Parabolic equation

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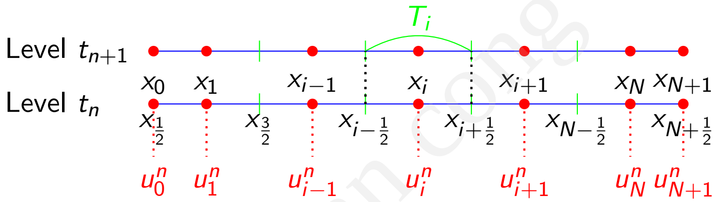
May 14, 2014

Introduction

The domain of the computation will be $\Omega =]0;1[$. Let the function $f \in L^2(\Omega)$, we will look for an appproximation of the following problem

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) & = f(x,t) \text{ in } \Omega \\ u(x,0) & = u_0(x), \quad x \in \Omega \\ u(x,t) & = g(x,t), \quad x \in \partial\Omega, t \in (0,T) \end{cases} \tag{1}$$



Let us choose $N+1$ points $\{x_{i+1/2}\}_{i\in\overline{0,N}}$ in $[0;1]$ such that

$$0 = x_{1/2} < x_{3/2} < \cdots < x_{N-1/2} < x_{N+1/2} = 1.$$

We set $T_i = [x_{i-1/2}, x_{i+1/2}], \quad \forall i \in \overline{1,N}, \quad h = \max_{i\in\overline{1,N}}\{|T_i|\}$

$$x_0 = 0, \quad x_{N+1} = 1, \quad x_i \in T_i, \ \forall i \in \overline{1,N}$$

We call $(T_i)_{i\in\overline{1,N}}$ control volumes and $(x_i)_{i\in\overline{0,N+1}}$ control points. We divide the interval $[0, T]$ into $N_k + 1$ sub-intervals of constant length k and denote $t_n = nk$. Denote by u_i^n the discrete unknowns; the value u_i^n is an approximation of $u(x_i, t_n)$.

Semi-discrete approximation

Integrating the first equation in (1) over control volume T_i, there holds

$$\frac{1}{|T_i|}\int_{T_i} u_t(x,t)dx + \frac{1}{|T_i|}\int_{T_i} -u_{xx}(x,t)dx = \frac{1}{|T_i|}\int_{T_i} f(x,t)dx \tag{2}$$

Applying the Green's formula, we obtain

$$\frac{-1}{|T_i|}\int_{T_i} -u_{xx}(x,t)dx = \frac{-u_x(x_{i+\frac{1}{2}},t) + u_x(x_{i-\frac{1}{2}},t)}{|T_i|} \tag{3}$$

and we put

$$f_i(t) = \frac{1}{|T_i|}\int_{T_i} f(x,t)dx \quad \text{mean-value of } f \text{ over } T_i \tag{4}$$

Thus

$$\frac{du_i}{dt}(t) + \frac{-u_x(x_{i+\frac{1}{2}},t) + u_x(x_{i-\frac{1}{2}},t)}{|T_i|} = f_i(t) \tag{5}$$

♦ Approximate $u_x(x_{i+\frac{1}{2}}, t)$: use Taylor series expansion

$$
\begin{aligned}
u(x_{i+1}, t) =& u(x_{i+\frac{1}{2}}, t) + u_x(x_{i+\frac{1}{2}}, t)(x_{i+1} - x_{i+\frac{1}{2}}) \\
&+ \frac{u_{xx}(x_{i+\frac{1}{2}}, t)}{2!}(x_{i+1} - x_{i+\frac{1}{2}})^2 + 0(h^3)
\end{aligned}
$$

$$
\begin{aligned}
u(x_i) =& u(x_{i+\frac{1}{2}}, t) + u_x(x_{i+\frac{1}{2}}, t)(x_i - x_{i+\frac{1}{2}}) \\
&+ \frac{u_{xx}(x_{i+\frac{1}{2}}, t)}{2!}(x_i - x_{i+\frac{1}{2}})^2 + 0(h^3)
\end{aligned}
$$

Thus

$$
\begin{aligned}
u(x_{i+1}, t) - u(x_i, t) =& (x_{i+1} - x_i)u_x(x_{i+\frac{1}{2}}, t) \\
&+ ((x_{i+1} - x_{i+\frac{1}{2}}, t)^2 - (x_i - x_{i+\frac{1}{2}})^2)\frac{u_{xx}(x_{i+\frac{1}{2}}, t)}{2!} + 0(h^3)
\end{aligned}
$$

Thus

$$u(x_{i+1},t)-u(x_i,t)=(x_{i+1}-x_i)u_x(x_{i+\frac{1}{2}},t)$$
$$+\left((x_{i+1}-x_{i+\frac{1}{2}})^2-(x_i-x_{i+\frac{1}{2}})^2\right)\frac{u_{xx}(x_{i+\frac{1}{2}},t)}{2!}+0(h^3)$$

We have two cases:

Case 1: $x_{i+\frac{1}{2}}$ is the midpoint of segment $[x_i,x_{i+1}]$ then

$$u_x(x_{i+\frac{1}{2}},t)=\frac{u(x_{i+1},t)-u(x_i,t)}{x_{i+1}-x_i}+0(h^2)$$

Case 2: Otherwise,

$$u_x(x_{i+\frac{1}{2}},t)=\frac{u(x_{i+1},t)-u(x_i,t)}{x_{i+1}-x_i}+0(h)$$

From two cases, we get the approximation of the term $u_x(x_{i+\frac{1}{2}}, t)$

$$u_x(x_{i+\frac{1}{2}}, t) = \frac{u_{i+1}(t) - u_i(t)}{x_{i+1} - x_i} \quad \forall i \in \overline{0, N+1} \qquad (6)$$

Substituting this approximation to the equation (5), we have

$$\frac{du_i(t)}{dt} - \frac{u_{i-1}(t)}{(x_i - x_{i-1})|T_i|} + \left(\frac{1}{(x_{i+1} - x_i)|T_i|} + \frac{1}{(x_i - x_{i-1})|T_i|}\right)u_i(t) - \frac{u_{i+1}(t)}{(x_{i+1} - x_i)|T_i|} = f_i(t) \quad \forall i \in \overline{1, N}$$

$$\frac{du_i(t)}{dt} - \frac{u_{i-1}(t)}{(x_i - x_{i-1})|T_i|} + \left(\frac{1}{(x_{i+1} - x_i)|T_i|} + \frac{1}{(x_i - x_{i-1})|T_i|}\right)u_i(t) - \frac{u_{i+1}(t)}{(x_{i+1} - x_i)|T_i|} = f_i(t) \qquad \forall i \in \overline{1, N}$$

We set, for all $i \in \overline{1, N}$,

$$\alpha_i = \frac{-1}{(x_i - x_{i-1})|T_i|}$$

$$\beta_i = \frac{1}{(x_{i+1} - x_i)|T_i|} + \frac{1}{(x_i - x_{i-1})|T_i|}$$

$$\gamma_i = \frac{-1}{(x_{i+1} - x_i)|T_i|}$$

Thus, we get

$$\frac{du_i(t)}{dt} + \alpha_i u_{i-1}(t) + \beta_i u_i(t) + \gamma_i u_{i+1}(t) = f_i(t) \quad \forall i \in \overline{1, N} \quad (7)$$

Linear system for the scheme

$$
\begin{cases}
\dfrac{du_1(t)}{dt} + \beta_1 u_1(t) + \gamma_1 u_2(t) & = f_1(t) \\
\dfrac{du_2(t)}{dt} + \alpha_2 u_1(t) + \beta_2 u_2(t) + \gamma_2 u_3(t) & = f_2(t) \\
\dfrac{du_3(t)}{dt} + \qquad \alpha_3 u_2(t) + \beta_3 u_3(t) + \gamma_3 u_4(t) & = f_3(t) \\
\qquad \cdots\cdots\cdots\cdots & \\
\dfrac{du_{N-1}(t)}{dt} + \quad \alpha_{N-1} u_{N-2}(t) + \beta_{N-1} u_{N-1}(t) + \gamma_{N-1} u_N(t) & = f_{N-1}(t) \\
\dfrac{du_N(t)}{dt} + \qquad \alpha_N u_{N-1}(t) \; + \; \beta_N u_N(t) & = f_N(t)
\end{cases}
\tag{8}
$$

If the spacing T_i is uniform, for each $i \in 1, .., N$ there holds

$$\frac{du_i(t)}{dt} = ru_{i+1}^n - 2ru_i^n + ru_{i-1}^n, \quad u_m^n \approx u(x_m, nk)$$

where $r = k/h^2$. Then we get the linear ODE system

$$\frac{dU(t)}{dt} = AU(t) + F(t) \tag{9}$$

where A is a discrete approximation of the differential operator ∂_{xx}^2.

$$A = \begin{bmatrix} r & -2r & 0 & 0 & 0 & 0 \\ r & -2r & r & 0 & 0 & 0 \\ 0 & r & -2r & r & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & r & -2r & r \\ 0 & 0 & 0 & 0 & r & -2r \end{bmatrix}, F = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ \vdots \\ f_{N-1}(t) \\ f_N(t) \end{bmatrix} \tag{10}$$

The matrix A is tridiagonal and symmetric positive definite

Fully-discrete approximation

The semi-discrete approximation leads to a system of ODEs

$$\frac{dU(t)}{dt} = AU(t) + F(t) \tag{11}$$

This can be solved by standard numerical methods for ODEs with a time step $\Delta t = k$, e.g. the Forward Euler method

$$U^{n+1} = U^n + k\left(AU^n + F^n\right), \quad U^n \approx U(nk) \tag{12}$$

or

$$U^{n+1} = (I + kA)U^n + kF^n, \quad U^n \approx U(nk) \tag{13}$$

This is an explicit method and the time step restriction is

$$r = \frac{k}{h^2} \leq \frac{1}{2} \tag{14}$$

In many cases, this restriction is too severe and we need to switch to implicit methods

The "θ-method"

$$U^{n+1} = U^n + kA\left[(\theta U^{n+1} + (1-\theta)U^n\right] + k\underbrace{\left[\theta F^{n+1} + (1-\theta)F^n\right]}_{F^n_\theta} \tag{15}$$

or

$$(I - \theta kA)U^{n+1} = (I + (1-\theta)kA)U^n + kF^n_\theta \tag{16}$$

This includes some common methods

- $\theta = 0 \Rightarrow$ Forward Euler (explicit, 1st order)
- $\theta = 1/2 \Rightarrow$ Crank-Nicolson (implicit, 2nd order)
- $\theta = 1 \Rightarrow$ Backward Euler (implicit, 1st order)

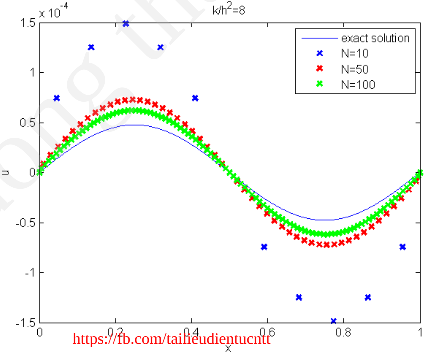
One can show the time step restriction

$$k \le h^2 \begin{cases} \frac{1}{2(1-2\theta)}, & \theta < 1/2 \\ \infty, & 1/2 \le \theta \le 1 \quad \text{(unconditionally stable)} \end{cases} \tag{17}$$

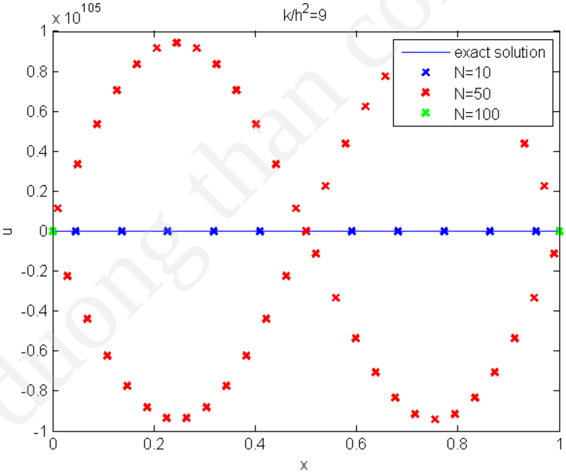
However, we need to solve a linear system in each time step.

Consider the PDE: $u_t = \frac{1}{16} u_{xx}, \quad x \in (0,1), t \in (0,T)$
Initial condition: $u_0(x) = sin(2\pi x)$
Boundary condition: $u(0,t) = u(1,t) = 0$
Exact solution: $u(x,t) = e^{-\frac{1}{4}\pi^2 t} sin(2\pi x)$
Stability condition: $k/h^2 \leq (1/2)/(1/16) = 8$

Exact solution vs numerical solution at $T = 4, \ k/h^2 = 8$



Exact solution vs numerical solution at $T = 4$, $k/h^2 = 9$



Lax equivalence theorem: The approximate numerical solution to a well-posed linear problem converges to the solution of the continuous equation if and only if the numerical scheme is linear, consistent and stable.

Let $u(x_i, t^n)$ be the exact solution of the PDE
$\overline{U}_i^n$ be the exact solution of the finite volume scheme
U_i^n be the actually computed solution of that scheme.
Then

$$|u_i^n - U_i^n| \le |u_i^n - \overline{U}_i^n| + |\overline{U}_i^n - U_i^n|$$

If the scheme is consistent then

$$|u_i^n - \overline{U}_i^n| \le O(h^\alpha, k^\beta), \forall i = 1, .., N; n = 0, 1, ...$$

If the scheme is stable then

$$|\overline{U}_i^n - U_i^n| \le O(h^\alpha, k^\beta), \forall i = 1, .., N; n = 0, 1, ...$$

Method 1: Matrix stability analysis
Consider the heat equation

$$u_t = u_{xx}$$

subject to a Dirichlet boundary condition.
After discretization by forward Euler scheme we obtain

$$U^{n+1} = (I + kA)U^n$$

Let $r = \frac{k}{h^2}$ and $B = I + kA$. For regular grid the matrix B has the form

$$\begin{pmatrix} 1-2r & r & 0 & . & . & 0 \\ r & 1-2r & r & 0 & . & 0 \\ . & . & . & . & . & . \\ 0 & . & 0 & r & 1-2r & r \\ 0 & . & . & 0 & r & 1-2r \end{pmatrix}$$

Iteration of the scheme will converge to a solution only if all the eigenvalues of B do not exceed 1 in magnitude. Indeed, if any of these eigenvalues exceeds 1 (say, $\lambda_1 > 1$), then $\|U^n = B^n U^0\|$ will grow as λ_1^n.

Proposition: Let B be an $N \times N$ matrix of the form

$$\begin{pmatrix} b & c & 0 & . & . & 0 \\ a & b & c & 0 & . & 0 \\ . & . & . & . & . & . \\ 0 & . & 0 & a & b & c \\ 0 & . & . & 0 & a & b \end{pmatrix}$$

The eigenvalues and the corresponding eigenvectors of B are

$$\lambda_j = b + 2\sqrt{ac}\cos\frac{\pi j}{N+1}, \quad \begin{pmatrix} \left(\frac{a}{c}\right)^{1/2} \sin\frac{1.\pi j}{N+1} \\ \left(\frac{a}{c}\right)^{2/2} \sin\frac{2.\pi j}{N+1} \\ \dots \\ \left(\frac{a}{c}\right)^{N/2} \sin\frac{N.\pi j}{N+1} \end{pmatrix}, \quad j = 1, .., N$$

We can immediately deduce that the eigenvalues of B are

$$\lambda_j = 1 - 2r + 2r\cos(\pi j/N), \quad j = 1, .., N-1$$

and hence

$$\lambda_{min} = \lambda_{N-1} = 1 - 2r + 2r\cos(\pi(N - 1/N)$$

$$\lambda_{max} = \lambda_1 = 1 - 2r + 2r\cos(\pi/N)$$

If $\pi/N \ll 1$ the preceding expressions reduce to

$$\lambda_{min} = \lambda_{N-1} = 1 - 4r + r\,(\pi/N)^2$$

$$\lambda_{max} = \lambda_1 = 1 - r\,(\pi/N)^2$$

The condition for convergence for λ_{min} yields

$$r \leq \frac{2}{4 - \left(\frac{\pi}{N}\right)^2} \approx \frac{1}{2}$$

With this condition, all the round-off errors will eventually decay, and the scheme is stable.

Method 2: Von Neumann stability analysis

It is rare that the eigenvalues of a matrix are available. We would like to deduce stability without finding those eigenvalues.

Let us denote the error at point (x_m, nk) by ϵ_m^n. Since the heat equation is linear, the error satisfies the same equation of the solution

$$\epsilon_m^{n+1} = r\epsilon_{i+1}^n + (1-2r)\epsilon_m^n + r\epsilon_{m-1}^n$$

At each time level, the error can be expanded as a linear superposition of Fourier harmonics:

$$\epsilon_m^n = \sum_l \rho^n e^{i\beta_l x_m}$$

Substituting the above expression into the equation for the error we obtain

$$\rho^{n+1} e^{i\beta x_m} = r\rho^n e^{i\beta x_{m+1}} + (1-2r)\rho^n e^{i\beta x_m} + r\rho^n e^{i\beta x_{m-1}}$$

We divide all terms by $\rho^n e^{i\beta x_m}$ to obtain

$$\rho = re^{i\beta h} + (1 - 2r) + re^{-i\beta h} = 1 - 2r + 2r\cos(\beta h)$$

Theorem (Von Neumann)
A numerical scheme for an evolution equation is stable if and only if the associated largest amplification factor satisfies

$$|\rho| \leq 1 + O(\Delta t)$$

Condition $|\rho| \leq 1$ yields

$$-1 \leq 1 - 2r + 2r\cos(\beta h) \leq 1$$

or

$$-1 \leq 1 - 4r\sin^2\left(\frac{\beta h}{2}\right) \leq 1$$

The LHS inequality implies

$$r \sin^2\left(\frac{\beta h}{2}\right) \leq \frac{1}{2}$$

In the worst case we must have

$$r \leq \frac{1}{2}$$

Compare method 1 and method 2

- The condition obtained by method 1

$$r \leq \frac{2}{4 - \left(\frac{\pi}{N}\right)^2}$$

is slightly different from that obtained by method 2

$$r \sin^2\left(\frac{\beta h}{2}\right) \leq \frac{1}{2}$$

because method 1 takes into account the boundary condition while method 2 ignores these conditions.

- Method 1 provides a sufficient condition for stability of the numerical scheme. A condition on r obtained by Von Neumann analysis is necessary, but not sufficient for the stability of a finite volume scheme. A scheme may found to be stable according to the Von Neumann analysis, but taking into account the boundary conditions may reveal that there still is an instability.

Existence and uniqueness of the solution

Proposition: For a given n, the discretized problem

$$u_i^{n+1} = u_i^n + k\left(\frac{(u_{i+1}^{n+1} - u_i^{n+1})}{|D_{i+1/2}||T_i|} + \frac{(u_{i-1}^{n+1} - u_i^{n+1})}{|D_{i-1/2}||T_i|}\right) + kf_i^{n+1} \quad (18)$$

for $i = 1, .., N$, $k = 1, .., N_k + 1$ with

$$u_i^0 = u_0(x_i), \quad i = 1, .., N$$

and

$$u_0^n = g(0, nk), \quad u_{N+1}^n = g(1, nk), \quad k = 1, .., N_k + 1$$

has a unique solution $U^n = (u_1^n, ..., u_N^n) \in \mathbb{R}^N$.

Proof

We only need to prove the uniqueness of the solution. For a given $n \in \{1, .., N_k + 1\}$, set $f_i^{n+1} = 0$ and $u_i^n = 0$ in (18), and $g(0, nk) = g(1, nk) = 0$ for all $i \in \{1, .., N\}$. Multiplying (18) by u_i^{n+1} and summing over $i \in \{1, .., N\}$ gives

$$\sum_{i=1}^{N} |T_i|(u_i^{n+1})^2 = k \sum_{i=1}^{N} \left(\frac{(u_{i+1}^{n+1} - u_i^{n+1})u_i^{n+1}}{|D_{i+1/2}|} + \frac{(u_{i-1}^{n+1} - u_i^{n+1})u_i^{n+1}}{|D_{i-1/2}|} \right)$$

or

$$\|u^{n+1}\|_{0,\mathcal{T}} + k \sum_{i=2}^{N} \frac{(u_{i-1}^{n+1} - u_i^{n+1})^2}{|D_{i-1/2}|} + k \frac{(u_N^{n+1})^2}{|D_{N+1/2}|} + k \frac{(u_1^{n+1})^2}{|D_{1/2}|} = 0$$

It yields that $u_i^{n+1} = 0$ for all $i = 1, .., N$.

Stability of the solution

Proposition: There exists c only depending on u_0, T, f and g such that

$$\|u^n\|_\infty, \quad n \in \{1,..,N_k+1\}\} \leq c$$

Moreover,

$$c = \|u_0\|_\infty + \|g\|_\infty + T\|f\|_\infty$$

Proof

Let $m_g = \min\{g(x,t), x \in \partial\Omega, t \in [0,T]\}$. Suppose that $\min\{u_i^{n+1}, \overline{1,N}\} < m_g$. Take i_0 such that $u_{i_0}^{n+1} = \min\{u_i^{n+1}, \overline{1,N}\}$. The discrete equation written for T_{i_0} is

$$u_{i_0}^{n+1} = u_{i_0}^n + k\left(\frac{(u_{i_0+1}^{n+1} - u_{i_0}^{n+1})}{|D_{i_0+1/2}||T_{i_0}|} + \frac{(u_{i_0-1}^{n+1} - u_{i_0}^{n+1})}{|D_{i_0-1/2}||T_{i_0}|}\right) + kf_{i_0}^{n+1}$$

Since $u_{i_0}^{n+1} < m_g$ we get

$$k\left(\frac{(u_{i_0+1}^{n+1} - u_{i_0}^{n+1})}{|D_{i+1/2}||T_i|} + \frac{(u_{i_0-1}^{n+1} - u_{i_0}^{n+1})}{|D_{i-1/2}||T_i|}\right) > 0$$

Therefore

$$u_{i_0}^{n+1} \geq u_{i_0}^{n} + kf_{i_0}^{n+1} \geq \min\{u_i^n, \overline{1,N}\} + km_f$$

where $m_f = \min\{f(x,t), x \in \bar{\Omega}, t \in [0,T]\}$
It implies that

$$\min\{u_i^{n+1}, \overline{1,N}\} \geq \min\{\min\{u_i^n, i = \overline{1,N}\} + km_f, m_g\}$$

By induction it yields

$$\min\{u_i^{n+1}, \overline{1,N}\} \geq \min\{\min\{u_i^0, \overline{1,N}\}, m_g\} + \min\{(n+1)km_f, 0\}$$

Similarly,

$$\max\{u_i^{n+1}, \overline{1,N}\} \geq \max\{\max\{u_i^0, \overline{1,N}\}, M_g\} + \max\{(n+1)kM_f, 0\}$$

where $M_f = \max\{f(x,t), x \in \bar{\Omega}, t \in [0,T]\}$ and
$M_g = \max\{g(x,t), x \in \partial\Omega, t \in [0,T]\}$.
It follows that

$$\|u^n\|_\infty \quad n \in \{1,..,N_k+1\}\} \leq c$$

where

$$c = \|u_0\|_\infty + \|g\|_\infty + T\|f\|_\infty$$

Consistency of the scheme

If $u \in \mathbb{C}^2([0,1],\mathbb{R})$, there exists $C \in \mathbb{R}_+$ only depending on u such that

$$|\tau_{i+1/2}^{n+1}| = |S_i^{n+1} + R_{i-1/2}^{n+1} - R_{i+1/2}^{n+1}| \leq C(h+k)$$

where

$$S_i^{n+1} = \int_{|T_i|} u_t(x, t_{n+1})dx - \frac{|T_i|(u(x_i, t_{n+1}) - u(x_i, t_n))}{k}$$

$$R_{i-1/2}^{n+1} = u_x(x_{i-1/2}, t_{n+1})dx - \frac{u(x_i, t_{n+1}) - u(x_{i-1}, t_{n+1})}{|D_{i-1/2}|}$$

$$R_{i+1/2}^{n+1} = u_x(x_{i+1/2}, t_{n+1})dx - \frac{u(x_{i+1}, t_{n+1}) - u(x_i, t_{n+1})}{|D_{i+1/2}|}|$$

Proof

We have proved for elliptic problems that

$$|R_{i-1/2}^{n+1}| \leq Ch$$

and

$$|R_{i+1/2}^{n+1}| \leq Ch$$

It suffices to prove

$$|S_i^{n+1}| \leq C(h+k)$$

Error estimate

Proposition: There exists $C \in \mathbb{R}_+$ only depending on u, Ω and T such that

$$\left(\sum_{i=1}^{N}(e_i^n)^2|T_i|\right)^{1/2} \leq C(h+k), \quad \forall n \in \{1,..,N_k+1\}$$

where $e_i^n = u(x_i, nk) - u_i^k$.

Proof

Integrating equation $u_t - u_{xx} = f$ over T_i yields

$$\int_{T_i} u_t(x, t_{n+1})dx - [u_x(x_{i+1/2}, t_{n+1}) - u_x(x_{i-1/2}, t_{n+1})] = \int_{T_i} f(x, t_{n+1})d$$

The approximate solution U satisfies

$$\frac{|T_i|(u_i^{n+1} - u_i^n)}{k} - \left(\frac{u_{i+1}^{n+1} - u_i^{n+1}}{|D_{i+1/2}|} + \frac{u_{i-1}^{n+1} - u_i^{n+1}}{|D_{i-1/2}|}\right) = \int_{T_i} f(x, t_{n+1})dx$$

Subtracting side by side these two equations gives

$$\frac{|T_i|(u_i^{n+1}-u_i^n)}{k}-\left(\frac{u_{i+1}^{n+1}-u_i^{n+1}}{|D_{i+1/2}|}+\frac{u_{i-1}^{n+1}-u_i^{n+1}}{|D_{i-1/2}|}\right)$$
$$-\int_{T_i} u_t(x,t_{n+1})dx+[u_x(x_{i+1/2},t_{n+1})-u_x(x_{i-1/2},t_{n+1})]=0$$

From here we make appear the error

$$\frac{|T_i|(e_i^{n+1}-e_i^n)}{k}-\left(\frac{e_{i+1}^{n+1}-e_i^{n+1}}{|D_{i+1/2}|}+\frac{e_{i-1}^{n+1}-e_i^{n+1}}{|D_{i-1/2}|}\right)$$
$$=-S_i^{n+1}+(R_{i+1/2}^{n+1}-R_{i-1/2}^{n+1})$$

Multiplying by e_i^{n+1} and summing all over i gives

$$\frac{1}{k}\|e^{n+1}\|_{0,T}^2 + \|e^{n+1}\|_{1,D}^2 = \frac{1}{k}\sum_{i=1}^{N}|T_i|e_i^n e_i^{n+1} - \sum_{i=1}^{N} S_i^{n+1}e_i^{n+1}$$

$$+\sum_{i=0}^{N} R_{i+1/2}(e_i^{n+1} - e_{i+1}^{n+1})$$

Using the same technique for elliptic problems we get

$$\frac{1}{k}\|e^{n+1}\|_{0,T}^2 + \frac{1}{2}\|e^{n+1}\|_{1,T}^2 \leq \frac{1}{k}\sum_{i=1}^{N}|T_i|e_i^n e_i^{n+1} - \sum_{i=1}^{N} S_i^{n+1}e_i^{n+1} + C_1h^2$$

We have the following inequalities

$$|\sum_{i=1}^{N} S_i^{n+1}e_i^{n+1}| \leq C_2(h+k)\sum_{i=1}^{N}|T_i||e_i^{n+1}| \leq C_2(h+k)|[0,1]|\|e^{n+1}\|_{0,T}$$

$$\frac{1}{k}\sum_{i=1}^{N}|T_i|e_i^n e_i^{n+1} \leq \frac{1}{2k}(\|e^{n+1}\|_{0,T} + \|e^n\|_{0,T})$$

Therefore

$$\frac{1}{2k}\|e^{n+1}\|_{0,T}^2 + \frac{1}{2}\|e^{n+1}\|_{1,D}^2 \leq \frac{1}{2k}\|e^n\|_{0,T}^2 + C_2(h+k)\|e^{n+1}\|_{0,T} + C_1 h^2$$

It implies that

$$\|e^{n+1}\|_{0,T}^2 \leq \|e^n\|_{0,T}^2 + C_3(kh^2 + k(h+k)\|e^{n+1}\|_{0,T}) \qquad (19)$$

where $C_3 = max\{2C_1, 2C_2\}$.
Applying Cauchy-Schwarz inequality gives

$$k(h+k)\|e^{n+1}\|_{0,T} \leq \epsilon^2\|e^{n+1}\|_{0,T}^2 + (1/\epsilon^2)C_3^2 k^2(k+h)^2 \qquad (20)$$

Taking $\epsilon^2 = k/(k+1)$ in (20) yields

$$k(h+k)\|e^{n+1}\|_{0,T} \le \frac{k}{k+1}\|e^{n+1}\|_{0,T}^2 + \frac{k+1}{k} C_3^2 k^2 (k+h)^2$$

Inequality (19) can now be rewritten as

$$\|e^{n+1}\|_{0,T}^2 \le (1+k)\|e^n\|_{0,T}^2 + C_3 k h^2 (1+k) + (1+k)^2 C_3^2 k (k+h)^2 \qquad (21)$$

Suppose that $\|e^n\|_{0,T}^2 \le c_n (h+k)^2$ then from (21) it yields

$$\begin{aligned}
\|e^{n+1}\|_{0,T}^2 &\le (1+k)(h+k)^2 c_n + C_3 k h^2 (1+k) + (1+k)^2 C_3^2 k (k+h)^2 \\
&\le (1+k)(h+k)^2 c_n + C_3 k (h+k)^2 (1+k) + (1+k)^2 C_3^2 k (k + \\
&\le (h+k)^2 [(1+k)c_n + k(C_3(1+k) + C_3^2 (1+k)^2)] \\
&\le (h+k)^2 [(1+k)c_n + k \underbrace{(C_3(1+T) + C_3^2(1+T)^2)}_{C_4}]
\end{aligned}$$

Setting $c_{n+1} = (1+k)c_n + C_4 k$ then

$$\|e^{n+1}\|^2_{0,T} \leq c_{n+1}(h+k)^2$$

Since $\|e^n\|^2_{0,T} = 0$ we can choose $c_0 = 0$. This choice gives

$$c_n = [(k+1)^n - 1]C_4$$

We will prove by induction that

$$c_n \leq e^{2nk} \leq C_4 e^{2T}$$

which gives

$$\|e^n\|^2_{0,T} \leq C_4 e^{2T}(h+k)^2$$