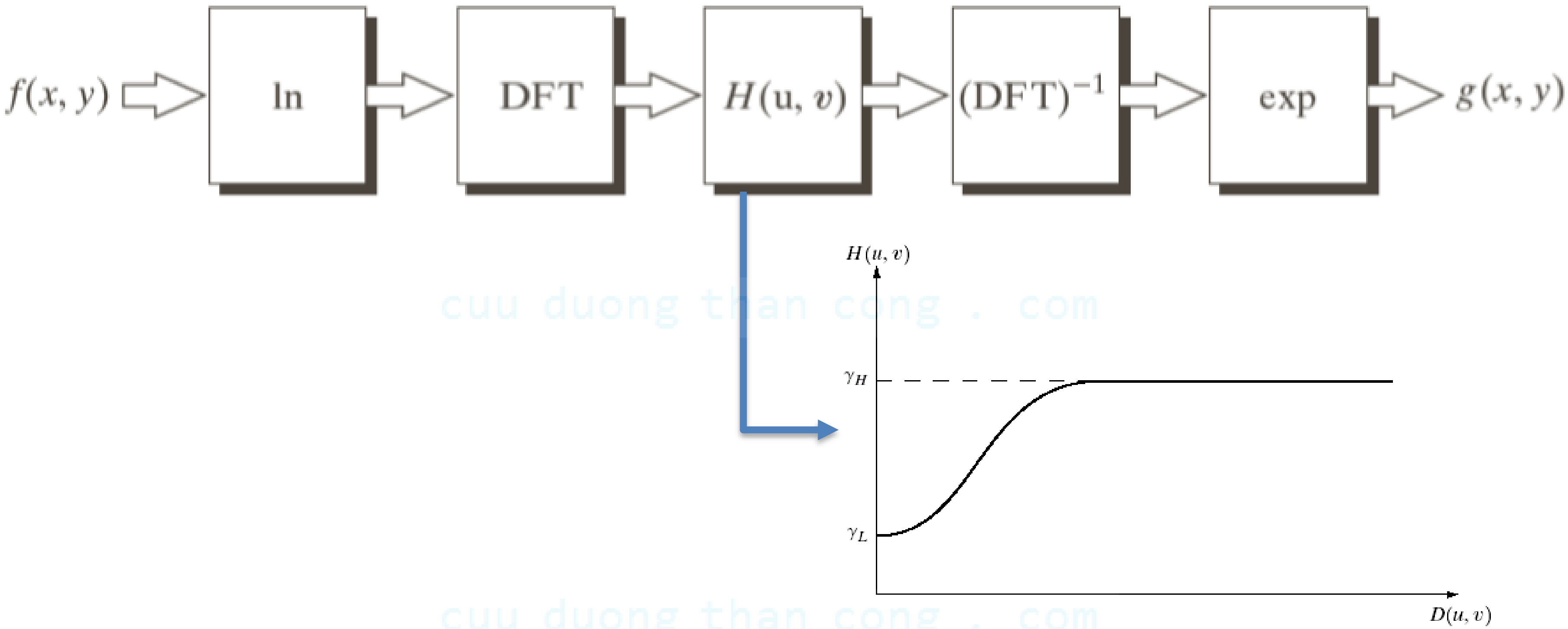
a	b
c	d

(a) A chest X-ray image. (b) Result of highpass filtering with a Gaussian filter with $D_0 = 40$. (c) Result of high-frequency-emphasis filtering using the same filter with $k_1 = 0.5$ and $k_2 = 0.75$. (d) Result of performing histogram equalization on (c).



Homomorphic filtering

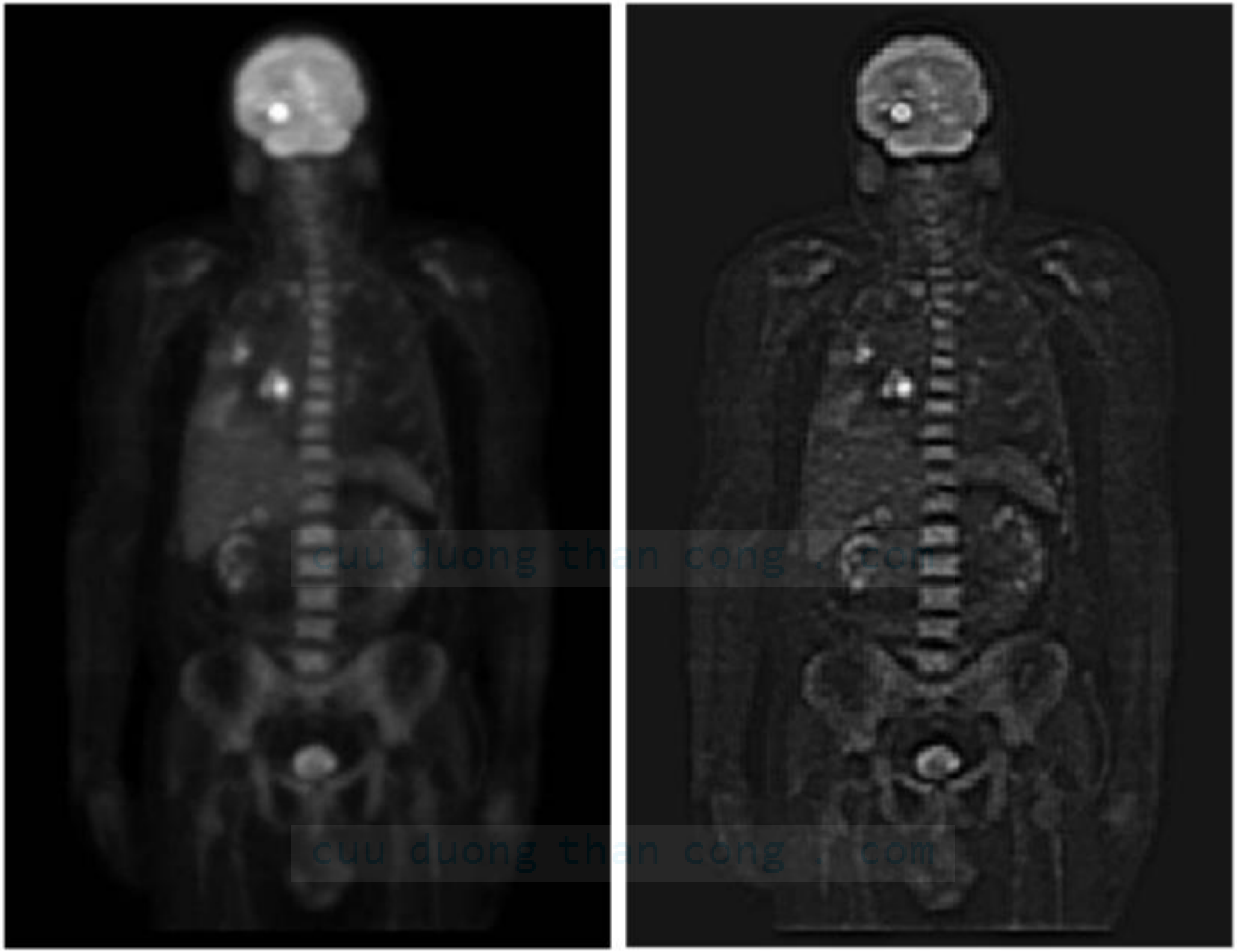
- The filtering approach is summarized as follows



$$H(u,v) = (\gamma_H - \gamma_L)\left[1 - e^{-c\left[\frac{D^2(u,v)}{D_0^2}\right]}\right] + \gamma_L$$

Homomorphic filtering

- **Key idea:** the separation of the illumination and reflectance components
  - The illumination component of an image generally is characterized by slow spatial variations, while the reflectance component tends to vary abruptly, particularly at the junctions of dissimilar objects.
- Tuning γ_L and γ_H affects the low and high frequency components of the FT in different, controllable ways
  - E.g., $\gamma_L < 1$ and $\gamma_H > 1 \rightarrow$ attenuate the contribution made by the low frequencies (illumination) while amplify the contribution made by high frequencies (reflectance)



a b (a) Full body PET scan. (b) Image enhanced using homomorphic filtering with $\gamma_L = 0.25,\ \ \gamma_H = 2,\ c = 1$ and $D_0 = 80$.

Section 4.4

SELECTIVE FILTERING

Bandreject filters

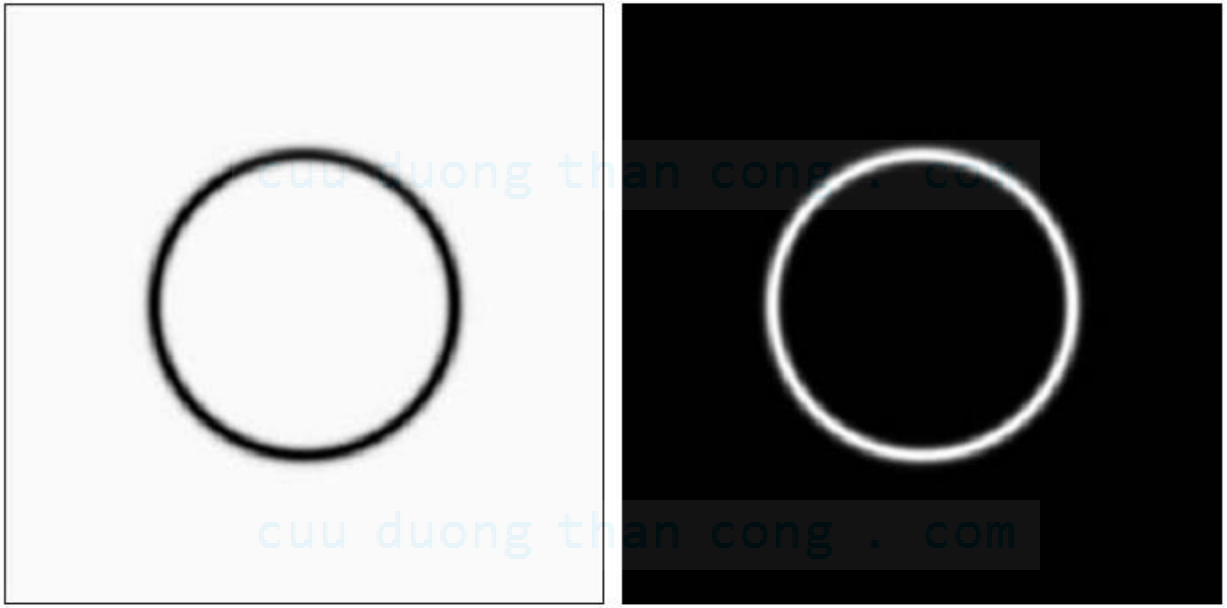
- Let D is the width of the band, i.e. the distance $D(u,v)$ from the center of the filter, and D_0 is the cut-off frequency

Name of the filter	Transfer function
Ideal bandreject filter	$H(u,v) = \begin{cases} 0 & if\, D_0 - \frac{W}{2} \leq D \leq D_0 + \frac{W}{2} \\ 1 & otherwise \end{cases}$
Butterworth bandreject filter	$H(u,v) = \frac{1}{1 + \left[\frac{DW}{D^2 - D_0^2}\right]^{2n}}$
Gaussian bandreject filter	$H(u,v) = 1 - e^{-\left[\frac{D^2 - D_0^2}{DW}\right]^2}$

Bandpass filters

- A **bandpass filter** can be obtained from a bandreject filter

$$H_{BP}(u,v) = 1 - H_{BR}(u,v)$$



a b (a) Bandreject Gaussian filter. (b) Corresponding bandpass filter. The thin black border in (a) was added for clarity; it is not part of the data.

Notch filters

- A **notch filter** rejects/passes frequencies in a predefined neighborhoods about a center frequency
- A **notch reject filter** is the product of highpass filters whose centers have been translated to the centers of the notches

$$H_{NR}(u,v) = \prod_{k=1}^{Q} H_k(u,v)H_{-k}(u,v)$$

  - where $H_k(u,v)$ and $H_{-k}(u,v)$ are highpass filters centered at (u_k, v_k) and $(-u_k, -v_k)$, respectively, and Q is the number of notch pairs

- A **notch pass filter** is obtained from a notch reject filter

$$H_{NP}(u,v) = 1 - H_{NR}(u,v)$$

Notch filters

- For example, the Butterworth notch reject filter of order n containing three notch pairs

$$H_{NR}(u,v) = \prod_{k=1}^{3} \left[\frac{1}{1 + [D_{0k}/D_k(u,v)]^{2n}} \right] \left[\frac{1}{1 + [D_{0k}/D_{-k}(u,v)]^{2n}} \right]$$

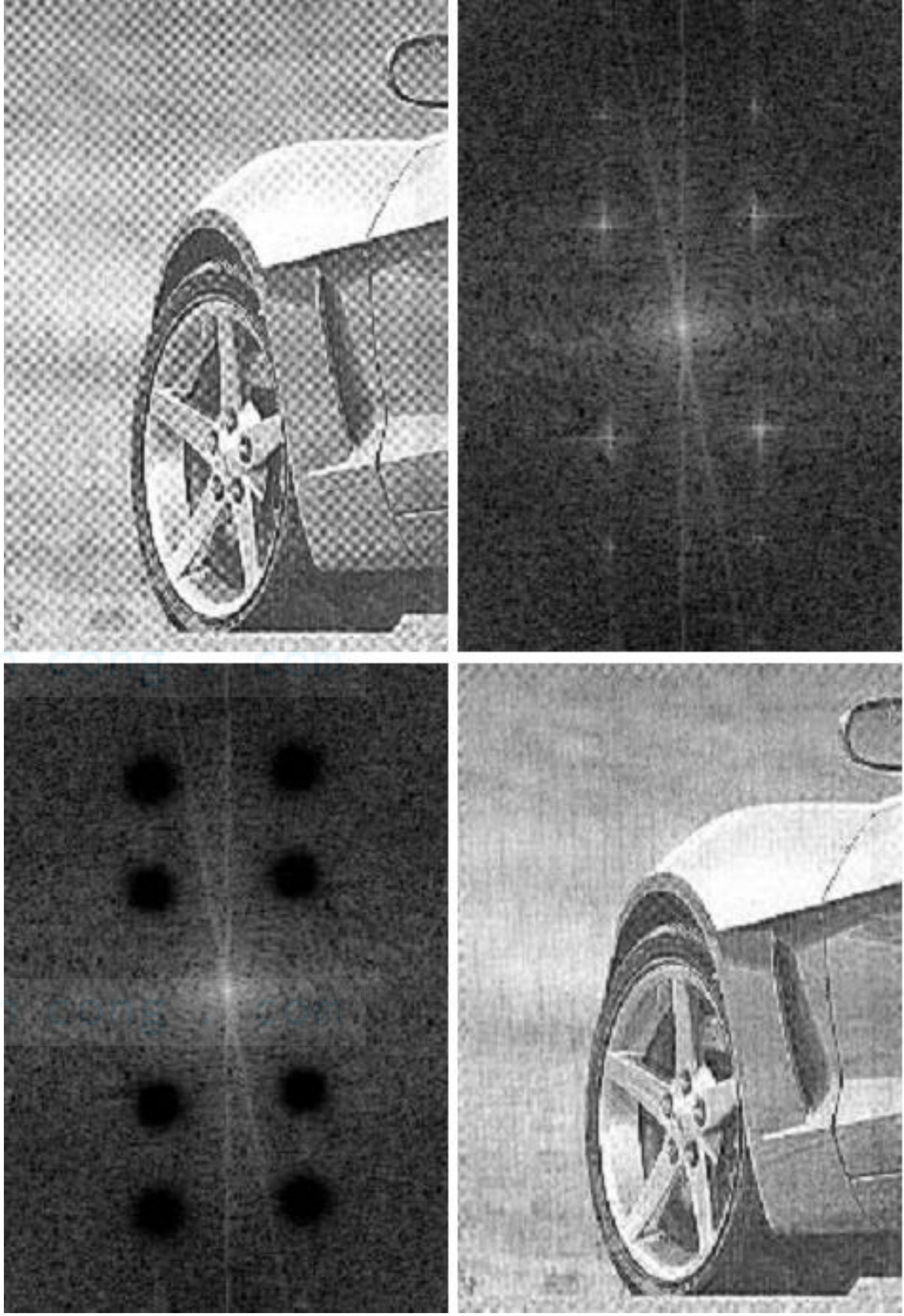
  - where D_{0k} is the same for each pair of notches,

$$D_k(u,v) = \left[\left(u - \frac{M}{2} - u_k \right)^2 + \left(v - \frac{N}{2} - v_k \right)^2 \right]^{1/2}$$

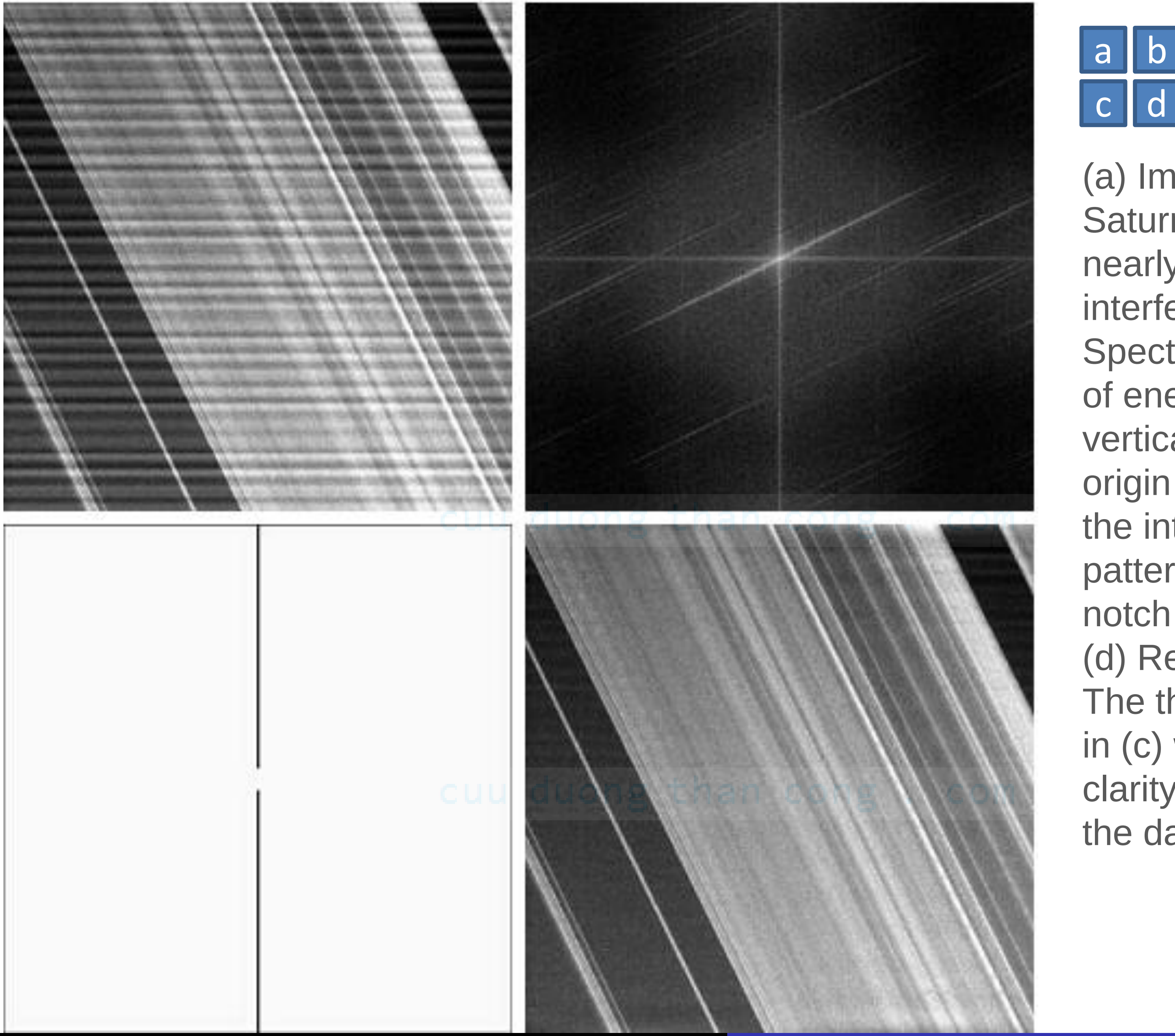
and

$$D_{-k}(u,v) = \left[\left(u - \frac{M}{2} + u_k \right)^2 + \left(v - \frac{N}{2} + v_k \right)^2 \right]^{1/2}$$

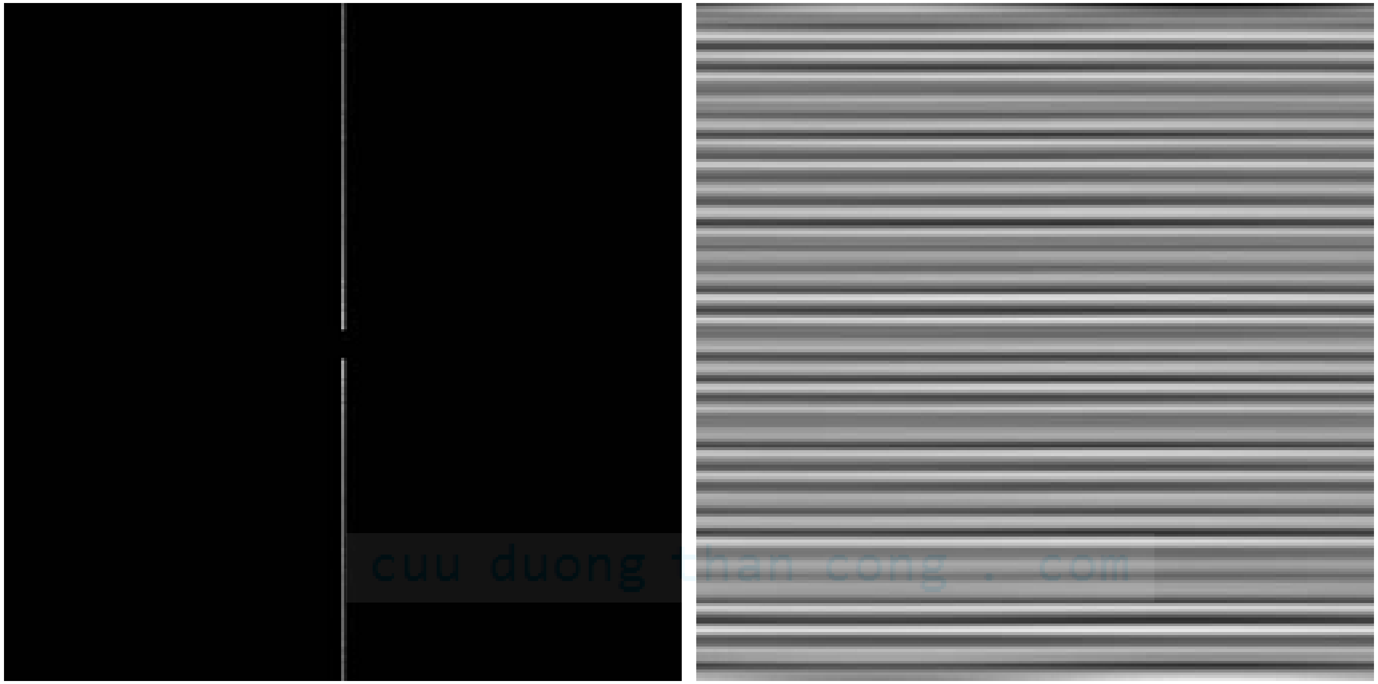
(a) Sampled newspaper image showing a moiré pattern. (b) Spectrum. (c) Butterworth notch reject filter ($D_0 = 3, n = 4$) multiplied by the Fourier transform. (d) Filtered image.



cuu duong than cong . com



(a) Image of the Saturn rings showing nearly periodic interference. (b) Spectrum: The bursts of energy in the vertical axis near the origin correspond to the interference pattern. (c) A vertical notch reject filter. (d) Result of filtering. The thin black border in (c) was added for clarity; it is not part of the data.



a b

(a) Result (spectrum) of applying a notch pass filter to the DFT in the previous side. (b) Spatial pattern obtained by computing the DFT of (a).

References

- Rafael C. Gonzalez, Richard E. Woods, “Digital Image Processing”, 3rd edition, 2008. Chapter 4
- Fourier Transform: http://www.thefouriertransform.com/
- Discrete Fourier Transform

  http://www.robots.ox.ac.uk/~sjrob/Teaching/SP/l7.pdf
- gear.kku.ac.th/~nawapak/178353/Chapter04.ppt
- Images are obtained from the above materials and Google

Some additional examples

- Filing:

  http://fourier.eng.hmc.edu/e161/demos/e161_demo0.html

- Fourier Transform Processing With ImageMagick – Practical Applications *(see the images of Fourier spectrum only, you do not need to understand the commands)*

  http://www.imagemagick.org/Usage/fourier/

- Image Filtering in the Frequency Domain

  http://paulbourke.net/miscellaneous/imagefilter/