Lecture notes on Topology

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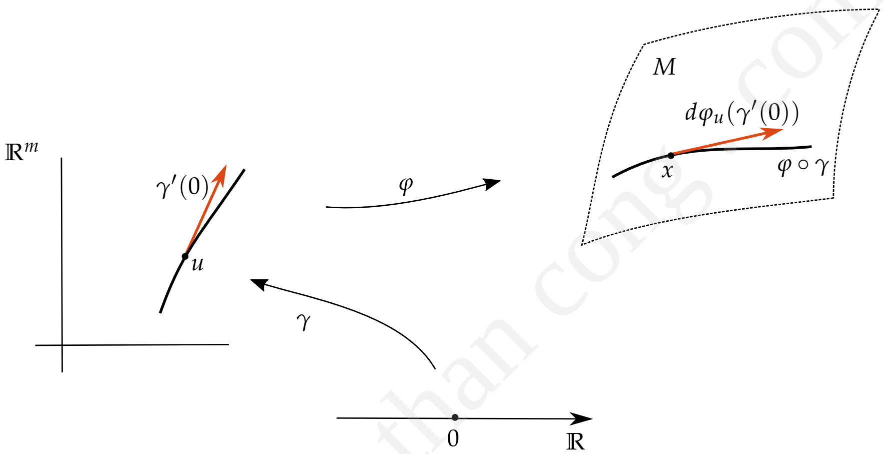

ABSTRACT. This is a set of lecture notes for a series of introductory courses in topology for undergraduate students at the University of Science, Ho Chi Minh City. It is written to be delivered by myself, tailored to my students. I did not write it with other lecturers or self-study readers in mind.

In writing these notes I intend that more explanations and discussions will be carried out in class. I hope by presenting only the essentials these notes will be more suitable for classroom use. Some details are left for students to fill in or to be discussed in class.

A sign $\surd$ in front of a problem notifies the reader that this is an important one although it might not appear to be so initially. A sign * indicates a relatively more difficult problem.

The latest version of this set of notes is available at
http://www.math.hcmus.edu.vn/∼hqvu/teaching/n.pdf, and the source file is at
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Introduction

Topology is a mathematical subject that studies shapes. A set becomes a topological space when each element of the set is given a collection of neighborhoods. Operations on topological spaces must be continuous, bringing certain neighborhoods into neighborhoods.

Unlike geometry, there is no notion of distance. So topology is "more general" than geometry. But usually people do not classify geometry as a subfield of topology. On the other hand, if one forgets the distance from geometrical objects, one gets topological space. This is a more prevalent point of view: topology is the part of geometry that does not concern distance.

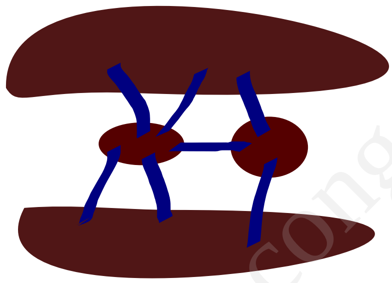


FIGURE 0.1. How to make a closed trip such that every bridge (a blue arc) is crossed exactly once? This is the problem Seven Bridges of Konigsberg, studied by Leonard Euler in the 18th century. It does not depend on the size of the bridges.

Characteristics of topology. Operations on topological objects are more relaxed: beside moving around (allowed in geometry), stretching or bending are allowed in topology (not allowed in geometry). For example, in topology circles - big or small, anywhere - are same. Ellipses and circles are same.

On the other hand in topology tearing or breaking are not allowed: circles are still different from lines.

While topological operations are more flexible they still retain some essential properties of spaces.

Contributions of topology. Topology provides basic notions to areas of mathematics where continuity appears.

Topology focuses on some essential properties of spaces. It can be used in qualitative study. It can be useful where metrics or coordinates are not available, not natural, or not necessary.

Topology often does not stand alone: there are fields such as algebraic topology, differential topology, geometric topology, combinatorial topology, quantum topology, ...

Topology often does not solve a problem by itself, but contributes important understanding, settings, and tools. Topology features prominently in differential geometry, global analysis, algebraic geometry, theoretical physics ...

General Topology

1. Infinite sets

General Topology is the part of Topology that studies basic settings, also called Point-set Topology.

In General Topology we often work in very general settings, in particular we often deal with infinite sets.

We will not define what a set is. In other words, we will work on the level of "naive set theory", pioneered by Georg Cantor in the late 19th century. We will use familiar notions such as maps, Cartesian product of two sets, … without recalling definitions. We will not go back to definitions of the natural numbers or the real numbers.

Still, we should be aware of certain problems in naive set theory.

EXAMPLE (Russell's paradox). Consider the set $S = \{x \mid x \notin x\}$ (the set of all sets which are not members of themselves). Then whether $S \in S$ or not is undecidable, since answering yes or no to this question leads to contradiction. [1]

Axiomatic systems for the theory of sets have been developed since then. In the Von Neumann-Bernays-Godel system a more general notion than set, called *class* (lớp), is used. In this course, we do not distinguish set, class, or *collection* (họ), but in occasions where we deal with "set of sets" we often prefer the term collection. For more one can read [**End77**, p. 6], [**Dug66**, p. 32].

Indexed collection. Suppose that A is a collection, I is a set and $f : I \to A$ is a map. The map f is called an *indexed collection*, or *indexed family* (họ được đánh chỉ số). We often write $f_i = f(i)$, and denote the indexed collection f by $(f_i)_{i\in I}$ $\{f_i\}_{i\in I}$. Notice that it can happen that $f_i = f_j$ for some $i \neq j$.

EXAMPLE. A sequence of elements in a set A is a collection of elements of A indexed by the set $\mathbb{Z}^+$ of positive integer numbers, written as $(a_n)_{n\in\mathbb{Z}^+}$.

Relation. A *relation* (quan hệ) R on a set S is a non-empty subset of the set $S \times S$.

When $(a, b) \in R$ we often say that a is related to b and often write $a \sim_R b$.

A relation said to be:

[1] Discovered in 1901 by Bertrand Russell. A famous version of this paradox is the barber paradox: In a village there is a barber; his job is to do hair cut for a villager if and only if the villager does not cut his hair himself. Consider the set of all villagers who had their hairs cut by the barber. Is the barber himself a member of that set?

(a) reflexive (phản xạ) if $\forall a \in S, (a,a) \in R$.
(b) symmetric (đối xứng) if $\forall a,b \in S, (a,b) \in R \Rightarrow (b,a) \in R$.
(c) antisymmetric (phản đối xứng) if $\forall a,b \in S, ((a,b) \in R \wedge (b,a) \in R) \Rightarrow a = b$.
(d) transitive (bắc cầu) if $\forall a,b,c \in S, ((a,b) \in R \wedge (b,c) \in R) \Rightarrow (a,c) \in R$.

An *equivalence relation* on S is a relation that is reflexive, symmetric and transitive.

If R is an equivalence relation on S then an *equivalence class* (lớp tương đương) represented by $a \in S$ is the subset $[a] = \{b \in S \mid (a,b) \in R\}$. Two equivalence classes are either coincident or disjoint. The set S is partitioned (phân hoạch) into the disjoint union of its equivalence classes.

Countable sets. Two sets are said to be *set-equivalent* if there is a bijection from one set to the other set. A set is said to be *finite* if it is equivalent to a subset $\{1,2,3,\ldots,n\}$ of all positive integers $\mathbb{Z}^+$ for some $n \in \mathbb{Z}^+$. If a set is not finite we say that it is *infinite*.

DEFINITION. A set is called *countably infinite* (vô hạn đếm được) if it is equivalent to the set of all positive integers. A set is called *countable* if it is either finite or countably infinite.

Intuitively, a countably infinite set can be "counted" by the positive integers. The elements of such a set can be indexed by the set of all positive integers as a sequence $a_1, a_2, a_3, \ldots$.

EXAMPLE. The set $\mathbb{Z}$ of all integer numbers is countable.

PROPOSITION. *A subset of a countable set is countable.*

PROOF. The statement is equivalent to the statement that a subset of $\mathbb{Z}^+$ is countable. Suppose that A is an infinite subset of $\mathbb{Z}^+$. Let a_1 be the smallest number in A. Let a_n be the smallest number in $A \setminus \{a_1, a_2, \ldots, a_{n-1}\}$. Then $a_{n-1} < a_n$ and the set $B = \{a_n | \; n \in \mathbb{Z}^+\}$ is a countably infinite subset of A.

We show that any element m of A is an a_n for some n, and therefore $B = A$.

Let $C = \{a_n \mid a_n \geq m\}$. Then $C \neq \emptyset$ since B is infinite. Let $a_{n_0} = \min C$. Then $a_{n_0} \geq m$. Further, since $a_{n_0-1} < a_{n_0}$ we have $a_{n_0-1} < m$. This implies $m \in A \setminus \{a_1, a_2, \ldots, a_{n_0-1}\}$. Since $a_{n_0} = \min\left(A \setminus \{a_1, a_2, \ldots, a_{n_0-1}\}\right)$ we must have $a_{n_0} \leq m$. Thus $a_{n_0} = m$. □

COROLLARY. *If there is an injective map from a set S to $\mathbb{Z}^+$ then S is countable.*

PROPOSITION 1.1. *If there is a surjective map from $\mathbb{Z}^+$ to a set S then S is countable.*

PROOF. Suppose that there is a surjective map $\phi : \mathbb{Z}^+ \to S$. For each $s \in S$ the set $\phi^{-1}(s)$ is non-empty. Let $n_s = \min \phi^{-1}(s)$. The map $s \mapsto n_s$ is an injective map from S to a subset of $\mathbb{Z}^+$, therefore S is countable. □

PROPOSITION. *$\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.*

PROOF. We can enumerate $\mathbb{Z}^+ \times \mathbb{Z}^+$ by the method shown in the following diagram:

$$\begin{tikzcd}
(1,1) \arrow[r] & (1,2) \arrow[dl] & (1,3) \arrow[dl] & (1,4) \arrow[dl] \\
(2,1) \arrow[urr] & (2,2) \arrow[dl] & (2,3) \arrow[dl] & \\
(3,1) \arrow[uurrr] & (3,2) \arrow[dl] & \dots & \\
(4,1) & & &
\end{tikzcd}$$

To prove in the detail we can derive the explicit formula by counting along the diagonals:

$$(m,n) \mapsto (1+2+\cdots+((m+n-1)-1)) + m = \frac{(m+n-2)(m+n-1)}{2} + m.$$

We will check that this map is injective. Let $k = m+n$. Suppose that $\frac{(k-2)(k-1)}{2} + m = \frac{(k'-2)(k'-1)}{2} + m'$. If $k = k'$ then the equation certainly leads to $m = m'$ and $n = n'$. If $k < k'$ then

$$\begin{aligned}
\frac{(k-2)(k-1)}{2} + m \quad &\leq \frac{(k-2)(k-1)}{2} + (k-1) = \frac{(k-1)k}{2} < \\
&< \frac{(k-1)k}{2} + 1 \leq \frac{(k'-2)(k'-1)}{2} + m',
\end{aligned}$$

a contradiction. □

THEOREM 1.2. *The union of a countable collection of countable sets is a countable set.*

PROOF. The collection can be indexed as $A_1, A_2, \dots, A_i, \dots$ (if the collection is finite we can let A_i be the same set for all i starting from a certain index). The elements of each set A_i can be indexed as $a_{i,1}, a_{i,2}, \dots, a_{i,j}, \dots$ (if A_i is finite we can let $a_{i,j}$ be the same element for all j starting from a certain index). This means there is a surjective map from the index set $\mathbb{Z}^+ \times \mathbb{Z}^+$ to the union $\bigcup_{i \in I} A_i$ by $(i,j) \mapsto a_{i,j}$. □

THEOREM. *The set $\mathbb{Q}$ of all rational numbers is countable.*

PROOF. One way to prove this result is to write $\mathbb{Q} = \bigcup_{q=1}^{\infty} \left\{ \frac{p}{q} \mid p \in \mathbb{Z} \right\}$, then use 1.2.

Another way is to observe that if we write each rational number in the form $\frac{p}{q}$ with $q > 0$ and $\gcd(p,q) = 1$ then the map $\frac{p}{q} \mapsto (p,q)$ from $\mathbb{Q}$ to $\mathbb{Z} \times \mathbb{Z}$ is injective. □

THEOREM 1.3. *The set $\mathbb{R}$ of all real numbers is uncountable.*

PROOF. The proof uses the *Cantor diagonal argument*.

Suppose that set of all real numbers in decimal form in the interval $[0,1]$ is countable, and is enumerated as a sequence $\{a_i \mid i \in \mathbb{Z}^+\}$. Let us write

$$a_1 = 0.a_{1,1}a_{1,2}a_{1,3}\ldots$$

$$a_2 = 0.a_{2,1}a_{2,2}a_{2,3}\ldots$$

$$a_3 = 0.a_{3,1}a_{3,2}a_{3,3}\ldots$$

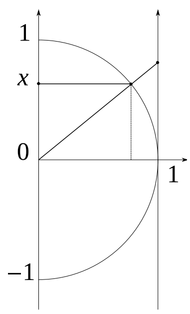
$$\vdots$$

There are real numbers whose decimal presentations are not unique, such as $\frac{1}{2} = 0.5000\ldots = 0.4999\ldots$. Choose a number $b = 0.b_1b_2b_3\ldots$ such that $b_n \neq 0,9$ and $b_n \neq a_{n,n}$. Choosing b_n differing from 0 and 9 will guarantee that $b \neq a_n$ for all n (see more at 1.14). Thus the number b is not in the above table, a contradiction. □

EXAMPLE. Two intervals $[a,b]$ and $[c,d]$ on the real number line are equivalent. The bijection can be given by a linear map $x \mapsto \frac{d-c}{b-a}(x-a)+c$. Similarly, two intervals (a,b) and (c,d) are equivalent.

The interval $(-1,1)$ is equivalent to $\mathbb{R}$ via a map related to the tan function:

$$x \mapsto \frac{x}{\sqrt{1-x^2}}.$$



REMARK. Whether there is a set which is "more" than $\mathbb{Z}$ but "less" than $\mathbb{R}$ cannot be answered. That there is no such set can be accepted as an axiom, called the *Continuum hypothesis*.

Given a set S the set of all subsets of S is denoted by $\mathcal{P}(S)$ or 2^S.

THEOREM 1.4. *Any non-empty set is not equivalent to the set of all of its subsets. Given a set S there is no surjective map from S to 2^S.*

PROOF. Let $S \neq \emptyset$. Let ϕ be any map from S to 2^S. Let $X = \{a \in S \mid a \notin \phi(a)\}$. Suppose that there is $x \in S$ such that $\phi(x) = X$. Then the truth of the statement

$x \in X$ (whether it is true or false) is undecidable, a contradiction. Therefore there is no $x \in S$ such that $\phi(x) = X$, so ϕ is not surjective. □

This result implies that any set is "smaller" than the set of all of its subsets. So there can not be a set that is "larger" than any other set. There is no "universal set", "set which contains everything", or "set of all sets".

REMARK. There is a notion of "sizes" of sets, called cardinality, but we will not present it here.

Order. An *order* (thứ tự) on a set S is a relation R on S that is reflexive, anti-symmetric and transitive.

Note that two arbitrary elements a and b do not need to be comparable; that is, the pair (a, b) may not belong to R. For this reason an order is often called a partial order.

When $(a, b) \in R$ we often write $a \le b$. When $a \le b$ and $a \ne b$ we write $a < b$.

If any two elements of S are related then the order is called a *total order* (thứ tự toàn phần) and $(S, \le)$ is called a *totally ordered set*.

EXAMPLE. The set $\mathbb{R}$ of all real numbers with the usual order $\le$ is totally ordered.

EXAMPLE. Let S be a set. Denote by 2^S the collection of all subsets of S. Then $(2^S, \subseteq)$ is a partially ordered set, but is not totally ordered if S has more than one element.

EXAMPLE (dictionary order). Let $(S_1, \le_1)$ and $(S_2, \le_2)$ be two ordered sets. The following is an order on $S_1 \times S_2$: $(a_1, b_1) \le (a_2, b_2)$ if $(a_1 < a_2)$ or $((a_1 = a_2) \wedge (b_1 \le b_2))$. This is called the *dictionary order* (thứ tự từ điển).

In an ordered set, the *smallest element* (phần tử nhỏ nhất) is the element that is smaller than all other elements. More concisely, if S is an ordered set, the smallest element of S is an element $a \in S$ such that $\forall b \in S, a \le b$. The smallest element, if exists, is unique.

A *minimal element* (phần tử cực tiểu) is an element which no element is smaller than. More concisely, a minimal element of S is an element $a \in S$ such that $\forall b \in S, b \le a \Rightarrow b = a$. There can be more than one minimal element.

A *lower bound* (chặn dưới) of a subset of an ordered set is an element of the set that is smaller than or equal to any element of the subset. More concisely, if $A \subset S$ then a lower bound of A in S is an element $a \in S$ such that $\forall b \in A, a \le b$.

The definitions of largest element, maximal element, and upper bound are similar.

The Axiom of choice.

THEOREM. *The following statements are equivalent:*

(a) *Axiom of choice: Given a collection of non-empty sets, there is a function defined on this collection, called a choice function, associating each set in the collection with an element of that set.*
(b) *Zorn lemma: If any totally ordered subset of an ordered set X has an upper bound then X has a maximal element.*

Zorn lemma is often a convenient form of the Axiom of choice.

Intuitively, a choice function "chooses" an element from each set in a given collection of non-empty sets. The Axiom of choice allows us to make infinitely many arbitrary choices. [2] This is also often used in constructions of functions, sequences, or nets, see one example at 5.4. One common application is the use of the product of an infinite family of sets – the Cartesian product, discussed below.

The Axiom of choice is needed for many important results in mathematics, such as the Tikhonov theorem in Topology, the Hahn-Banach theorem and Banach-Alaoglu theorem in Functional analysis, the existence of a Lebesgue unmeasurable set in Real analysis, ...

There are cases where this axiom could be avoided. For example in the proof of 1.1 we used the well-ordered property of $\mathbb{Z}^+$ instead. See for instance [**End77**, p. 151] for further material on this subject.

Cartesian product. Let $(A_i)_{i\in I}$ be a collection of sets indexed by a set I. The *Cartesian product* (tích Decartes) $\prod_{i\in I} A_i$ of this indexed collection is defined to be the collection of all maps $a : I \to \bigcup_{i\in I} A_i$ such that $a(i) \in A_i$ for every $i \in I$. The existence of such a map is a consequence of the Axiom of choice. An element a of $\prod_{i\in I} A_i$ is often denoted by $(a_i)_{i\in I}$, with $a_i = a(i) \in A_i$ being the coordinate of index i, in analog to the finite product case.

Problems.

1.5. Let f be a function. Show that:
(a) $f(\bigcup_i A_i) = \bigcup_i f(A_i)$.
(b) $f(\bigcap_i A_i) \subset \bigcap_i f(A_i)$. If f is injective (one-one) then equality happens.
(c) $f^{-1}(\bigcup_i A_i) = \bigcup_i f^{-1}(A_i)$.
(d) $f^{-1}(\bigcap_i A_i) = \bigcap_i f^{-1}(A_i)$.

1.6. Let f be a function. Show that:
(a) $f(f^{-1}(A)) \subset A$. If f is surjective (onto) then equality happens.
(b) $f^{-1}(f(A)) \supset A$. If f is injective then equality happens.

1.7. If A is countable and B is infinite then $A \cup B$ is equivalent to B.

1.8. Give another proof of 1.2 by checking that the map $\mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$, $(m,n) \mapsto 2^m 3^n$ is injective.

[2] Bertrand Russell said that choosing one shoe from each pair of shoes from an infinite collection of pairs of shoes does not need the Axiom of choice (because in a pair of shoes the left shoe is different from the right one so we can define our choice), but usually in a pair of socks the two socks are identical, so choosing one sock from each pair of socks from an infinite collection of pairs of socks needs the Axiom of choice.

1.9. Show that the set of points in $\mathbb{R}^n$ with rational coordinates is countable.

1.10. Show that if A has n elements then $|2^A| = 2^n$.

1.11. Show that the set of all functions $f : A \to \{0,1\}$ is equivalent to 2^A.

1.12. A real number α is called an algebraic number if it is a root of a polynomial with integer coefficients. Show that the set of all algebraic numbers is countable.

A real number which is not algebraic is called transcendental. For example it is known that π and e are transcendental. Show that the set of all transcendental numbers is uncountable.

1.13. A continuum set is a set which is equivalent to $\mathbb{R}$. Show that a countable union of continuum sets is a continuum set.

1.14. Show that any real number could be written in base d with any $d \in \mathbb{Z}$, $d \geq 2$. However two forms in base d could represent the same real number, as seen in 1.3. This happens only if starting from certain digits, all digits of one form are 0 and all digits of the other form are $d-1$. (This result is used in 1.3.)

1.15 ($2^{\aleph_0} = c$). * We prove that $2^{\mathbb{N}}$ is equivalent to $\mathbb{R}$.

(a) Show that $2^{\mathbb{N}}$ is equivalent to the set of all sequences of binary digits.
(b) Using 1.14, deduce that $|[0,1]| \leq |2^{\mathbb{N}}|$.
(c) Consider a map $f : 2^{\mathbb{N}} \to [0,2]$, for each binary sequence $a = a_1a_2a_3\cdots$ define $f(a)$ as follows. If starting from a certain digit, all digits are 1, then let $f(a) = 1.a_1a_2a_3\cdots$. Otherwise let $f(a) = 0.a_1a_2a_3\cdots$. Show that f is injective.

Deduce that $|2^{\mathbb{N}}| \leq |[0,2]|$.

1.16 ($\mathbb{R}^2$ is equivalent to $\mathbb{R}$). * Here we prove that $\mathbb{R}^2$ is equivalent to $\mathbb{R}$, in other words, a plane is equivalent to a line. As a corollary, $\mathbb{R}^n$ is equivalent to $\mathbb{R}$.

(a) First method: Construct a map from $[0,1) \times [0,1)$ to $[0,1)$ as follows. In view of 1.14, we only allow decimal presentations in which not all digits are 9 starting from a certain digit. The pair of two real numbers $0.a_1a_2\ldots$ and $0.b_1b_2\ldots$ corresponds to the real number $0.a_1b_1a_2b_2\ldots$. Check that this map is injective.
(b) Second method: Construct a map from $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ to $2^{\mathbb{N}}$ as follows. The pair of two binary sequences $a_1a_2\ldots$ and $b_1b_2\ldots$ corresponds to the binary sequence $a_1b_1a_2b_2\ldots$. Check that this map is injective. Then use 1.15.

1.17 (Cantor set). Deleting the open interval $(\frac{1}{3}, \frac{2}{3})$ from the interval of real numbers $[0,1]$, one gets a space consisting of two intervals $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Continuing, delete the intervals $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. In general on each of the remaining intervals, delete the middle open interval of $\frac{1}{3}$ the length of that interval. The Cantor set is the set of remaining points. It can be described as the set of real numbers $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$, $a_n = 0, 2$. In other words, it is the set of real numbers in $[0,1]$ which in base 3 could be written without the digit 1.

Show that the total length of the deleted intervals is 1. Is the Cantor set countable?

1.18 (transfinite induction principle). An ordered set S is *well-ordered* (được sắp tốt) if every non-empty subset A of S has a smallest element, i.e. $\exists a \in A, \forall b \in A, a \leq b$. For example with the usual order, $\mathbb{N}$ is well-ordered while $\mathbb{R}$ is not. Notice that a well-ordered set must be totally ordered. Ernst Zermelo proved in 1904, based on the Axiom of choice, that any set can be well-ordered.

Prove the following generalization of the principle of induction. Let A be a well-ordered set. Let $P(a)$ be a statement whose truth depends on $a \in A$. Suppose that if $P(a)$ is true for all $a < b$ then $P(b)$ is true. Then $P(a)$ is true for all $a \in A$.

2. Topological space

The reason we study topological space is that this is a good setting for discussions on continuity of maps. Briefly, a topology is a system of open sets.

DEFINITION. A *topology* on a set X is a collection τ of subsets of X satisfying:

(a) The sets $\emptyset$ and X are elements of τ.
(b) A union of elements of τ is an element of τ.
(c) A finite intersection of elements of τ is an element of τ.

Elements of τ are called *open sets* of X in the topology τ.

In short, a topology on a set X is a collection of subsets of X which includes $\emptyset$ and X and is "closed" under unions and finite intersections.

EXAMPLE. On any set X there is the *trivial topology* (tôpô hiển nhiên) $\{\emptyset, X\}$. There is also the *discrete topology* (tôpô rời rạc) whereas any subset of X is open. Thus on a set there can be many topologies.

A set X together with a topology τ is called a *topological space*, denoted by (X, τ) or X alone if we do not need to specify the topology. An element of X is often called a *point*.

A *neighborhood* (lân cận) of a point $x \in X$ is a subset of X which contains an open set containing x. Note that a neighborhood does not need to be open.[3]

REMARK. The statement "intersection of finitely many open sets is open" is equivalent to the statement "intersection of two open sets is open".

Metric space. Recall that, briefly, a metric space is a set equipped with a distance between every two points. Namely, a metric space is a set X with a map $d : X \times X \mapsto \mathbb{R}$ such that for all $x, y, z \in X$:

(a) $d(x, y) \geq 0$ (distance is non-negative),
(b) $d(x, y) = 0 \iff x = y$ (distance is zero if and only if the two points coincide),
(c) $d(x, y) = d(y, x)$ (distance is symmetric),
(d) $d(x, y) + d(y, z) \geq d(x, z)$ (triangular inequality).

A ball is a set of the form $B(x, r) = \{y \in X \mid d(y, x) < r\}$ where $r \in \mathbb{R}, r > 0$.

In the theory of metric spaces, a subset U of X is said to be open if for all x in U there is $\epsilon > 0$ such that $B(x, \epsilon)$ is contained in U. This is equivalent to saying that a non-empty open set is a union of balls.

To check that this is indeed a topology, we only need to check that the intersection of two balls is a union of balls. Let $z \in B(x, r_x) \cap B(y, r_y)$, let $r_z = \min\{r_x - d(z, x), r_y - d(z, y)\}$. Then the ball $B(z, r_z)$ will be inside both $B(x, r_x)$ and $B(y, r_y)$.

[3]Be careful that not everyone uses this convention. For instance Kelley [**Kel55**] uses this convention but Munkres [**Mun00**] requires a neighborhood to be open.

Thus a metric space is canonically a topological space with the topology generated by the metric. *When we speak about topology on a metric space we mean this topology*.

EXAMPLE (normed spaces). Recall that a normed space (không gian định chuẩn) is briefly a vector spaces equipped with lengths of vectors. Namely, a normed space is a set X with a structure of vector space over the real numbers and a real function $X \to \mathbb{R}$, $x \mapsto ||x||$, called a *norm* (chuẩn), satisfying:

(a) $||x|| \geq 0$ and $||x|| = 0 \iff x = 0$ (length is non-negative),
(b) $||cx|| = |c|||x||$ for $c \in \mathbb{R}$ (length is proportionate to vector),
(c) $||x + y|| \leq ||x|| + ||y||$ (triangle inequality).

A normed space is canonically a metric space with metric $d(x, y) = ||x - y||$. Therefore a normed space is canonically a topological space with the topology generated by the norm.

EXAMPLE (Euclidean topology). In $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\}$, the Euclidean norm of a point $x = (x_1, x_2, \ldots, x_n)$ is $\|x\| = \left[\sum_{i=1}^n x_i^2\right]^{1/2}$. The topology generated by this norm is called the *Euclidean topology* (tôpô Euclid) of $\mathbb{R}^n$.

A complement of an open set is called a *closed set*.

PROPOSITION (dual description of topology). *In a topological space X:*

(a) *$\emptyset$ and X are closed.*
(b) *A finite union of closed sets is closed.*
(c) *An intersection of closed sets is closed.*

Interior – Closure – Boundary. Let X be a topological space and let A be a subset of X. A point x in X is said to be:

- an *interior point* (điểm trong) of A in X if there is an open set of X containing x that is contained in A.
- a *contact point* (điểm dính) (or point of closure) of A in X if any open set of X containing x contains a point of A.
- a *limit point* (điểm tụ) (or cluster point, or accumulation point) of A in X if any open set of X containing x contains a point of A other than x. Of course a limit point is a contact point. We can see that a contact point of A which is not a point of A is a limit point of A.
- a *boundary point* (điểm biên) of A in X if every open set of X containing x contains a point of A and a point of the complement of A. In other words, a boundary point of A is a contact point of both A and the complement of A.

With these notions we define:

- The set of all interior points of A is called the *interior* (phần trong) of A in X, denoted by $\mathring{A}$ or $\mathrm{int}(A)$.

- The set of all contact points of A in X is called the *closure* (bao đóng) of A in X, denoted by $\overline{A}$ or $\mathrm{cl}(A)$.
- The set of all boundary points of A in X is called the *boundary* (biên) of A in X, denoted by ∂A.

EXAMPLE. On the Euclidean line $\mathbb{R}$, consider the subset $A = [0,1) \cup \{2\}$. Its interior is $\mathrm{int}A = (0,1)$, the closure is $\mathrm{cl}A = [0,1] \cup \{2\}$, the boundary is $\partial A = \{0,1,2\}$, the set of all limit points is $[0,1]$.

Bases of a topology.

DEFINITION. Given a topology, a collection of open sets is a *basis* (cơ sở) for that topology if every non-empty open set is a union of members of that collection.

More concisely, let τ be a topology of X, then a collection $B \subset \tau$ is called a basis for τ if for any $\emptyset \neq V \in \tau$ there is $C \subset B$ such that $V = \bigcup_{O \in C} O$.

So a basis of a topology is a subset of the topology that generates the entire topology via unions. Specifying a basis is a more "efficient" way to give a topology.

EXAMPLE. In a metric space the collection of all balls is a basis for the topology.

EXAMPLE. The Euclidean plane has a basis consisting of all open disks. It also has a basis consisting of all open rectangles.

DEFINITION. A collection $S \subset \tau$ is called a *subbasis* (tiền cơ sở) for the topology τ if the collection of all finite intersections of members of S is a basis for τ.

Clearly a basis for a topology is also a subbasis for that topology. Briefly, given a topology, a subbasis is a subset of the topology that can generate the entire topology by unions and finite intersections.

EXAMPLE. Let $X = \{1,2,3\}$. The topology $\tau = \{\emptyset, \{1,2\}, \{2,3\}, \{2\}, \{1,2,3\}\}$ has a basis $\{\{1,2\}, \{2,3\}, \{2\}\}$ and a subbasis $\{\{1,2\}, \{2,3\}\}$.

EXAMPLE 2.1. The collection of all open rays, that are, sets of the forms (a,∞) and $(-\infty,a)$, is a subbasis for the Euclidean topology of $\mathbb{R}$.

Comparing topologies.

DEFINITION. Let τ_1 and τ_2 be two topologies on X. If $\tau_1 \subset \tau_2$ we say that τ_2 is *finer* (mịn hơn) (or stronger, bigger) than τ_1 and τ_1 is *coarser* (thô hơn) (or weaker, smaller) than τ_2.

EXAMPLE. On a set the trivial topology is the coarsest topology and the discrete topology is the finest one.

Generating topologies. Suppose that we have a set and we want a topology such that certain subsets of that set are open sets, how do find a topology for that purpose?

THEOREM. *Let S be a collection of subsets of X. The collection τ consisting of $\varnothing$, X, and all unions of finite intersections of members of S is the coarsest topology on X that contains S, called the topology generated by S. The collection $S \cup \{X\}$ is a subbasis for this topology.*

REMARK. In several textbooks to avoid adding the element X to S it is required that the union of all members of S is X.

PROOF. Let B be the collection of all finite intersections of members of S, that is, $B = \{\bigcap_{O \in I} O \mid I \subset S, |I| < \infty\}$. Let τ be the collection of all unions of members of B, that is, $\tau = \{\bigcup_{U \in F} U \mid F \subset B\}$. We check that τ is a topology.

First we check that τ is closed under unions. Let $\sigma \subset \tau$, consider $\bigcup_{A \in \sigma} A$. We write $\bigcup_{A \in \sigma} A = \bigcup_{A \in \sigma} \left(\bigcup_{U \in F_A} U\right)$, where $F_A \subset B$. Since

$$\bigcup_{A \in \sigma} \left(\bigcup_{U \in F_A} U \right) = \bigcup_{U \in (\bigcup_{A \in \sigma} F_A)} U,$$

and since $\bigcup_{A \in \sigma} F_A \subset B$, we conclude that $\bigcup_{A \in \sigma} A \in \tau$.

We only need to check that τ is closed under intersections of two elements. Let $\bigcup_{U \in F} U$ and $\bigcup_{V \in G} V$ be two elements of τ, where $F, G \subset B$. We can write

$$(\bigcup_{U \in F} U) \cap (\bigcup_{V \in G} V) = \bigcup_{U \in F, V \in G} (U \cap V).$$

Let $J = \{U \cap V \mid U \in F, V \in G\}$. Then $J \subset B$, and we can write

$$(\bigcup_{U \in F} U) \cap (\bigcup_{V \in G} V) = \bigcup_{W \in J} W,$$

showing that $(\bigcup_{U \in F} U) \cap (\bigcup_{V \in G} V) \in \tau$. $\square$

By this theorem, *given a set, any collection of subsets generates a topology.*

EXAMPLE. Let $X = \{1, 2, 3, 4\}$. The set $\{\{1\}, \{2, 3\}, \{3, 4\}\}$ generates the topology $\{\varnothing, \{1\}, \{3\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$. A basis for this topology is $\{\{1\}, \{3\}, \{2, 3\}, \{3, 4\}\}$.

EXAMPLE (ordering topology). Let $(X, \leq)$ be a totally ordered set. The collection of subsets of the forms $\{\beta \in X \mid \beta < \alpha\}$ and $\{\beta \in X \mid \beta > \alpha\}$ generates a topology on X, called the *ordering topology*.

EXAMPLE. The Euclidean topology on $\mathbb{R}$ is the ordering topology with respect to the usual order of real numbers. (This is just a different way to state 2.1.)

Problems.

2.2 (finite complement topology). The *finite complement topology* on X consists of the empty set and all subsets of X whose complements are finite. Check that this is indeed a topology. Is it true if "finite" is replaced by "countable"?

2.3. Let X be a set and $p \in X$. Show that the collection consisting of $\varnothing$ and all subsets of X containing p is a topology on X. This topology is called the Particular Point Topology on X, denoted by PPX_p. Describe the closed sets in this space.

2.4. (a) The interior of A in X is the largest open subset of X that is contained in A. A subset is open if all of its points are interior points.
(b) The closure of A in X is the smallest closed subset of X containing A. A subset is closed if and only if it contains all of its contact points.

2.5. Show that $\overline{A}$ is the disjoint union of $\mathring{A}$ and ∂A. Show that X is the disjoint union of $\mathring{A}$, ∂A, and $X \setminus \overline{A}$.

2.6. The set $\{x \in \mathbb{Q} \mid -\sqrt{2} \leq x \leq \sqrt{2}\}$ is both closed and open in $\mathbb{Q}$ under the Euclidean topology of $\mathbb{R}$.

2.7. In a metric space X, a point $x \in X$ is a limit point of the subset A of X if and only if there is a sequence in $A \setminus \{x\}$ converging to x. (This is not true in general topological spaces, see 5.4.)

2.8. (a) In a normed space, show that the boundary of the ball $B(x,r)$ is the sphere $\{y \mid \|x-y\| = r\}$, and so the ball $B'(x,r) = \{y \mid \|x-y\| \leq r\}$ is the closure of $B(x,r)$.
(b) In a metric space, show that the boundary of the ball $B(x,r)$ is a subset of the sphere $\{y \mid d(x,y) = r\}$. Is the ball $B'(x,r) = \{y \mid d(x,y) \leq r\}$ the closure of $B(x,r)$?

2.9. Let $O_n = \{k \in \mathbb{Z}^+ \mid k \geq n\}$. Check that $\{\emptyset\} \cup \{O_n \mid n \in \mathbb{Z}^+\}$ is a topology on $\mathbb{Z}^+$. Find the closure of the set $\{5\}$. Find the closure of the set of all even positive integers.

2.10. Show that an open set in $\mathbb{R}$ is a countable union of open intervals.

2.11. In the real number line with the Euclidean topology, is the Cantor set (see 1.17) closed or open, or neither? Find the boundary and the interior of the Cantor set (see 2.11).

2.12. Show that the intersection of a collection of topologies on a set X is a topology on X. If S is a subset of X, then the intersection of all topologies of X containing S is the smallest topology that contains S. Show that this is exactly the topology generated by S.

2.13. A collection B of open sets is a basis if for each point x and each open set O containing x there is a U in B such that U contains x and U is contained in O.

2.14. Show that two bases generate the same topology if and only if each member of one basis is a union of members of the other basis.

2.15. Let B be a collection of subsets of X. Then $B \cup \{X\}$ is a basis for a topology on X if and only if the intersection of two members of B is either empty or is a union of some members of B. (In several textbooks to avoid adding the element X to B it is required that the union of all members of B is X.)

2.16. In a metric space the set of all balls with rational radii is a basis for the topology. The set of all balls with radii $\frac{1}{2^m}$, $m \geq 1$ is another basis.

2.17 ($\mathbb{R}^n$ has a countable basis). $\surd$ The set of all balls each with rational radius whose center has rational coordinates forms a basis for the Euclidean topology of $\mathbb{R}^n$.

2.18. Let d_1 and d_2 be two metrics on X. If there are $\alpha, \beta > 0$ such that for all $x, y \in X$, $\alpha d_1(x,y) \leq d_2(x,y) \leq \beta d_1(x,y)$ then the two metrics are said to be equivalent. Show that two equivalent metrics generate same topologies.

2.19 (all norms in $\mathbb{R}^n$ generate the Euclidean topology). In $\mathbb{R}^n$ denote by $\|\cdot\|_2$ the Euclidean norm, and let $\|\cdot\|$ be any norm.

(a) Check that the map $x \mapsto \|x\|$ from $(\mathbb{R}^n, \|\cdot\|_2)$ to $(\mathbb{R}, \|\cdot\|_2)$ is continuous.
(b) Let S^n be the unit sphere under the Euclidean norm. Show that the restriction of the map above to S^n has a maximum value β and a minimum value α. Hence $\alpha \leq \left\|\frac{x}{\|x\|_2}\right\| \leq \beta$ for all $x \neq 0$.
(c) Deduce that any two norms in $\mathbb{R}^n$ generate equivalent metrics, hence all norms in $\mathbb{R}^n$ generate the Euclidean topology.

2.20. Let (X, d) be a metric space.

(a) Let $d_1(x, y) = \min\{d(x, y), 1\}$. Show that d_1 is a metric on X generating the same topology as that generated by d. Is d_1 equivalent to d?
(b) Let $d_2(x, y) = \frac{d(x,y)}{1+d(x,y)}$. Show that d_2 is a metric on X generating the same topology as the topology generated by d. Is d_2 equivalent to d?

2.21. Is the Euclidean topology on $\mathbb{R}^2$ the same as the ordering topology on $\mathbb{R}^2$ with respect to the dictionary order? If it is not the same, can the two be compared?

2.22 (the Sorgenfrey's line). The collection of all intervals of the form $[a, b)$ generates a topology on $\mathbb{R}$. Is it the Euclidean topology?

2.23. * On the set of all integer numbers $\mathbb{Z}$, consider arithmetic progressions

$$S_{a,b} = a + b\mathbb{Z},$$

where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$.

(a) Show that these sets form a basis for a topology on $\mathbb{Z}$.
(b) Show that with this topology each set $S_{a,b}$ is closed.
(c) Show that if there are only finitely many prime numbers then the set $\{\pm 1\}$ is open.
(d) Conclude that *there are infinitely many prime numbers*. (This proof was given by Hillel Furstenberg in 1955.)

3. Continuity

Continuous maps. Previously in metric spaces a function f is considered continuous at x if $f(y)$ can be arbitrarily close to $f(x)$ provided that y is sufficiently close to x. This notion is generalized to the following:

DEFINITION. Let X and Y be topological spaces. We say a map $f : X \to Y$ is *continuous* at a point x in X if for any open set U of Y containing $f(x)$ there is an open set V of X containing x such that $f(V)$ is contained in U.

We say that f is continuous (on X) if it is continuous at every point in X.

THEOREM. *A map is continuous if and only if the inverse image of an open set is an open set.*

PROOF. ($\Rightarrow$) Suppose that $f : X \to Y$ is continuous. Let U be an open set in Y. Let $x \in f^{-1}(U)$. Since f is continuous at x and U is an open neighborhood of $f(x)$, there is an open set V_x containing x such that V_x is contained in $f^{-1}(U)$. Therefore $f^{-1}(U) = \bigcup_{x \in f^{-1}(U)} V_x$ is open.

($\Leftarrow$) Suppose that the inverse image of any open set is an open set. Let $x \in X$. Let U be an open neighborhood of $f(x)$. Then $V = f^{-1}(U)$ is an open set containing x, and $f(V)$ is contained in U. Therefore f is continuous at x. □

EXAMPLE. Let X and Y be topological spaces.

(a) The identity function, $\mathrm{id}_X : X \to X$, $x \mapsto x$, is continuous.
(b) The constant function, with a given $a \in Y$, $x \mapsto a$, is continuous.
(c) If Y has the trivial topology then any map $f : X \to Y$ is continuous.
(d) If X has the discrete topology then any map $f : X \to Y$ is continuous.

EXAMPLE (metric space). Let (X, d_1) and (Y, d_2) be metric spaces. Recall that in the theory of metric spaces, a map $f : (X, d_1) \to (Y, d_2)$ is continuous at $x \in X$ if and only if

$$\forall \epsilon > 0, \exists \delta > 0, d_1(y, x) < \delta \Rightarrow d_2(f(y), f(x)) < \epsilon.$$

In other words, given any ball $B(f(x), \epsilon)$ centered at $f(x)$, there is a ball $B(x, \delta)$ centered at x such that f brings $B(x, \delta)$ into $B(f(x), \epsilon)$. It is apparent that this definition is equivalent to the definition of continuity in topological spaces where the topologies are generated by the metrics. In other words, if we look at a metric space as a topological space with the topology generated by the metric then continuity in the metric space is the same as continuity in the topological space. Therefore *we inherit all results concerning continuity in metric spaces*.

Homeomorphism. A map from one topological space to another is said to be a *homeomorphism* (phép đồng phôi) if it is a bijection, is continuous and its inverse map is also continuous. Two spaces are said to be *homeomorphic* (đồng phôi) if there is a homeomorphism from one to the other. This is a basic relation among topological spaces.

PROPOSITION. *A homeomorphism between two spaces induces a bijection between the two topologies.*

PROOF. A homeomorphism $f : X \to Y$ induces a bijection

$$\begin{aligned} \tilde{f} : \tau_X &\to \tau_Y \\ O &\mapsto f(O). \end{aligned}$$

□

Roughly speaking, in the field of Topology, when two spaces are homeomorphic they are considered the same. For example a "topological sphere" means a topological space which is homeomorphic to a sphere.

Topology generated by maps. Let (X, τ_X) be a topological space, Y be a set, and $f : X \to Y$ be a map, we want to find a topology on Y such that f is continuous. The requirement for such a topology τ_Y is that if $U \in \tau_Y$ then $f^{-1}(U) \in \tau_X$. The trivial topology on Y satisfies that requirement. It is the coarsest topology satisfying that requirement. On the other hand the collection $\{U \subset Y \mid f^{-1}(U) \in \tau_X\}$ is actually a topology on Y. This is the finest topology satisfying that requirement.

In another situation, let X be a set, (Y, τ_Y) be a topological space, and $f : X \to Y$ be a map, we want to find a topology on X such that f is continuous. The requirement for such a topology τ_X is that if $U \in \tau_Y$ then $f^{-1}(U) \in \tau_X$. The discrete topology on X is the finest topology satisfying that requirement. The collection $\tau_X = \{f^{-1}(U) \mid U \in \tau_Y\}$ is the coarsest topology satisfying that requirement. We can observe further that if the collection S_Y generates τ_Y then τ_X is generated by the collection $\{f^{-1}(U) \mid U \in S_Y\}$.

Subspace. Let (X, τ) be a topological space and let Y be a subset of X. We want to define a topology on Y that can be naturally considered as being "inherited" from X. Thus any open set of X that is contained in Y should be considered open in Y. If an open set of X is not contained in Y then its restriction to Y should be considered open in Y. We can easily check that the collection of restrictions of the open sets in X to Y is a topology on Y.

DEFINITION. Let Y be a subset of the topological space X. The *subspace topology* on Y, also called the *relative topology* (tôpô tương đối) with respect to X is defined to be the collection of restrictions of the open sets of X to Y, that is, the set $\{O \cap Y \mid O \in \tau\}$. With this topology we say that Y is a *subspace* (không gian con) of X.

In brief, a subset of a subspace Y of X is open in Y if and only if it is a restriction of a open set of X to Y.

REMARK. An open or a closed subset of a subspace Y of a space X is not necessarily open or closed in X. For example, under the Euclidean topology of $\mathbb{R}$, the set $[0, \frac{1}{2})$ is open in the subspace $[0, 1]$, but is not open in $\mathbb{R}$. *When we say that a set is open, we must know which topology we are using*.

PROPOSITION. *Let X be a topological space and let $Y \subset X$. The subspace topology on Y is the coarsest topology on Y such that the inclusion map $i : Y \hookrightarrow X$, $x \mapsto x$ is continuous. In other words, the subspace topology on Y is the topology generated by the inclusion map from Y to X.*

PROOF. If O is a subset of X then $i^{-1}(O) = O \cap Y$. Thus the topology generated by i is $\{O \cap Y \mid O \in \tau_X\}$, exactly the subspace topology of Y. $\square$

EXAMPLE (Subspaces of a metric space). It's not hard to see that the notion of topological subspaces is compatible with the earlier notion of metric subspaces. Let (X, d) be a metric space and $Y \subset X$. Then Y is a metric space with the metric inherited from X. A ball in Y is a set of the form, for $a \in Y$, $r > 0$:

$$B_Y(a, r) = \{y \in Y \mid d(y, a) < r\} = \{x \in X \mid d(x, a) < r\} \cap Y = B_X(a, r) \cap Y.$$

Any open set A in Y is the union of a collection of balls in Y, i.e.

$$A = \bigcup_{i \in I,\, a_i \in Y} B_Y(a_i, r_i) = \bigcup_{i \in I} (B_X(a_i, r_i) \cap Y) = \left(\bigcup_{i \in I} B_X(a_i, r_i)\right) \cap Y,$$

thus A is the intersection of an open set of X with Y. Conversely, if B is open in X, then for each $x \in B$ there is $r_x > 0$ such that $B_X(x, r_x) \subset B$, so

$$\begin{aligned} B \cap Y &= \left(\bigcup_{x \in B} B_X(x, r_x)\right) \cap Y = \left(\bigcup_{x \in B \cap Y} B_X(x, r_x)\right) \cap Y \\ &= \bigcup_{x \in B \cap Y} (B_X(x, r_x) \cap Y) = \bigcup_{x \in B \cap Y} B_Y(x, r_x). \end{aligned}$$

This implies that $B \cap Y$ is an open set in Y.

EXAMPLE. For $n \in \mathbb{Z}^+$ define the *sphere* S^n to be the subspace of the Euclidean space $\mathbb{R}^{n+1}$ given by $\{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$.

Embedding. An *embedding* (or *imbedding*) (phép nhúng) from a topological space X to a topological space Y is a homeomorphism from X to a subspace of Y, i.e. it is a map $f : X \to Y$ such that the restriction $\tilde{f} : X \to f(X)$ is a homeomorphism. If there is an imbedding from X to Y then we say that X can be *embedded* in Y.

EXAMPLE. With the subspace topology the inclusion map is an embedding.

EXAMPLE. The Euclidean line $\mathbb{R}$ can be embedded in the Euclidean plane $\mathbb{R}^2$ as a line in the plane.

EXAMPLE. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous under the Euclidean topology. Then $\mathbb{R}$ can be embedded into the plane as the graph of f.

EXAMPLE (stereographic projection). The space $S^n \setminus \{(0, 0, \ldots, 0, 1)\}$ is homeomorphic to the Euclidean space $\mathbb{R}^n$ via the *stereographic projection* (phép chiếu nổi). Each point x on the sphere minus the North Pole corresponds to the inter-

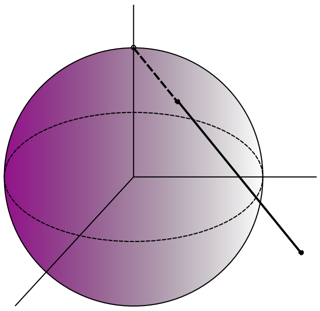


FIGURE 3.1. The stereographic projection.

section between the straight line from the North Pole to x with the plane through the equator. By solving an equation intersection we can find the formula for this projection to be:

$$\begin{array}{rcl} S^n \setminus \{(0,0,\ldots,0,1)\} & \to & \mathbb{R}^n \times \{0\} \\ (x_1, x_2, \ldots, x_{n+1}) & \mapsto & (y_1, y_2, \ldots, y_n, 0) \end{array}$$

where $y_i = \dfrac{1}{1 - x_{n+1}} x_i$. The inverse map is given by $x_i = \dfrac{2y_i}{1 + \sum_{i=1}^n y_i^2}$, $1 \le i \le n$, and $x_{n+1} = \dfrac{-1 + \sum_{i=1}^n y_i^2}{1 + \sum_{i=1}^n y_i^2}$. Both maps are continuous, thus the Euclidean space $\mathbb{R}^n$ can be embedded onto the n-sphere minus one point.

Problems.

3.2. If $f : X \to Y$ and $g : Y \to Z$ are continuous then $g \circ f$ is continuous.

3.3. A map is continuous if and only if the inverse image of a closed set is a closed set.

3.4. $\surd$ Suppose that $f : X \to Y$ and S is a subbasis for the topology of Y. Show that f is continuous if and only if the inverse image of any element of S is an open set in X.

3.5. Define an *open map* to be a map such that the image of an open set is an open set. A *closed map* is a map such that the image of a closed set is a closed set. Show that a homeomorphism is an open map and is also a closed map.

3.6. Show that a continuous bijection is a homeomorphism if and only if it is an open map.

3.7. Show that (X, PPX_p) and (X, PPX_q) (see 2.3) are homeomorphic.

3.8. $\surd$ Let X be a set and (Y, τ) be a topological space. Let $f_i : X \to Y$, $i \in I$ be a collection of maps. Find the coarsest topology on X such that all maps f_i, $i \in I$ are continuous.

In Functional Analysis this construction is used to construct the weak topology on a normed space. It is the coarsest topology such that all linear functionals which are continuous under the norm are still continuous under the topology. See for instance **[Con90]**.

3.9. Suppose that X is a normed space. Prove that the topology generated by the norm is exactly the coarsest topology on X such that the norm and the translations (maps of the form $x \mapsto x + a$) are continuous.

3.10 (isometry). An *isometry* (phép đẳng cấu metric, phép đẳng cấu hình học, hay phép đẳng cự) from a metric space X to a metric space Y is a surjective map $f : X \to Y$ that preserves distance, that is $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. If there exists such an isometry then X is said to be *isometric* to Y.

(a) Show that an isometry is a homeomorphism.
(b) Show that being isometric is an equivalence relation among metric spaces.
(c) Show that $(\mathbb{R}^2, \|\cdot\|_\infty)$ and $(\mathbb{R}^2, \|\cdot\|_1)$ are isometric, but they are not isometric to $(\mathbb{R}^2, \|\cdot\|_2)$, although the three spaces are homeomorphic (see 2.19). (For higher dimensions one may use the Mazur-Ulam theorem.)

3.11. $\surd$ Verify the formulas for the stereographic projection and its inverse. Check that the stereographic projection is indeed a homeomorphism.

3.12. $\surd$ Show that a subset of a subspace Y of X is closed in Y if and only if it is a restriction of a closed set in X to Y.

3.13. $\surd$ Suppose that X is a topological space and $Z \subset Y \subset X$. Then the relative topology of Z with respect to Y is the same as the relative topology of Z with respect to X.

3.14. $\surd$ Let X and Y be topological spaces and let $f : X \to Y$.

(a) If Z is a subspace of X, denote by $f|_Z$ the restriction of f to Z. Show that if f is continuous then $f|_Z$ is continuous.
(b) Let Z be a space containing Y as a subspace. Consider f as a function from X to Z, that is, let $\tilde{f} : X \to Z$, $\tilde{f}(x) = f(x)$. Show that f is continuous if and only if $\tilde{f}$ is continuous.

3.15 (gluing continuous functions). $\surd$ Let $X = A \cup B$ where A and B are both open or are both closed in X. Suppose $f : X \to Y$, and $f|_A$ and $f|_B$ are both continuous. Then f is continuous.

Another way to phrase this is the following. Let $g : A \to Y$ and $h : B \to Y$ be continuous and $g(x) = h(x)$ on $A \cap B$. Define

$$f(x) = \begin{cases} g(x), & x \in A \\ h(x), & x \in B. \end{cases}$$

Then f is continuous.

Is it still true if the restriction that A and B are both open or are both closed in X is removed?

3.16. $\surd$ Any two balls in a normed space are homeomorphic. Any ball in a normed space is homeomorphic to the whole space.

Is it true that any two balls in a metric space are homeomorphic?

3.17. Any two finite-dimensional normed spaces of same dimensions are homeomorphic.

3.18. In the Euclidean plane an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is homeomorphic to a circle.

3.19. In the Euclidean plane the upper half-plane $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ is homeomorphic to the plane.

3.20. Deduce that in $\mathbb{R}^n$ balls with respect to different norms are homemorphic (see 2.19). In particular a Euclidean n-dimensional ball is homemorphic to an n-dimensional rectangle.

3.21. $\surd$ If $f : X \to Y$ is a homeomorphism and $Z \subset X$ then $X \setminus Z$ and $Y \setminus f(Z)$ are homeomorphic.

3.22. On the Euclidean plane $\mathbb{R}^2$, show that:

(a) $\mathbb{R}^2 \setminus \{(0,0)\}$ and $\mathbb{R}^2 \setminus \{(1,1)\}$ are homeomorphic.
(b) $\mathbb{R}^2 \setminus \{(0,0),(1,1)\}$ and $\mathbb{R}^2 \setminus \{(1,0),(0,1)\}$ are homeomorphic.

Can you generalize these results?

3.23. Show that $\mathbb{N}$ and $\mathbb{Z}$ are homeomorphic under the Euclidean topology. Further, prove that any two set-equivalent discrete spaces are homeomorphic.

3.24. Among the following spaces, which one is homeomorphic to another? $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, each with the Euclidean topology, and $\mathbb{R}$ with the finite complement topology.

3.25. Show that any homeomorphism from S^{n-1} onto S^{n-1} can be extended to a homeomorphism from the unit disk $D^n = B'(0,1)$ onto D^n.

3.26. Under the Euclidean topology the map $\varphi : [0, 2\pi) \to S^1$ given by $t \mapsto (\cos t, \sin t)$ is a bijection but is not a homeomorphism.

3.27. Find the closures, interiors and the boundaries of the interval $[0,1)$ under the Euclidean, discrete and trivial topologies of $\mathbb{R}$.

4. Connectedness

A topological space is said to be *connected* (liên thông) if it is not a union of two non-empty disjoint open subsets.

Equivalently, a topological space is connected if and only if its only subsets which are both closed and open are the empty set and the space itself.

REMARK. When we say that a subset of a topological space is connected we mean that the subset under the subspace topology is a connected space.

EXAMPLE. A space containing only one point is connected.

EXAMPLE. The Euclidean real number line minus a point is not connected.

PROPOSITION (continuous image of connected space is connected). *If $f : X \to Y$ is continuous and X is connected then $f(X)$ is connected.*

PROOF. Suppose that U and V are non-empty disjoint open subset of $f(X)$. Since $f : X \to f(X)$ is continuous (see 3.14), $f^{-1}(U)$ and $f^{-1}(V)$ are open in X, and are non-empty and disjoint. This contradicts the connectedness of X. □

Connected component.

PROPOSITION 4.1. *If a collection of connected subspaces of a space has non-empty intersection then its union is connected.*

PROOF. Consider a topological space and let F be a collection of connected subspaces whose intersection is non-empty. Let A be the union of the collection, $A = \bigcup_{D \in F} D$. Suppose that C is subset of A that is both open and closed in A. If $C \neq \emptyset$ then there is $D \in F$ such that $C \cap D \neq \emptyset$. Then $C \cap D$ is a subset of D, both open and closed in D (we are using 3.13 here). Since D is connected and $C \cap D \neq \emptyset$, we must have $C \cap D = D$. This implies C contains the intersection of F. Therefore $C \cap D \neq \emptyset$ for all $D \in F$. The argument above shows that C contains all D in F, that is, $C = A$. We conclude that A is connected. □

Let X be a topological space. Define a relation on X whereas two points are related if both belong to a connected subspace of X (we say that the two points are connected). Then this relation is an equivalence relation, by 4.1.

PROPOSITION. *Under the above equivalence relation the equivalence class containing a point x is equal to the union of all connected subspaces containing x, thus it is the largest connected subspace containing x.*

PROOF. Consider the equivalence class $[a]$ represented by a point a. By the definition, $b \in [a]$ if and only if there is a connected set O_b containing both a and b. Thus $[a] = \bigcup_{b \in [a]} O_b$. By 4.1, $[a]$ is connected. □

DEFINITION. Under the above equivalence relation, the equivalence classes are called the *connected components* of the space.

Thus a space is a disjoint union of its connected components.

THEOREM. *If two spaces are homeomorphic then there is a bijection between the collections of connected components of the two spaces.*

PROOF. Let $f : X \to Y$ be a homeomorphism. Since $f([x])$ is connected, we have $f([x]) \subset [f(x)]$. For the same reason, $f^{-1}([f(x)]) \subset [f^{-1}(f(x))] = [x]$. Apply f to both sides we get $[f(x)] \subset f([x])$. Therefore $f([x]) = [f(x)]$. Similarly $f^{-1}([f(x)]) = [x]$. Thus f brings connected components to connected components, inducing a bijection on the collections of connected components. $\square$

For the above reason we say that *connectedness is a topological property*. We also say that *the number of connected components is a topological invariant*. If two spaces have different numbers of connected components then they must be different (not homeomorphic).

EXAMPLE (line not homeomorphic to plane). Suppose that $\mathbb{R}$ and $\mathbb{R}^2$ under the Euclidean topologies are homeomorphic via a homeomorphism f. Delete any point x from $\mathbb{R}$. By 3.21 the subspaces $\mathbb{R} \setminus \{x\}$ and $\mathbb{R}^2 \setminus \{f(x)\}$ are homeomorphic. But $\mathbb{R} \setminus \{x\}$ is not connected while $\mathbb{R}^2 \setminus \{f(x)\}$ is connected (see 4.24).

PROPOSITION 4.2. *A connected subspace with a limit point added is still connected. Consequently the closure of a connected subspace is connected, and any connected component is closed.*

PROOF. Let A be a connected subspace of a space X and let $a \notin A$ be a limit point of A, we show that $A \cup \{a\}$ is connected. Suppose that $A \cup \{a\} = U \cup V$ where U and V are non-empty disjoint open subsets of $A \cup \{a\}$. Suppose that $a \in U$. Then $a \notin V$, so $V \subset A$. Since a is a limit point of A, $U \cap A$ is non-empty. Then $U \cap A$ and V are open subsets of A, by 3.13, which are non-empty and disjoint. This contradicts the assumption that A is connected. $\square$

Connected sets in the Euclidean real number line.

THEOREM. *A subspace of the Euclidean real number line is connected if and only if it is an interval.*

PROOF. Suppose that a subset A of $\mathbb{R}$ is connected. Suppose that $x, y \in A$ and $x < y$. If $x < z < y$ we must have $z \in A$, otherwise the set $\{a \in A \mid a < z\} = \{a \in A \mid a \leq z\}$ will be both closed and open in A. Thus A contains the interval $[x, y]$.

Let $a = \inf A$ if A is bounded from below and $a = -\infty$ otherwise. Similarly let $b = \sup A$ if A is bounded from above and $b = \infty$ otherwise. Suppose that A contains more than one element. There are sequences $\{a_n\}_{n\in\mathbb{Z}^+}$ and $\{b_n\}_{n\in\mathbb{Z}^+}$ of elements in A such that $a < a_n < b_n < b$, and $a_n \to a$ while $b_n \to b$. By the above argument, $[a_n, b_n] \subset A$ for all n. So $(a, b) = \bigcup_{n=1}^{\infty}[a_n, b_n] \subset A \subset [a, b]$. It follows that A is either (a, b) or $[a, b)$ or $(a, b]$ or $[a, b]$.

We prove that any interval is connected. By homeomorphisms we just need to consider the intervals $(0,1)$, $(0,1]$, and $[0,1]$. Since $[0,1]$ is the closure of $(0,1)$, and $(0,1] = (0,3/4) \cup [1/2,1]$, it is sufficient to prove that $(0,1)$ is connected. Instead we prove that $\mathbb{R}$ is connected.

Suppose that $\mathbb{R}$ contains a non-empty, proper, open and closed subset C. Let $x \notin C$ and let $D = C \cap (-\infty, x) = C \cap (-\infty, x]$. Then D is both open and closed in $\mathbb{R}$, and is bounded from above.

If $D \neq \emptyset$, consider $s = \sup D$. Since D is closed and s is a contact point of D, $s \in D$. Since D is open s must belong to an open interval contained in D. But then there are points in D which are bigger than s, a contradiction.

If $D = \emptyset$ we let $E = C \cap (x, \infty)$, consider $t = \inf E$ and proceed similarly. □

EXAMPLE. Since the Euclidean $\mathbb{R}^n$ is the union of all lines passing through the origin, it is connected.

Below is a simple application, a form of Intermediate value theorem (4.8):

THEOREM (Borsuk-Ulam theorem). *For any continuous real function on the sphere S^n there must be antipodal (i.e. opposite) points where the values of the function are same.* [4]

PROOF. Let $f : S^n \to \mathbb{R}$ be continuous. Let $g(x) = f(x) - f(-x)$. Then g is continuous and $g(-x) = -g(x)$. If there is an x such that $g(x) \neq 0$ then $g(x)$ and $g(-x)$ have opposite signs. Since S^n is connected (see 4.7), the range $g(S^n)$ is a connected subset of the Euclidean $\mathbb{R}$, and so is an interval, containing the interval between $g(x)$ and $g(-x)$. Therefore 0 is in the range of g. □

Path-connected space. Path-connectedness is a more intuitive notion than connectedness. Shortly, a space is path-connected if for any two points there is a path connecting them.

DEFINITION. A *path* (đường đi) in a topological space X from a point x to a point y is a continuous map $\alpha : [a,b] \to X$ such that $\alpha(a) = x$ and $\alpha(b) = y$, where the interval of real numbers $[a,b]$ has the Euclidean topology. The space X is said to be *path-connected* (liên thông đường) if for any two different points x and y in X there is a path in X from x to y.

EXAMPLE. A normed space is path-connected, and so is any convex subspace of that space: any two points x and y are connected by a straight line segment $x + t(y - x)$, $t \in [0,1]$.

EXAMPLE. In a normed space, the sphere $S = \{x \mid \|x\| = 1\}$ is path-connected. One way to show this is as follow. If two points x and y are not opposite then they can be connected by the arc $\frac{x+t(y-x)}{||x+t(y-x)||}$, $t \in [0,1]$. If x and y are opposite, we can take a third point z, then compose a path from x to z with a path from z to y.

[4] On the surface of the Earth at any moment there two opposite places where temperatures are same!

LEMMA. *The relation on a topological space X whereas a point x is related to a point y if there is a path in X from x to y is an equivalence relation.*

PROOF. If α is a path defined on $[a, b]$ then there is a path β defined on $[0, 1]$ with the same images (also called the traces of the paths): we can just use the linear homeomorphism $(1-t)a + tb$ from $[0, 1]$ to $[a, b]$ and let $\beta(t) = \alpha((1-t)a + tb)$. For convenience we can assume that the domains of paths is the interval $[0, 1]$.

If there is a path $\alpha : [0, 1] \to X$ from x to y then there is a path from y to x, for example $\beta : [0, 1] \to X$, $\beta(t) = \alpha(1-t)$.

If $\alpha : [0, 1] \to X$ is a path from x to y and $\beta : [0, 1] \to X$ is a path from y to z then there is a path from x to z, for example

$$\gamma(t) = \begin{cases} \alpha(2t), & 0 \le t \le \frac{1}{2}, \\ \beta(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

This path follows α at twice the speed, then follow β at twice the speed and at half of a unit time later. It is continuous by 3.15. □

An equivalence class under the above equivalence relation is called a *path-connected component*.

THEOREM (path-connected ⇒ connected). *Any path-connected space is connected.*

PROOF. This is a consequence of the fact that an interval on the Euclidean real number line is connected. Let X be path-connected. Let $x, y \in X$. There is a path from x to y. The image of this path is a connected subspace of X. That means every point y belongs to the connected component containing x. Therefore X has only one connected component. □

A topological space is said to be *locally path-connected* if every neighborhood of a point contains an open path-connected neighborhood of that point.

EXAMPLE. Open sets in a normed space are locally path-connected.

PROPOSITION 4.3. *A connected, locally path-connected space is path-connected.*

PROOF. Suppose that X is connected and is locally path-connected. Let C be a path-connected component of X. If $x \in X$ is a contact point of C then there is a path-connected neighborhood U in X of x such that $U \cap C \neq \emptyset$. By 4.20, $U \cup C$ is path-connected , thus $U \subset C$. This implies that C is both open and closed in X. Hence $C = X$. □

Topologist's sine curve. The closure in the Euclidean plane of the graph of the function $y = \sin \frac{1}{x}$, $x > 0$ is often called the *Topologist's sine curve*. This is a classic example of a space which is connected but is not path-connected.

Denote $A = \{(x, \sin \frac{1}{x}) \mid x > 0\}$ and $B = \{0\} \times [-1, 1]$. Then the Topologist's sine curve is $X = A \cup B$.

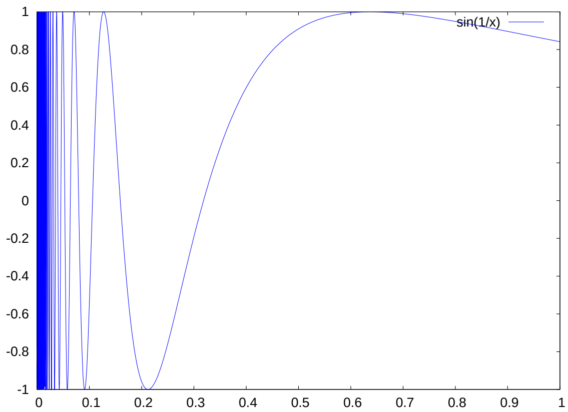


FIGURE 4.4. Topologist's sine curve.

PROPOSITION (connected $\neq$ path-connected). *The Topologist's sine curve is connected but is not path-connected.*

PROOF. By 4.9 the set A is connected. Each point of B is a limit point of A, so by 4.2 X is connected.

Suppose that there is a path $\gamma(t) = (x(t), y(t))$, $t \in [0,1]$ from the origin $(0,0)$ on B to a point on A, we show that there is a contradiction.

Let $t_0 = \sup\{t \in [0,1] \mid x(t) = 0\}$. Then $x(t_0) = 0$, $t_0 < 1$, and $x(t) > 0$ for all $t > t_0$. Thus t_0 is the moment when the path γ departs from B. We can see that the path jumps immediately when it departs from B. Thus we will show that $\gamma(t)$ cannot be continuous at t_0 by showing that for any $\delta > 0$ there are $t_1, t_2 \in (t_0, t_0 + \delta)$ such that $y(t_1) = 1$ and $y(t_2) = -1$.

To find t_1, note that the set $x([t_0, t_0 + \frac{\delta}{2}])$ is an interval $[0, x_0]$ where $x_0 > 0$. There exists an $x_1 \in (0, x_0)$ such that $\sin \frac{1}{x_1} = 1$: we just need to take $x_1 = \frac{1}{\frac{\pi}{2} + k2\pi}$ with sufficiently large k. There is $t_1 \in (t_0, t_0 + \frac{\delta}{2}]$ such that $x(t_1) = x_1$. Then $y(t_1) = \sin \frac{1}{x(t_1)} = 1$. We can find t_2 similarly. $\square$

Problems.

4.5. A space is connected if whenever it is a union of two non-empty disjoint subsets, then at least one set must contain a contact point of the other set.

4.6. Here is a different proof that any interval of real numbers is connected. Suppose that A and B are non-empty, disjoint subsets of $(0,1)$ whose union is $(0,1)$. Let $a \in A$ and $b \in B$. Let $a_0 = a$, $b_0 = b$, and for each $n \geq 1$ consider the middle point of the segment from a_n to b_n. If $\frac{a_n + b_n}{2} \in A$ then let $a_{n+1} = \frac{a_n + b_n}{2}$ and $b_{n+1} = b_n$; otherwise let $a_{n+1} = a_n$ and $b_{n+1} = \frac{a_n + b_n}{2}$. Then:

(a) The sequence $\{a_n \mid n \geq 1\}$ is a Cauchy sequence, hence is convergent to a number a.

(b) The sequence $\{b_n \mid n \geq 1\}$ is also convergent to a. This implies that $(0,1)$ is connected.

4.7. Show that the sphere S^n is connected.

4.8 (intermediate value theorem). If X is a connected space and $f : X \to \mathbb{R}$ is continuous, where $\mathbb{R}$ has the Euclidean topology, then the image $f(X)$ is an interval.

A consequence is the following familiar theorem in Calculus: Let $f : [a,b] \to \mathbb{R}$ be continuous under the Euclidean topology. If $f(a)$ and $f(b)$ have opposite signs then the equation $f(x) = 0$ has a solution.

4.9. $\checkmark$ If $f : \mathbb{R} \to \mathbb{R}$ is continuous under the Euclidean topology then its graph is connected in the Euclidean plane. Moreover the graph is homeomorphic to $\mathbb{R}$.

4.10. Let X be a topological space and let A_i, $i \in I$ be connected subspaces. If $A_i \cap A_j \neq \emptyset$ for all $i, j \in I$ then $\bigcup_{i \in I} A_i$ is connected.

4.11. Let X be a topological space and let A_i, $i \in \mathbb{Z}^+$ be connected subsets. If $A_i \cap A_{i+1} \neq \emptyset$ for all $i \geq 1$ then $\bigcup_{i=1}^{\infty} A_i$ is connected.

4.12. Let A be a subspace of X with the particular point topology (X, PPX_p) (see 2.3). Find the connected components of A.

4.13. Let X be connected and let $f : X \to Y$ be continuous. If f is locally constant on X (meaning that every point has a neighborhood on which f is a constant map) then f is constant on X.

4.14. Let X be a topological space. A map $f : X \to Y$ is called a *discrete map* if Y has the discrete topology and f is continuous. Show that X is connected if and only if all discrete maps on X are constant.

4.15. What are the connected components of $\mathbb{N}$ and $\mathbb{Q}$ under the Euclidean topology?

4.16. What are the connected components of $\mathbb{Q}^2$ as a subspace of the Euclidean plane?

4.17. Find the connected components of the Cantor set (see 2.11).

4.18. Show that if a space has finitely many components then each component is both open and closed. Is it still true if there are infinitely many components?

4.19. Suppose that a space X has finitely many connected components. Show that a map defined on X is continuous if and only if it is continuous on each components. Is it still true if X has infinitely many components?

4.20. If a collection of path-connected subspaces of a space has non-empty intersection then its union is path-connected.

4.21. If $f : X \to Y$ is continuous and X is path-connected then $f(X)$ is path-connected.

4.22. The path-connected component containing a point x is the union of all path-connected subspaces containing x, thus it is the largest path-connected subspace containing x.

4.23. If two space are homeomorphic then there is a bijection between the collections of path-connected components of the two spaces. In particular, if one space is path-connected then the other space is also path-connected.

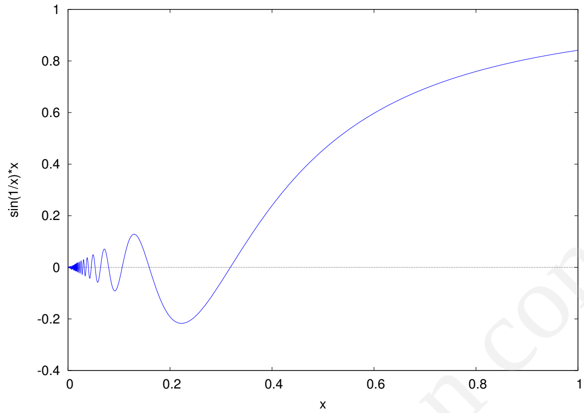
4.24. The plane with countably many points removed is path-connected under the Euclidean topology.

4.25. Show that $\mathbb{R}$ with the finite complement topology and $\mathbb{R}^2$ with the finite complement topology are homeomorphic.

4.26. Find as many ways as you can to prove that S^n is path-connected.

4.27. A topological space is locally path-connected if and only if the collection of all open path-connected subsets is a basis for the topology.

4.28. Let $X = \{(x, x\sin\frac{1}{x}) \mid x > 0\} \cup \{(0,0)\}$, that is, the graph of the function $x\sin\frac{1}{x}$, $x > 0$ with the origin added. Under the Euclidean topology of the plane, is the space X connected or path-connected?



4.29. The Topologist's sine curve is not locally path-connected.

4.30. * Classify the alphabetical characters up to homeomorphisms, that is, which of the following characters are homeomorphic to each other as subspaces of the Euclidean plane? Try to provide rigorous arguments.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

Note that the result depends on the font you use!

Do the same for the Vietnamese alphabetical characters:

A Ă Â B C D Đ E Ê G H I K L M N O Ô Ơ P Q R S T U Ư V X Y

5. Convergence

Separation. In this section we begin to put restrictions on topologies in terms of separation properties.

For example, we know that any metric on a set induces a topology on that set. If a topology can be induced from a metric, we say that the topological space is *metrizable*. Is there anything special about a metrizable space?

EXAMPLE. On any set X, the discrete topology is generated by the following metric:

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y. \end{cases}$$

Indeed for any $x \in X$, the set containing one point $\{x\} = B(x,1)$ is open, therefore any subset of X is open. On the other hand no metric can generate the trivial topology on X if X has more than one element. Indeed, if X has two different elements x and y then the ball $B(x, \frac{d(x,y)}{2})$ is a non-empty open proper subset of X.

DEFINITION. We define:

T_1: A topological space is called a T_1-space if for any two points $x \neq y$ there is an open set containing x but not y and an open set containing y but not x.

T_2: A topological space is called a T_2-space or *Hausdorff* if for any two points $x \neq y$ there are disjoint open sets U and V such that $x \in U$ and $y \in V$.

T_3: A T_1-space is called a T_3-space or *regular* (chính tắc) if for any point x and a closed set F not containing x there are disjoint open sets U and V such that $x \in U$ and $F \subset V$. [5]

T_4: A T_1-space is called a T_4-space or *normal* (chuẩn tắc) if for any two disjoint closed sets F and G there are disjoint open sets U and V such that $F \subset U$ and $G \subset V$.

These definitions are often called separation axioms because they involve "separating" certain sets by open sets.

PROPOSITION. *A space is a T_1 space if and only if any subset containing exactly one point is a closed set.*

If a space is T_1, given $x \in X$, for any $y \neq x$ there is an open set U_y that does not contain x. Then $X \setminus \{x\} = \bigcup_{y \neq x} U_y$

COROLLARY ($T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$). *If a space is T_i then it is T_{i-1}, for $2 \leq i \leq 4$.*

EXAMPLE. Any space with the discrete topology is normal.

Any metric space is clearly a Hausdorff space. In other words, if a space is not a Hausdorff space then it cannot be metrizable. We have a stronger result:

[5] We include T_1 requirement for regular and normal spaces, as in Munkres **[Mun00]**. Some authors such as Kelley **[Kel55]** do not include the T_1 requirement.

PROPOSITION. *Any metric space is normal.*

PROOF. Let (X, d) be a metric space. For $x \in X$ and $A \subset X$ we define the distance from x to A to be $d(x, A) = \inf\{d(x, y) \mid y \in A\}$. This is a continuous function with respect to x, see 5.12.

Suppose that A and B are disjoint closed subsets of X. Let $U = \{x \mid d(x, A) < d(x, B)\}$ and $V = \{x \mid d(x, A) > d(x, B)\}$. Then $A \subset U$, $B \subset V$ (using the fact that A and B are closed, see 5.12), $U \cap V = \emptyset$, and both U and V are open. $\square$

EXAMPLE 5.1. The set of all real number under the finite complement topology is T_1 but is not T_2.

There are examples of a T_2-space which is not T_3, and a T_3-space which is not T_4, but they are rather difficult, see 5.11, 8.14, [**Mun00**, p. 197] and [**SJ70**].

PROPOSITION 5.2. *A T_1-space X is normal if and only if given a closed set C and an open set U containing C there is an open set V such that $C \subset V \subset \overline{V} \subset U$.*

PROOF. Suppose that X is normal. Since $X \setminus U$ is closed and disjoint from C there is an open set V containing C and an open set W containing $X \setminus U$ such that V and W are disjoint. Then $V \subset (X \setminus W)$, so $\overline{V} \subset (X \setminus W) \subset U$.

In the reverse direction, given a closed set F and a closed set G disjoint from F, let $U = X \setminus G$. There is an open set V containing F such that $V \subset \overline{V} \subset U$. Then V and $X \setminus \overline{V}$ separate F and G. $\square$

Sequence. Recall that a sequence in a set X is a map $\mathbb{Z}^+ \to X$, a countably indexed family of elements of X. Given a sequence $x : \mathbb{Z}^+ \to X$ the element $x(n)$ is often denoted as x_n, and sequence is often denoted by $(x_n)_{n \in \mathbb{Z}^+}$ or $\{x_n\}_{n \in \mathbb{Z}^+}$.

When X is a metric space, the sequence is said to be convergent to x if x_n can be as closed to x as we want provided n is sufficiently large, i.e.

$$\forall \epsilon > 0, \exists N \in \mathbb{Z}^+, n \geq N \implies d(x_n, x) < \epsilon.$$

It is easy to see that this statement is equivalent to:

$$\forall U \text{ open} \ni x, \exists N \in \mathbb{Z}^+, n \geq N \implies x_n \in U.$$

This definition can be used in topological spaces.

However, in general topological spaces we need to use a notion more general than sequence. Roughly speaking, sequences (having countable indexes) might not be sufficient for describing the neighborhood systems at a point, we need something of arbitrary index (see 5.21).

Net. A *directed set* (tập được định hướng) is a (partially) ordered set such that for any two indices there is an index greater than or equal to both. In symbols: $\forall i, j \in I, \exists k \in I, k \geq i \wedge k \geq j$.

A *net* (lưới) (also called a generalized sequence) in a space is a map from a directed set to that space. In symbols, a net on a space X with index set a directed set I is a map $x : I \to X$. It is an element of the Cartesian product set $\prod_{i\in I} X$. Writing $x_i = x(i)$ we often denote the net as $(x_i)_{i\in I}$ or $\{x_i\}_{i\in I}$.

EXAMPLE. Nets with index set $I = \mathbb{N}$ with the usual order are exactly sequences.

EXAMPLE. Let X be a topological space and $x \in X$. Let I be the collection of all open neighborhoods of x. Define an order on I by $U \leq V \iff U \supset V$. Then I becomes a directed set.

DEFINITION. A net $(x_i)_{i\in I}$ is said to be *convergent* (hội tụ) to $x \in X$ if for each neighborhood U of x there is an index $i \in I$ such that if $j \geq i$ then x_j belongs to U. The point x is called a limit of the net $(x_i)_{i\in I}$ and we often write $x_i \to x$.

EXAMPLE. Convergence of nets with index set $I = \mathbb{N}$ with the usual order is exactly convergence of sequences.

EXAMPLE. Let $X = \{x_1, x_2, x_3\}$ with topology $\{\emptyset, X, \{x_1, x_3\}, \{x_2, x_3\}, \{x_3\}\}$. The net (x_3) converges to x_1, x_2, x_3. The net (x_1, x_2) converges to x_2.

EXAMPLE. If X has the trivial topology then any net in X is convergent to any point in X.

PROPOSITION 5.3. *A space is a Hausdorff space if and only if any net has at most one limit.*

PROOF. Suppose the space is a Hausdorff space. Suppose that a net (x_i) is convergent to two different limits x and y. Since the space is Hausdorff, there are disjoint open neighborhoods U and V of x and y. There is $i \in I$ such that for $\gamma \geq i$ we have $x_\gamma \in U$, and there is $j \in I$ such that for $\gamma \geq j$ we have $x_\gamma \in U$. Since there is a $\gamma \in I$ such that $\gamma \geq i$ and $\gamma \geq j$, the point x_γ will be in $U \cap V$, a contradiction.

Suppose that the space is not a Hausdorff space, then there are two points x and y that could not be separated by open sets. Consider the index set I whose elements are pairs (U, V) of open neighborhoods of x and y, with the order $(U_1, V_1) \leq (U_2, V_2)$ if $U_1 \supset U_2$ and $V_1 \supset V_2$. With this order the index set is directed. Since $U \cap V \neq \emptyset$, take $z_{(U,V)} \in U \cap V$. Then the net $\left(z_{(U,V)}\right)_{(U,V)\in I}$ is convergent to both x and y, contradicting the uniqueness of limit. $\square$

PROPOSITION 5.4. *A point $x \in X$ is a contact point of a subset $A \subset X$ if and only if there is a net in A convergent to x. Consequently a subset is closed if and only if any limit of any net in that set belongs to that set. A subset is open if and only if no limit of a net outside of that set belong to that set.*

This proposition allows us to describe topologies in terms of convergences. With it many statements about convergence in metric spaces could be carried to topological spaces by simply replacing sequences by nets.

PROOF. ($\Leftarrow$) Suppose that there is a net $(x_i)_{i\in I}$ in A convergent to x. Let U be an open neighborhood of x. There is an $i \in I$ such that for $j \geq i$ we have $x_j \in U$, in particular $x_i \in U \cap A$. Thus x is a contact point of A.

($\Rightarrow$) Suppose that x is a contact point of A. Consider the directed set I consisting of all the open neighborhoods of x with the partial order $U \leq V$ if $U \supset V$. For any open neighborhood U of x there is an element $x_U \in U \cap A$. For any $V \geq U$, $x_V \in V \subset U$. Thus $\{x_U\}_{U\in I}$ is a net in A convergent to x. (This construction of the net $\{x_U\}_{U\in I}$ involves the Axiom of choice.) $\square$

REMARK. When can nets be replaced by sequences? By examining the above proof we can see that the term net can be replaced by the term sequence if there is a countable collection F of neighborhoods of x such that any neighborhood of x contains a member of F. In this case the point x is said to have a countable *neighborhood basis*. A space having this property at every point is said to be a *first countable space*. A metric space is such a space, where for example each point has a neighborhood basis consisting of balls of rational radii. See also 5.21.

PROPOSITION 5.5. *Let τ_1 and τ_2 be two topologies on X. If convergence in τ_1 implies convergence in τ_2 then τ_1 is finer than τ_2. In symbols: if for all nets x_i and all points x, $x_i \xrightarrow{\tau_1} x \Rightarrow x_i \xrightarrow{\tau_2} x$, then $\tau_2 \subset \tau_1$. As a consequence, if convergences are same then topologies are same.*

PROOF. If convergence in τ_1 implies convergence in τ_2 then contact points in τ_1 are contact points in τ_2. Therefore closed sets in τ_2 are closed sets in τ_1, and so are open sets. $\square$

Similarly to the case of metric spaces, we have:

THEOREM. *A function f is continuous at x if and only if for all nets (x_i), $x_i \to x \Rightarrow f(x_i) \to f(x)$.*

PROOF. The proof is simply a repeat of the proof for the case of metric spaces.

($\Rightarrow$) Suppose that f is continuous at x. Let U is a neighborhood of $f(x)$. Then $f^{-1}(U)$ is a neighborhood of x in X. Since (x_i) is convergent to x, there is an $i \in I$ such that for all $j \geq i$ we have $x_j \in f^{-1}(U)$, which implies $f(x_j) \in U$.

($\Leftarrow$) We will show that if U is an open neighborhood in Y of $f(x)$ then $f^{-1}(U)$ is a neighborhood in X of x. Suppose the contrary, then x is not an interior point of $f^{-1}(U)$, so it is a limit point of $X \setminus f^{-1}(U)$. By 5.4 there is a net (x_i) in $X \setminus f^{-1}(U)$ convergent to x. Since f is continuous, $f(x_i) \in Y \setminus U$ is convergent to $f(x) \in U$. This contradicts the assumption that U is open. $\square$

Problems.

5.6. If a finite set is a T_1-space then the topology is the discrete topology.

5.7. Is the space (X, PPX_p) (see 2.3) a Hausdorff space?

5.8. A T_1-space X is regular if and only if given a point x and an open set U containing x there is an open set V such that $x \in V \subset \overline{V} \subset U$.

5.9. Show that a subspace of a Hausdorff space is a Hausdorff space.

5.10. Show that a closed subspace of a normal space is normal.

5.11 (T_2 but not T_3). Show that the set $\mathbb{R}$ with the topology generated by all the subsets of the form (a, b) and $(a, b) \cap \mathbb{Q}$ is a Hausdorff space but is not a regular space.

5.12. Let (X, d) be a metric space. For $x \in X$ and $A \subset X$ we define the distance from x to A to be $d(x, A) = \inf\{d(x, y) \mid y \in A\}$. More generally, for two subsets A and B of X, define $d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$.

(a) Check that $d(x, A)$ is a continuous function with respect to x.
(b) Check that $d(x, A) = 0 \iff x \in \overline{A}$.
(c) Check that $d(A, B) = \inf\{d(a, B) \mid a \in A\} = \inf\{d(A, b) \mid b \in B\}$.
(d) When is $d(A, B) = 0$? Is d a metric?

5.13 (Hausdorff distance). The Hausdorff distance between two bounded subsets A and B of a metric space X is defined to be $d_H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}$. Show that d_H is a metric on the set of all closed bounded subsets of X. (For more, see [**BBI01**, Chapter 7].)

5.14. Let X be a normal space. Suppose that U_1 and U_2 are open sets in X satisfying $U_1 \cup U_2 = X$. Show that there is an open sets V_1 such that $\overline{V_1} \subset U_1$ and $V_1 \cup U_2 = X$.

5.15. * Let X be normal, let $f : X \to Y$ be a surjective, continuous, and closed map. Prove that Y is a normal space.

5.16. Let $I = (0, \infty) \subset \mathbb{R}$. For $i, j \in I$, define $i \leq_I j$ if $i \geq_{\mathbb{R}} j$ (i is less than or is equal to j as indexes if i is greater than or is equal to j as real numbers). On $\mathbb{R}$ with the Euclidean topology, consider the net $(x_i = i)_{i \in I}$. Is this net convergent?

5.17. On $\mathbb{R}$ with the finite complement topology and the usual order, consider the net $(x_i = i)_{i \in \mathbb{R}}$. Where does this net converge to?

5.18. Reconsider Problems 2.18, 2.20 using 5.5.

5.19. Let Y be a T_1-space, and let $f : X \to Y$ be continuous. Suppose that $A \subset X$ and $f(x) = c$ on A, where c is a constant. Show that $f(x) = c$ on $\overline{A}$, by:

(a) using nets.
(b) not using nets.

5.20. Let Y be a Hausdorff space and let $f, g : X \to Y$ be continuous. Show that the set $\{x \in X \mid f(x) = g(x)\}$ is closed in X, by:

(a) using nets.
(b) not using nets.

Show that, as a consequence, if f and g agree on a *dense* (trù mật) subspace of X (meaning the closure of that subspace is X) then they agree on X.

5.21 (sequence is not adequate for convergence). * Let $(A, \leq)$ be a well-ordered uncountable set (see 1.18). The smallest element of A is be denoted by 0. If A does not have a biggest element then add an element to A and define that element to be the biggest one, denoted by ∞. For $a, b \in A$ denote $[a, b] = \{x \in A \mid a \leq x \leq b\}$ and $[a, b) = \{x \in A \mid a \leq x < b\}$. Thus we can write $A = [0, \infty]$.

Let Ω be the smallest element of the set $\{a \in A \mid [0,a]$ is uncountable$\}$ (this set is non-empty since it contains ∞).

(a) Show that $[0,\Omega)$ is uncountable, and for all $a \in A$, $a < \Omega$ the set $[0,a]$ is countable.
(b) Consider $[0,\Omega]$ with the order topology. Show that Ω is a limit point of $[0,\Omega]$.
(c) Show that every countable subset of $[0,\Omega)$ is bounded in $[0,\Omega)$, therefore a sequence in $[0,\Omega)$ cannot converge to Ω.

5.22 (filter). A *filter* (lọc) on a set X is a collection F of non-empty subsets of X such that:

(a) if $A, B \in F$ then $A \cap B \in F$,
(b) if $A \subset B$ and $A \in F$ then $B \in F$.

For example, given a point, the collection of all neighborhoods of that point is a filter.

A filter is said to be convergent to a point if every neighborhood of that point is an element of the filter.

A *filter-base* (cơ sở lọc) is a collection G of non-empty subsets of X such that if $A, B \in G$ then there is $C \in G$ such that $G \subset (A \cap B)$.

If G is a filter-base in X then the filter generated by G is defined to be the collection of all subsets of X each containing an element of G: $\{A \subset X \mid \exists B \in G,\ B \subset A\}$.

For example, in a metric space, the collection of all open balls centered at a point is the filter-base for the filter consisting of all neighborhoods of that point.

A filter-base is said to be convergent to a point if the filter generated by it converges to that point.

(a) Show that a filter-base is convergent to x if and only if every neighborhood of x contains an element of the filter-base.
(b) Show that a point $x \in X$ is a limit point of a subset A of X if and only if there is a filter-base in $A \setminus \{x\}$ convergent to x.
(c) Show that a map $f : X \to Y$ is continuous at x if and only if for any filter-base F that is convergent to x, the filter-base $f(F)$ is convergent to $f(x)$.

Filter gives an alternative way to net for describing convergence. For more see [**Dug66**, p. 209], [**Eng89**, p. 49], [**Kel55**, p. 83].

6. Compact space

A *cover* (phủ) of a set X is a collection of subsets of X whose union is X. A subset of a cover which is itself a cover is called a *subcover* (phủ con). A cover is said to be an *open cover* if each member of the cover is an open subset of X.

DEFINITION. A space is *compact* if every open cover has a finite subcover. In symbols: a space (X, τ) is compact if

$$(I \subset \tau, \bigcup_{O \in I} O = X) \implies (\exists J \subset I, |J| < \infty, \bigcup_{O \in J} O = X).$$

EXAMPLE. Any finite space is compact. Any space whose topology is finite (that is, the space has finitely many open sets) is compact.

EXAMPLE. On the Euclidean line $\mathbb{R}$ the collection $\{(-n, n) \mid n \in \mathbb{Z}^+\}$ is an open cover without a finite subcover. Therefore the Euclidean line $\mathbb{R}$ is not compact.

REMARK. Let A be a subspace of a topological space X. Let I be an open cover of A. Each $O \in I$ is an open set of A, so it is the restriction of an open set U_O of X. Thus we have a collection $\{U_O \mid O \in I\}$ of open sets of X whose union contains A. On the other hand if we have a collection I of open sets of X whose union contains A then the collection $\{U \cap A \mid U \in I\}$ is an open cover of A. For this reason we often use the term open cover of a subspace A of X in both senses: either as an open cover of A or as a collection of open subsets of the space X whose union contains A.

THEOREM (continuous image of compact space is compact). *If X is compact and $f : X \to Y$ is continuous then $f(X)$ is compact.*

PROOF. Let I be a cover of $f(X)$ by open sets of Y (see the above remark). Then $\{f^{-1}(O) \mid O \in I\}$ is an open cover of X. Since X is compact there is a finite subcover, so there is a finite set $J \subset I$ such that $\{f^{-1}(O) \mid O \in J\}$ covers X. This implies $f^{-1}(\bigcup_{O \in J} O) = X$, so $\bigcup_{O \in J} O \supset Y$, hence J is a subcover of I. $\square$

In particular, compactness is preserved under homeomorphism. We say that *compactness is a topological property*.

PROPOSITION. *Any closed subspace of a compact space is compact.*

PROOF. Suppose that X is compact and $A \subset X$ is closed. Let I be an open cover of A by open set of X. By adding the open set $X \setminus A$ to I we get an open cover of X. This open cover has a finite subcover. This subcover of X must contain $X \setminus A$, thus omitting this set we get a finite subcover of A from I. $\square$

PROPOSITION 6.1. *Any compact subspace of a Hausdorff space is closed.*

PROOF. Let A be a compact set in a Hausdorff space X. We show that $X \setminus A$ is open.

Let $x \in X \setminus A$. For each $a \in A$ there are disjoint open sets U_a containing x and V_a containing a. The collection $\{V_a \mid a \in A\}$ covers A, so there is a finite subcover $\{V_{a_i} \mid 1 \leq i \leq n\}$. Let $U = \bigcap_{i=1}^n U_{a_i}$ and $V = \bigcup_{i=1}^n V_{a_i}$. Then U is an open neighborhood of x disjoint from V, a neighborhood of A. □

EXAMPLE. Any subspace of $\mathbb{R}$ with the finite complement topology is compact. Note that this space is not Hausdorff (5.1).

Characterization of compact spaces in terms of closed subsets. In the definition of compact spaces by writing open sets as complements of closed sets, we get a dual statement: A space is compact if for every collection of closed subsets whose intersection is empty there is a a finite subcollection whose intersection is empty. We will say that a collection of subsets of a set is having the *finite intersection property* (tính giao hữu hạn) if the intersection of every finite subcollection is non-empty. We get:

PROPOSITION 6.2. *A space is compact if and only if every collection of closed subsets with the finite intersection property has non-empty intersection.*

Compact metric spaces. A space is called *sequentially compact* (compắc dãy) if every sequence has a convergent subsequence.

LEMMA 6.3 (Lebesgue's number). *In a sequentially compact metric space, for any open cover there exists a number $\epsilon > 0$ such that any ball of radius ϵ is contained in an element of the cover.*

PROOF. Let O be a cover of a sequentially compact metric space X. Suppose the opposite of the conclusion, that is for any number $\epsilon > 0$ there is a ball $B(x, \epsilon)$ not contained in any of the element of O. Take a sequence of such balls $B(x_n, 1/n)$. The sequence $\{x_n\}_{n \in \mathbb{Z}^+}$ has a subsequence $\{x_{n_k}\}_{k \in \mathbb{Z}^+}$ converging to x. There is $\epsilon > 0$ such that $B(x, 2\epsilon)$ is contained in an element U of O. Take k sufficiently large such that $n_k > 1/\epsilon$ and x_{n_k} is in $B(x, \epsilon)$. Then $B(x_{n_k}, 1/n_k) \subset B(x_{n_k}, \epsilon) \subset U$, a contradiction. □

THEOREM. *A metric space is compact if and only if it is sequentially compact.*

PROOF. ($\Rightarrow$) Let $\{x_n\}_{n \in \mathbb{Z}^+}$ be a sequence in a compact metric space X. Suppose that this sequence has no convergent subsequence. This implies that for any point $x \in X$ there is an open neighborhood U_x of x and $N_x \in \mathbb{Z}^+$ such that if $n \geq N_x$ then $x_n \notin U_x$. Because the collection $\{U_x \mid x \in X\}$ covers X, it has a finite subcover $\{U_{x_k} \mid 1 \leq k \leq m\}$. Let $N = \max\{N_{x_k} \mid 1 \leq k \leq m\}$. If $n \geq N$ then $x_n \notin U_{x_k}$ for all k, a contradiction.

($\Leftarrow$) First we show that for any $\epsilon > 0$ the space X can be covered by finitely many balls of radii ϵ (a property called *total boundedness* or *pre-compact* (tiền compắc)). Suppose the contrary. Let $x_1 \in X$, and inductively let $x_{n+1} \notin \bigcup_{1 \leq i \leq n} B(x_i, \epsilon)$.

Since $d(x_m, x_n) \geq \epsilon$ if $m \neq n$, the sequence $(x_n)_{n\geq 1}$ cannot have any convergent subsequence, a contradiction.

Now let O be any open cover of X. By 6.3 there is a corresponding Lebesgue's number ϵ such that a ball of radius ϵ is contained in an element of O. The space X is covered by finitely many balls of radii ϵ. The collection of finitely many corresponding elements of O covers X. Thus O has a finite subcover. □

The above theorem shows that compactness in metric space as defined in previous courses agrees with compactness in topological spaces. We inherit all results obtained previously on compactness in metric spaces.

In particular we have the following results, which were proved using sequential compactness (it should be helpful to review the previous proofs).

PROPOSITION. *If a subspace of a metric space is compact then it is closed and bounded.*

PROOF. We give a proof using compactness. Suppose that X is a metric space and suppose that Y is a compact subspace of X. Let $x \in Y$. Consider the open cover of Y by balls centered at x, that is, $\{B(x,r\} \mid r > 0\}$. Since there is a finite subcover, there is an $r > 0$ such that $Y \subset B(x,r)$, thus Y is bounded. That Y is closed in X follows from 6.1. □

The following result is well-known from previous courses, we include it here for convenience, with a proof using open covering compactness.

THEOREM (Heine-Borel). *A subspace of the Euclidean space $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

PROOF. It is sufficient to prove that the unit rectangle $I = [0,1]^n$ is compact. Suppose that O is an open cover of I. Suppose that no finite subset of O can cover I. Divide each dimension of I by half, we get 2^n subrectangles. Let I_1 be one of these rectangles that cannot be covered by a finite subset of O. Inductively, divide I_k to 2^n equal subrectangles and let I_{k+1} be a subrectangle that is not covered by a finite subset of O. We have a family of descending rectangles $(I_k)_{k\in\mathbb{Z}^+}$. The dimension of I_k is $1/2^k$, going to 0 as k goes to infinity.

We claim that the intersection of this family is non-empty. Let $I_k = \prod_{i=1}^n [a_k^i, b_k^i]$. For each i, the sequence $\{a_i^k\}_{k\in\mathbb{Z}^+}$ is increasing and is bounded from above. Let $x^i = \lim_{k\to\infty} a_k^i = \sup\{a_k^i \mid k \in \mathbb{Z}^+\}$. Then $a_k^i \leq x^i \leq b_k^i$ for all $k \geq 1$. Thus the point $x = (x^i)_{1\leq i\leq n}$ is in the intersection of $(I_k)_{k\in\mathbb{Z}^+}$.

There is $U \in O$ that contains x. There is a number $\epsilon > 0$ such that $B(x,\epsilon) \subset U$. Then for k sufficiently large $I_k \subset B(x,\epsilon) \subset U$. This is a contradiction. □

In general topological spaces compactness and sequentially compactness are different notions, none implies the other. An example of a compact space which is not sequentially compact can be constructed based on the spaces in problem 5.21.

Compactification. A *compactification* (compắc hóa) of a space X is a compact space Y such that X is homeomorphic to a dense subspace of Y.

EXAMPLE. A compactification of the Euclidean interval $(0,1)$ is the Euclidean interval $[0,1]$. Another is the circle S^1. Yet another is the Topologist's sine curve $\{(x, \sin\frac{1}{x}) \mid 0 < x \le 1\} \cup \{(0,y) \mid -1 \le y \le 1\}$ (see 4.4).

EXAMPLE. A compactification of the Euclidean plane $\mathbb{R}^2$ is the sphere S^2. When $\mathbb{R}^2$ is identified with the complex plane $\mathbb{C}$ then S^2 is often called the *Riemann sphere*.

In some cases it is possible to compactify a non-compact space by adding just one point, obtaining a *one-point compactification*. For example the Euclidean interval $[0,1]$ is a one-point compactification of the Euclidean interval $[0,1)$.

Let X be a non-empty space. Since the set $\mathcal{P}(X)$ of all subsets of X cannot be contained in X there is an element of $\mathcal{P}(X)$ that is not in X. Let us denote that element by ∞, and let $X^\infty = X \cup \{\infty\}$. Let us see what a topology on X^∞ should be in order for X^∞ to contain X as a subspace and to be compact. If an open subset U of X^∞ does not contain ∞ then U is contained in X, therefore U is an open subset of X in the subspace topology of X, which is the same as the original topology of X. If U contains ∞ then its complement $X^\infty \setminus U$ must be a closed subset of X^∞, hence is compact, furthermore $X^\infty \setminus U$ is contained in X and is therefore a closed subset of X.

THEOREM (Alexandroff compactification). *The collection consisting of all open subsets of X and all complements in X^∞ of closed compact subsets of X is the finest topology on X^∞ such that X^∞ is compact and contains X as a subspace. If X is not compact then X is dense in X^∞, and X^∞ is called the Alexandroff compactification of X.*[6]

PROOF. We go through several steps.

(a) We check that we really have a topology.

Let I be a collection of closed compact sets in X. Then $\bigcup_{C\in I}(X^\infty \setminus C) = X^\infty \setminus \bigcap_{C\in I} C$, where $\bigcap_{C\in I} C$ is closed compact.

If O is open in X and C is closed compact in X then $O \cup (X^\infty \setminus C) = X^\infty \setminus (C \setminus O)$, where $C \setminus O$ is a closed and compact subset of X.

Also $O \cap (X^\infty \setminus C) = O \cap (X \setminus C)$ is open in X.

If C_1 and C_2 are closed compact in X then $(X^\infty \setminus C_1) \cap (X^\infty \setminus C_2) = X^\infty \setminus (C_1 \cup C_2)$, where $C_1 \cup C_2$ is closed compact.

So we do have a topology. With this topology X is a subspace of X^∞.

(b) We show that X^∞ is compact. Let F be an open cover of X^∞. Then an element $O \in F$ will cover ∞. The complement of O in X^∞ is a closed compact set C in X. Then $F \setminus \{O\}$ is an open cover of C. From this cover

[6]Proved in the early 1920s by Pavel Sergeyevich Alexandrov. Alexandroff is another way to spell his name.

there is a finite cover. This finite cover together with O is a finite cover of X^∞.

(c) Since X is not compact and X^∞ is compact, X cannot be closed in X^∞, therefore the closure of X in X^∞ is X^∞.

□

A space X is called *locally compact* if every point has a compact neighborhood.

EXAMPLE. The Euclidean space $\mathbb{R}^n$ is locally compact.

PROPOSITION. *The Alexandroff compactification of a locally compact Hausdorff space is Hausdorff.*

PROOF. Suppose that X is locally compact and is Hausdorff. We check that ∞ and $x \in X$ can be separated by open sets. Since X is locally compact there is a compact set C containing an open neighborhood O of x. Since X is Hausdorff, C is closed in X. Then $X^\infty \setminus C$ is open in the Alexandroff compactification X^∞. So O and $X^\infty \setminus C$ separate x and ∞. □

The need for the locally compact assumption is discussed in 6.26.

PROPOSITION. *If X is homeomorphic to Y then a Hausdorff one-point compactification of X is homeomorphic to a Hausdorff one-point compactification of Y.*

In particular, Hausdorff one-point compactification is unique up to homeomorphisms. For this reason we can talk about *the* one-point compactification of a locally compact Hausdorff space.

PROOF. Suppose that $h : X \to Y$ is a homeomorphism. Let $X \cup \{a\}$ and $Y \cup \{b\}$ be Hausdorff one-point compactifications of X and Y. Let $\tilde{h} : X \cup \{a\} \to Y \cup \{b\}$ be defined by $\tilde{h}(x) = h(x)$ if $x \neq a$ and $\tilde{h}(a) = b$. We show that $\tilde{h}$ is a homeomorphism. We will prove that $\tilde{h}$ is continuous, that the inverse map is continuous is similar, or we can use 6.10 instead.

Let U be an open subset of $Y \cup \{b\}$. If U does not contain b then U is open in Y, so $h^{-1}(U)$ is open in X, and so is open in $X \cup \{a\}$. If U contains b then $(Y \cup \{b\}) \setminus U$ is closed in $Y \cup \{b\}$, which is compact, so $(Y \cup \{b\}) \setminus U = Y \setminus U$ is compact. Then $\tilde{h}^{-1}((Y \cup \{b\}) \setminus U) = h^{-1}(Y \setminus U)$ is a compact subspace of X and therefore of $X \cup \{a\}$. Since $X \cup \{a\}$ is a Hausdorff space, $\tilde{h}^{-1}((Y \cup \{b\}) \setminus U)$ is closed in $X \cup \{a\}$. Thus $\tilde{h}^{-1}(U)$ must be open in $X \cup \{a\}$. □

EXAMPLE. The Euclidean line $\mathbb{R}$ is homeomorphic to the circle S^1 minus a point. The circle is of course a Hausdorff one-point compactification of the circle minus a point. Thus a Hausdorff one-point compactification (in particular, the Alexandroff compactification) of the Euclidean line is homeomorphic to the circle.

Problems.

6.4. Any discrete compact topological space is finite.

6.5. In a topological space a finite unions of compact subsets is compact.

6.6. In a Hausdorff space an intersection of compact subsets is compact.

6.7 (extension of Cantor lemma in Calculus). Let X be compact and $X \supset A_1 \supset A_2 \supset \cdots \supset A_n \supset \ldots$ be a descending sequence of closed, non-empty sets. Then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$. (This is a special case of 6.2.)

6.8 (extreme value theorem). If X is a compact space and $f : X \to (\mathbb{R}, \text{Euclidean})$ is continuous then f has a maximum value and a minimum value.

6.9 (uniformly continuous). A function f from a metric space to a metric space is uniformly continuous if for any $\epsilon > 0$, there is $\delta > 0$ such that if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$. Show that a continuous function from a compact metric space to a metric space is uniformly continuous.

6.10. $\surd$ If X is compact, Y is Hausdorff, $f : X \to Y$ is bijective and continuous, then f is a homeomorphism.

6.11. In a compact space any infinite set has a limit point.

6.12. In a Hausdorff space a point and a disjoint compact set can be separated by open sets.

6.13. In a regular space a closed set and a disjoint compact set can be separated by open sets.

6.14. In a Hausdorff space two disjoint compact sets can be separated by open sets.

6.15. $\surd$ Any compact Hausdorff space is normal.

6.16. Prove 6.2.

6.17. Prove the existence of Lebesgue's number 6.3 by using open coverings.

6.18. Find the one-point compactification of $(0, 1) \cup (2, 3)$ with the Euclidean topology, that is, describe this space more concretely.

6.19. Find the one-point compactification of $\{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$ under the Euclidean topology?

6.20. Find the one-point compactification of $\mathbb{Z}^+$ under the Euclidean topology? How about $\mathbb{Z}$?

6.21. Show that $\mathbb{Q}$ is not locally compact (under the Euclidean topology of $\mathbb{R}$). Is its Alexandroff compactification Hausdorff?

6.22. What is the one-point compactification of the Euclidean open ball $B(0, 1)$? Find the one-point compactification of the Euclidean space $\mathbb{R}^n$.

6.23. What is the one-point compactification of the Euclidean annulus $\{(x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\}$?

6.24. Define a topology on $\mathbb{R} \cup \{\pm\infty\}$ such that it is a compactification of the Euclidean $\mathbb{R}$.

6.25. Consider $\mathbb{R}$ with the Euclidean topology. Find a necessary and sufficient condition for a continuous function from $\mathbb{R}$ to $\mathbb{R}$ to have an extension to a continuous function from the one-point compactification $\mathbb{R} \cup \{\infty\}$ to $\mathbb{R}$.

6.26. If there is a topology on the set $X^\infty = X \cup \{\infty\}$ such that it is compact, Hausdorff, and containing X as a subspace, then X must be Hausdorff, locally compact, and there is only one such topology – the topology of the Alexandroff compactification.

6.27. We could have noticed that the notion of local compactness as we have defined is not apparently a local property. For a property to be local, every neighborhood of any point must contain a neighborhood of that point with the given property (as in the cases of local connectedness and local path-connectedness). Show that for Hausdorff spaces local compactness is indeed a local property, i.e., every neighborhood of any point contains a compact neighborhood of that point.

6.28. Any locally compact Hausdorff space is a regular space.

6.29. In a locally compact Hausdorff space, if K is compact, U is open, and $K \subset U$, then there is an open set V such that $\overline{V}$ is compact and $K \subset V \subset \overline{V} \subset U$. (Compare with 5.2.)

6.30. A space is locally compact Hausdorff if and only if it is homeomorphic to an open subspace of a compact Hausdorff space.

6.31. * Let X be a compact Hausdorff space. Let $X_1 \supset X_2 \supset \cdots$ be a nested sequence of closed connected subsets of X. Show that $Y = \bigcap_{i=1}^{\infty} X_i$ is connected.

Is the statement correct if connected is replaced by path-connected?

6.32. The set of $n \times n$-matrix with real coefficients, denoted by $M(n,\mathbb{R})$, could be naturally considered as a subset of the Euclidean space $\mathbb{R}^{n^2}$ by considering entries of a matrix as coordinates, via the map

$$(a_{i,j}) \longmapsto (a_{1,1}, a_{2,1}, \ldots, a_{n,1}, a_{1,2}, a_{2,2}, \ldots, a_{n,2}, a_{1,3}, \ldots, a_{n-1,n}, a_{n,n}).$$

Let $\mathrm{GL}(n,\mathbb{R})$ be the set of all invertible $n \times n$-matrices with real coefficients.

(a) Show that taking product of two matrices is a continuous map on $\mathrm{GL}(n,\mathbb{R})$.
(b) Show that taking inverse of a matrix is a continuous map $\mathrm{GL}(n,\mathbb{R})$.
(c) A set with both a group structure and a topology such that the group operations are continuous is called a *topological group*. Show that $\mathrm{GL}(n,\mathbb{R})$ is a topological group. It is called the *General Linear Group.*

6.33. * The *Orthogonal Group* $\mathrm{O}(n)$ is defined to be the group of matrices representing orthogonal linear maps of $\mathbb{R}^n$, that is, linear maps that preserve inner product. Thus

$$\mathrm{O}(n) = \{A \in M(n,\mathbb{R}) \mid A \cdot A^T = I_n\}.$$

The *Special Orthogonal Group* $\mathrm{SO}(n)$ is the subgroup of $\mathrm{O}(n)$ consisting of all orthogonal matrices with determinant 1.

(a) Show that any element of $\mathrm{SO}(2)$ is of the form

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

This is a rotation in the plane around the origin with an angle φ. Thus $\mathrm{SO}(2)$ is the group of rotations on the plane around the origin.

(b) Show that $\mathrm{SO}(2)$ is path-connected.

(c) How many connected components does $\mathrm{O}(2)$ have?

(d) Is $\mathrm{SO}(n)$ compact?

(e) It is known [F. Gantmacher, Theory of matrices, vol. 1, Chelsea, 1959, p. 285], that for any matrix $A \in \mathrm{O}(n)$ there is a matrix $P \in \mathrm{O}(n)$ such that $A = PBP^{-1}$ where B is a matrix of the form

$$\begin{pmatrix} R(\varphi_1) & & & & & \\ & \ddots & & & & \\ & & R(\varphi_k) & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

Using this fact, prove that $\mathrm{SO}(n)$ is path-connected.

7. Product of spaces

Finite products of spaces. Let X and Y be two topological spaces, and consider the Cartesian product $X \times Y$. The *product topology* on $X \times Y$ is the topology generated by the collection F of sets of the form $U \times V$ where U is an open set of X and V is an open set of Y. Since the intersection of two members of F is also a member of F, the collection F is a basis for the product topology. Thus every open set in the product topology is a union of products of open sets of X with open sets of Y.

REMARK. Notice *a common error*: to assume that an arbitrary open set in the product topology is a product $U \times V$.

The product topology on $\prod_{i=1}^{n}(X_i, \tau_i)$ is defined similarly to be the topology generated by the collection $\{\prod_{i=1}^{n} U_i \mid U_i \in \tau_i\}$.

PROPOSITION. *If each b_i is a basis for X_i then $\{\prod_{i=1}^{n} U_i \mid U_i \in b_i\}$ is a basis for the product topology on $\prod_{i=1}^{n} X_i$.*

PROOF. Consider an element in the above basis of the product topology, which is of the form $\prod_{i=1}^{n} V_i$ where $V_i \in \tau_i$. Each V_i can be written $V_i = \bigcup_{i_j \in I_i} U_{i_j}$, where $U_{i_j} \in b_i$. Then

$$\prod_{i=1}^{n} V_i = \bigcup_{i_j \in I_i, 1 \le i \le n} \prod_{i=1}^{n} U_{i_j}.$$

This proves our assertion. □

EXAMPLE (Euclidean topology). Recall that $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ copies of } \mathbb{R}}$. Let $\mathbb{R}$ have Euclidean topology, generated by open intervals. An open set in the product topology of $\mathbb{R}^n$ is a union of products of open intervals. Since a product of open intervals is an open rectangle, and an open rectangle is a union of open balls and vice versa, *the product topology on $\mathbb{R}^n$ is exactly the Euclidean topology*.

Arbitrary products of spaces.

DEFINITION. Let $\{(X_i, \tau_i)\}_{i \in I}$ be a family of topological spaces. The *product topology* on the set $\prod_{i \in I} X_i$ is the topology generated by the collection F consisting of all sets of the form $\prod_{i \in I} U_i$, where $U_i \in \tau_i$ and $U_i = X_i$ for all except finitely many $i \in I$. In symbols:

$$F = \{\prod_{i \in I} U_i \mid U_i \in \tau_i, \exists J_i \subset \tau_i, |J_i| < \infty, \forall i \in I \setminus J_i, U_i = X_i\}.$$

Notice that the collection F above is a basis of the product topology. The subcollection of all sets of the form $\prod_{i \in I} U_i$, where $U_i \in \tau_i$ and $U_i = X_i$ for all except one $i \in I$ is a subbasis for the product topology.

Recall that an element of the set $\prod_{i \in I} X_i$ is written $(x_i)_{i \in I}$. For $j \in I$ the *projection to the j-coordinate* is defined by $p_j : \prod_{i \in I} X_i \to X_j$, $p_j((x_i)) = x_j$.

The definition of the product topology is explained in the following:

THEOREM 7.1 (product topology is the topology such that projections are continuous). *The product topology is the coarsest topology on $\prod_{i\in I} X_i$ such that all the projection maps p_i are continuous. In other words, the product topology is the topology generated by the projection maps.*

PROOF. Notice that if $O_j \in X_j$ then $p_j^{-1}(O_j) = \prod_{i\in I} U_i$ with $U_i = X_i$ for all i except j, and $U_j = O_j$. The topology generated by all the maps p_i is the topology generated by all sets of the form $p_i^{-1}(O_i)$ with $O_i \in \tau_i$, see 3.8. $\square$

THEOREM 7.2 (map to a product space is continuous if and only if each component map is continuous). *A map $f : Y \to \prod_{i\in I} X_i$ is continuous if and only if each component $f_i = p_i \circ f$ is continuous.*

PROOF. If $f : Y \to \prod_{i\in I} X_i$ is continuous then $p_i \circ f$ is continuous because p_i is continuous.

On the other hand, let us assume that every f_i is continuous. Let $U = \prod_{i\in I} U_i$ – an element of the basis of $\prod_{i\in I} X_i$ – where $U_i \in \tau_i$ and $\exists J \subset I$ such that $I \setminus J$ is finite and $U_i = X_i, \forall i \in J$. Then

$$f^{-1}(U) = \bigcap_{i\in I} f_i^{-1}(U_i) = \bigcap_{i\in I\setminus J} f_i^{-1}(U_i),$$

which is an open set. So f is continuous. $\square$

THEOREM 7.3 (convergence in product topology is coordinate-wise convergence). *A net $n : J \to \prod_{i\in I} X_i$ is convergent if and only if all of its projections $p_i \circ n$ are convergent.*

PROOF. ($\Leftarrow$) Suppose that each $p_i \circ n$ is convergent to a_i, we show that n is convergent to $a = (a_i)_{i\in I}$.

A neighborhood of a contains an open set of the form $U = \prod_{i\in I} O_i$ with O_i are open sets of X_i and $O_i = X_i$ except for $i \in K$, where K is a finite subset of I.

For each $i \in K$, $p_i \circ n$ is convergent to a_i, therefore there exists an index $j_i \in J$ such that for $j \geq j_i$ we have $p_i(n(j)) \in O_i$. Take an index j_0 such that $j_0 \geq j_i$ for all $i \in K$. Then for $j \geq j_0$ we have $n(j) \in U$. $\square$

Tikhonov theorem.

THEOREM (Tikhonov theorem). *The product of any family of compact spaces is compact. More concisely, if X_i is compact for all $i \in I$ then $\prod_{i\in I} X_i$ is compact.*[7]

EXAMPLE. Let $[0,1]$ have the Euclidean topology. The space $\prod_{i\in\mathbb{Z}^+}[0,1]$ is called the *Hilbert cube*. By Tikhonov theorem the Hilbert cube is compact.

Applications of Tikhonov theorem include the Banach-Alaoglu theorem in Functional Analysis and the Stone-Cech compactification.

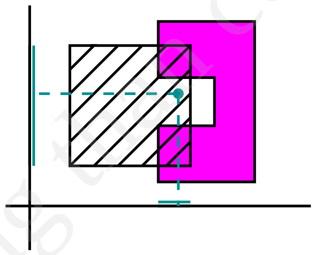
[7]Proved by Andrei Nicolaievich Tikhonov around 1926. The product topology was defined by him. His name is also spelled as Tychonoff.

Tikhonov theorem is equivalent to the Axiom of choice. The proofs we have are rather difficult. However in the case of finite product it can be proved more easily (7.18). Different techniques can be used in special cases of this theorem (7.21 and 7.8).

PROOF OF TIKHONOV THEOREM. Let X_i be compact for all $i \in I$. We will show that $X = \prod_{i\in I} X_i$ is compact by showing that if a collection of closed subsets of X has the finite intersection property then it has non-empty intersection (see 6.2).[8]

Let F be a collection of closed subsets of X that has the finite intersection property. We will show that $\bigcap_{A\in F} A \neq \emptyset$.

Have a look at the following argument, which suggests that proving the Tikhonov theorem might not be easy. If we take the closures of the projections of the collection F to the i-coordinate then we get a collection $\{\overline{p_i(A)}, A \in F\}$ of closed subsets of X_i having the finite intersection property. Since X_i is compact, this collection has non-empty intersection. From this it is tempting to conclude that F must have non-empty intersection itself. But that is not true, see the figure.



In what follows we will overcome this difficulty by first enlarging the collection F.

(a) We show that there is a maximal collection $\tilde{F}$ of subsets of X such that $\tilde{F}$ contains F and still has the finite intersection property. We will use Zorn lemma for this purpose.[9]

Let K be the collection of collections G of subsets of X such that G contains F and has the finite intersection property. On K we define an order by the usual set inclusion.

Now suppose that L is a totally ordered subcollection of K. Let $H = \bigcup_{G\in L} G$. We will show that $H \in K$, therefore H is an upper bound of L.

First H contains F. We need to show that H has the finite intersection property. Suppose that $H_i \in H$, $1 \leq i \leq n$. Then $H_i \in G_i$ for some $G_i \in L$. Since L is totally ordered, there is an i_0, $1 \leq i_0 \leq n$ such that G_{i_0} contains all G_i, $1 \leq i \leq n$. Then $H_i \in G_{i_0}$ for all $1 \leq i \leq n$, and since G_{i_0} has the finite intersection property, we have $\bigcap_{i=1}^n H_i \neq \emptyset$.

[8]A proof based on open covers is also possible, see [**Kel55**, p. 143].

[9]This is a routine step; it might be easier for the reader to carry it out instead of reading.

(b) Since $\tilde{F}$ is maximal, it is closed under finite intersection. Moreover if a subset of X has non-empty intersection with every element of $\tilde{F}$ then it belongs to $\tilde{F}$.

(c) Since $\tilde{F}$ has the finite intersection property, for each $i \in I$ the collection $\{p_i(A) \mid A \in \tilde{F}\}$ also has the finite intersection property, and so does the collection $\{\overline{p_i(A)} \mid A \in \tilde{F}\}$. Since X_i is compact, $\bigcap_{A\in\tilde{F}} \overline{p_i(A)}$ is non-empty.

(d) Let $x_i \in \bigcap_{A\in\tilde{F}} \overline{p_i(A)}$ and let $x = (x_i)_{i\in I} \in \prod_{i\in I}[\bigcap_{A\in\tilde{F}} \overline{p_i(A)}]$. We will show that $x \in \overline{A}$ for all $A \in \tilde{F}$, in particular $x \in A$ for all $A \in F$.

We need to show that any neighborhood of x has non-empty intersection with every $A \in \tilde{F}$. It is sufficient to prove this for neighborhoods of x belonging to the basis of X, namely finite intersections of sets of the form $p_i^{-1}(O_i)$ where O_i is an open neighborhood of $x_i = p_i(x)$. For any $A \in \tilde{F}$, since $x_i \in \overline{p_i(A)}$ we have $O_i \cap p_i(A) \neq \emptyset$. Therefore $p_i^{-1}(O_i) \cap A \neq \emptyset$. By the maximality of $\tilde{F}$ we have $p_i^{-1}(O_i) \in \tilde{F}$, and the desired result follows.

□

Stone-Cech compactification. Let X be a topological space. Denote by $C(X)$ the set of all bounded continuous functions from X to $\mathbb{R}$ where $\mathbb{R}$ has the Euclidean topology. By Tikhonov theorem the space $\prod_{f\in C(X)}[\inf f, \sup f]$ is compact. Define

$$\begin{aligned}\Phi : X &\to \prod_{f\in C(X)} [\inf f, \sup f]\\ x &\mapsto (f(x))_{f\in C(X)}.\end{aligned}$$

Thus for each $x \in X$ and each $f \in C(X)$, the f-coordinate of the point $\Phi(x)$ is $\Phi(x)_f = f(x)$. This means the f-component of Φ is f, i.e. $p_f \circ \Phi = f$, where p_f is the projection to the f-coordinate.

Notice that the closure $\overline{\Phi(X)}$ is compact.

THEOREM 7.4. *If X is completely regular then $\Phi : X \to \Phi(X)$ is a homeomorphism, i.e. Φ is an embedding. In this case $\overline{\Phi(X)}$ is called the Stone-Cech compactification of X. It is a Hausdorff space.*

Here, a space is said to be *completely regular* (also called a $T_{3\frac{1}{2}}$-space) if it is a T_1-space and for each point x and each closed set A with $x \notin A$ there is a map $f \in C(X)$ such that $f(x) = a$ and $f(A) = \{b\}$ where $a \neq b$. Thus in a completely regular space a point and a closed set disjoint from it can be separated by a continuous real function.

PROOF. We go through several steps.

(a) Φ is injective: If $x \neq y$ then since X is completely regular there is $f \in C(X)$ such that $f(x) \neq f(y)$, therefore $\Phi(x) \neq \Phi(y)$.

(b) Φ is continuous: Since the f-component of Φ is f, which is continuous, the result follows from 7.2.

(c) Φ^{-1} is continuous: We prove that Φ brings an open set onto an open set. Let U be an open subset of X and let $x \in U$. There is a function $f \in C(X)$ that separates x and $X \setminus U$. In particular there is an interval (a, b) containing $f(x)$ such that $f^{-1}((a,b)) \cap (X \setminus U) = \emptyset$. We have $f^{-1}((a,b)) = (p_f \circ \Phi)^{-1}((a,b)) = \Phi^{-1}(p_f^{-1}((a,b))) \subset U$. Apply Φ to both sides, we get $p_f^{-1}((a,b)) \cap \Phi(X) \subset \Phi(U)$. Since $p_f^{-1}((a,b)) \cap \Phi(X)$ is an open set in $\Phi(X)$ containing $\Phi(x)$, we see that $\Phi(x)$ is an interior point of $\Phi(U)$. We conclude that $\Phi(U)$ is open.

That $\overline{\Phi(X)}$ is a Hausdorff space follows from that Y is a Hausdorff space, by 7.15, and 5.9. □

THEOREM. *A bounded continuous real function on a completely regular space has a unique extension to the Stone-Cech compactification of the space.*

More concisely, if X is a completely regular space and $f \in C(X)$ then there is a unique function $\tilde{f} \in C(\overline{\Phi(X)})$ such that $f = \tilde{f} \circ \Phi$.

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "f"'] & \overline{\Phi(X)} \arrow[dl, "\tilde{f}"] \\ \mathbb{R} & \end{tikzcd}$$

PROOF. A continuous extension of f, if exists, is unique, by 5.20.

Since $p_f \circ \Phi = f$ the obvious choice for $\tilde{f}$ is the projection p_f. □

Problems.

7.5. Note that, as sets:

(a) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

(b) $(A \times B) \cup (C \times D) \subsetneq (A \cup C) \times (B \cup D) = (A \times B) \cup (A \times D) \cup (C \times B) \cup (C \times D)$.

7.6. Check that in topological sense (i.e. up to homeomorphisms):

(a) $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

(b) More generally, is the product associative? Namely, is $(X \times Y) \times Z = X \times (Y \times Z)$? Is $(X \times Y) \times Z = X \times Y \times Z$?

7.7. Show that the sphere S^2 with the North Pole and the South Pole removed is homeomorphic to the infinite cylinder $S^1 \times \mathbb{R}$.

7.8. Let $(X_i, d_i), 1 \le i \le n$ be metric spaces. Let $X = \prod_{i=1}^n X_i$. For $x = (x_1, x_2, \dots, x_n) \in X$ and $y = (y_1, y_2, \dots, y_n) \in X$, define

$$\delta_1(x, y) = \max\{d_i(x_i, y_i) \mid 1 \le i \le n\},$$

$$\delta_2(x, y) = \left(\sum_{i=1}^n d_i(x_i, y_i)^2\right)^{1/2}.$$

Show that δ_1 and δ_2 are metrics on X generating the product topology.

7.9. Show that a space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \in X \times X\}$ is closed in $X \times X$, by:

(a) using nets,
(b) not using nets.

7.10. Show that if Y is Hausdorff and $f : X \to Y$ is continuous then the graph of f (the set $\{(x, f(x)) \mid x \in X\}$) is closed in $X \times Y$.

7.11. If for each $i \in I$ the space X_i is homeomorphic to the space Y_i then $\prod_{i \in I} X_i$ is homeomorphic to $\prod_{i \in I} Y_i$.

7.12. Show that each projection map p_i is a an open map, mapping an open set onto an open set. Is it a closed map?

7.13 (disjoint union). $\surd$ Let A and B be topological spaces. On the set $(A \times \{0\}) \cup (B \times \{1\})$ consider the topology generated by subsets of the form $U \times \{0\}$ and $V \times \{1\}$ where U is open in A and V is open in B. Show that $A \times \{0\}$ is homeomorphic to A, while $B \times \{1\}$ is homeomorphic to B. Notice that $(A \times \{0\}) \cap (B \times \{1\}) = \emptyset$. The space $(A \times \{0\}) \cup (B \times \{1\})$ is called the *disjoint union* (hội rời) of A and B, denoted by $A \sqcup B$. We use this construction when for example we want to consider a space consisting of two disjoint circles.

7.14. $\surd$ Fix a point $O = (O_i) \in \prod_{i \in I} X_i$. Define the inclusion map $f : X_i \to \prod_{i \in I} X_i$ by

$$x \mapsto f(x) \text{ with } f(x)_j = \begin{cases} O_j & \text{if } j \neq i \\ x & \text{if } j = i \end{cases}.$$

Show that f is a homeomorphism onto its image $\tilde{X}_i$ (an embedding of X_i). Thus $\tilde{X}_i$ is a copy of X_i in $\prod_{i \in I} X_i$. The spaces $\tilde{X}_i$ have O as the common point. This is an analog of the coordinate system Oxy on $\mathbb{R}^2$.

7.15. Show that

(a) If each X_i, $i \in I$ is a Hausdorff space then $\prod_{i \in I} X_i$ is a Hausdorff space.
(b) If $\prod_{i \in I} X_i$ is a Hausdorff space then each X_i is a Hausdorff space.

7.16. Show that

(a) If $\prod_{i \in I} X_i$ is path-connected then each X_i is path-connected.
(b) If each X_i, $i \in I$ is path-connected then $\prod_{i \in I} X_i$ is path-connected.

7.17. Show that

(a) If $\prod_{i \in I} X_i$ is connected then each X_i is connected.
(b) If X and Y are connected then $X \times Y$ is connected.
(c) * If each X_i, $i \in I$ is connected then $\prod_{i \in I} X_i$ is connected.

7.18. Show that

(a) If $\prod_{i \in I} X_i$ is compact then each X_i is compact.
(b) * If X and Y are compact then $X \times Y$ is compact (of course without using the Tikhonov theorem).

7.19. Let X be a normed space over a field $\mathbb{F}$ which is $\mathbb{R}$ or $\mathbb{C}$. Check that the addition $(x, y) \mapsto x + y$ is a continuous map from the product space $X \times X$ to X, while the scalar multiplication $(c, x) \mapsto c \cdot x$ is a continuous map from $\mathbb{F} \times X$ to X. This is an example of a *topological vector space*, and is a special case of topological groups (see 6.32).

7.20 (Zariski topology). * Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

A polynomial in n variables on $\mathbb{F}$ is a function from $\mathbb{F}^n$ to $\mathbb{F}$ that is a finite sum of terms of the form $ax_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}$, where $a, x_i \in \mathbb{F}$ and $m_i \in \mathbb{N}$. Let P be the set of all polynomials in n variables on $\mathbb{F}$.

If $S \subset P$ then define $Z(S)$ to be the set of all common zeros of all polynomials in S, thus $Z(S) = \{x \in \mathbb{F}^n \mid \forall p \in S, p(x) = 0\}$. Such a set is called an *algebraic set*.

(a) Show that if we define that a subset of $\mathbb{F}^n$ is closed if it is algebraic, then this gives a topology on $\mathbb{F}^n$, called the *Zariski topology*.
(b) Show that the Zariski topology on $\mathbb{F}$ is exactly the finite complement topology.
(c) Show that if both $\mathbb{F}$ and $\mathbb{F}^n$ have the Zariski topology then all polynomials on $\mathbb{F}^n$ are continuous.
(d) Is the Zariski topology on $\mathbb{F}^n$ the product topology?

The Zariski topology is used in Algebraic Geometry.

7.21. Using the characterization of compact subsets of Euclidean spaces, prove the Tikhonov theorem for finite products of compact subsets of Euclidean spaces.

Using the characterization of compact metric spaces in terms of sequences, prove the Tikhonov theorem for finite products of compact metric spaces.

7.22. Any completely regular space is a regular space.

7.23. Prove 7.4 using nets.

7.24. A space is completely regular if and only if it is homeomorphic to a subspace of a compact Hausdorff space. As a corollary, a locally compact Hausdorff space is completely regular.

By 7.24 if a space has a Hausdorff Alexandroff compactification then it also has a Hausdorff Stone-Cech compactification. In a certain sense, for a noncompact space the Alexandroff compactification is the "smallest" Hausdorff compactification of the space and the Stone-Cech compactification is the "largest" one. For more discussions on this topic see for instance [**Mun00**, p. 237].

8. Real functions and Spaces of functions

In this section the set $\mathbb{R}$ is assumed to have the Euclidean topology.

Urysohn lemma.

THEOREM 8.1 (Urysohn lemma). *If X is normal, F is closed, U is open, and $F \subset U$, then there exists a continuous map $f : X \to [0,1]$ such that $f(x) = 0$ on F and $f(x) = 1$ on $X \setminus U$.*

Equivalently, if X is normal, A and B are two disjoint closed subsets of X, then there is a continuous function f from X to $[0,1]$ such that $f(x) = 0$ on A and $f(x) = 1$ on B.

Thus in a normal space two disjoint closed subsets can be separated by a continuous real function.

EXAMPLE. For metric space we can take

$$f(x) = \frac{d(x,A)}{d(x,A) + d(x,B)}.$$

PROOF OF URYSOHN LEMMA. The proof goes through the following steps:

(a) Construct a family of open sets in the following manner (recalling 5.2):
Let $U_1 = U$.
$n = 0$: $F \subset U_0 \subset \overline{U_0} \subset U_1$.
$n = 1$: $\overline{U_0} \subset U_{\frac{1}{2}} \subset \overline{U_{\frac{1}{2}}} \subset U_1$.
$n = 2$: $\overline{U_0} \subset U_{\frac{1}{4}} \subset \overline{U_{\frac{1}{4}}} \subset U_{\frac{2}{4}} = U_{\frac{1}{2}} \subset \overline{U_{\frac{2}{4}}} \subset U_{\frac{3}{4}} \subset \overline{U_{\frac{3}{4}}} \subset U_{\frac{4}{4}} = U_1$.
Inductively,

$$F \subset U_0 \subset \overline{U_0} \subset U_{\frac{1}{2^n}} \subset \overline{U_{\frac{1}{2^n}}} \subset U_{\frac{2}{2^n}} \subset \overline{U_{\frac{2}{2^n}}} \subset U_{\frac{3}{2^n}} \subset \overline{U_{\frac{3}{2^n}}} \subset \cdots \subset$$
$$\subset U_{\frac{2^n-1}{2^n}} \subset \overline{U_{\frac{2^n-1}{2^n}}} \subset U_{\frac{2^n}{2^n}} = U_1.$$

(b) Let $I = \{\frac{m}{2^n} \mid m, n \in \mathbb{N}; 0 \leq m \leq 2^n\}$. We have a family of open sets $\{U_r \mid r \in I\}$ having the property $r < s \Rightarrow \overline{U_r} \subset U_s$. We can check that I is dense in $[0,1]$ (this is really the same thing as that any real number in $[0,1]$ can be written in binary form, compare 1.14).

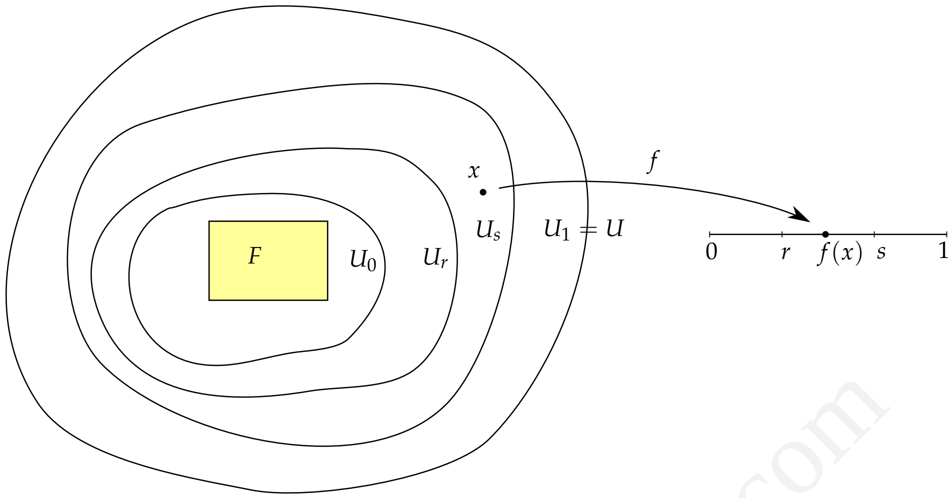
(c) Define $f : X \to [0,1]$,

$$f(x) = \begin{cases} \inf\{r \in I \mid x \in U_r\} & \text{if } x \in U, \\ 1 & \text{if } x \notin U. \end{cases}$$

In this way if $x \in U_r$ then $f(x) \leq r$, while if $x \notin U_r$ then $f(x) \geq r$. So $f(x)$ gives the "level" of x on the scale from 0 to 1, while U_r is like a sublevel set of f.

We prove that f is continuous, so f is the function we are looking for. It is enough to prove that sets of the form $\{x \mid f(x) < a\}$ and $\{x \mid f(x) > a\}$ are open.

(d) If $a \leq 1$ then $f(x) < a$ if and only if there is $r \in I$ such that $r < a$ and $x \in U_r$. Thus $\{x \mid f(x) < a\} = \{x \in U_r \mid r < a\} = \bigcup_{r<a} U_r$ is open.



(e) If $a < 1$ then $f(x) > a$ if and only if there is $r \in I$ such that $r > a$ and $x \notin U_r$. Thus $\{x \mid f(x) > a\} = \{x \in X \setminus U_r \mid r > a\} = \bigcup_{r>a} X \setminus U_r$.

Now we show that $\bigcup_{r>a} X \setminus U_r = \bigcup_{r>a} X \setminus \overline{U_r}$, which implies that $\bigcup_{r>a} X \setminus U_r$ is open. Indeed, if $r \in I$ and $r > a$ then there is $s \in I$ such that $r > s > a$. Then $\overline{U_s} \subset U_r$, therefore $X \setminus U_r \subset X \setminus \overline{U_s}$.

□

Partition of unity. An important application of the Urysohn lemma is the existence of *partion of unity* (phân hoạch đơn vị).

If $f : X \to \mathbb{R}$ then the *support* of f, denoted by $\text{supp}(f)$ is defined to be the closure of the subset $\{x \in X \mid f(x) \neq 0\}$.

THEOREM 8.2 (partition of unity). *Let X be a a normal space. Suppose that X has a finite open cover O. Then there is a collection of continuous maps $(f_U : X \to [0,1])_{U \in O}$ such that $\text{supp}(f_U) \subset U$ and for every $x \in X$ we have $\sum_{U \in O} f_U(x) = 1$.*

PROOF. From problem 5.14, there is an open cover $(U''_U)_{U \in O}$ of X such that for each $U \in O$ there is an open U'_U satisfying $U''_U \subset \overline{U''_U} \subset U'_U \subset \overline{U'_u} \subset U$. By Urysohn lemma there is a continuous map $\varphi_U : X \to [0,1]$ such that $\varphi_U|_{U''_U} = 1$ and $\varphi_U|_{X \setminus U'_U} = 0$. This implies $\text{supp}(\varphi_U) \subset \overline{U'_u} \subset U$. For each $x \in X$ there is $U \in O$ such that U''_U contains x, therefore $\varphi_U(x) = 1$. Let

$$f_U = \frac{\varphi_U}{\sum_{U \in O} \varphi_U}.$$

□

Partition of unity allows us to extend some local properties to global ones, by "patching" neighborhoods. It is needed for such important results as the existence of a Riemannian metric on a manifold or the definition of integration on manifolds.

The compact-open topology. Let X and Y be two topological spaces. We say that a net $(f_i)_{i \in I}$ of functions from X to Y *converges point-wise* to a function $f : X \to Y$ if for each $x \in X$ the net $(f_i(x))_{i \in I}$ converges to $f(x)$.

Now let Y be a metric space. Recall that a function $f : X \to Y$ is said to be *bounded* if the set of values $f(X)$ is a bounded subset of Y. We consider the set $B(X,Y)$ of all bounded functions from X to Y. If $f,g \in B(X,Y)$ then we define a metric on $B(X,Y)$ by $d(f,g) = \sup\{d(f(x),g(x)) \mid x \in X\}$. The topology generated by this metric is called the *topology of uniform convergence*. If a net $(f_i)_{i\in I}$ converges to f in the metric space $B(X,Y)$ then we say that $(f_i)_{i\in I}$ converges to f *uniformly*.

PROPOSITION. *Suppose that $(f_i)_{i\in I}$ converges to f uniformly. Then:*

(a) *$(f_i)_{i\in I}$ converges to f point-wise.*
(b) *If each f_i is continuous then f is continuous.*

PROOF. The proof of the second statement is the same as the proof for metric spaces. Suppose that each f_i is continuous. Let $x \in X$, we prove that f is continuous at x. The key step is the following inequality:

$$d(f(x),f(y)) \leq d(f(x),f_i(x)) + d(f_i(x),f_i(y)) + d(f_i(y),f(y)).$$

Given $\epsilon > 0$, fix an $i \in I$ such that $d(f_i,f) < \epsilon$. For this i, there is a neighborhood U of x such that if $y \in U$ then $d(f_i(x),f_i(y)) < \epsilon$. The above inequality implies that for $y \in U$ we have $d(f(x),f(y)) < 3\epsilon$. $\square$

DEFINITION. Let X and Y be two topological spaces. Let $C(X,Y)$ be the set of all continuous functions from X to Y. The topology generated by all sets of the form

$$S(A,U) = \{f \in C(X,Y) \mid f(A) \subset U\}$$

where $A \subset X$ is compact and $U \subset Y$ is open is called the *compact-open topology* on $C(X,Y)$.

PROPOSITION. *If X is compact and Y is a metric space then on $C(X,Y)$ the compact-open topology is the same as the uniform convergence topology.*

This says that the compact-open topology is a generalization of the uniform convergence topology to topological spaces.

PROOF. Given $f \in C(X,Y)$ and $\epsilon > 0$, we show that the ball $B(f,\epsilon) \subset C(X,Y)$ in the uniform metric contains an open neighborhood of f in the compact-open topology. For each $x \in X$ there is an open set U_x containing x such that $f(U_x) \subset B(f(x),\epsilon/3)$. Since X is compact, there are finitely many x_i, $1 \leq i \leq n$, such that $\bigcup_{i=1}^n U_{x_i} \supset X$ and $f(U_{x_i}) \subset B(f(x_i),\epsilon/3)$. Then

$$f \in \bigcap_{i=1}^{n} S(\overline{U}_{x_i}, B(f(x_i),\epsilon/2)) \subset B(f,\epsilon).$$

In the opposite direction, we need to show that every open neighborhood of f in the compact-open topology contains a ball $B(f,\epsilon)$ in the uniform metric. It is suffcient to show that for open neighborhood of f of the form $S(A,U)$. For each $x \in A$ there is a ball $B(f(x),\epsilon_x) \subset U$. Since $f(A)$ is compact, there are

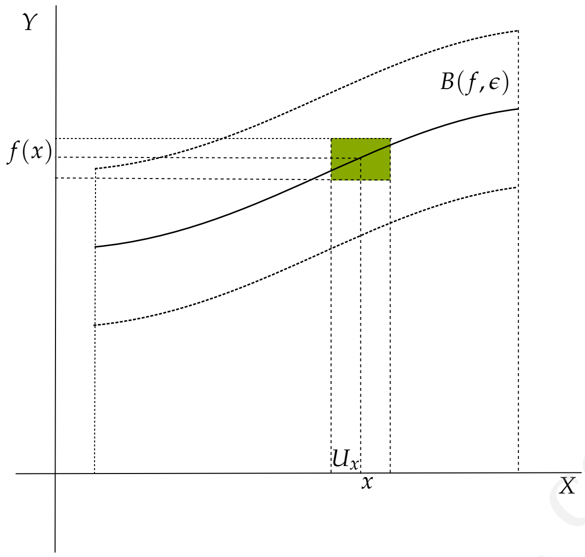


FIGURE 8.3. Notice that $f \in S(A,U)$ means $\mathrm{graph}(f|_A) \subset A \times U$.

finitely many $x_i \in A$ and $\epsilon_i > 0$, $1 \le i \le n$, such that $B(f(x_i),\epsilon_i) \subset U$ and $\bigcup_{i=1}^n B(f(x_i),\epsilon_i/2) \supset f(A)$. Let $\epsilon = \min\{\epsilon_i/2 \mid 1 \le i \le n\}$. Suppose that $g \in B(f,\epsilon)$. For each $x \in A$, there is an i such that $f(x) \in B(f(x_i),\epsilon_i/2)$. Then

$$d(g(x),f(x_i)) \le d(g(x),f(x_i)) + d(f(x_i),f(x)) < \epsilon + \frac{\epsilon_i}{2} \le \epsilon_i,$$

so $g(x) \in U$. Thus $g \in S(A,U)$. $\square$

Tiestze extension theorem. Below is another application of Urysohn lemma.

THEOREM (Tiestze extension theorem). *Let X be a normal space. Let F be closed in X. Let $f : F \to \mathbb{R}$ be continuous. Then there is a continuous map $g : X \to \mathbb{R}$ such that $g|_F = f$.*

Thus *in a normal space a continuous real function on a closed subspace can be extended continuously to the whole space*.

PROOF. First consider the case where f is bounded.

(a) The general case can be reduced to the case when $\inf_F f = 0$ and $\sup_F f = 1$. We will restrict our attention to this case.

(b) By Urysohn lemma, there is a continuous function $g_1 : X \to [0, \frac{1}{3}]$ such that

$$g_1(x) = \begin{cases} 0 & \text{if } x \in f^{-1}([0,\frac{1}{3}]) \\ \frac{1}{3} & \text{if } x \in f^{-1}([\frac{2}{3},1]). \end{cases}$$

Let $f_1 = f - g_1$. Then $\sup_X g_1 = \frac{1}{3}$, $\sup_F f_1 = \frac{2}{3}$, and $\inf_F f_1 = 0$.

(c) Inductively, once we have a function $f_n : F \to \mathbb{R}$, for a certain $n \geq 1$ we will obtain a function $g_{n+1} : X \to [0, \frac{1}{3}\left(\frac{2}{3}\right)^n]$ such that

$$g_{n+1}(x) = \begin{cases} 0 & \text{if } x \in f_n^{-1}([0, \frac{1}{3}\left(\frac{2}{3}\right)^n]) \\ \frac{1}{3}\left(\frac{2}{3}\right)^n & \text{if } x \in f_n^{-1}([\left(\frac{2}{3}\right)^{n+1}, \left(\frac{2}{3}\right)^n]). \end{cases}$$

Let $f_{n+1} = f_n - g_{n+1}$. Then $\sup_X g_{n+1} = \frac{1}{3}\left(\frac{2}{3}\right)^n$, $\sup_F f_{n+1} = \left(\frac{2}{3}\right)^{n+1}$, and $\inf_F f_{n+1} = 0$.

(d) The series $\sum_{n=1}^{\infty} g_n$ converges uniformly to a continuous function g.

(e) Since $f_n = f - \sum_{i=1}^{n} g_i$, the series $\sum_{n=1}^{n} g_n|_F$ converges uniformly to f. Therefore $g|_F = f$.

(f) Note that with this construction $\inf_X g = 0$ and $\sup_X g = 1$.

Now consider the case when f is not bounded.

(a) Suppose that f is neither bounded from below nor bounded from above. Let h be a homeomorphism from $(-\infty, \infty)$ to $(0, 1)$. Then the range of $f_1 = h \circ f$ is a subset of $(0, 1)$, therefore it can be extended as in the previous case to a continuous function g_1 such that $\inf_{x \in X} g_1(x) = \inf_{x \in F} f_1(x) = 0$ and $\sup_{x \in X} g_1(x) = \sup_{x \in F} f_1(x) = 1$.

If the range of g_1 includes neither 0 nor 1 then $g = h^{-1} \circ g_1$ will be the desired function.

It may happens that the range of g_1 includes either 0 or 1. In this case let $C = g_1^{-1}(\{0, 1\})$. Note that $C \cap F = \emptyset$. By Urysohn lemma, there is a continuous function $k : X \to [0, 1]$ such that $k|_C = 0$ and $k|_F = 1$. Let $g_2 = kg_1 + (1 - k)\frac{1}{2}$. Then $g_2|_F = g_1|_F$ and the range of g_2 is a subset of $(0, 1)$ ($g_2(x)$ is a certain convex combination of $g_1(x)$ and $\frac{1}{2}$). Then $g = h^{-1} \circ g_2$ will be the desired function.

(b) If f is bounded from below then similarly to the previous case we can use a homeomorphism $h : [a, \infty) \to [0, 1)$, and we let $C = g_1^{-1}(\{1\})$.

The case when f is bounded from above is similar.

□

Problems.

8.4. Show that a normal space is completely regular. So: normal ⇒ completely regular ⇒ regular. In other words: $T_4 \Rightarrow T_{3\frac{1}{2}} \Rightarrow T_3$.

8.5. Show that a space is completely regular if and only if it is homeomorphic to a subspace of a compact Hausdorff space. As a corollary, a locally compact Hausdorff space is completely regular.

8.6. Prove the following version of Urysohn lemma, as stated in [**Rud86**]. Suppose that X is a locally compact Hausdorff space, V is open in X, $K \subset V$, and K is compact. Then there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ for $x \in K$ and $\mathrm{supp}(f) \subset V$.

8.7. Show that the Tiestze extension theorem implies the Urysohn lemma.

8.8. The Tiestze extension theorem is not true without the condition that the set F is closed.

8.9. Show that the Tiestze extension theorem can be extended to maps to the space $\prod_{i\in I}\mathbb{R}$ where $\mathbb{R}$ has the Euclidean topology.

8.10. Let X be a normal space and F be a closed subset of X. Then any continuous map $f : F \to S^n$ can be extended to an open set containing F.

8.11 (point-wise convergence topology). Now we view a function from X to Y as an element of the set $Y^X = \prod_{x\in X} Y$. In this view a function $f : X \to Y$ is an element $f \in Y^X$, and for each $x \in X$ the value $f(x)$ is the x-coordinate of the element f.

(a) Let $(f_i)_{i\in I}$ be a net of functions from X to Y, i.e. a net of points in Y^X. Show that $(f_i)_{i\in I}$ converges to a function $f : X \to Y$ point-wise if and only if the net of points $(f_i)_{i\in I}$ converges to the point f in the product topology of Y^X.

(b) Define the *point-wise convergence topology* on the set Y^X of functions from X to Y as the topology generated by sets of the form

$$S(x, U) = \{f \in Y^X \mid f(x) \in U\}$$

with $x \in X$ and $U \subset Y$ is open. Show that the point-wise convergence topology is exactly the product topology on Y^X.

8.12. Let X and Y be two topological spaces. Let $C(X,Y)$ be the set of all continuous functions from X to Y. Show that if a net $(f_i)_{i\in I}$ converges to f in the compact-open topology of $C(X,Y)$ then it converges to f point-wise.

8.13 (continuity of functions of two variables). * Let Y, Z be topological spaces and let X be a locally compact Hausdorff space. Recall that it is not true that a map on a product space is continuous if it is continuous on each variable. Prove that a map

$$\begin{aligned} f : X \times Z &\to Y \\ (x,t) &\mapsto f(x,t) \end{aligned}$$

is continuous if and only if the map

$$\begin{aligned} f_t : X &\to Y \\ x &\mapsto f_t(x) = f(x,t) \end{aligned}$$

is continuous for each $t \in Z$ and the map

$$\begin{aligned} Z &\to C(X,Y) \\ t &\mapsto f_t \end{aligned}$$

is continuous with the compact-open topology on $C(X,Y)$.

8.14 (Niemytzki space). * Let $\mathbb{H} = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$ be the upper half-plane. Equip $\mathbb{H}$ with the topology generated by the Euclidean open disks (i.e. open balls) in $K = \{(x,y) \in \mathbb{R}^2 | \ y > 0\}$, together with sets of the form $\{p\} \cup D$ where p is a point on the line $L = \{(x,y) \in \mathbb{R}^2 | \ y = 0\}$ and D is an open disk in K tangent to L at p. This is called the *Niemytzki space*.

(a) Check that this is a topological space.

(b) What is the subspace topology on L?

(c) What are the closed sets in $\mathbb{H}$?
(d) Show that $\mathbb{H}$ is Hausdorff.
(e) Show that $\mathbb{H}$ is regular.
(f) Show that $\mathbb{H}$ is not normal.

9. Quotient space

We consider the operation of gluing parts of a space to form a new space. For example when we glue the two endpoints of a line segment together we get a circle.

Mathematically speaking, gluing elements mean to let them be equivalent, to identify them as one. For a set X and an equivalence relation $\sim$ on X, the quotient set $X/\sim$ is exactly what we need.

We also want the gluing to be continuous. That means when X is a topological space we equip the quotient set $X/\sim$ with a topology such that the gluing map $p : X \to X/\sim, x \mapsto [x]$ is continuous. Namely, a subset U of $X/\sim$ is open if and only if the preimage $p^{-1}(U)$ is open in X (see 3.8). The set $X/\sim$ with this topology is called the *quotient space* of X by the equivalence relation.

In a special case, if A is a subspace of X then there is this equivalence relation on X: $x \sim x$ if $x \in X$, and $x \sim y$ if $x, y \in A$. The quotient space $X/\sim$ is often written as X/A, and we can think of it as being obtained from X by collapsing the whole subset A to one point.

PROPOSITION 9.1. *Let Y be a topological space. A map $f : X/\sim \to Y$ is continuous if and only if $f \circ p$ is continuous.*

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[dr, "f\circ p"'] & X/\sim \arrow[d, "f"] \\ & Y \end{tikzcd}$$

PROOF. The map $f \circ p$ is continuous if and only if for each open subset U of Y, the set $(f \circ p)^{-1}(U) = p^{-1}(f^{-1}(U))$ is open in X. The latter statement is equivalent to that $f^{-1}(U)$ is open for every U, that is, f is continuous. □

The following result will provide us a tool for identifying quotient spaces:

THEOREM. *Suppose that X is compact and $\sim$ is an equivalence relation on X. Suppose that Y is Hausdorff, and $f : X \to Y$ is continuous and onto. Suppose that $f(x_1) = f(x_2)$ if and only if $x_1 \sim x_2$. Then f induces a homeomorphism from $X/\sim$ onto Y.*

PROOF. Define $h : X/\sim \to Y$ by $h([x]) = f(x)$. Then h is onto and is injective, thus it is a bijection.

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[dr, "f"'] & X/\sim \arrow[d, "h"] \\ & Y \end{tikzcd}$$

Notice that $f = h \circ p$ (in such a case people often say that the above diagram is *commutative*, and that the map f can be factored). By 9.1 h is continuous. By 6.10, h is a homeomorphism. □

EXAMPLE (gluing the two end-points of a line segment gives a circle). More precisely $[0,1]/0 \sim 1$ (see 9.24) is homeomorphic to S^1:

$$\begin{tikzcd}
{[0,1]} \arrow[r, "p"] \arrow[dr, "f"'] & {[0,1]/0 \sim 1} \arrow[d, "h"] \\
& S^1
\end{tikzcd}$$

Here f is the map $t \mapsto (\cos(2\pi t), \sin(2\pi t))$. The map f is continuous, onto, and it fails to be injective only at $t = 0$ and $t = 1$. Since in the quotient set 0 and 1 are identified, the induced map h on the quotient set becomes a bijection. The above theorem allows us to check that h is a homeomorphism.

EXAMPLE (gluing a pair of opposite edges of a square gives a cylinder). Let $X = [0,1] \times [0,1]/\sim$ where $(0,t) \sim (1,t)$ for all $0 \leq t \leq 1$. Then X is homeomorphic to the cylinder $[0,1] \times S^1$. The homeomorphism is induced by the map $(s,t) \mapsto (s, \cos(2\pi t), \sin(2\pi t))$.

EXAMPLE (gluing opposite edges of a square gives a torus). Let $X = [0,1] \times [0,1]/\sim$ where $(s,0) \sim (s,1)$ and $(0,t) \sim (1,t)$ for all $0 \leq s,t \leq 1$, then X is homeomorphic to the *torus*[10] (mặt xuyến) T^2.

FIGURE 9.2. The torus.

The torus T^2 is homeomorphic to a subspace of $\mathbb{R}^3$, in other words, the torus can be embedded in $\mathbb{R}^3$. A of subspace $\mathbb{R}^3$ homeomorphic to T^2 can be obtained as the surface of revolution obtained by revolving a circle around a line not intersecting it.

Suppose that the circle is on the Oyz-plane, the center is on the y-axis and the axis for the rotation is the z-axis. Let a be the radius of the circle, b be the distance from the center of the circle to O, ($a < b$). Let S be the surface of revolution, then the embedding can be given by

$$\begin{tikzcd}
{[0,2\pi] \times [0,2\pi]} \arrow[r] \arrow[dr, "f"'] & T^2 \arrow[d, "h"] \\
& S
\end{tikzcd}$$

where $f(\phi,\theta) = ((b + a\cos\theta)\cos\phi, (b + a\cos\theta)\sin\phi, a\sin\theta)$.

[10]The plural form of the word torus is tori.

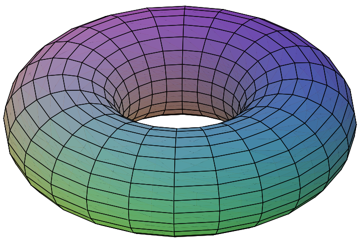
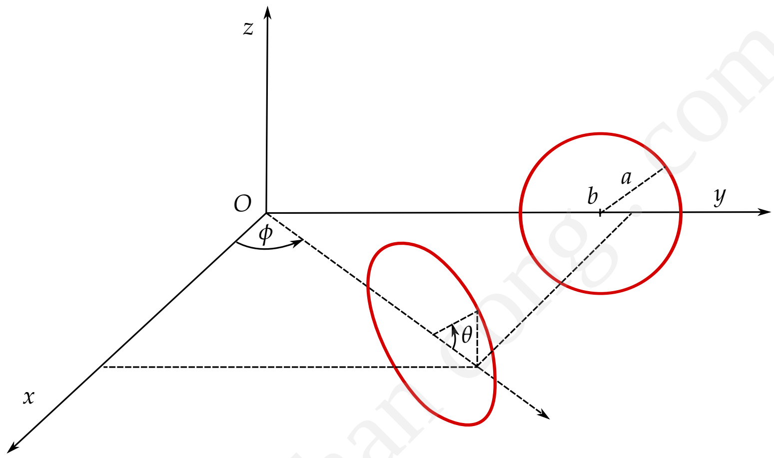
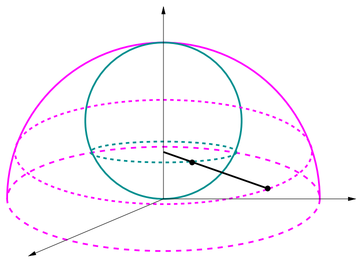


FIGURE 9.3. The torus embedded in $\mathbb{R}^3$.



EXAMPLE (gluing the boundary circle of a disk together gives a sphere). More precisely $D^2/\partial D^2$ is homeomorphic to S^2. We only need to construct a continuous map from D^2 onto S^2 such that after quotient out by the boundary ∂D^2 it becomes injective.



EXAMPLE 9.4 (the Mobius band). Gluing a pair of opposite edges of a square in opposite directions gives the *Mobius band* (dải, lá, mặt Mobius[11]). More precisely the Mobius band is $X = [0,1] \times [0,1]/\sim$ where $(0,t) \sim (1, 1-t)$ for all $0 \le t \le 1$.

The Mobius band could be embedded in $\mathbb{R}^3$. It is homeomorphic to a subspace of $\mathbb{R}^3$ obtained by rotating a straight segment around the z-axis while also turning that segment "up side down". The embedding can be induced by the map (see

[11]Möbius or Moebius are other spellings for this name.

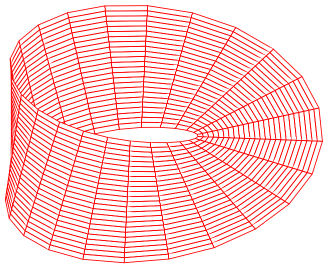


FIGURE 9.5. The Mobius band embedded in $\mathbb{R}^3$.

Figure 9.6)

$$(s,t) \mapsto ((a + t\cos(s/2))\cos s, (a + t\cos(s/2))\sin s, t\sin(s/2)),$$

with $0 \le s \le 2\pi$ and $-1 \le t \le 1$.

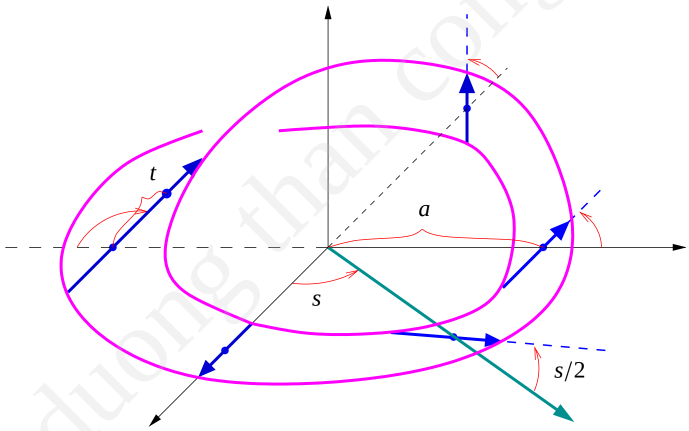


FIGURE 9.6. The embedding of the Mobius band in $\mathbb{R}^3$.

The Mobius band is famous as an example of *unorientable* surfaces. It is also *one-sided*. A proof is in 25.2.

EXAMPLE (the projective plane). Identifying opposite points on the boundary of a disk (they are called *antipodal points*) we get a topological space called the *projective plane* (mặt phẳng xạ ảnh) $\mathbb{RP}^2$. The real projective plane cannot be embedded in $\mathbb{R}^3$. It can be embedded in $\mathbb{R}^4$.

More generally, identifying antipodal boundary points of D^n gives us *the projective space* (không gian xạ ảnh) $\mathbb{RP}^n$. With this definition $\mathbb{RP}^1$ is homeomorphic to S^1. See also 9.19.

EXAMPLE (gluing a disk to the Mobius band gives the projective plane). In other words, deleting a disk from the projective plane gives the Mobius band. See Figure 9.7

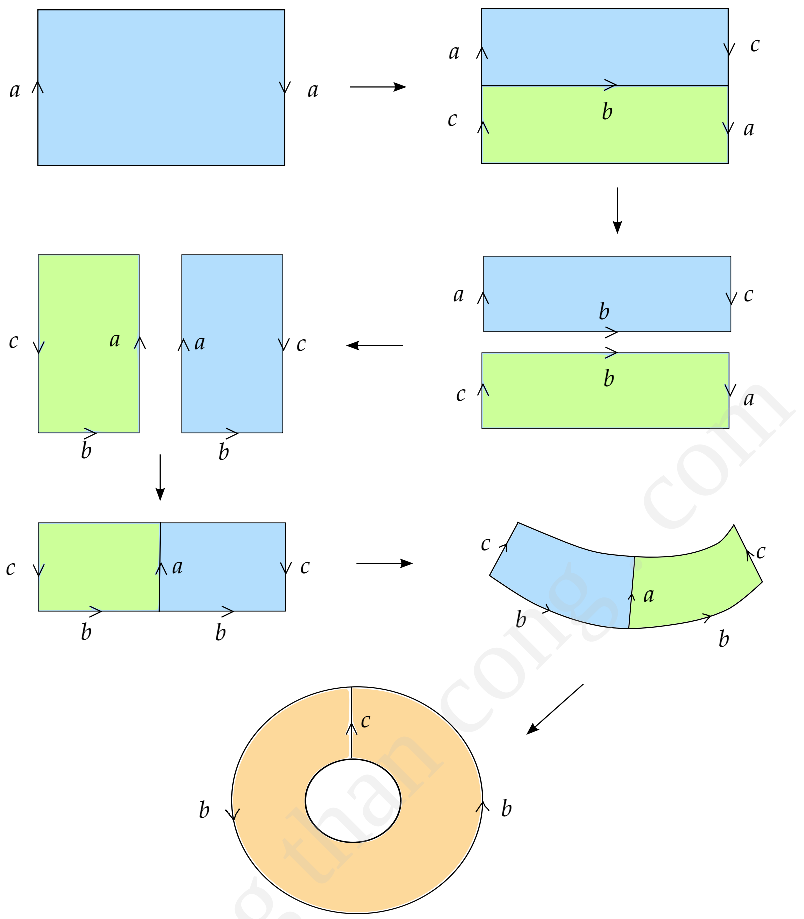


FIGURE 9.7. Gluing a disk to the Mobius band gives the projective plane.

EXAMPLE (the Klein bottle). Identifying one pair of opposite edges of a square and the other pair in opposite directions gives a topological space called the *Klein bottle.* More precisely it is $[0,1] \times [0,1]/\sim$ with $(0,t) \sim (1,t)$ and $(s,0) \sim (1-s,1)$.

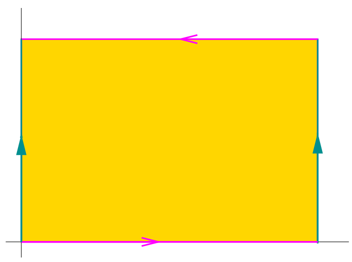


FIGURE 9.8. The Klein bottle.

This space cannot be embedded in $\mathbb{R}^3$, but it can be *immersed* in $\mathbb{R}^3$. An *immersion* (phép nhúng chìm) is a local embedding. More concisely, $f : X \to Y$ is an immersion if each point in X has a neighborhood U such that $f|_U : U \to f(U)$ is a homeomorphism. Intuitively, an *immersion allows self-intersection* (tự cắt).

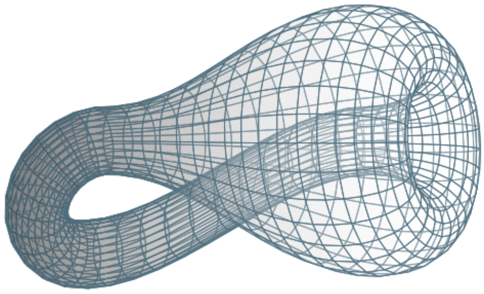


FIGURE 9.9. The Klein bottle immersed in $\mathbb{R}^3$.

EXAMPLE 9.10 (the three-dimensional torus). Consider a cube $[0,1]^3$. Identifying opposite faces by $(x,y,0) \sim (x,y,1)$, $(x,0,z) \sim (x,1,z)$, $(0,y,z) \sim (1,y,z)$ we get a space called the *three-dimensional torus*.

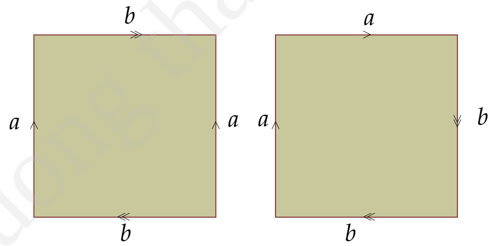
Problems.

9.11. Describe the space $[0,1]/0 \sim \frac{1}{2} \sim 1$.

9.12. Describe the space that is the quotient of the sphere S^2 by its equator S^1.

9.13. Show that the torus T^2 is homeomorphic to $S^1 \times S^1$.

9.14. Show that the following spaces are homeomorphic (one of them is the Klein bottle).



9.15. What do we obtain after we cut a Mobius band along its middle circle? Try it with an experiment.

To cut a subset S from a space X means to delete S from X, the resulting space is the subspace $X \setminus S$. In figure 9.15 the curve CC' is deleted.

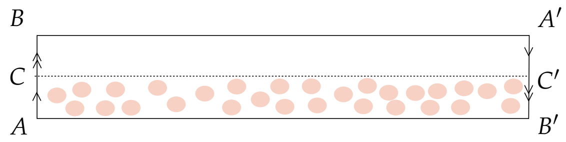
9.17. If X is connected then $X/\sim$ is connected.

9.18. The one-point compactification of the open Mobius band (the Mobius band without the boundary circle) is the projective space $\mathbb{R}P^2$.

9.19. * Show that identifying antipodal boundary points of D^n is equivalent to identifying antipodal points of S^n. In other words, the projective space $\mathbb{R}P^n$ is homeomorphic to $S^n/x \sim -x$.

9.20. Show that the projective space $\mathbb{R}P^n$ is a Hausdorff space.

9.21. In order for the quotient space $X/\sim$ to be a Hausdorff space, a necessary condition is that each equivalence class $[x]$ must be a closed subset of X. Is this condition sufficient?



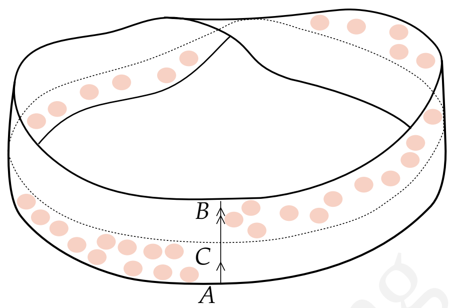


FIGURE 9.16. Cutting a Mobius band along the middle circle.

9.22. On the Euclidean $\mathbb{R}$ define $x \sim y$ if $x - y \in \mathbb{Z}$. Show that $\mathbb{R}/\sim$ is homeomorphic to S^1. The space $\mathbb{R}/\sim$ is also described as "$\mathbb{R}$ quotiented by the action of the group $\mathbb{Z}$".

9.23. On the Euclidean $\mathbb{R}^2$, define $(x_1, y_1) \sim (x_2, y_2)$ if $(x_1 - x_2, y_1 - y_2) \in \mathbb{Z} \times \mathbb{Z}$. Show that $\mathbb{R}^2/\sim$ is homeomorphic to T^2.

9.24 (minimal equivalence relation). Given a set X and a set $Y \subset X \times X$, show that there is an equivalence relation on X that contains Y and is contained in every equivalence relation that contains Y, called the *minimal equivalence relation* containing Y.

For example, when we write $[0,1]/0 \sim 1$ we mean the quotient of the set $[0,1]$ by the minimal equivalence relation on $[0,1]$ such that $0 \sim 1$. In this case that minimal equivalence relation is clearly $\{(0,1),(1,0),(x,x) \mid x \in [0,1]\}$.

9.25. * A question can be raised: In quotient spaces, if identifications are carried out in steps rather than simultaneously, will the results be different? More precisely, let R_1 and R_2 be two equivalence relations on a space X and let R be the minimal equivalence relation containing $R_1 \cup R_2$ is also an equivalence relation on space X. On the space X/R_1 we define an equivalence relation $\tilde{R}_2$ induced from R_2 by: $[x]_{R_1} \sim_{\tilde{R}_2} [y]_{R_1}$ if $(x \sim_{R_1 \cup R_2} y)$. Prove that the map

$$\begin{aligned} X/(R_1 \cup R_2) &\to (X/R_1)/\tilde{R}_2 \\ [x]_{R_1 \cup R_2} &\mapsto [[x]_{R_1}]_{\tilde{R}_2} \end{aligned}$$

is a homeomorphism. Thus in this sense the results are same.

Other topics

Below are several more advanced topics. Though we do not present them in detail, we think it is useful for the reader to have some familiarity with them. At the end is a guide for further reading.

Invariance of dimension. That the Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic is not easy. It is a consequence of the following difficult theorem of L. Brouwer in 1912:

THEOREM 9.26 (invariance of dimension). *If two subsets of the Euclidean $\mathbb{R}^n$ are homeomorphic and one set is open then the other is also open.*

This theorem is often proved using Algebraic Topology, see for instance [**Mun00**, p. 381], [**Vic94**, p. 34], [**Hat01**, p. 126].

COROLLARY. *The Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic if $m \neq n$.*

PROOF. Suppose that $m < n$. It is easy to check that the inclusion map $\mathbb{R}^m \to \mathbb{R}^n$, $(x_1, x_2, \ldots, x_m) \mapsto (x_1, x_2, \ldots, x_m, 0, \ldots, 0)$ is a homeomorphism onto its image $A \subset \mathbb{R}^n$. If A is homeomorphic to $\mathbb{R}^n$ then by Invariance of dimension, A is open in $\mathbb{R}^n$. But A is clearly not open in $\mathbb{R}^n$. $\square$

This result allows us to talk about topological dimension.

Jordan curve theorem. The following is an important and deep result of plane topology.

THEOREM (Jordan curve theorem). *A simple, continuous, closed curve separates the plane into two disconnected regions. More concisely, if C is a subset of the Euclidean plane homeomorphic to the circle then $\mathbb{R}^2 \setminus C$ has two connected components.*

Nowadays this theorem is usually proved in a course in Algebraic Topology.

Space filling curves. A rather curious and surprising result is:

THEOREM. *There is a continuous curve filling a rectangle on the plane. More concisely, there is a continuous map from the interval $[0,1]$ onto the square $[0,1]^2$ under the Euclidean topology.*

Note that this map cannot be injective, in other words the curve cannot be simple.

Such a curve is called a *Peano curve*. It could be constructed as a limit of an iteration of piecewise linear curves.

Strategy for a proof of Tikhonov theorem based on net. The proof that we will outline here is based on further developments of the theory of nets and a characterization of compactness in terms of nets.

DEFINITION (subnet). Let I and I' be directed sets, and let $h : I' \to I$ be a map such that

$$\forall k \in I, \exists k' \in I', (i' \geq k' \Rightarrow h(i') \geq k).$$

If $n : I \to X$ is a net then $n \circ h$ is called a *subnet* of n.

The notion of subnet is an extension of the notion of subsequence. If we take $n_i \in \mathbb{Z}^+$ such that $n_i < n_{i+1}$ then (x_{n_i}) is a subsequence of (x_n). In this case the map $h : \mathbb{Z}^+ \to \mathbb{Z}^+$ given by $h(i) = n_i$ is a strictly increasing function. Thus a subsequence of a sequence is a subnet of that sequence. On the other hand a subnet of a sequence does not need to be a subsequence, since for a subnet the map h is only required to satisfy $\lim_{i\to\infty} h(i) = \infty$.

A net $(x_i)_{i\in I}$ is called *eventually* in $A \subset X$ if there is $j \in I$ such that $i \geq j \Rightarrow x_i \in A$.

Universal net A net n in X is *universal* if for any subset A of X either n is eventually in A or n is eventually in $X \setminus A$.

PROPOSITION. *If $f : X \to Y$ is continuous and n is a universal net in X then $f(n)$ is a universal net.*

PROPOSITION. *The following statements are equivalent:*

(a) *X is compact.*
(b) *Every universal net in X is convergent.*
(c) *Every net in X has a convergent subnet.*

The proof of the two propositions above could be found in [**Bre93**].

Then we finish the proof of Tikhonov theorem as follows.

PROOF OF TIKHONOV THEOREM. Let $X = \prod_{i\in I} X_i$ where each X_i is compact. Suppose that $(x_j)_{j\in J}$ is a universal net in X. By 7.3 the net (x_j) is convergent if and only if the projection $(p_i(x_j))$ is convergent for all i. But that is true since $(p_i(x_j))$ is a universal net in the compact set X_i. $\square$

Metrizability.

THEOREM 9.27 (Urysohn metrizability theorem). *A regular space with a countable basis is metrizable.*

The proof uses the Urysohn lemma [**Mun00**].

Guide for further reading. The book by Kelley [**Kel55**] has been a classics and a standard reference although it was published in 1955. Its presentation is rather abstract. The book contains no figure!

Munkres' book [**Mun00**] is presently a standard textbook. The treatment there is somewhat more modern than that in Kelley's book, with many examples, figures and exercises. It also has a section on Algebraic Topology.

Hocking and Young's book [**HY61**] contains many deep and difficult results. This book together with Kelley's and Munkres' books contain many topics not discussed in our lectures.

For General Topology as a service to Analysis, [**KF75**] is an excellent textbook. [**Cai94**] and [**VINK08**] are other good books on General Topology.

A more recent textbook by Roseman [**Ros99**] works mostly in $\mathbb{R}^n$ and is more down-to-earth. The newer textbook [**AF08**] contains many interesting recent applications of topology.

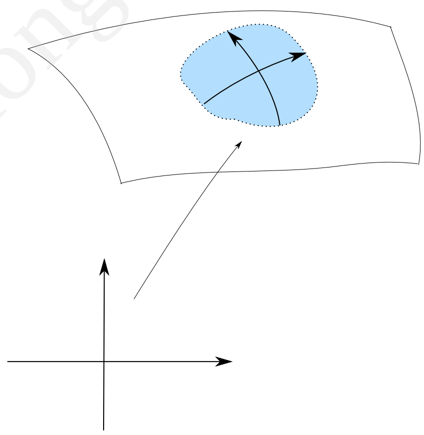
Algebraic Topology

10. Structures on topological spaces

Topological manifold . If we only stay around our small familiar neighborhood then we might not be able to recognize that the surface of the Earth is curved, and to us it is indistinguishable from a plane. When we begin to travel farther and higher, we can realize that the surface of the Earth is a sphere, not a plane. In mathematical language, a sphere and a plane are locally same but globally different.

Briefly, *a manifold is a space that is locally Euclidean*. In a manifold each element can be described by a list of independent parameters. The description can vary from one part to another part of the manifold. [12]

DEFINITION. A *topological manifold* (đa tạp tôpô) of dimension n is a topological space each point of which has a neighborhood homeomorphic to the Euclidean space $\mathbb{R}^n$.



We can think of a manifold as a space that could be covered by a collection of open subsets each of which homeomorphic to $\mathbb{R}^n$.

REMARK. In this chapter we assume $\mathbb{R}^n$ has the Euclidean topology unless we mention otherwise.

The statement below is often convenient in practice:

[12] Bernard Riemann proposed the idea of manifold in his Habilitation dissertation [**Spi99**].

PROPOSITION. *A manifold of dimension n is a space such that each point has an open neighborhood homeomorphic to an open subset of* $\mathbb{R}^n$.

PROOF. Let M be a space, $x \in M$, and suppose that x has an open neighborhood U in M with a homeomorphism $h : U \to V$ where V is an open subset of $\mathbb{R}^n$. We will show that x has a neighborhood homeomorphic to $\mathbb{R}^n$. There is a ball $B(f(x), \epsilon) \subset V$. Since h is a homeomorphism, $h^{-1}(B(f(x), \epsilon))$ is open in U, so is open in M. So x has an open neighborhood $h^{-1}(V)$ homeomorphic to a ball $B(f(x), \epsilon)$, which in turns homeomorphic to $\mathbb{R}^n$ (3.16). $\square$

REMARK. By Invariance of dimension (9.26), $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic unless $m = n$, therefore a manifold has a unique dimension.

EXAMPLE. Any open subspace of $\mathbb{R}^n$ is a manifold of dimension n.

EXAMPLE. Let $f : D \to \mathbb{R}$ be a continuous function where $D \subset \mathbb{R}^n$ is an open set, then the graph of f, the set $\{(x, f(x)) \mid x \in D\}$, as a subspace of $\mathbb{R}^{n+1}$, is homeomorphic to D, therefore is an n-dimensional manifold.

Thus manifolds generalizes curves and surfaces.

EXAMPLE. The sphere S^n is an n-dimensional manifold. One way to show this is by covering S^n with two neighborhoods $S^n \setminus \{(0, \ldots, 0, 1)\}$ and $S^n \setminus \{(0, \ldots, 0, -1)\}$. Each of these neighborhoods is homeomorphic to $\mathbb{R}^n$ via stereographic projections. Another way is by covering S^n by hemispheres $\{(x_1, x_2, \ldots, x_{n+1}) \in S^n \mid x_i > 0\}$ and $\{(x_1, x_2, \ldots, x_{n+1}) \in S^n \mid x_i < 0\}$, $1 \leq i \leq n+1$. Each of these hemispheres is a graph, homeomorphic to an open n-dimensional unit ball.

EXAMPLE. The torus is a two-dimensional manifold. Let us consider the torus as the quotient space of the square $[0, 1]^2$ by identifying opposite edges. Each point has a neighborhood homeomorphic to an open disk, as can be seen easily in the following figure, though explicit description would be time consuming. We can

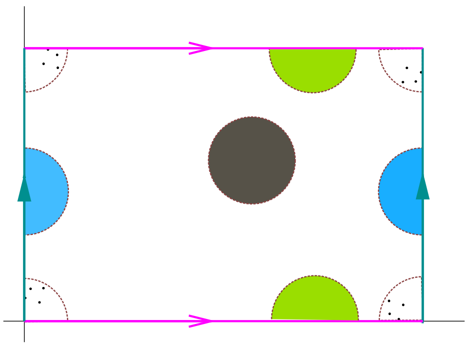


FIGURE 10.1. The sets with same colors are glued to form a neighborhood of a point on the torus. Each such neighborhood is homeomorphic to an open ball.

also view the torus as a surface in $\mathbb{R}^3$, given by the equation $(\sqrt{x^2 + y^2} - a)^2 +$

$z^2 = b^2$. As such it can be covered by the open subsets of $\mathbb{R}^3$ corresponding to $z > 0, z < 0, x^2 + y^2 < a^2, x^2 + y^2 > a^2$.

REMARK. The interval $[0,1]$ is not a manifold, it is a "manifold with boundary". We will not give a precise definition of manifold with boundary here.

A two-dimensional manifold is often called a *surface*.

Simplicial complex. For an integer $n \geq 0$, an n-dimensional *simplex* (đơn hình) is a subspace of a Euclidean space $\mathbb{R}^m$, $m \geq n$ which is the convex linear combination of $(n+1)$ points in $\mathbb{R}^m$ that do not belong to any n-dimensional hyperplane. As a set it is given by $\{t_0v_0 + t_1v_1 + \cdots + t_nv_n \mid t_0, t_1, \ldots, t_n \in [0,1],\ t_0 + t_1 + \cdots + t_n = 1\}$ where $v_0, v_1, \ldots, v_n \in \mathbb{R}^m$ and $v_1 - v_0, v_2 - v_0, \ldots, v_n - v_0$ are n linearly independent vectors (it can be checked that this condition does not depend on the order of the points). The points $v_0, v_1, \ldots, v_n$ are called the *vertices* of the simplex.

EXAMPLE. A 0-dimensional simplex is just a point. A 1-dimensional simplex is a straight segment in $\mathbb{R}^m$, $m \geq 1$. A 2-dimensional simplex is a triangle in $\mathbb{R}^m$, $m \geq 2$. A 3-dimensional simplex is a tetrahedron in $\mathbb{R}^m$, $m \geq 3$.

In particular, the *standard n-dimensional simplex* (đơn hình chuẩn) Δ_n is the convex linear combination of the $(n+1)$ vectors $(1,0,0,\ldots)$, $(0,1,0,0,\ldots)$, $\ldots$, $(0,0,\ldots,0,1)$ in the standard linear basis of $\mathbb{R}^{n+1}$. Thus

$$\Delta_n = \{(t_0, t_1, \ldots, t_n) \mid t_0, t_1, \ldots, t_n \in [0,1],\ t_0 + t_1 + \cdots + t_n = 1\}.$$

The convex linear combination of any subset of the set of vertices of a simplex is called a *face* of the simplex.

EXAMPLE. For a 2-dimensional simplex (a triangle) its faces are the vertices, the edges, and the triangle itself.

DEFINITION. An n-dimensional *simplicial complex* (phức đơn hình) in $\mathbb{R}^m$ is a finite collection S of simplexes in $\mathbb{R}^m$ of dimensions at most n and at least one simplex is of dimension n and such that:

(a) any face of an element of S is an element of S,
(b) the intersection of any two elements of S is a common face.

The union of all elements of S is called its *underlying space*, denoted by $|S|$, a subspace of $\mathbb{R}^m$. A space which is the underlying space of a simplicial complex is also called a *polyhedron* (đa diện).

EXAMPLE. A 1-dimensional simplicial complex is a graph.

Triangulation. A *triangulation* (phép phân chia tam giác) of a topological space X is a homeomorphism from the underlying space of a simplicial complex to X, the space X is then said to be *triangulated*.

For example, a triangulation of a surface is an expression of the surface as the union of finitely many triangles, with a requirement that two triangles are either disjoint, or have one common edge, or have one common vertex.

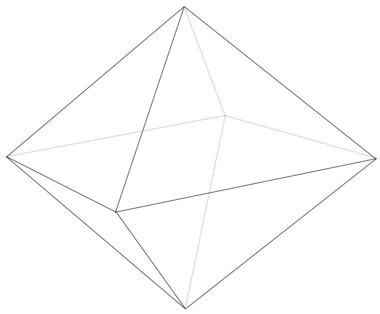


FIGURE 10.2. A triangulation of the 2-dimensional sphere.

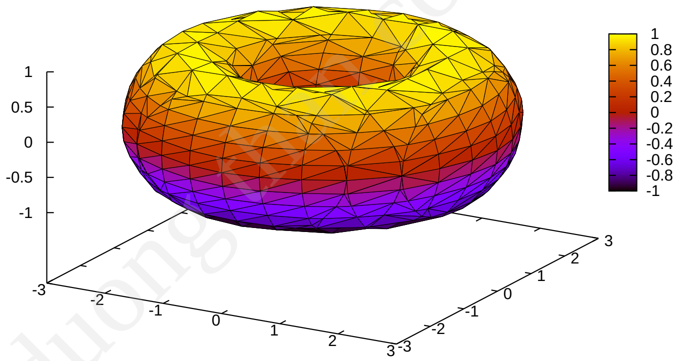


FIGURE 10.3. A triangulation of the torus.

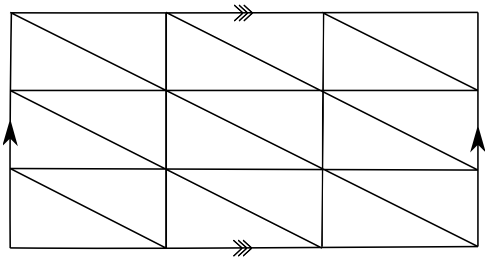


FIGURE 10.4. Description of a triangulation of the torus.

It is known that any two or three dimensional manifold can be triangulated, and that there exists a 4-dimensional manifold with no triangulation. The situations in higher dimensions are still being studied.

A simplicial complex is specified by a finite set of points, if a space can be triangulated then we can study that space combinatorially, using constructions and computations in finitely many steps.

Cell complex. For $n \geq 1$ a *cell* (ô) is an open ball in the Euclidean space $\mathbb{R}^n$. A 0-dimensional cell is a point (this is consistent with the general case with the convention that $\mathbb{R}^0 = \{0\}$).

Recall a familiar term that an n-dimensional *disk* is a closed ball in $\mathbb{R}^n$, in particular when $n = 0$ it is a point. The unit disk centered at the origin $B'(0,1)$ is denoted by D^n. Thus for $n \geq 1$ the boundary ∂D^n is the sphere S^{n-1} and the interior $\mathrm{int}(D^n)$ is an n-cell.

By *attaching a cell* to a topological space X we mean taking a continuous function $f : \partial D^n \to X$ then forming the quotient space $(X \sqcup D^n)/(x \sim f(x),\ x \in \partial D^n)$ (for disjoint union see 7.13). Intuitively, we attach a cell to the space by gluing each point on the boundary of the disk to a point on the space in a certain way.

We can attach finitely many cells to X in the same manner. Precisely, attaching k n-cells to X means taking the quotient space $\left(X \sqcup \left(\bigsqcup_{i=1}^k D_i^n\right)\right)/(x \sim f_i(x),\ x \in \partial D_i^n, 1 \leq i \leq k)$ where $f_i : \partial D_i^n \to X$ is continuous, $1 \leq i \leq k$.

DEFINITION. A (finite) n-dimensional *cell complex* (phức ô) or *CW-complex* X is a topological space built as follows:

(a) X^0 is a finite collection of 0-cells, with the discrete topology,
(b) for $1 \leq i \leq n \in \mathbb{Z}^+$, X^i is obtained by attaching finitely many i-cells to X^{i-1}, and $X^n = X$.

Briefly, a cell complex is a topological space with an instruction for building it by attaching cells. The subspaces X^i are called the i-dimensional *skeleton* (khung) of X.[13]

EXAMPLE. A topological circle has a cell complex structure as a triangle with three 0-cells and three 1-cells. There is another cell complex structure with only one 0-cell and one 1-cell.

EXAMPLE. The 2-dimensional sphere has a cell complex structure as a tetrahedron with four 0-cells, six 1-cells, and four 2-cells. There is another cell complex structure with only one 0-cell and one 2-cell.

EXAMPLE. The torus, as we can see directly from its definition (figure 9), has a cell complex structure with one 0-cells, two 1-cells, and one 2-cells.

A simplicial complex gives rise to a cell complex:

PROPOSITION. *Any triangulated space is a cell complex.*

[13]The term CW-complex is more general than the term cell-complex, can be used when there are infinitely many cells.

PROOF. Let X be a simplicial complex. Let X^i be the union of all simplexes of X of dimensions at most i. Then X^{i+1} is the union of X^i with finitely many $(i+1)$-dimensional simplexes. Let Δ^{i+1} be such an $(i+1)$-dimensional simplex. The i-dimensional faces of Δ^{i+1} are simplexes of X, so the union of those faces, which is the boundary of Δ^{i+1}, belongs to X^i. There is a homeomorphism from an $(i+1)$-disk to Δ^{i+1} (see 10.15), bringing the boundary of the disk to the boundary of Δ^{i+1}. Thus including Δ^{i+1} in X implies attaching an $(i+1)$-cell to X^i. $\square$

The example of the torus indicates that cell complexes may require less cells than simplicial complexes. On the other hand we loose the combinatorial setting, because we need to specify the attaching maps.

It is known that any compact manifold of dimension different from 4 has a cell complex structure. Whether that is true or not in dimension 4 is not known yet [**Hat01**, p. 529].

Euler characteristic.

DEFINITION. The *Euler characteristic* (đặc trưng Euler) of a cell complex is defined to be the alternating sum of the number of cells in each dimension of that cell complex. Namely, let X be an n-dimensional cell complex and let c_i, $0 \leq i \leq n$, be the number of i-dimensional cells of X, then the Euler characteristic of X is

$$\chi(X) = \sum_{i=0}^{n} (-1)^i c_i.$$

EXAMPLE. For a triangulated surface, its Euler characteristic is the number v of vertices minus the number e of edges plus the number f of triangles (faces):

$$\chi(S) = v - e + f.$$

THEOREM 10.5. *The Euler characteristic of homeomorphic cell complexes are equal.*

We will discuss a proof of this result in 17.

In particular two cell complex structures on a topological space have same Euler characteristic. The Euler characteristic is a *topological invariant* (bất biến tôpô).

We have $\chi(S^2) = 2$. A consequence is the famous formula of Leonhard Euler:

THEOREM (Euler's formula). *For any polyhedron homeomorphic to a 2-dimensional sphere, $v - e + f = 2$.*

EXAMPLE. The Euler characteristic of a surface is defined and does not depend on the choice of triangulation. If two surfaces have different Euler characteristic they are not homeomorphic.

From any triangulation of the torus, we get $\chi(T^2) = 0$. For the projective plane, $\chi(\mathbb{R}P^2) = 1$. As a consequence, the sphere, the torus, and the projective plane are not homeomorphic to each other: they are different surfaces.

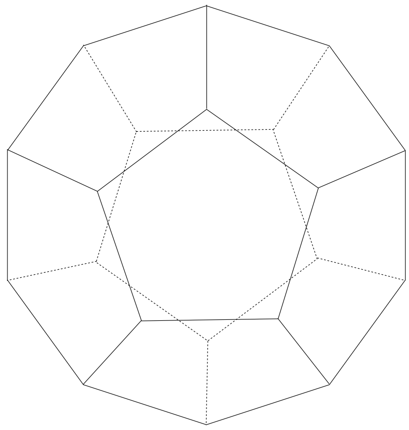


FIGURE 10.6. The Euler characteristic of the dodecahedron is 2.

Problems.

10.7. Show that if two spaces are homeomorphic and one space is an n-dimensional manifold then the other is also an n-dimensional manifold.

10.8. Show that an open subspace of a manifold is a manifold.

10.9. Show that if X and Y are manifolds then $X \times Y$ is also a manifold.

10.10. Show that a connected manifold must be path-connected. Thus for manifold connectedness and path-connectedness are same.

10.11. Show that $\mathbb{RP}^n$ is an n-dimensional topological manifold.

10.12. Show that a manifold is a locally compact space.

10.13. Show that the Mobius band, without its boundary circle, is a manifold.

10.14. Give examples of topological spaces which are not manifolds. Discuss necessary conditions for a topological space to be a manifold.

10.15. $\checkmark$ Show that any n-dimensional simplex is homeomorphic to an n-dimensional disk.

10.16. Give examples of different spaces (not homeomorphic) but with same Euler characteristic.

10.17. Give a triangulation for the Mobius band, find a simpler cell complex structure for it, and compute its Euler characteristic.

10.18. Give a triangulation for the Klein bottle, find a simpler cell complex structure for it, and compute its Euler characteristic.

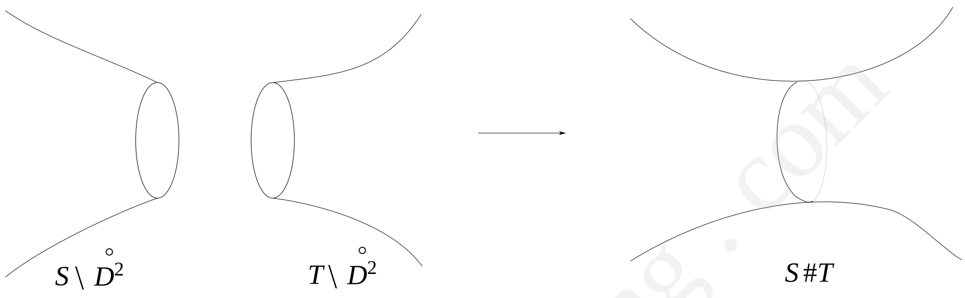
10.19. Draw a cell complex structure on the torus with two holes.

10.20. Find a cell complex structure on $\mathbb{RP}^n$.

11. Classification of compact surfaces

In this section by a *surface* (mặt) we mean a two-dimensional topological manifold (without boundary).

Connected sum. Let S and T be two surfaces. From each surface deletes an open disk, then glue the two boundary circles. The resulting surface is called the *connected sum* (tổng liên thông) of the two surfaces, denoted by $S\#T$.



It is known that the connected sum does not depend on the choices of the disks.

EXAMPLE. If S is any surface then $S\#S^2 = S$.

Classification.

THEOREM (classification of compact surfaces). *A connected compact surface is homeomorphic to either the sphere, or a connected sum of tori, or a connected sum of projective planes.*

We denote by T_g the connected sum of g tori, and by M_g the connected sum of g projective planes. The number g is called the *genus* (giống) of the surface.

The sphere and the surfaces T_g are *orientable* (định hướng được) surface , while the surfaces M_g are *non-orientable* (không định hướng được) surfaces. We will not give a precise definition of orientability here (compare 25).

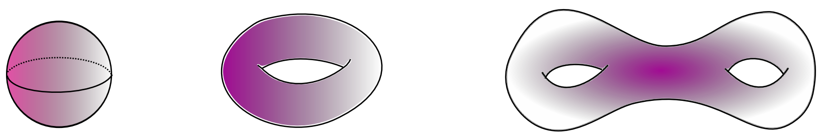


FIGURE 11.1. Orientable surfaces: $S^2, T_1, T_2, \ldots$

Recall that a torus T_1 is the quotient space of a square by the identification on the boundary described by the word $aba^{-1}b^{-1}$. In general, given a polygon on the plane (the underlying space that is homeomorphic to a disk of a simplicial complex), and suppose that the edges of the polygon are labeled and oriented. Choose one edge as the initial one then follow the edge of the polygon in a predetermined direction (the boundary of the polygon is homemorphic to a circle). If an edge a is met in the opposite direction then write it down as a^{-1}. In this way we associate each polygon with a word. We also write aa as a^2.

In the reverse direction, a word gives a polygon with labeled and oriented edges. If a label appears more than once, then the edges with this label can be identified in the orientations assigned to the edges. Let us consider two words *equivalent* if they give rise to homeomorphic spaces. For examples, changing labels and cyclic permutations are equivalence operations on words.

The classification theorem is a direct consequence of the following result:

THEOREM 11.2. *A connected compact surface is homeomorphic to the space obtained by identifying the edges of a polygon in one of the following ways:*

(a) aa^{-1},
(b) $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}$,
(c) $c_1^2c_2^2\cdots c_g^2$.

Proof of 11.2. We will use the following fact: any compact surface can be triangulated (a proof is available in [**Moi77**]).

Take a triangulation X of the compact connected surface S. Let T be the set of triangles in X. Label the edges of the triangles and mark an orientation for each edge. Each edge will appear twice, on two different triangles.

We now build a new space P out of these triangles. Let P_1 be any element of T, which is homeomorphic to a disk. Inductively suppose that we have built a space P_i from a subset T_i of T, where P_i is homeomorphic to a disk and its boundary consists of labeled and directed edges. If $T_i \neq T$ then there exists an element $\Delta \in T \setminus T_i$ such that Δ has at least one edge with the same label α as an edge on the boundary of P_i, otherwise $|X|$ could not be connected. Glue Δ to P_i via this edge to get P_{i+1}, in other words $P_{i+1} = (P_i \sqcup \Delta)/\alpha \sim \alpha$. Since two disks glued along a common arc on the boundaries is still a disk, P_{i+1} is homeomorphic to a disk. Also, let $T_{i+1} = T_i \cup \{\Delta\}$. When this process stops we get a space P homemorphic to a disk, whose boundary consists of labeled, directed edges. If these edges are identified following the instruction by the labels and the directions, we get a space homemorphic to the original surface S. The polygon P is called a *fundamental polygon* of the surface S. See an example in figure 11.3.

Theorem 11.2 is a direct consequence of the following:

PROPOSITION 11.4. *An associated word to a connected compact surface is equivalent to a word of the forms:*

(a) aa^{-1},

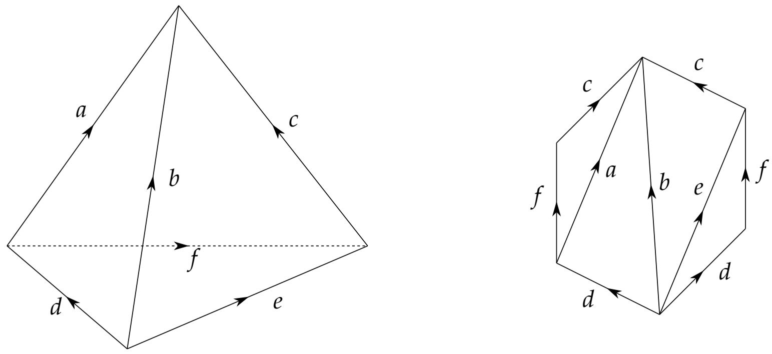


FIGURE 11.3. A triangulation of the sphere and an associated fundamental polygon.

(b) $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}$,
(c) $c_1^2c_2^2\cdots c_g^2$.

We will prove 11.4 through a series of lemmas.

Let w a the word associated to a fundamental polygon of a compact connected surface. If w_1 and w_2 are two equivalent words then we write $w_1 \sim w_2$.

LEMMA 11.5. *$aa^{-1}\alpha \sim \alpha$, if $\alpha \neq \emptyset$.*

PROOF. See figure 11.6. □

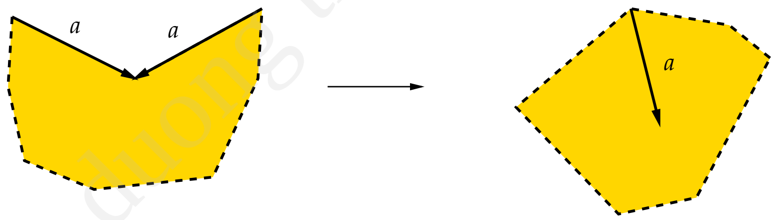


FIGURE 11.6. Lemma 11.5.

LEMMA 11.7. *The word w is equivalent to a word whose all of the vertices of the associated polygon is identified to a single point on the associated surface (w is said to be "reduced").*

PROOF. When we do the operation in figure 11.8, the number of P vertices is decreased. When there is only one P vertex left, we arrive at the situation in lemma 11.5. □

LEMMA 11.9. *$a\alpha a\beta \sim aa\alpha\beta^{-1}$.*

PROOF. See figure 11.10. □

LEMMA 11.11. *Suppose that w is reduced. If $w = a\alpha a^{-1}\beta$ where $\alpha \neq \emptyset$ then $\exists b \in \alpha$ such that $b \in \beta$ or $b^{-1} \in \beta$.*

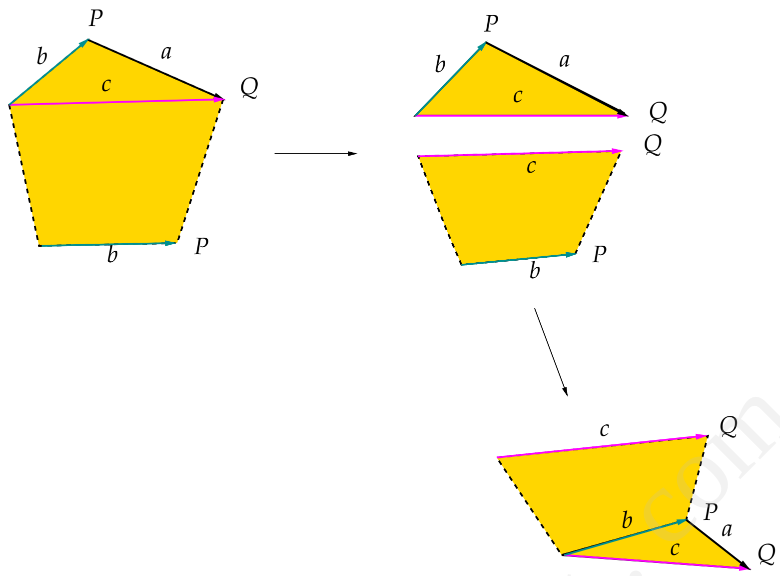


FIGURE 11.8. Lemma 11.7.

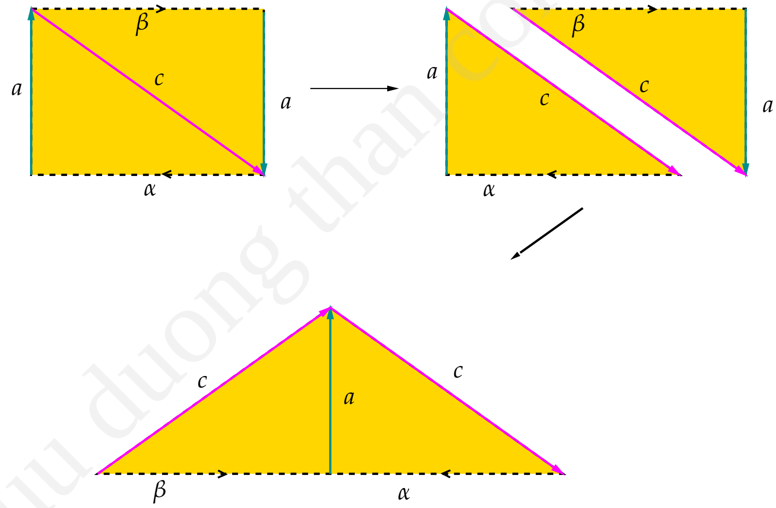


FIGURE 11.10. Lemma 11.9.

PROOF. If all labels in α appear in pairs then the vertices in the part of the polygon associated to α are identified only with themselves, and are not identified with a vertex outside of that part. This contradicts the assumption that w is reduced. $\square$

LEMMA 11.12. $a\alpha b\beta a^{-1}\gamma b^{-1}\delta \sim aba^{-1}b^{-1}\alpha\delta\beta\gamma$.

PROOF. See figure 11.13. $\square$

LEMMA 11.14. $aba^{-1}b^{-1}\alpha c^2\beta \sim a^2b^2\alpha c^2\beta$.

PROOF. Do the operation in figure 11.15, after that we are in a situation where we can apply lemma 11.9 three times. $\square$

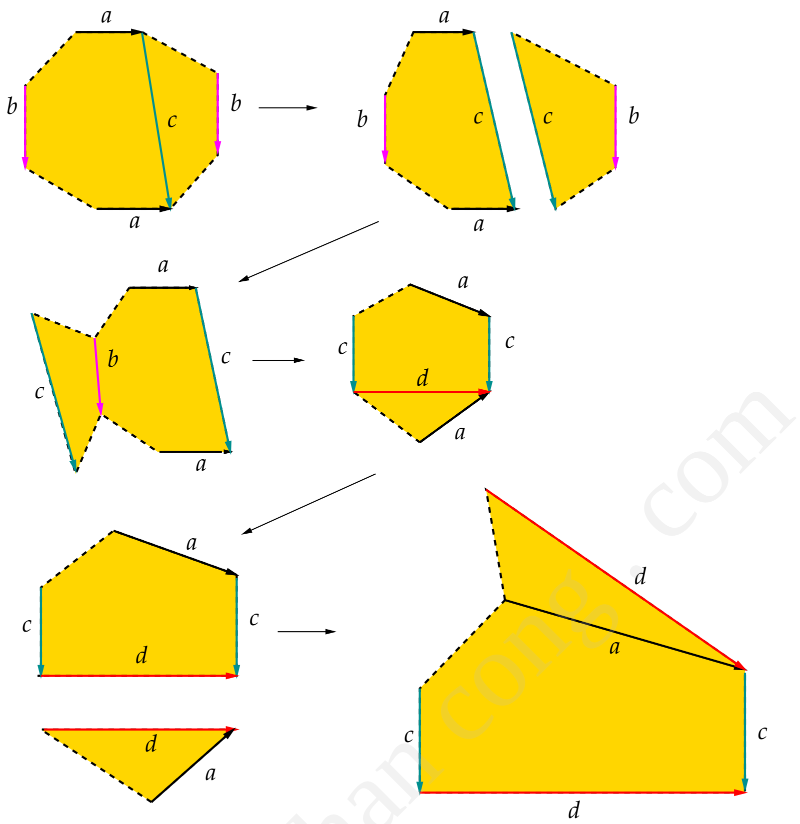


FIGURE 11.13. Lemma 11.12.

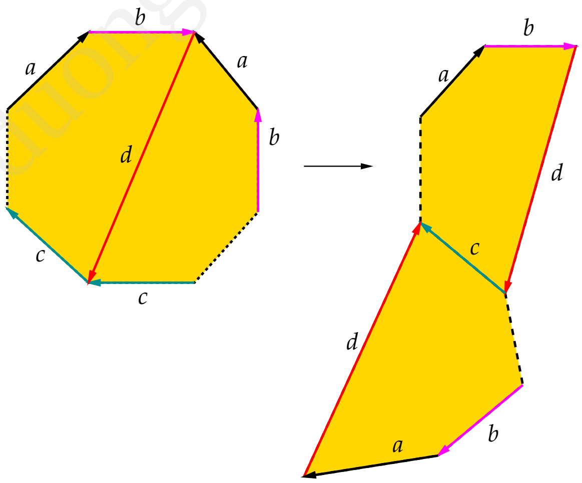


FIGURE 11.15. Lemma 11.14.

PROOF OF 11.4. The proof follows the following steps.

1. Bring w to reduced form by using 11.7 finitely many times.
2. If w has the form $aa^{-1}-$ then go to 2.1, if not go to 3.
2.1. If w has the form aa^{-1} then stop, if not go to 2.2.

2.2. w has the form $aa^{-1}\alpha$ where $\alpha \neq \emptyset$. Repeatedly apply 11.5 finitely many times, deleting pairs of the form aa^{-1} in w until no such pair is left or w has the form aa^{-1}. If no such pair is left go to 3.

3. w does not have the form $-aa^{-1}-$. If we apply 11.9 then a pair of of the form $-a\alpha a-$ with $\alpha \neq \emptyset$ could become a pair of the form $-a - a^{-1}-$, but a pair of the form $-aa-$ will not be changed. Therefore 11.9 could be used finitely many times until there is no pair $-a\alpha a-$ with $\alpha = \emptyset$ left. Notice from the proof of 11.9 that this step will not undo the steps before it.

4. If there is no pair of the form $-a\alpha a^{-1}$ where $\alpha \neq \emptyset$, then stop: w has the form $a_1^2 a_2^2 \cdots a_g^2$.

5. w has the form $-a\alpha a^{-1}$ where $\alpha \neq \emptyset$. By 11.11 w must has the form $-a - b - a^{-1} - b^{-1}-$, since after Step 3 there could be no $-b - a^{-1} - b-$.

6. Apply 11.12 finitely many times until w no longer has the form $-a\alpha b\beta a^{-1}\gamma b^{-1}-$ where at least one of α, β, or γ is non-empty. If w is not of the form $-aa-$ then stop: w has the form $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$.

7. w has the form $-aba^{-1}b^{-1} - cc-$. Use 11.14 finitely many times to transform w to the form $a_1^2 a_2^2 \cdots a_g^2$.

□

EXAMPLE. What is the topological space in figure 11.16?

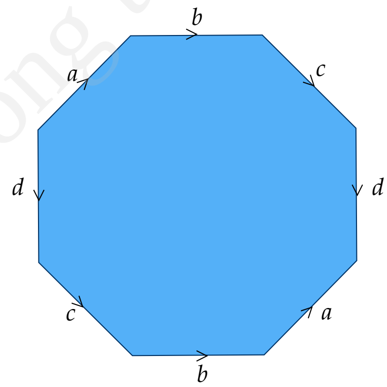


FIGURE 11.16.

Applying lemma 11.12 we get $abcda^{-1}b^{-1}c^{-1}d^{-1} \sim aba^{-1}b^{-1}cdc^{-1}d^{-1}$. Thus this space is homemorphic to the genus 2 torus.

For more on this topic one can read [**Mas91**, ch. 1].

Problems.

11.17. Show that gluing two Mobius bands along their boundaries gives the Klein bottle. In other words, $\mathbb{R}P^2 \# \mathbb{R}P^2 = K$.[14] (See figure 9.7.)

[14]There is a humorous poem:
A mathematician named Klein

11.18. Show that $T^2 \# \mathbb{R}P^2 = K \# \mathbb{R}P^2$, where K is the Klein bottle.

11.19. What is the topological space in figure 11.20?

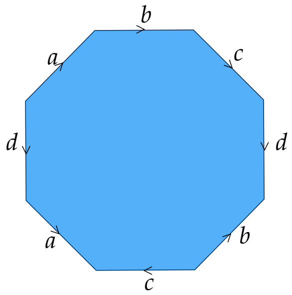


FIGURE 11.20.

11.21. (a) Show that $T_g \# T_h = T_{g+h}$.
(b) Show that $M_g \# M_h = M_{g+h}$.
(c) $M_g \# T_h = ?$

11.22. Show that for any two compact surfaces S_1 and S_2 we have $\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$.

11.23. Compute the Euler characteristics of all connected compact surfaces. Deduce that the orientable surfaces S^2 and T_g, for different g, are distinct, meaning not homeomorphic to each other. Similarly the non-orientable surfaces M_g are all distinct.

11.24. From 11.2, describe a cell complex structure on any compact surface. From that compute the Euler characteristics.

11.25. Let S be a triangulated compact connected surface with v vertices, e edges, and f triangles. Prove that:

(a) $2e = 3f$.
(b) $v \leq f$.
(c) $e \leq \frac{1}{2}v(v-1)$.
(d) $v \geq \left\lfloor \frac{1}{2}\left(7 + \sqrt{49 - 24\chi(S)}\right) \right\rfloor = H(S)$. (The integer $H(S)$, called the *Heawood number*, gives the minimal number of colors needed to color a map on the surface S, except in the case of the Klein bottle. When S is the sphere this is *the four colors problem* [**MT01**, p. 230].)
(e) Show that a triangulation of the torus needs at least 14 triangles. Indeed 14 is the minimal number: there are triangulations of the torus with exactly 14 triangles, see e.g. [**MT01**, p. 142].

11.26 (surfaces are homogeneous). A space is *homogeneous* (đồng nhất) if given two points there exists a homeomorphism from the space to itself bringing one point to the other point.

(a) Show that the sphere S^2 is homogeneous.

Thought the Mobius band was divine
Said he, "If you glue
The edges of two,
You'll get a weird bottle like mine."

(b) Show that the torus T^2 is homogeneous.

For more see 26.1.

12. Homotopy

Homotopy of maps.

DEFINITION. Let X and Y be topological spaces and $f, g : X \to Y$ be continuous. We say that f and g are *homotopic* (đồng luân) if there is a continuous map

$$\begin{aligned} F : X \times [0,1] &\to Y \\ (x,t) &\mapsto F(x,t) \end{aligned}$$

such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$. The map F is called a *homotopy* (phép đồng luân) from f to g.

We can think of t as a time parameter and F as a continuous process in time that starts with f and ends with g. To suggest this view $F(x,t)$ is often written as $F_t(x)$. So $F_0 = f$ and $F_1 = g$.

Here we are looking at $[0,1] \subset \mathbb{R}$ with the Euclidean topology and $X \times [0,1]$ with the product topology.

PROPOSITION 12.1. *Homotopic relation on the set of continuous maps between two given topological spaces is an equivalence relation.*

PROOF. If $f : X \to Y$ is continuous then f is homotopic to itself via the map $F_t = f$, $\forall t \in [0,1]$. If f is homotopic to g via F then g is homotopic to f via $G(x,t) = F(x, 1-t)$. We can think of G is an inverse process to H. If f is homotopic to g via F and g is homotopic to h via G then f is homotopic to h via

$$H_t = \begin{cases} F_{2t}, & 0 \le t \le \frac{1}{2} \\ G_{2t-1}, & \frac{1}{2} \le t \le 1. \end{cases}$$

Thus we obtain H by following F at twice the speed, then continuing with a copy of G also at twice the speed. To check the continuity of H it is better to write it as

$$H(x,t) = \begin{cases} F(x,2t), & (x,t) \in X \times [0, \frac{1}{2}], \\ G(x, 2t-1), & (x,t) \in X \times [\frac{1}{2}, 1], \end{cases}$$

then use 3.15. Another, more advanced way to check continuity of H is by using the property 8.13 about the compact-open topology. $\square$

In this section, and whenever it is clear from the context, we indicate this relation by the usual notation for equivalence relation $\sim$.

LEMMA 12.2. *If $f, g : X \to Y$ and $f \sim g$ then $f \circ h \sim g \circ h$ for any $h : X' \to X$, and $k \circ f \sim k \circ g$ for any $k : Y \to Y'$.*

PROOF. Let F be a homotopy from f to g then $G_t = F_t \circ h$ is a homotopy from $f \circ h$ to $g \circ h$. Rewriting $G(x', t) = F(h(x'), t)$ we see that G is continuous. $\square$

Homotopic spaces.

DEFINITION. Two topological spaces X and Y are *homotopic* if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to Id_X and $f \circ g$ is homotopic to Id_Y. Each of the maps f and g is called a *homotopy equivalence*.

Immediately we have:

PROPOSITION (homeomorphic $\Rightarrow$ homotopic). *Homeomorphic spaces are homotopic.*

PROOF. If f is a homeomorphism then we can take $g = f^{-1}$. □

PROPOSITION. *Homotopic relation on the set of all topological spaces is an equivalence relation.*

PROOF. Any space is homotopic to itself. From the definition it is clear that this relation is symmetric. Now we check that it is transitive. Suppose that X is homotopic to Y via $f : X \to Y$, $g : Y \to X$ and suppose that Y is homotopic to Z via $h : Y \to Z$, $k : Z \to Y$. By 12.2:

$$(g \circ k) \circ (h \circ f) = (g \circ (k \circ h)) \circ f \sim (g \circ \mathrm{Id}_Y) \circ f = g \circ f \sim \mathrm{Id}_X.$$

Similarly

$$(h \circ f) \circ (g \circ k) = (h \circ (f \circ g)) \circ k \sim (h \circ \mathrm{Id}_Y) \circ k = h \circ k \sim \mathrm{Id}_Z.$$

Thus X is homotopic to Z. □

Deformation retraction. Let X be a space, and let A be a subspace of X. We say that A is a *retract* (rút) of X if there is a continuous map $r : X \to A$ such that $r|_A = \mathrm{id}_A$, called a *retraction* (phép rút) from X to A. In other words A is a retract of X if the identity map id_A can be extended to X.

A *deformation retraction* (phép co rút) from X to A is a homotopy F that starts with id_X, ends with a retraction from X to A, and fixes A throughout, i.e., $F_0 = \mathrm{id}_X$, $F_1(X) = A$, and $F_t|_A = \mathrm{id}_A$, $\forall t \in [0,1]$. If there is such a deformation retraction we say that A is a *deformation retract* (co rút) of X.

In such a deformation retraction each point $x \in X \setminus A$ "moves" along the path $F_t(x)$ to a point in A, while every point of A is fixed.

EXAMPLE. A subset A of a normed space is called a *star-shaped* region if there is a point $x_0 \in A$ such that for any $x \in A$ the straight segment from x to x_0 is contained in A. For example, any convex subset of the normed space is a star-shaped region. The map $F_t = (1-t)x + tx_0$ is a deformation retraction from A to x_0, so a star-shaped region has a deformation retraction to a point.

EXAMPLE. A normed space minus a point has a deformation retraction to a sphere. For example, a normed space minus the origin has a deformation retraction $F_t(x) = (1-t)x + t\frac{x}{||x||}$ to the unit sphere at the origin.

EXAMPLE. An annulus $S^1 \times [0,1]$ has a deformation retraction to one of its circle boundary $S^1 \times \{0\}$.

PROPOSITION. *If a space X has a deformation retraction to a subspace A then X is homotopic to A.*

PROOF. Suppose that F_t is a deformation retraction from X to A. Consider $F_1 : X \to A$ and the inclusion map $g : A \to X$, $g(x) = x$. Then id_X is homotopic to $g \circ F_1$ via F_t, while $F_1 \circ g = \mathrm{id}_A$. □

EXAMPLE. The circle, the annulus, and the Mobius band are homotopic to each other, although they are not homeomorphic to each other.

A space which is homotopic to a space containing only one point is said to be *contractible* (thắt được).

COROLLARY. *A space which has a deformation retraction to a point is contractible.*

The converse is not true, see [**Hat01**, p. 18].

Homotopy of paths. Recall that a *path* (đường đi) in a space X is a continuous map α from the Euclidean interval $[0,1]$ to X. The point $\alpha(0)$ is called the initial end point, and $\alpha(1)$ is called the final end point. In this section for simplicity of presentation we assume the domain of a path is the Euclidean interval $[0,1]$ instead of any Euclidean closed interval as before.

A *loop* (vòng) or a *closed path* (đường đi kín) based at a point $a \in X$ is a path whose initial point and end point are a. In other words it is a continuous map $\alpha : [0,1] \to X$ such that $\alpha(0) = \alpha(1) = a$. The *constant loop* at a is the loop $\alpha(t) = a$ for all $t \in [0,1]$.

DEFINITION. Let α and β be two paths from a to b in X. A *path-homotopy* (phép đồng luân đường) from α to β is a continuous map $F : [0,1] \times [0,1] \to X$, $F(s,t) = F_t(s)$, such that $F_0 = \alpha$, $F_1 = \beta$, and for each t the path F_t goes from a to b, i.e. $F_t(0) = a$, $F_t(1) = b$.

If there is a path-homotopy from α to β we say that α is *path-homotopic* (đồng luân đường) to β.

REMARK. A homotopy of path is a homotopy of maps defined on $[0,1]$, *with the further requirement that the homotopy fixes the initial point and the terminal point*. To emphasize this we have used the word path-homotopy, but some sources (e.g. [**Hat01**, p. 25]) simply use the term homotopy, taking this further requirement implicitly.

EXAMPLE. In a convex subset of a normed space any two paths α and β with the same initial points and end points are homotopic, via the homotopy $(1-t)\alpha + t\beta$.

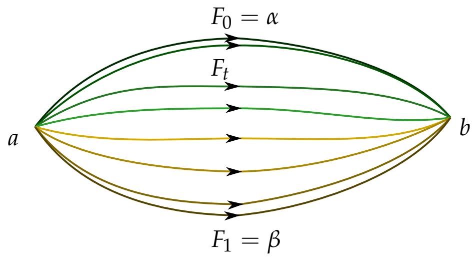


FIGURE 12.3. We can think of a path-homotopy from α to β as a way to continuously brings α to β, keeping the endpoints fixed, similar to a motion picture.

PROPOSITION. *Path-homotopic relation on the set of all paths from a to b is an equivalence relation.*

The proof is the same as the proof of 12.1.

Problems.

12.4. Show in detail that the Mobius band has a deformation retraction to a circle.

12.5. Show that contractible spaces are path-connected.

12.6. Let X be a topological space and let Y be a retract of X. Show that any continuous map from Y to a topological space Z can be extended to X.

12.7. Let X be a topological space and let Y be a retract of X. Show that if every continuous map from X to itself has a fixed point then every continuous map from Y to itself has a fixed point.

12.8. Show that if B is contractible then $A \times B$ is homotopic to A.

12.9. Show that if a space X is contractible then all continuous maps from a space Y to X are homotopic. Is the converse correct?

12.10. Show that if $f : S^n \to S^n$ is not homotopic to the identity map then there is $x \in S^n$ such that $f(x) = -x$.

12.11. Classify the alphabetical characters according to homotopy types, that is, which of the characters are homotopic to each other as subspaces of the Euclidean plane? Do the same for the Vietnamese alphabetical characters. Note that the result depends on the font you use.

13. The fundamental group

If $\alpha(t)$, $0 \leq t \leq 1$ is a path from a to b then we write as α^{-1} the path given by $\alpha^{-1}(t) = \alpha(1-t)$, called the *inverse path* of α, going from b to a.

If α is a path from a to b, and β is a path from b to c, then we define the *composition* (hợp) of α with β, written $\alpha \cdot \beta$, to be the path

$$\alpha \cdot \beta(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Notice that $\alpha \cdot \beta$ is continuous by 3.15.

LEMMA 13.1. *If α is path-homotopic to α_1 and β is path-homotopic to β_1 then $\alpha \cdot \beta$ is path-homotopic to $\alpha_1 \cdot \beta_1$.*

PROOF. Let F be a path-homotopy from $\alpha$to β and G be a homotopy from β to γ. Let $H_t = F_t \cdot G_t$, that is:

$$H(s,t) = \begin{cases} F(2s,t), & 0 \leq s \leq \frac{1}{2},\ 0 \leq t \leq 1 \\ G(2s-1,t), & \frac{1}{2} \leq s \leq 1,\ 0 \leq t \leq 1. \end{cases}$$

Then H is continuous by 3.15 and is a homotopy from $\alpha \cdot \beta$ to $\alpha_1 \cdot \beta_1$. □

LEMMA 13.2. *If α is a path from a to b then $\alpha \cdot \alpha^{-1}$ is path-homotopic to the constant loop at a.*

PROOF. A homotopy F from $\alpha \cdot \alpha^{-1}$ to the constant loop at a can be described as follows. At a fixed t, the loop F_t starts at time 0 at a, goes along α but at twice the speed of α, until time $\frac{1}{2} - \frac{t}{2}$, stays there until time $\frac{1}{2} + \frac{t}{2}$, then catches the inverse path α^{-1} at twice its speed to come back to a. More precisely,

$$F(s,t) = \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2} - \frac{t}{2} \\ \alpha(1-t), & \frac{1}{2} - \frac{t}{2} \leq s \leq \frac{1}{2} + \frac{t}{2} \\ \alpha^{-1}(2s-1) = \alpha(2-2s), & \frac{1}{2} + \frac{t}{2} \leq s \leq 1. \end{cases}$$

Again by 3.15, F is continuous. □

LEMMA 13.3 (reparametrization does not change homotopy class). *With a continuous map $\varphi : [0,1] \to [0,1]$, $\varphi(0) = 0$, $\varphi(1) = 1$, for any path α the path $\alpha \circ \varphi$ is path-homotopic to α.*

PROOF. Let $F_t = (1-t)\varphi + t\mathrm{Id}_{[0,1]}$, a homotopy from φ to $\mathrm{Id}_{[0,1]}$ on $[0,1]$. We can check that $G_t = \alpha \circ F_t$, i.e., $G = \alpha \circ F$ is a path-homotopy from $\alpha \circ \varphi$ to α. □

COROLLARY 13.4. *$(\alpha \cdot \beta) \cdot \gamma$ is path-homotopic to $\alpha \cdot (\beta \cdot \gamma)$.*

PROOF. We can check directly that $(\alpha \cdot \beta) \cdot \gamma$ is a reparametrization of $\alpha \cdot (\beta \cdot \gamma)$. (It is likely easier for the reader to find out a formula for a reparametrization himself/herself than to read about one). □

The fundamental group. For elementary algebra **[Gal10]** is a very good textbook. Recall that a *group* is a set G with a map

$$\begin{aligned} G \times G &\to G \\ (a, b) &\mapsto a \cdot b, \end{aligned}$$

callled a multiplication, such that:

(a) multiplication is associative: $(ab)c = a(bc)$, $\forall a, b, c \in G$,
(b) there exists an element of G called the unit element or the identity element $e$satisfying $ae = ea = a$, $\forall a \in G$,
(c) for each element a of G there exists an element of G called the inverse element of a, denoted by a^{-1}, satisfying $a \cdot a^{-1} = a^{-1} \cdot a = e$.

Consider the set of loops of X based at a point x_0 under the path-homotopy relation. On this set we define a multiplication operation $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$. By 13.1 this operation is well-defined.

THEOREM 13.5. *The set of all path-homotopy classes of loops of X based at a point x_0 is a group under the above operation.*

This group is called *the fundamental group* (nhóm cơ bản) of X at x_0, denoted by $\pi_1(X, x_0)$. The point x_0 is called the *base point*.

PROOF. Denote by 1 the constant loop at x_0. By lemma 13.3, $1 \cdot \alpha$ is path-homotopic to α, thus $[1] \cdot [\alpha] = [\alpha]$. So $[1]$ is the identity in $\pi_1(X, x_0)$.

We define $[\alpha]^{-1} = [\alpha^{-1}]$. It is easy to check that this is well-defined. By 13.2 $[\alpha]^{-1}$ is indeed the inverse element of $[\alpha]$.

By 13.4 we have associativity:

$$([\alpha] \cdot [\beta]) \cdot [\gamma] = [\alpha \cdot \beta] \cdot [\gamma] = [(\alpha \cdot \beta) \cdot \gamma] = [\alpha \cdot (\beta \cdot \gamma)] = [\alpha] \cdot ([\beta] \cdot [\gamma]).$$

□

EXAMPLE. The fundamental group of a space containing only one point is trivial. Namely, if $X = \{x_0\}$ then $\pi_1(X, x_0) = 1$, the group with only one element.

Recall that a map $h : G_1 \to G_2$ between two groups is called a group *homomorphism* if it preserves group operations, i.e. $h(a \cdot b) = h(a) \cdot h(b)$, $\forall a, b \in G_1$. It is a group *isomorphism* if is a bijective homomorphism (in this case the inverse map is also a homomorphism).

PROPOSITION 13.6 (dependence on base point). *If there is a path from x_0 to x_1 then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.*

PROOF. Let α be a path from x_0 to x_1. Consider the map

$$\begin{aligned} h_\alpha : \quad \pi_1(X, x_1) &\to \pi_1(X, x_0) \\ [\gamma] &\mapsto [\alpha \cdot \gamma \cdot \alpha^{-1}]. \end{aligned}$$

Using 13.1 and 13.4 we can check that this is a well-defined map, and is a group homomorphism with an inverse homomorphism:

$$\begin{array}{rrcl} h_\alpha^{-1}: & \pi_1(X, x_1) & \to & \pi_1(X, x_0) \\ & [\gamma] & \mapsto & [\alpha^{-1} \cdot \gamma \cdot \alpha]. \end{array}$$

□

Thus for a path-connected space the fundamental group is the same up to group isomorphisms for any choice of base point. Therefore if X is a path-connected space we often drop the base point in the notation and just write $\pi_1(X)$.

Induced homomorphism. Let X and Y be topological spaces, and $f : X \to Y$. Then f induces the following map

$$\begin{array}{rcl} f_* : \pi_1(X, x_0) & \to & \pi_1(Y, f(x_0)) \\ [\gamma] & \mapsto & [f \circ \gamma]. \end{array}$$

This is a well-defined map (problem 13.8). Notice that f_* depends on the base point x_0. Furthermore f_* is a group homomorphism, indeed:

$$f_*([\gamma_1] \cdot [\gamma_2]) = f_*([\gamma_1 \cdot \gamma_2]) = [f \circ (\gamma_1 \cdot \gamma_2)] = (f \circ \gamma_1) \cdot (f \circ \gamma_2) = f_*([\gamma_1]) \cdot f_*([\gamma_2]).$$

PROPOSITION ($(g \circ f)_* = g_* \circ f_*$). *If $f : X \to Y$ and $g : Y \to Z$ then $(g \circ f)_* = g_* \circ f_*$ as maps $\pi_1(X, x_0) \to \pi_1(Z, g(f(x_0)))$.*

PROOF.

$$(g \circ f)_*([\gamma]) = [(g \circ f) \circ \gamma] = [g \circ (f \circ \gamma)] = g_*([f \circ \gamma]) = g_*(f_*([\gamma])).$$

□

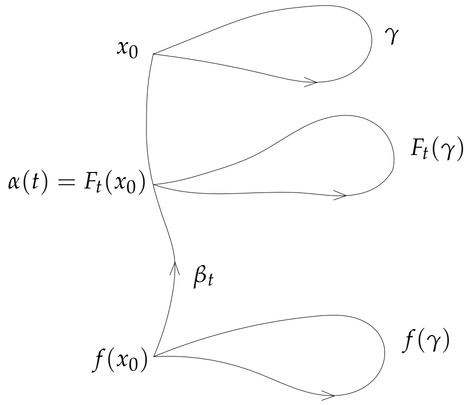
LEMMA. *If $f : X \to X$ is homotopic to the identity map then $f_* : \pi_1(X, x_0) \to \pi_1(X, f(x_0))$ is an isomorphism.*

PROOF. From the assumption there is a homotopy F from f to id_X. Then $\alpha(t) = F_t(x_0)$, $0 \le t \le 1$, is a continuous path from $f(x_0)$ to x_0. We will show that $f_* = h_\alpha$ where h_α is the map used in the proof of 13.6, which was shown there to be an isomorphism.

For each fixed $0 \le t \le 1$, let β_t be the path that goes along α from $\alpha(0) = f(x_0)$ to $\alpha(t)$, given by $\beta_t(s) = \alpha(st)$, $0 \le s \le 1$. For any loop γ at x_0, let $G_t = \beta_t \cdot F_t(\gamma) \cdot \beta_t^{-1}$. That G is continuous can be checked by writing down the formula for G explicitly. Then G gives a path-homotopy from $f(\gamma)$ to $\alpha \cdot \gamma \cdot \alpha^{-1}$, thus $f_*([\gamma]) = [f \circ \gamma] = [\alpha \cdot \gamma \cdot \alpha^{-1}] = h_\alpha(\gamma)$. □

THEOREM. *If $f : X \to Y$ is a homotopy equivalence then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.*

PROOF. Since f is a homotopy equivalence there is $g : Y \to X$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y. By the above lemma, the



composition

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, g(f(x_0)))$$

is an isomorphism, which implies that g_* is surjective. Similarly the composition

$$\pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, g(f(x_0))) \xrightarrow{f'_*} \pi_1(Y, f(g(f(x_0))))$$

is an isomorphism, where f'_* denotes the induced homomorphism by f with base point $g(f(x_0))$. This implies that g_* is injective. Since g_* is bijective, from the first composition we see that f_* is bijective. □

COROLLARY (homotopy invariance). *If two path-connected spaces are homotopic then their fundamental groups are isomorphic.*

We say that for path-connected spaces, *the fundamental group is a homotopy invariant*.

EXAMPLE. The fundamental group of a contractible space is trivial.

Problems.

13.7. Show that if X_0 is a path-connected component of X and $x_0 \in X_0$ then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X_0, x_0)$.

13.8. Let X and Y be topological spaces, $f : X \to Y$, $f(x_0) = y_0$. Show that the induced map

$$\begin{aligned} f_* : \pi_1(X, x_0) &\to \pi_1(Y, y_0) \\ [\gamma] &\mapsto [f \circ \gamma] \end{aligned}$$

is a well-defined.

13.9. Suppose that $f : X \to Y$ is a homeomorphism. Show that the induces homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism, directly without using the general result about homotopy equivalence.

13.10. Suppose that Y is a retract of X through a retraction $r : X \to Y$. Let $i : Y \hookrightarrow X$ be the inclusion map. With $y_0 \in Y$, show that $r_* : \pi_1(X, y_0) \to \pi_1(Y, y_0)$ is surjective and $i_* : \pi_1(Y, y_0) \to \pi_1(X, y_0)$ is injective.

13.11. Show that

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

13.12. Show that in a space X the following statements are equivalent:

(a) All continuous maps $S^1 \to X$ are homotopic.
(b) Any continuous map $S^1 \to X$ has a continuous extension to a map $D^2 \to X$.
(c) $\pi_1(X, x_0) = 1, \forall x_0 \in X$.

13.13. A space is said to be *simply connected* (đơn liên) if it is path-connected and any loop is path-homotopic to a constant loop. Show that a space is simply-connected if and only if it is path-connected and its fundamental group is trivial.

13.14. Show that any contractible space is simply connected.

13.15. Show that a space is simply connected if and only if all paths with same initial points and same terminal points are path-homotopic, in other words, there is exactly one path-homotopy class from one point to another point.

14. The fundamental group of the circle

THEOREM ($\pi_1(S^1) \cong \mathbb{Z}$). *The fundamental group of the circle is infinite cyclic.*

Let γ_n be the loop $(\cos(n2\pi t), \sin(n2\pi t))$, $0 \le t \le 1$, the loop on the circle S^1 based at the point $(1,0)$ that goes n times around the circle at uniform speed in the counter-clockwise direction if $n > 0$ and in the clockwise direction if $n < 0$. Consider the map

$$\begin{aligned} \Phi : \mathbb{Z} &\to \pi_1(S^1, (1,0)) \\ n &\mapsto [\gamma_n]. \end{aligned}$$

This map associates each integer n with the path-homotopy class of γ_n. We will show that Φ is a group isomorphism, where $\mathbb{Z}$ has the usual additive structure. This implies that the fundamental group of the circle is generated by a loop that goes once around the circle in the counter-clockwise direction, and the homotopy class of a loop in the circle corresponds to an integer representing the "number of times" that the loop goes around the circle, with the counter-clockwise direction being the positive direction.

Φ is a group homomorphism. This means γ_{m+n} is path-homotopic to $\gamma_m \cdot \gamma_n$. If $m = -n$ then we can verify from the formulas that $\gamma_n(t) = \gamma_{-m}(t) = \gamma_m(1-t) = \gamma_m^{-1}(t)$. We have checked earlier 13.2 that $\gamma_m \cdot \gamma_m^{-1}$ is path-homotopic to the constant loop γ_0.

If $m \neq -n$ then $\gamma_m \cdot \gamma_n$ is a reparametrization of γ_{m+n} so they are path-homotopic by 13.3. Indeed, with

$$\gamma_{m+n}(t) = (\cos((m+n)2\pi t), \sin((m+n)2\pi t)),\ 0 \le t \le 1,$$

and

$$\gamma_m \cdot \gamma_n(t) = \begin{cases} (\cos(m2\pi 2t), \sin(m2\pi 2t)), & 0 \le t \le \frac{1}{2}, \\ (\cos(n2\pi(2t-1)), \sin(n2\pi(2t-1))), & \frac{1}{2} \le t \le 1, \end{cases}$$

we can find φ such that $\gamma_{m+n} \circ \varphi = \gamma_m \cdot \gamma_n$, namely,

$$\varphi(t) = \begin{cases} \frac{m}{m+n} 2t, & 0 \le t \le \frac{1}{2}, \\ \frac{n(2t-1)+m}{m+n}, & \frac{1}{2} \le t \le 1. \end{cases}$$

Covering spaces. Let $p : \mathbb{R} \to S^1$, $p(t) = (\cos(2\pi t), \sin(2\pi t))$, a map that wraps the line around the circle countably infinitely many times in the counter-clockwise direction, called the projection map. This is related to the usual parametrization of the circle by angle. The map p is called the *covering map* associated with of the *covering space* $\mathbb{R}$ of S^1. For a path $\gamma : [0,1] \to S^1$, a path $\tilde{\gamma} : [0,1] \to \mathbb{R}$ such that $p \circ \tilde{\gamma} = \gamma$ is called a *lift* of γ. For example $\gamma_n(t) = p(nt)$, $0 \le t \le 1$, so γ_n has a lift $\widetilde{\gamma_n}$ with $\widetilde{\gamma_n}(t) = nt$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ {[0,1]} \arrow[ur, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & S^1 \end{tikzcd}$$

An idea is that paths in S^1 are projections of paths in $\mathbb{R}$, while $\mathbb{R}$ is simpler than S^1.

As a demonstration, we use this idea to show again that γ_{m+n} is path-homotopic to $\gamma_m \cdot \gamma_n$. Consider the case $m+n>0$. Let

$$\begin{aligned} \widetilde{\gamma_{m+n}} : [0,1] &\to [0, m+n] \\ t &\mapsto (m+n)t, \end{aligned}$$

then this is a lift of γ_{m+n}. Let

$$\begin{aligned} \widetilde{\gamma_m \cdot \gamma_n} : [0,1] &\to [0, m+n] \\ t &\mapsto \begin{cases} m2t, & 0 \le t \le \frac{1}{2}, \\ n(2t-1)+m, & \frac{1}{2} \le t \le 1, \end{cases} \end{aligned}$$

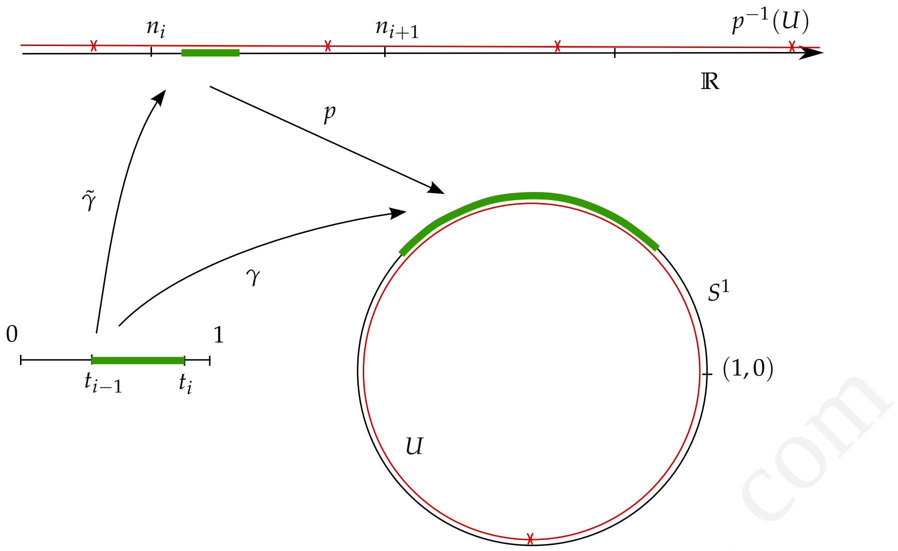
then this is a lift of $\gamma_m \cdot \gamma_n$. The two paths $\widetilde{\gamma_{m+n}}$ and $\widetilde{\gamma_m \cdot \gamma_n}$ are paths in $\mathbb{R}$ (a normed vector space) with same endpoints, so they are path-homotopic via a path-homotopy such as $F_s = (1-s)\widetilde{\gamma_{m+n}} + s\widetilde{\gamma_m \cdot \gamma_n}$, $0 \le s \le 1$. Then γ_{m+n} is path-homotopic to $\gamma_m \cdot \gamma_n$ via the path-homotopy $p \circ F$.

Φ is surjective. This means every loop γ on the circle based at $(1,0)$ is path-homotopic to a loop γ_n. Our is based on the fact that there is a path $\tilde{\gamma}$ on $\mathbb{R}$ starting at 0 such that $\gamma = p \circ \tilde{\gamma}$. This is an important result in its own right and will be proved separately below at 14.1. Then $\tilde{\gamma}(1)$ is an integer n. On $\mathbb{R}$ the path $\tilde{\gamma}$ is path-homotopic to the path $\widetilde{\gamma_n}$, via a path-homotopy F, then γ is path-homotopic to γ_n via the path-homotopy $p \circ F$.

LEMMA 14.1 (existence of lift). *Every path in S^1 has a lift to $\mathbb{R}$. Furthermore if the initial point of the lift is specified then the lift is unique.*

PROOF. Let us write $S^1 = U \cup V$ with $U = S^1 \setminus \{(0,-1)\}$ and $V = S^1 \setminus \{(0,1)\}$. Then $p^{-1}(U) = \bigcup_{n\in\mathbb{Z}}(n - \frac{1}{4}, n + \frac{3}{4})$, consisting of infinitely many disjoint open subsets of $\mathbb{R}$, each of which is homeomorphic to U via p, i.e. $p : (n-\frac{1}{4}, n+\frac{3}{4}) \to U$ is a homeomorphism, in particular the inverse map exists and is continuous. The same thing happens with respect to V.

Let $\gamma : [0,1] \to S^1$, $\gamma(0) = (1,0)$. We can divide $[0,1]$ into sub-intervals with endpoints $0 = t_0 < t_1 < \cdots < t_n = 1$ such that on each sub-interval $[t_{i-1}, t_i]$, $1 \le i \le n$, the path γ is either contained in U or in V. This is guaranteed by the existence of a Lebesgue number (6.3) with respect to the open cover $\gamma^{-1}(U) \cup \gamma^{-1}(V)$ of $[0,1]$.



Suppose a lift $\tilde{\gamma}(0)$ is chosen, an integer. Suppose that $\tilde{\gamma}$ has been constructed on $[0, t_{i-1}]$ for a certain $1 \leq i \leq n$. If $\gamma([t_{i-1}, t_i]) \subset U$ then there is a unique $n_i \in \mathbb{Z}$ such that $\tilde{\gamma}(t_{i-1}) \in (n_i - \frac{1}{4}, n_i + \frac{3}{4})$. There is only one way to continuously extend $\tilde{\gamma}$ to $[t_{i-1}, t_i]$, that is by defining $\tilde{\gamma} = p|^{-1}_{(n_i - \frac{1}{4}, n_i + \frac{3}{4})} \circ \gamma$. In this way $\tilde{\gamma}$ is extended continuously to $[0,1]$. $\square$

Examining the proof above we can see that the key property of the covering space $p : \mathbb{R} \to S^1$ is the following: each point on the circle has an open neighborhood U such that the preimage $p^{-1}(U)$ is the disjoint union of open subsets of $\mathbb{R}$, each of which is homeomorphic to U via p. This is the defining property of general covering spaces.

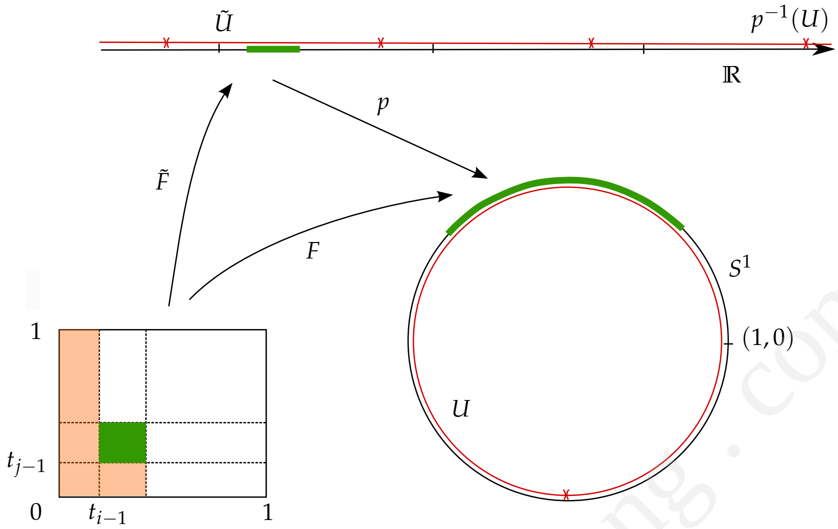
Φ is injective. This is reduced to showing that if γ_m is path-homotopic to γ_n then $m = n$. Our proof is based on another important result below, 14.2, which says that if γ_m is path-homotopic to γ_n then $\widetilde{\gamma_m}$ is path-homotopic to $\widetilde{\gamma_n}$. This implies the terminal point m of $\widetilde{\gamma_m}$ must be the same as the terminal point n of $\widetilde{\gamma_n}$.

LEMMA 14.2 (homotopy of lifts). *Lifts of path-homotopic paths with same initial points are path-homotopic.*

PROOF. The proof is similar to the above proof of 14.1. Let $F : [0,1] \times [0,1] \to S^1$ be a path-homotopy from the path F_0 to the path F_1. If the two lifts $\widetilde{F_0}$ and $\widetilde{F_1}$ have same initial points then that initial point is the lift of the point $F((0,0))$.

As we noted earlier, the circle has an open cover O such that each $U \in O$ we have $p^{-1}(U)$ is the disjoint union of open subsets of $\mathbb{R}$, each of which is homeomorphic to U via p. The collection $F^{-1}(O)$ is an open cover of the square $[0,1] \times [0,1]$. By the existence of Lebesgue's number, there is a partition of $[0,1] \times [0,1]$ into sub-rectangles such that each sub-rectangle is contained in an element of $F^{-1}(O)$. More concisely, we can divide $[0,1]$ into sub-intervals with endpoints

$0 = t_0 < t_1 < \cdots < t_n = 1$ such that for each $1 \leq i, j \leq n$ there is $U \in O$ such that $F([t_{i-1}, t_i] \times [t_{j-1}, t_j]) \subset U$.



We will build up $\tilde{F}$ sub-rectangle by sub-rectangle, going from left to right then below to above. We already have $\tilde{F}((0,0))$. Suppose that for some $1 \leq i, j \leq n$ the point $\tilde{F}((t_{i-1}, t_{j-1}))$ is already defined. We have $F([t_{i-1}, t_i] \times [t_{j-1}, t_j]) \subset U$ for some $U \in O$. Let $\tilde{U}$ be the unique open subset of $\mathbb{R}$ such that $\tilde{U}$ contains the point $\tilde{F}((t_{i-1}, t_{j-1}))$ and $p|_{\tilde{U}} : \tilde{U} \to U$ is a homeomorphism. Then we define $\tilde{F}$ on the sub-rectangle $[t_{i-1}, t_i] \times [t_{j-1}, t_j]$ to be $p|_{\tilde{U}}^{-1} \circ F$.

We need to check $\tilde{F}$ is continuous on the new domain. Since we extend one sub-rectangle at a time in this way, the intersection of the previous domain of $\tilde{F}$ and the sub-rectangle $[t_{i-1}, t_i] \times [t_{j-1}, t_j]$ is connected, being one edge or the union of two edges with a common vertex. That implies $\tilde{F}$ must bring the entire common domain to $\tilde{U}$, therefore on this common domain $\tilde{F}$ must be $p|_{\tilde{U}}^{-1} \circ F$, agreeing with the new definition. The continuity of $\tilde{F}$ now follows from gluing of continuous functions 3.15.

Thus we obtained a continuous lift $\tilde{F}$ of F. Since the initial point is given, by uniqueness of lifts of paths in 14.1, the restriction of $\tilde{F}$ to $[0,1] \times \{0\}$ is $\widetilde{F_0}$ while the restriction of $\tilde{F}$ to $[0,1] \times \{1\}$ is $\widetilde{F_1}$. Thus $\tilde{F}$ is a path-homotopy from $\widetilde{F_0}$ to $\widetilde{F_1}$. □

That the fundamental group of the circle is non-trivial gives us:

COROLLARY. *The circle is not contractible.*

COROLLARY 14.3. *There cannot be any retraction from the disk D^2 to its boundary S^1.*

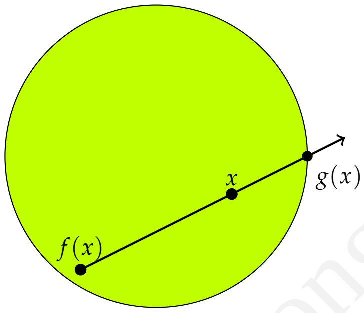
PROOF. Suppose there is a retraction $r : D^2 \to S^1$. Let $i : S^1 \hookrightarrow D^2$ be the inclusion map. From the diagram $S^1 \xrightarrow{i} D^2 \xrightarrow{r} S^1$ we have $r \circ i = \mathrm{id}_{S^1}$, therefore on the fundamental groups $(r \circ i)_* = r_* \circ i_* = \mathrm{id}_{\pi_1(S^1)}$. A consequence is that r_* is onto, but this is not possible since $\pi_1(D^2)$ is trivial while $\pi_1(S^1)$ is non-trivial. □

A proof of this result for higher dimensions is presented in 16.2.

The important Brouwer fixed point theorem follows from that simple result:

THEOREM 14.4 (Brouwer fixed point theorem for dimension two). *Any continuous map from the disk D^2 to itself has a fixed point.*

PROOF. Suppose that $f : D^2 \to D^2$ does not have a fixed point, i.e. $f(x) \neq x$ for all $x \in D^2$. The straight line from $f(x)$ to x will intersect the boundary ∂D^2 at a point $g(x)$.



We can check that g is continuous (see 14.8). Then $g : D^2 \to \partial D^2$ is a retraction, contradicting 14.3. □

Problems.

14.5. Find the fundamental groups of:

(a) the Mobius band,

(b) the cylinder.

14.6. Show that the plane minus a point is not simply connected.

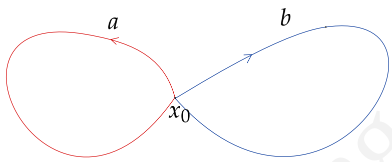
14.7. Derive from the fact that the torus T^2 is homeomorphic to $S^1 \times S^1$ (9.13) the fundamental group of the torus.

14.8. Check that the map g in the proof of the Brouwer fixed point theorem is indeed continuous.

15. Van Kampen theorem

Van Kampen theorem is about giving the fundamental group of a union of subspaces from the fundamental groups of the subspaces.

EXAMPLE ($S^1 \vee S^1$). Two circles with one common point (the figure 8) is called a wedge product $S^1 \vee S^1$. Let x_0 be the common point, let a be a loop starting at x_0 going once around the first circle and let b the a loop starting at x_0 going once around the second circle. Then a and b generate the fundamental groups of the two circles with based points at x_0. Intuitively we can see that $\pi_1(S^1 \vee S^1, x_0)$ consists of path-homotopy classes of loops like a, ab, bba, $aabab^{-1}a^{-1}a^{-1}, \ldots$ This is a group called the free group generated by a and b, denoted by $\langle a, b\rangle$.



Free group. Let S be a set. Let S^{-1} be a set having a bijection with S. Corresponding to each element $x \in S$ is an element in S^{-1} denoted by x^{-1}. A *word with letters in S* is a finite sequence of elements in $S \cup S^{-1}$. The sequence with no element is called the *empty word*. Given two words we form a new word by juxtaposition (đặt kề): $(s_1s_2\cdots s_n)\cdot(s'_1s'_2\cdots s'_m) = s_1s_2\cdots s_ns'_1s'_2\cdots s'_m$. With this operation the set of all words with letters in S becomes a group. The identity element 1 is the empty word. The inverse element of a word $s_1s_2\cdots s_n$ is the word $s_n^{-1}s_{n-1}^{-1}\cdots s_1^{-1}$. This group is called the *free group generated by the set S*, denoted by $\langle S\rangle$.

EXAMPLE. The free group $\langle\{a\}\rangle$ generated by the set $\{a\}$ is often written as $\langle a\rangle$. As a set $\langle a\rangle$ can be written as $\{a^n \mid n \in \mathbb{Z}\}$. The product is given by $a^m \cdot a^n = a^{m+n}$. The identity is a^0. Thus as a group $\langle a\rangle$ is an infinite cyclic group, isomorphic to $(\mathbb{Z}, +)$.

Let G be a set and let R be a set of words with letters in G, i.e. R is a finite subset of the free group $\langle G\rangle$. Let N be the smallest normal subgroup of $\langle G\rangle$ containing R. The quotient group $\langle G\rangle / N$ is written $\langle G \mid R\rangle$. Elements of G are called *generators* of this group and elements of R are called *relators* of this group. We can think of $\langle G \mid R\rangle$ as consisting of words in G subjected to the *relations* $r = 1$ for all $r \in R$.

EXAMPLE. $\langle a \mid a^2\rangle = \{a^0, a\} \cong \mathbb{Z}_2$.

Free product of groups. Let G and H be groups. Form the set of all words with letters in G or H. In such a word, two consecutive elements from the same group can be reduced by the group operation. For example $ba^2ab^3b^{-5}a^4 = ba^3b^{-2}a^4$. In particular if x and x^{-1} are next to each other then they will be canceled. The identities of G and H are also reduced. For example $abb^{-1}c = a1c = ac$.

As with free groups, given two words we form a new word by juxtaposition. For example $(a^2b^3a^{-1}) \cdot (a^3ba) = a^2b^3a^{-1}a^3ba = a^2b^3a^2ba$. This is a group operation, with the identity element 1 being the empty word, the inverse of a word $s_1s_2\cdots s_n$ is the word $s_n^{-1}s_{n-1}^{-1}\cdots s_1^{-1}$. This group is called the *free product* of G with H.

EXAMPLE ($G * H \neq G \times H$). We have

$$\langle g\rangle * \langle h\rangle = \langle g, h\rangle = \{g^{m_1}h^{n_1}g^{m_2}h^{n_2}\cdots g^{m_k}h^{n_k} \mid m_1, n_1, \ldots, m_k, n_k \in \mathbb{Z},\ k \in \mathbb{Z}^+\}.$$

Compare that to $\langle g\rangle \times \langle h\rangle = \{(g^m, h^n) \mid m, n \in \mathbb{Z}\}$ with component-wise multiplication. This group can be identified with $\langle g, h \mid gh = hg\rangle = \{g^m h^n \mid m, n \in \mathbb{Z}\}$. Thus $\mathbb{Z} * \mathbb{Z} \neq \mathbb{Z} \times \mathbb{Z}$.

For more details on free group and free product, see textbooks on Algebra such as **[Gal10]** or **[Hun74]**.

Van Kampen theorem.

THEOREM (Van Kampen theorem). *Suppose that $U, V \subset X$ are open, path-connected, $U \cap V$ is path-connected, and $x_0 \in U \cap V$. Let $i_U : U \cap V \hookrightarrow U$ and $i_V : U \cap V \hookrightarrow V$ be inclusion maps. Then*

$$\pi_1(U \cup V, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(V, x_0)}{\langle\{(i_U)_*(\alpha)(i_V)_*(\alpha)^{-1} \mid \alpha \in \pi_1(U \cap V, x_0)\}\rangle}.$$

PROOF. Let γ be any loop in X at x_0. By the existence of a Lebesgue number in association with the open cover $\{\gamma^{-1}(U), \gamma^{-1}(V)\}$ of $[0,1]$, there is a partition of $[0,1]$ by $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\gamma([t_{i-1}, t_i]) \subset U$ or $\gamma([t_{i-1}, t_i]) \subset V$ for every $1 \leq i \leq n$, and furthermore we can arrange so that $\gamma(t_i) \in U \cap V$ for every $1 \leq i \leq n$. Let γ_i be the path $\gamma\big|_{[t_{i-1}, t_i]}$ reparametrized to the domain $[0,1]$. Then γ has a reparametrization as $\gamma_1 \cdot \gamma_2 \cdots \gamma_n$. Let β_i be a path in $U \cap V$ from $\gamma(t_i)$ to x_0, $1 \leq i \leq n-1$. In terms of path-homotopy in $U \cup V$ we have:

$$\begin{aligned} \gamma &\sim \gamma_1 \cdot \gamma_2 \cdots \gamma_n \\ &\sim (\gamma_1 \cdot \beta_1) \cdot (\beta_1^{-1} \cdot \gamma_2 \cdot \beta_2) \cdots (\beta_{n-2}^{-1} \cdot \gamma_{n-1} \cdot \beta_{n-1})(\beta_{n-1}^{-1} \cdot \gamma_n). \end{aligned}$$

Thus every loop at x_0 in $U \cup V$ is path-homotopic to a product of loops at x_0 each of which is contained entirely either in U or in V.

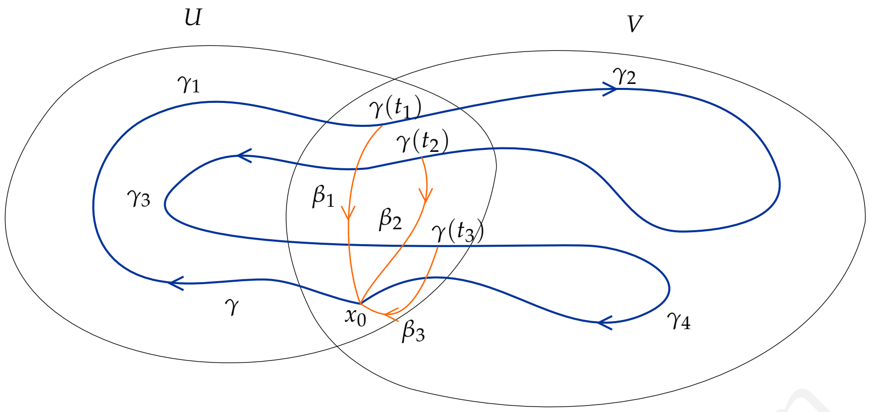
Let $j_U : U \hookrightarrow U \cup V$ and $j_V : V \hookrightarrow U \cup V$ be inclusion maps. Let

$$\begin{aligned} \Phi : \pi_1(U, x_0) * \pi_1(V, x_0) &\to \pi_1(U \cup V, x_0) \\ a_1 b_1 \cdots a_n b_n &\mapsto (j_U)_*(a_1)(j_V)_*(b_1) \cdots (j_U)_*(a_n)(j_V)_*(b_n). \end{aligned}$$

It is immediate that Φ is a homomorphism. The above disscusion implies that Φ is surjective.

For the kernel of Φ, let $\alpha \in \pi_1(U \cap V, x_0)$, then

$$\begin{aligned} \Phi((i_U)_*(\alpha)(i_V)_*(\alpha)^{-1}) &= (j_U)_*((i_U)_*(\alpha))(j_V)_*((i_V)_*(\alpha)^{-1})) \\ &= (j_U \circ i_U)_*(\alpha)(j_V \circ i_V)_*(\alpha)^{-1} = 1, \end{aligned}$$



by noticing that $j_U \circ i_U = j_V \circ i_V$. Thus $\ker \Phi$ contains the normal subgroup generated by all elements $(i_U)_*(\alpha)(i_V)_*(\alpha)^{-1}$, $\alpha \in \pi_1(U \cap V, x_0)$.

That $\ker \Phi$ is equal to that group is more difficult to prove, we will not present a proof. The reader can read for instance in **[Vic94]**. □

COROLLARY. *The spheres of dimensions greater than one are simply connected:*

$$\pi_1(S^n) \cong \begin{cases} \mathbb{Z}, & n = 1 \\ 1, & n > 1. \end{cases}$$

PROOF. Let $A = S^n \setminus \{(0,0,\ldots,0,1)\}$ and $B = S^n \setminus \{(0,0,\ldots,0,-1)\}$. Then A and B are contractible. If $n \geq 2$ then $A \cap B$ is path-connected. By Van Kampen theorem, $\pi_1(S^2) \cong \pi_1(A) * \pi_1(B) = 1$. □

COROLLARY. *If $X = U \cup V$ with U, V open, path-connected, $U \cap V$ is simply connected, and $x_0 \in U \cap V$, then $\pi_1(X, x_0) \cong \pi_1(U, x_0) * \pi_1(V, x_0)$.*

EXAMPLE ($S^1 \vee S^1$). Let U be the union of the first circle with an open arc on the second circle containing the common point. Similarly let V be the union of the second circle with an open arc on the first circle containing the common point. Clearly U and V have deformation retractions to the first and the second circles respectively, while $U \cap V$ is ismply connected (has a deformation retraction to the common point). Applying the Van Kampen theorem, or the corollary above, we get

$$\pi_1(S^1 \vee S^1) \cong \pi_1(S^1) * \pi_1(S^1) \cong \mathbb{Z} * \mathbb{Z}.$$

The fundamental group of a cell complex.

THEOREM 15.1. *Let X be a path-connected topological space and consider the space $X \sqcup_f D^n$ obtained by attaching an n-dimensional cell to X via the map $f : \partial D^n = S^{n-1} \to X$.*

(a) *If $n > 2$ then $\pi_1(X \sqcup_f D^n) \cong \pi_1(X)$.*
(b) *If $n = 2$: Let γ_1 be the loop $(\cos 2\pi t, \sin 2\pi t)$ on the circle ∂D^2. Let $x_0 = f(1,0)$. Then $\pi_1(X \sqcup_f D^n, x_0) \cong \pi_1(X, x_0) / \langle [f \circ \gamma_1] \rangle$.*

Intuitively, *gluing a 2-disk destroys the boundary circle of the disk homotopically*, while *gluing disks of dimensions greater than 2 does not affect the fundamental group*.

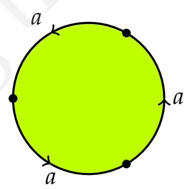
PROOF. Let $Y = X \sqcup_f D^n$. Let $U = X \sqcup_f \{x \in D^n \mid \|x\| > \frac{1}{2}\} \subset Y$. There is a deformation retraction from U to X. Let V be the image of the imbedding of the interior of D^n in Y. Then V is contractible. Also, $U \cap V$ is homeomorphic to $\{x \in D^n \mid \frac{1}{2} < \|x\| < 1\}$, which has a deformation retraction to S^{n-1}. We now apply Van Kampen theorem to the pair (U, V).

When $n > 2$ the fundamental group of $U \cap V$ is trivial, therefore $\pi_1(Y) \cong \pi_1(U) \cong \pi_1(X)$.

Consider the case $n = 2$. Let $y_0 \in U \cap V$ be the image of the point $(\frac{2}{3}, 0) \in D^2$, let γ be imbedding of the loop $\frac{2}{3}(\cos 2\pi t, \sin 2\pi t)$ starting at y_0 going once around the annulus $U \cap V$. Then $[\gamma]$ is a generator of $\pi_1(U \cap V, y_0)$. In V the loop γ is homotopically trivial, therefore $\pi_1(Y, y_0) \cong \pi_1(U, y_0) / \langle [\gamma] \rangle$. Under the deformation retraction from U to X, $[\gamma]$ becomes $[f \circ \gamma_1]$. Therefore $\pi_1(Y, x_0) \cong \pi_1(X, x_0) / \langle [f \circ \gamma_1] \rangle$. $\square$

This result shows that the fundamental group only gives information about the two-dimensional skeleton of a cell complex, it does not give information on cells of dimensions greater than 2.

EXAMPLE 15.2. Consider the space below. As a cell-complex it consists of



one 0-cell, one 1-cell (represented by a), forming the 1-dimensional skeleton (a circle, also denoted by a), and one 2-cell attached to the 1-dimensional skeleton by wrapping the boundary of the disk around $a$three times. Thus the fundamental group of the space is isomorphic to $\langle a \mid a^3 = 1 \rangle \cong \mathbb{Z}_3$. In practice, we get the fundamental group immediately by seeing that the boundary of the disk is a^3 and gluing this disk homotopically destroys a^3.

The fundamental groups of surfaces. By the classification theorem, any compact without boundary surface is obtained by identifying the edges of a polygon following a word as in 11.2. As such it has a cell complex structure with a two-dimensional disk glued to the boundary of the polygon under the equivalence relation, which is a wedge of circles. An application of 15.1 gives us:

THEOREM. *The fundamental group of a connected compact surface S is isomorphic to one of the following groups:*

(a) *trivial group, if $S = S^2$,*

(b) $\left\langle a_1, b_1 a_2, b_2, \ldots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \right\rangle$, *if S is the orientable surface of genus g,*

(c) $\left\langle c_1, c_2, \ldots, c_g \mid c_1^2 c_2^2 \cdots c_g^2 \right\rangle$, *if S is the unorientable surface of genus g.*

EXAMPLE. For the torus: $\pi_1(T_1) \cong \langle a, b \mid ab = ba \rangle \cong \mathbb{Z} \times \mathbb{Z}$.
For the projective plane: $\pi_1(\mathbb{RP}^2) \cong \langle c \mid c^2 = 1 \rangle \cong \mathbb{Z}_2$.

Problems.

15.3. Use the Van Kampen theorem to find the fundamental groups of the following spaces:

(a) A wedge of finitely many circles.
(b) $S^1 \vee S^2$.
(c) $S^2 \vee S^3$.
(d) The plane minus finitely many points.
(e) The Euclidean space $\mathbb{R}^3$ minus finitely many points.

15.4. Give a rigorous definition of the wedge product of two spaces. For example, what really is $S^1 \vee S^1$?

15.5. By problem 9.14 we have two different presentations for the fundamental group of the Klein bottle: $\pi_1(K) \cong \langle a, b \mid aba^{-1}b \rangle$ and $\pi_1(K) \cong \langle a, b \mid a^2 b^2 \rangle$. Show directly that $\langle a, b \mid aba^{-1}b \rangle \cong \langle a, b \mid a^2 b^2 \rangle$.

15.6. Is the fundamental group of the Klein bottle abelian?

15.7. Show that the fundamental groups of the one-hole torus and the two-holes torus are not isomorphic. Therefore the two surfaces are different.

15.8. Find a space whose fundamental group is isomorphic to $\mathbb{Z} * \mathbb{Z}_2 * \mathbb{Z}_3$.

15.9. Show that if $X = A \cup B$, where A and B are open, simply connected, and $A \cap B$ is path-connected, then X is simply connected.

15.10. Is $\mathbb{R}^3 \setminus \{(0,0,0)\}$ simply connected? How about $\mathbb{R}^3 \setminus \{(0,0,0), (1,0,0)\}$?

15.11 (two-dimensional Poincaré conjecture). If a compact surface is simply connected then it is homeomorphic to the two-dimensional sphere.

15.12. Consider the three-dimensional torus T^3, obtained from the cube $[0,1]^3$ by identifying opposite faces by projection maps, that is $(x,y,0) \sim (x,y,1)$, $(0,y,z) \sim (1,y,z)$, $(x,0,z) \sim (x,1,z)$, $\forall (x,y,z) \in [0,1]^3$.

(a) Show that T^3 is homeomorphic to $S^1 \times S^1 \times S^1$.
(b) Show that T^3 is a 3-dimensional manifold.
(c) Construct a cellular structure on T^3.
(d) Compute the fundamental group of T^3.

15.13. * Consider the space X obtained from the cube $[0,1]^3$ by rotating each lower face (i.e. faces on the planes xOy, yOz, zOx) an angle of $\pi/2$ about the normal line that goes through the center of the face, then gluing this face to the opposite face. For example the point (1,1,0) will be glued to the point $(0,1,1)$.

(a) Show that X has a cellular structure consisting of 2 0-cells, 4 1-cells, 3 2-cells, 1 3-cell.
(b) Compute the fundamental group of X.
(c) Check that the fundamental group of X is isomorphic to the quaternion group.

15.14. If

$$G = \langle g_1, g_2, \ldots, g_{m_1} \mid r_1, r_2, \ldots, r_{n_1} \rangle$$

and

$$H = \langle h_1, h_2, \ldots, h_{m_2} \mid s_1, s_2, \ldots, s_{n_2} \rangle$$

then

$$G * H = \langle g_1, g_2, \ldots, g_{m_1}, h_1, h_2, \ldots, h_{m_2} \mid r_1, r_2, \ldots, r_{n_1}, s_1, s_2, \ldots, s_{n_2} \rangle .$$

16. Homology

In homology theory we associate each space with a sequence of groups.

A *singular simplex* (đơn hình suy biến, kì dị) is a continuous map from a standard simplex to a topological space. More precisely, an n-dimensional singular simplex in a topological space X is a continuous map $\sigma : \Delta_n \to X$, where Δ_n is the standard n-simplex.

Let $S_n(X)$ be the free abelian group generated by all n-dimensional singular simplexes in X. As a set

$$S_n(X) = \left\{ \sum_{i=1}^{m} n_i\sigma_i \mid \sigma_i : \Delta_n \to X,\ n_i \in \mathbb{Z},\ m \in \mathbb{Z}^+ \right\}.$$

Each element of $S_n(X)$ is a finite sum of integer multiples of n-dimensional singular simplexes, called a *singular n-chain.*

Let σ be an n-dimensional singular simplex in X, i.e., a map

$$\begin{aligned} \sigma : \Delta_n &\to X \\ (t_0, t_1, \dots, t_n) &\mapsto \sigma(t_0, t_1, \dots, t_n). \end{aligned}$$

For $0 \le i \le n$ define the *ith face* of σ to be the $(n-1)$-singular simplex

$$\begin{aligned} \partial_n^i(\sigma) : \Delta_{n-1} &\to X \\ (t_0, t_1, \dots, t_{n-1}) &\mapsto \sigma(t_0, t_1, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}). \end{aligned}$$

Define the *boundary* of σ to be the singular $(n-1)$-chain

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \partial_n^i(\sigma).$$

Intuitively one can think of $\partial_n(\sigma)$ as $\sigma|_{\partial\Delta_n}$. The map ∂_n is extended linearly to become a group homomorphism from $S_n(X)$ to $S_{n-1}(X)$.

EXAMPLE. A 0-dimensional singular simplex in X is a point in X.

A 1-dimensional singular simplex is a continuous map $\sigma(t_0, t_1)$ with $t_0, t_1 \in [0,1]$ and $t_0 + t_1 = 1$. Its image is a curve between the points $A = \sigma(1,0)$ and $B = \sigma(0,1)$. Its boundary is $-A + B$.

A 2-dimensional singular simplex is a continuous map $\sigma(t_0, t_1, t_2)$ with $t_0, t_1, t_2 \in [0,1]$ and $t_0 + t_1 + t_2 = 1$. Its image is a “curved triangle” between the points $A = \sigma(1,0,0)$, $B = \sigma(0,1,0)$, and $C = (0,0,1)$. Intuitively, the image of the face ∂^0 is the “curved edge” BC, the image of ∂^1 is AC, and the image of ∂^2 is AB. The boundary is $\partial^0 - \partial^1 + \partial^2$. Intuitively, it is $BC - AC + AB$.

PROPOSITION 16.1. $\partial_{n-1} \circ \partial_n = 0$, $\forall n \ge 2$.

PROOF. Let σ be a singular n-simplex. From definition:

$$
\begin{aligned}
(\partial_{n-1}\partial_n)(\sigma) &= \partial_{n-1}\left(\sum_{i=0}^{n}(-1)^i\partial_n^i(\sigma)\right) \\
&= \sum_{j=0}^{n-1}(-1)^j\partial_{n-1}^j\left(\sum_{i=0}^{n}(-1)^i\partial_n^i(\sigma)\right) \\
&= \sum_{0\le i\le n, 0\le j\le n-1}(-1)^{i+j}\partial_{n-1}^j\partial_n^i(\sigma) \\
&= \sum_{0\le j<i\le n}(-1)^{i+j}\partial_{n-1}^j\partial_n^i(\sigma) + \\
&\quad + \sum_{0\le i\le j\le n-1}(-1)^{i+j}\partial_{n-1}^j\partial_n^i(\sigma) \\
&= \sum_{0\le j<i\le n}(-1)^{i+j}\partial_{n-1}^j\partial_n^i(\sigma) + \\
&\quad + \sum_{0\le i<k\le n,\ (k=j+1)}(-1)^{i+k-1}\partial_{n-1}^{k-1}\partial_n^i(\sigma) \\
&= \sum_{0\le j<i\le n}(-1)^{i+j}\partial_{n-1}^j\partial_n^i(\sigma) + \\
&\quad + \sum_{0\le j<i\le n}-(-1)^{i+j}\partial_{n-1}^{i-1}\partial_n^j(\sigma).
\end{aligned}
$$

For $0 \le j < i \le n$:

$$
\begin{aligned}
\partial_{n-1}^j\partial_n^i(\sigma)(t_0,\ldots,t_{n-2}) &= \partial_n^i\sigma(t_0,\ldots,t_{j-1},0,t_j,\ldots,t_{n-2}) \\
&= \sigma(t_0,\ldots,t_{j-1},0,t_j,\ldots,t_{i-2},0,t_{i-1},\ldots,t_{n-2}),
\end{aligned}
$$

while

$$
\begin{aligned}
\partial_{n-1}^{i-1}\partial_n^j(\sigma)(t_0,\ldots,t_{n-2}) &= \partial_n^j\sigma(t_0,\ldots,t_{i-2},0,t_{i-1},\ldots,t_{n-2}) \\
&= \sigma(t_0,\ldots,t_{j-1},0,t_j,\ldots,t_{i-2},0,t_{i-1},\ldots,t_{n-2}),
\end{aligned}
$$

thus $\partial_{n-1}^j\partial_n^i = \partial_{n-1}^{i-1}\partial_n^j$. So $\partial_{n-1}\circ\partial_n = 0$. □

That $\partial_{n-1}\circ\partial_n = 0$ can be interpreted as

$$\operatorname{Im}(\partial_{n+1}) \subset \ker(\partial_n).$$

Elements of $\ker(\partial_n)$ are often called *closed chains* or *cycles*, while elements of $\operatorname{Im}(\partial_{n+1})$ are called *exact chains* or *boundaries*. We have just observed that

exact ⇒ closed

boundaries are cycles

Homology group. In general, a sequence of groups and homomorphisms

$$\ldots \xrightarrow{\partial_{n+2}} S_{n+1} \xrightarrow{\partial_{n+1}} S_n \xrightarrow{\partial_n} S_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_1} S_0$$

satisfying $\operatorname{Im}(\partial_{n+1}) \subset \ker(\partial_n)$, i.e. $\partial_n \circ \partial_{n+1} = 0$ for all $n \geq 0$ is called a *chain complex* (phức xích). If furthermore $\operatorname{Im}(\partial_{n+1}) = \ker(\partial_n)$, $\forall n \geq 0$ then the chain complex is called *exact* (khớp).

Notice that if the group S_n are abelian then $\operatorname{Im}(\partial_{n+1})$ is a normal subgroup of $\ker(\partial_n)$.

DEFINITION. The n-dimensional *singular homology group* of a topological space X is defined to be the quotient group

$$H_n(X) = \frac{\ker(\partial_n)}{\operatorname{Im}(\partial_{n+1})}.$$

EXAMPLE. Denoting by $\{pt\}$ a space containing only one point, then $H_n(\{pt\}) = 0$, $n \geq 1$ and $H_0(\{pt\}) = \langle pt \rangle \cong \mathbb{Z}$.

PROPOSITION. *If X is path-connected then $H_0(X) \simeq \mathbb{Z}$, generated by any point of X. In general $H_0(X)$ is generated by one point in each path-connected components of X.*

PROOF. Let x_0 and x_1 be two points in X. A continuous path from x_0 to x_1 gives rise to a singular 1-simplex $\sigma : \Delta_1 \to X$ such that $\sigma(0,1) = x_0$ and $\sigma(1,0) = x_1$. The boundary of this singular simplex is $\partial\sigma = x_1 - x_0 \in \operatorname{Im}\partial_1$. Thus $[x_0] = [x_1] \in H_0(X) = S_0(X)/\operatorname{Im}\partial_1$. $\square$

Induced homomorphism. Let X and Y be topological spaces and let $f : X \to Y$ be continuous. For any n-singular simplex σ let $f_\#(\sigma) = f \circ \sigma$, then extend $f_\#$ linearly, we get a group homomorphism $f_\# : S_n(X) \to S_n(Y)$.

EXAMPLE. $\mathrm{Id}_* = \mathrm{Id}$.

LEMMA. $\partial \circ f_\# = f_\# \circ \partial$. *As a consequence $f_\#$ brings cycles to cycles, boundaries to boundaries.*

PROOF. Because both $f_\#$ and ∂ are linear we only need to prove $\partial(f_\#(\sigma)) = (f_\#(\partial(\sigma))$ for any n-singular simplex $\sigma : \Delta_n \to X$. We have

$$f_\#(\partial(\sigma)) = f_\#\left(\sum_{i=0}^{n}(-1)^i\partial^i(\sigma)\right) = \sum_{i=0}^{n}(-1)^i f \circ \partial^i(\sigma).$$

On the other side: $\partial(f_\#(\sigma)) = \sum_{i=0}^{n}(-1)^i\partial^i(f \circ \sigma)$. Notice that

$$\begin{aligned}
\partial^i(f \circ \sigma)(t_0, \ldots, t_{n-1}) &= (f \circ \sigma)(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}) \\
&= f(\sigma(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})) \\
&= f(\partial^i\sigma(t_0, \ldots, t_{n-1})).
\end{aligned}$$

Thus $\partial^i(f \circ \sigma) = f \circ \partial^i(\sigma)$. From this the result follows. $\square$

As a consequence $f_\#$ induces a group homomorphism:

$$\begin{aligned}
f_* : H_n(X) &\to H_n(Y). \\
[c] &\mapsto [f_\#(c)].
\end{aligned}$$

LEMMA. $(g \circ f)_* = g_* \circ f_*$.

PROOF. Since the maps involved are linear it is sufficient to check this property on each n-singular simplex σ:

$$(g \circ f)_*([\sigma]) = [(g \circ f)_\#(\sigma)] = [(g \circ f) \circ \sigma] = [g \circ (f \circ (\sigma))] = g_*([f \circ \sigma]) = \\ = g_*(f_*([\sigma])).$$

□

A simple application of the above lemma gives us an important result:

THEOREM (topological invariance of homology). *If $f : X \to Y$ is a homeomorphism then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism.*

PROOF. Apply the above lemma to the pair f and f^{-1}. □

We will use, but will not prove, the following stronger result :

THEOREM (homotopy invariance of homology). *If $f : X \to Y$ is a homotopy equivalence then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism.*

Mayer-Vietoris sequence.

THEOREM. *Let X be a topological space. Suppose $U, V \subset X$ and $int(U) \cup int(V) = X$. Then there is an exact chain complex, called the Mayer-Vietoris sequence:*

$$\cdots \to H_n(U \cap V) \xrightarrow{(i_*, j_*)} H_n(U) \oplus H_n(V) \xrightarrow{\psi_*} H_n(U \cup V) \xrightarrow{\Delta} H_{n-1}(U \cap V) \to \cdots \\ \cdots \to H_0(U \cup V) \to 0.$$

Here i and j are the inclusion maps from $U \cap V$ to U and V respectively.

The Mayer-Vietoris sequence allows us to study the homology of a space from homologies of subspaces, in a similar manner to the Van Kampen theorem.

We will use the Mayer-Vietoris sequence to compute the homology of the sphere:

THEOREM. *For $m \geq 1$,*

$$H_n(S^m) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, m \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. Let U and V be the upper hemisphere and the lower hemisphere slightly enlarged, for example, $U = S^m \setminus \{(0, \ldots, 0, -1)\}$ and $V = S^m \setminus \{(0, \ldots, 0, 1)\}$. The Mayer-Vietoris sequence for this pair gives an exact short sequence:

$$H_n(U) \oplus H_n(V) \to H_n(S^m) \to H_{n-1}(U \cap V) \to H_{n-1}(U) \oplus H_n(V).$$

Notice that for $m \geq 1$, U and V are contractible, while $U \cap V$ is homotopic to S^{m-1}. We get for $n \geq 2$ a short exact sequence

$$0 \to H_n(S^m) \to H_{n-1}(S^{m-1}) \to 0.$$

This implies $H_n(S^m) \cong H_{n-1}(S^{m-1})$ for $n \geq 2$. The problem now reduces to computation of $H_1(S^m)$.

Consider the exact sequence:

$$H_1(U) \oplus H_1(V) \to H_1(S^m) \xrightarrow{\Delta} H_0(U \cap V) \xrightarrow{(i_*, j_*)} H_0(U) \oplus H_0(V).$$

Since $H_1(U) = H_1(V) = 0$, it follows that Δ is injective.

For $m \geq 2$ a point $x \in U \cap V$ generates $H_0(U \cap V)$ as well as $H_0(U)$ and $H_0(V)$. Therefore the maps i_* and j_* are injective. This implies $\operatorname{Im}(\Delta) = 0$. This can happen only when $H_1(S^m) = 0$.

For $m = 1$ the intersection $U \cap V$ has two path-connected components. Let x and y be points in each connected component. If $mx + ny \in H_0(U \cap V)$ then $i_*(mx + ny) = j_*(mx + ny) = mx + nx = (m + n)x$. Thus $\ker(i_*, j_*) = \{mx - my = m(x - y) \mid m \in \mathbb{Z}\}$. This implies $H_1(S^1) \cong \operatorname{Im}(\Delta) = \ker(i_*, j_*) \cong \mathbb{Z}$. □

COROLLARY 16.2. *For $n \geq 2$ there cannot be any retraction from the disk D^n to its boundary S^{n-1}.*

The proof is similar to the 2-dimensional case (14.3).

PROOF. Suppose there is a retraction $r : D^n \to S^{n-1}$. Let $i : S^{n-1} \hookrightarrow D^n$ be the inclusion map. From the diagram $S^{n-1} \xrightarrow{i} D^n \xrightarrow{r} S^{n-1}$ we have $r \circ i = \mathrm{id}_{S^{n-1}}$, therefore on the $(n-1)$-dimensional homology groups $(r \circ i)_* = r_* \circ i_* = \mathrm{id}_{H_{n-1}(S^{n-1})}$, implying that $r_* : H_{n-1}(D^n) \to H_{n-1}(S^{n-1})$ is onto. But this is not possible for $n \geq 2$, since $H_{n-1}(D^n)$ is trivial while $H_{n-1}(S^{n-1})$ is not. □

A proof of this result in differentiable setting is presented in 24.1.

Just as in the case of dimension two (14.4), the Brouwer fixed point theorem follows, with the same proof:

THEOREM 16.3 (Brouwer fixed point theorem). *Any continuous map from the disk D^n to itself has a fixed point.*

By 16.16 and 16.15 we get a more general version:

THEOREM (generalized Brouwer fixed point theorem). *Any continuous map from a compact convex subset of a Euclidean space to itself has a fixed point.*

For more on singular homology, one can read **[Vic94]** and **[Hat01]**.

Problems.

16.4. Prove 16.1.

16.5. Compute the homology groups of $S^2 \times [0, 1]$.

16.6. Compute the fundamental group and the homology groups of the Euclidean space $\mathbb{R}^3$ minus a straight line.

16.7. Compute the fundamental group and the homology groups of the Euclidean space $\mathbb{R}^3$ minus two intersecting straight lines.

16.8. Compute the fundamental group and the homology groups of $\mathbb{R}^3 \setminus S^1$.

16.9. Show that if X has k connected components X_i, $1 \leq i \leq k$, then $S_n(X) \cong \bigoplus_{i=1}^k S_n(X_i)$ and $H_n(X) \cong \bigoplus_{i=1}^k H_n(X_i)$.

16.10. Show that if A and B are open, $A \cap B$ is contractible then $H_i(A \cup B) \cong H_i(A) \oplus H_i(B)$ for $i \geq 2$. Is this true if $i = 0, 1$?

16.11. Using problem 16.10, compute the homology groups of $S^2 \vee S^4$.

16.12. Compute the fundamental group and the homology groups of $S^2 \cup \{(0,0,z) \mid -1 \leq z \leq 1\}$.

16.13. Let U be an open subset of $\mathbb{R}^2$ and $a \in U$. Show that

$$H_1(U \setminus \{a\}) \cong H_1(U) \oplus \mathbb{Z}.$$

16.14. Let A be a retract of X via a retraction $r : X \to A$. Let $i : A \hookrightarrow X$ be the inclusion map. Show that the induced homomorphism $i_* : H_n(A) \to H_n(X)$ is injective while $r_* : H_n(X) \to H_n(A)$ is surjective.

16.15. Show that if every continuous map from X to itself has a fixed point and Y is homeomorphic to X then every continuous map from Y to itself has a fixed point.

16.16. * Show that:

(a) Any convex compact subset of $\mathbb{R}^n$ with empty interior is homemorphic to the closed interval D^1.
(b) Any convex compact subset of $\mathbb{R}^n$ with non-empty interior is homemorphic to the disk D^n.

16.17. Is the Brouwer fixed point theorem correct for open balls? for spheres? for tori?

17. Homology of cell complexes

In this section we consider homology for a special class of topological spaces: cell complexes, including simplicial complexes.

Degrees of maps on spheres. A continuous map $f : S^n \to S^n$ induces a homomorphism $f_* : H_n(S^n) \to H_n(S^n)$. We know $H_n(S^n) \cong \mathbb{Z}$, so there is a generator a such that $H_n(S^n) = \langle a \rangle$. Then $f_*(a) = ma$ for a certain integer m, called the *topological degree* of f, denoted by $\deg f$.

EXAMPLE. If f is the identity map then $\deg f = 1$. If f is the constant map then $\deg f = 0$.

Relative homology groups. Let A be a subspace of X. Viewing each singular simplex in A as a singular simplex in X, we have a natural inclusion $S_n(A) \hookrightarrow S_n(X)$. In this way $S_n(A)$ is a normal subgroup of $S_n(X)$. The boundary map ∂_n induces a homomorphism $\partial_n : S_n(X)/S_n(A) \to S_{n-1}(X)/S_{n-1}(A)$, giving a chain complex

$$\cdots \to S_n(X)/S_n(A) \xrightarrow{\partial_n} S_{n-1}(X)/S_{n-1}(A) \xrightarrow{\partial_{n-1}} S_{n-2}(X)/S_{n-2}(A) \to \cdots$$

The homology groups of this chain complex is called the *relative homology groups* of the pair (X, A), denoted by $H_n(X, A)$.

If $f : X \to Y$ is continuous and $f(A) \subset B$ then as before it induces a homomorphism $f_* : H_n(X, A) \to H_n(Y, B)$.

Homology of cell complexes. Let X be a cellular complex. Recall that X^n denote the n-dimensional skeleton of X. Suppose that X^n is obtained from X^{n-1} by attaching the n-dimensional disks $D_1^n, D_2^n, \dots, D_{c_n}^n$. Let $e_1^n, e_2^n, \dots, e_{c_n}^n$ be the corresponding cells. Then

$$H_n(X^n, X^{n-1}) \cong \langle e_1^n, e_2^n, \dots, e_{c_n}^n \rangle = \{\sum_{i=1}^{c_n} m_i e_i^n \mid m_i \in \mathbb{Z}\}.$$

If $c_n = 0$ then let the group be 0.

Consider the sequence

$$C(X) = \cdots \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \xrightarrow{d_{n-1}} \cdots$$
$$\cdots \xrightarrow{d_2} H_1(X^1, X^0) \xrightarrow{d_1} H_0(X).$$

Here the map d_n is given by

$$d_n(e_i^n) = \sum_{j=1}^{c_{n-1}} d_{i,j} e_j^{n-1},$$

where the integer number $d_{i,j}$ is the degree of the following map on spheres:

$$S_i^{n-1} = \partial D_i^n \to X^{n-1} \to X^{n-1}/X^{n-2} = S_1^{n-1} \vee S_2^{n-1} \vee \cdots \vee S_{c_{n-1}}^{n-1} \to S_j^{n-1}.$$

Of course if $c_{n-1} = 0$ then $d_n = 0$.

Below is the main tool for computing homology of cellular complexes:

THEOREM. *The sequence $C(X)$ is a chain complex and its homology coincides with the homology of X.*

. As an application we get:

THEOREM (homology groups of surfaces). *The homology groups of a connected compact orientable surface S of genus $g \geq 0$ is*

$$H_n(S) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 2 \\ \mathbb{Z}^g & \text{if } n = 1. \end{cases}$$

For more on cellular homology one can read [**Hat01**, p. 137].

Homology of simplicial complexes. Now we look at a special class of cell complexes: simplicial complexes. For simplicial complexes we will develop an algorithm for computing homology.

Consider a simplex of dimension greater than 0. Consider the relation on the collection of ordered sets of vertices of this simplex whereas two order sets of vertices are related if they differ by an even permutation. This is an equivalence relation. Each of the two equivalence classes is called an *orientation* of the simplex. If we choose an orientation, then the simplex is said to be *oriented*.

In more details, an orientation for a set A consisting of n elements is represented by bijective map from the set of integers $I = \{1, 2, \ldots, n\}$ to A. If o and o' are two such maps, then $o = o' \circ \sigma$ where $\sigma = o'^{-1} \circ o : I \to I$ is bijective, a permutation. Any permutation falls into one of tow classes: even or odd permutations. We say that o and o' represent the same orientations if σ is an even permutation.

For each simplex of dimension greater than 0 there are two oriented simplexes. For convenience we say that for a 0-dimensional simplex (a vertex) there is only one orientation.

REMARK. In this section, to be clearer we reserve the notation $v_0v_1 \cdots v_n$ for an un-oriented simplex with the set of vertices $\{v_0, v_1, \ldots, v_n\}$, and the notation $[v_0, v_1, \ldots, v_n]$ for an oriented simplex with the same set of vertices, i.e. the simplex $v_0v_1 \cdots v_n$ with this particular order of vertices. (Later, and in other sources, this convention can be dropped.)

EXAMPLE. A 1-dimensional simplex in $\mathbb{R}^n$ is a straight segment connecting two points. Choosing one point as the first point and the other point as the second gives an orientation to this simplex. Intuitively, this means to give a direction to the straight segment. If the two vertices are labeled v_0 and v_1, then an ordered pair $[v_0, v_1]$ gives an orientation for this simplex, while an ordered pair $[v_1, v_0]$ gives a simplex with the opposite orientation.

Consider a 2-dimensional simplex. Let v_0, v_1, v_2 be the vertices. The oriented simplexes $[v_0, v_1, v_2]$, $[v_1, v_2, v_0]$, $[v_2, v_0, v_1]$ have the same orientations, opposite to the orientations of the oriented simplexes $[v_1, v_0, v_2]$, $[v_2, v_1, v_0]$, $[v_0, v_2, v_1]$.

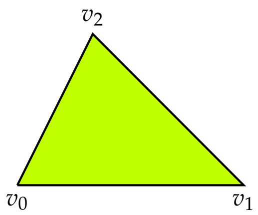


FIGURE 17.1.

Let X be a simplicial complex in a Euclidean space. For each integer n let $S_n(X)$ be the free abelian group generated by all n-dimensional oriented simplexes in X modulo the relation that if σ and σ' are oriented simplexes with opposite orientations then $\sigma = -\sigma'$. Each element of $S_n(X)$, called an n-dimensional *chain* (xích), is a finite sum of integer multiples of n-dimensional oriented simplexes, i.e. of the form $\sum_{i=1}^{m} n_i\sigma_i$ where σ_i is an n-dimensional oriented simplex of X and $n_i \in \mathbb{Z}$.

If $n < 0$ or $n > \dim X$ then $S_n(X)$ is assigned to be the trivial group 0.

The group $S_n(X)$ is generated by the set of all n-simplexes of X with each simplex given any orientation.

EXAMPLE 17.2. Consider the simplicial complex whose underlying space is the triangle in figure 17.1:

$$X = \{v_0v_1v_2, v_0v_1, v_1v_2, v_2v_0, v_0, v_1, v_2\}.$$

For $n = 0$:

$$S_0(X) = \{n_0v_0 + n_1v_1 + n_2v_2 \mid n_0, n_1, n_2 \in \mathbb{Z}\}.$$

For $n = 1$:

$$S_1(X) = \{n_0[v_0, v_1] + n_1[v_1, v_2] + n_2[v_2, v_0] \mid n_0, n_1, n_2 \in \mathbb{Z}\}.$$

In this group $S_1(X)$ we have the relations $[v_1, v_0] = -[v_0, v_1]$, $[v_2, v_1] = -[v_1, v_2]$, $[v_0, v_2] = -[v_2, v_0]$.

For $n = 2$:

$$S_2(X) = \{n_0[v_0, v_1, v_2] \mid n_0 \in \mathbb{Z}\}.$$

Here for example $[v_0, v_1, v_2] = -[v_2, v_1, v_0] = [v_1, v_2, v_0]$.

Let $\sigma = [v_0, v_1, \dots, v_n]$ be an n-dimensional oriented simplex. Define the *boundary* of σ to be the following $(n-1)$-dimensional chain, the alternating sum of the $(n-1)$-dimensional faces of σ:

$$\partial_n\sigma = \sum_{i=0}^{n}(-1)^i[v_0, v_1, \dots, v_{i-1}, \widehat{v_i}, v_{i+1}, \dots, v_n],$$

where the notation $\widehat{v_i}$ is traditionally used to indicate that this element is dropped. This map is extended linearly to become a map from $S_n(X)$ to $S_{n-1}(X)$, namely

$$\partial_n \left(\sum_{i=1}^{m} n_i \sigma_i \right) = \sum_{i=1}^{m} n_i \partial_n (\sigma_i).$$

It can be checked directly that this boundary map does not depend on choices of orientations of simplexes.

REMARK. If $n < 0$ or $n > \dim X$ then $\partial_n = 0$ since $S_n(X) = 0$.

EXAMPLE. Continuing example 17.2:

$$\partial_1([v_0, v_1]) = v_1 - v_0,$$
$$\partial_1([v_1, v_2]) = v_2 - v_1,$$
$$\partial_1([v_2, v_0]) = v_0 - v_2,$$
$$\partial_2([v_0, v_1, v_2]) = [v_1, v_2] - [v_0, v_2] + [v_0, v_1] = [v_1, v_2] + [v_2, v_0] + [v_0, v_1].$$

Notice that:

$$\begin{aligned} \partial_1(\partial_2([v_0, v_1, v_2])) &= \partial_1([v_0, v_1] + [v_1, v_2] + [v_2, v_0]) \\ &= (v_1 - v_0) + (v_2 - v_1) + (v_0 - v_2) = 0. \end{aligned}$$

By 17, we have:

THEOREM (simplicial homology). *The n-dimensional group of a simplicial complex is isomorphic to the quotient group*

$$\frac{\ker(\partial_n)}{\operatorname{Im}(\partial_{n+1})}.$$

EXAMPLE. Continuing example 17.2, we compute the homology groups of X. From

$$\partial_1(n_0[v_0, v_1] + n_1[v_1, v_2] + n_2[v_2, v_0]) = n_0(v_1 - v_0) + n_1(v_2 - v_1) + n_2(v_0 - v_2),$$

we see that an element of $\operatorname{Im} \partial_1$ has the form

$$\begin{aligned} n_0(v_1 - v_0) + n_1(v_2 - v_1) + n_2(v_0 - v_2) &= n_0(v_1 - v_0) + n_1(v_2 - v_0 + v_0 - v_1) + n_2(v_0 - v_2) \\ &= (n_0 - n_1)(v_1 - v_0) + (n_1 - n_2)(v_2 - v_0). \end{aligned}$$

Thus $\operatorname{Im} \partial_1 = \langle \{v_1 - v_0, v_2 - v_0\} \rangle$. So

$$\begin{aligned} H_0(X) &= S_0(X) / \operatorname{Im} \partial_1 \cong \langle v_0, v_1, v_2 \mid v_1 = v_0, v_2 = v_0 \rangle \\ &\cong \langle v_0 \rangle \cong \langle v_1 \rangle \cong \langle v_2 \rangle \cong \mathbb{Z}. \end{aligned}$$

Since

$$n_0(v_1 - v_0) + n_1(v_2 - v_1) + n_2(v_0 - v_2) = (n_2 - n_0)v_0 + (n_0 - n_1)v_1 + (n_1 - n_2)v_2$$

we see that $\partial_1(n_0[v_0, v_1] + n_1[v_1, v_2] + n_2[v_2, v_0]) = 0$ if and only if $n_0 = n_1 = n_2$. So

$$\ker \partial_1 = \{n([v_0, v_1] + [v_1, v_2] + [v_2, v_0]) \mid n \in \mathbb{Z}\} = \langle([v_0, v_1] + [v_1, v_2] + [v_2, v_0])\rangle .$$

Since

$$\partial_2(n[v_0, v_1, v_2]) = n([v_0, v_1] + [v_1, v_2] + [v_2, v_0])$$

we see that

$$\operatorname{Im} \partial_2 = \{n([v_0, v_1] + [v_1, v_2] + [v_2, v_0]) \mid n \in \mathbb{Z}\} = \langle([v_0, v_1] + [v_1, v_2] + [v_2, v_0])\rangle .$$

Thus $\operatorname{Im} \partial_2 = \ker \partial_1$, so

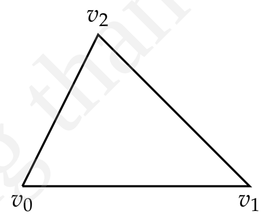
$$H_1(X) = 0.$$

We also see immediately that $\partial_2(n[v_0, v_1, v_2]) = 0$ if and only if $n = 0$. Thus $\ker \partial_2 = 0$ and

$$H_2(X) = \ker \partial_2 = 0.$$

EXAMPLE. Instead of a 2-dimensional triangle in example 17.2, let us consider a 1-dimensional triangle, namely

$$X = \{v_0v_1, v_1v_2, v_2v_0, v_0, v_1, v_2\}.$$



In this case $H_1(X) = \ker \partial_1 = \langle([v_0, v_1] + [v_1, v_2] + [v_2, v_0])\rangle \cong \mathbb{Z}$, $H_0(X) = \langle v_0 \rangle \cong \mathbb{Z}$, and $H_n(X) = 0$ for $n \neq 0, 1$.

For more on simplicial homology one can read [**Mun84**], [**Cro78**], or [**Ams83**]. In [**Hat01**] Hatcher used a modified notion called Δ-complex, different from simplicial complex. For algorithms for computation of simplicial homology, see [**KMM04**, chapter 3].

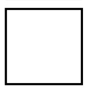
Problems.

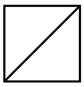
17.3. Using cellular homology compute the homology groups of the following spaces:

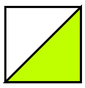
(a) The Klein bottle.
(b) $S^1 \vee S^1$.
(c) $S^1 \vee S^2$.
(d) $S^2 \vee S^3$.

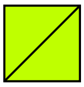
17.4. Compute the homology of the simplicial complex described in the picture:

(a)

(b) 

(c) 

(d) 

(e) 

17.5. Show that if the underlying space of a simplical complex is connected then for any two vertices there is a continuous path whose trace is a union of some edges connecting the two vertices.

17.6. Given a simplicial complex X.

(a) Show that if the underlying space $|X|$ is connected then $H_0(X) \cong \langle v \rangle \cong \mathbb{Z}$ where v is any vertex of X.

(b) Show that if $|X|$ has k connected components then $H_0(X) \cong \bigoplus_{i=1}^{k} \langle v_i \rangle \cong \mathbb{Z}^k$, where v_i, $1 \leq i \leq k$ are arbitrary vertices in different components of $|X|$.

17.7. Show that if the underlying space of a simplicial complex X has k connected components X_i, $1 \leq i \leq k$, then $H_n(X) \cong \bigoplus_{i=1}^{k} H_n(X_i)$.

17.8. Show that two compact connected surfaces are homeomorphic if and only if their homologies are isomorphic.

Other topics

The Poincaré conjecture. One of the most celebrated achievements in topology is the resolution of the Poincaré conjecture:

THEOREM (Poincaré conjecture). *A compact manifold which is homotopic to the sphere is homeomorphic to the sphere.*

The proof of this statement is the result of a cumulative effort of many mathematicians, including Stephen Smale (for dimension ≥ 5, early 1960s), Michael Freedman (for dimension 4, early 1980s), and Grigory Perelman (for dimension 3, early 2000s). For dimension 2 it is elementary, see problem 15.11.

Guide for further study. There are several books presenting aspects of algebraic topology to undergraduate students, such as [**Cro78**], [**Ams83**], [**J84**].

The book of Munkres [**Mun00**], also aims at undergraduate students, has a part on algebraic topology, but stops before homology.

The book [**Vas01**] gives a modern overview of many aspects of both algebraic and differential topology, aiming at undergraduate students. Although it often provides only sketches of proofs, it introduces general ideas very well.

For cellular homology the book of Hatcher [**Hat01**] is popular, however it aims at graduate students.

It turns out that our choice of topics is similar to that in the book [**Lee11**], which aims at graduate students. The reader can find more details and more advanced treatments there.

Recently algebraic topology has begun to be applied to science and engineering. One can read about a new field called "Computational Topology" in [**EH10**].

Differential Topology

18. Smooth manifolds

In this chapter we always assume that $\mathbb{R}^n$ has the Euclidean topology.

Roughly, a smooth manifold is a space that is locally diffeomorphic to $\mathbb{R}^n$. This allows us to bring the differential and integral calculus from $\mathbb{R}^n$ to manifolds. Smooth manifolds provide a setting for generalizations of vector calculus of curves and surfaces (see e.g. [**Vugt3**]) to higher dimensions. Smooth manifolds constitute a subset of the set of topological manifolds, introduced in section 10.

Smooth maps on $\mathbb{R}^n$. For a function f from a subset D of $\mathbb{R}^k$ to $\mathbb{R}^l$ we say that f is *smooth* (or infinitely differentiable) at an *interior point* x of D if f is continuous and partial derivatives of all orders of f exist and are continuous at x. (In common notation in analysis, f is smooth means $f \in C^\infty = \bigcap_{k=0}^{\infty} C^k$.)

If x is a boundary point of D, then f is said to be *smooth* at x if f can be extended to be a function which is smooth at every point in an open neighborhood in $\mathbb{R}^k$ of x. Precisely, f is smooth at x if there is an open set $U \subset \mathbb{R}^k$ containing x, and function $F : U \to \mathbb{R}^l$ such that F is smooth at every point of U and $F|_{U \cap D} = f$.

If f is smooth at every point of D then we say that f is smooth on D, or $f \in C^\infty(D)$.

Let $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^l$. Then $f : X \to Y$ is a *diffeomorphism* if it is bijective and both f and f^{-1} are smooth. If there is a diffeomorphism from X to Y then we say that X and Y are *diffeomorphic*.

If a function is smooth at a point, it must be continuous at that point. Therefore a diffeomorphism is also a homeomorphism.

EXAMPLE. Any open ball $B(x, r)$ in $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$ (compare 3.16).

Smooth manifolds.

DEFINITION. A subspace $M \subset \mathbb{R}^k$ is a *smooth manifold* of dimension $m \in \mathbb{Z}^+$ if every point in M has a neighborhood in M which is diffeomorphic to $\mathbb{R}^m$.

REMARK. By convention, a manifold of dimension 0 is a discrete subspace of a Euclidean space.

Recall that by Invariance of dimension 9.26, $\mathbb{R}^m$ cannot be homeomorphic to $\mathbb{R}^n$ if $m \neq n$, therefore a manifold has a unique dimension.

REMARK. A diffeomorphism is a homeomorphism, therefore a smooth manifold is a topological manifold. In differential topology unless stated otherwise manifolds mean smooth manifolds.

The following is a simple but convenient observation, although it seems to be less intuitive than our original definition, it is technically more convenient to use:

PROPOSITION. *A subspace $M \subset \mathbb{R}^k$ is a smooth manifold of dimension m if every point in M has an open neighborhood in M which is diffeomorphic to an open subset of $\mathbb{R}^m$.*

PROOF. Suppose that (U, ϕ) is a local coordinate on M where U is a neighborhood of x in M and $\phi : U \to \mathbb{R}^m$ is a diffeomorphism. There is an open subset U' of M such that $x \in U' \subset U$. Since ϕ is a homeomorphism, $\phi(U')$ is an open neighborhood of $\phi(x)$. There is a ball $B(\phi(x), r) \subset \phi(U')$. Let $U'' = \phi^{-1}(B(\phi(x), r))$. Then U'' is open in U', so is open in M. Furthermore $\phi|_{U''} : U'' \to B(\phi(x), r)$ is a diffeomorphism.

We have just shown that any point in the manifold has an open neighborhood diffeomorphic to an open ball in $\mathbb{R}^m$. For the reverse direction, we recall that any open ball in $\mathbb{R}^m$ is diffeomorphic to $\mathbb{R}^m$. □

By this result, each point x in a manifold has an open neighborhood U in M and a diffeomorphism $\varphi : U \to V$ where V is an open subset of $\mathbb{R}^m$. The pair (U, φ) is called a *local coordinate* at x. The pair (V, φ^{-1}) is called a *local parametrization* at x. A set of pairs (U_x, φ) for $x \in M$ is called an *atlas* (tập bản đồ) for M, in this way each pair (U_x, φ) can be called a *chart* (tờ bản đồ) of this atlas. One can think of a manifold as a space together with an atlas.

EXAMPLE. Any open subset of $\mathbb{R}^m$ is a smooth manifold of dimension m.

PROPOSITION 18.1. *The graph of a smooth function $f : D \to \mathbb{R}^l$, where $D \subset \mathbb{R}^k$ is an open set, is a smooth manifold of dimension k.*

PROOF. Let $G = \{(x, f(x)) \mid x \in D\} \subset \mathbb{R}^{k+l}$ be the graph of f. The map $x \mapsto (x, f(x))$ from D to G is smooth. Its inverse is the projection $(x, y) \mapsto x$. This projection is the restriction of the projection given by the same formula from $\mathbb{R}^{k+l}$ to $\mathbb{R}^k$, which is a smooth map. Therefore D is diffeomorphic to G. □

EXAMPLE (curves and surfaces). The graph of a smooth function $y = f(x)$ for $x \in (a, b)$ is a 1-dimensional smooth manifold, often called a smooth curve.

The graph of a smooth function $z = f(x, y)$ for $x, y \in D$, where D is an open set in $\mathbb{R}^2$ is a 2-dimensional smooth manifold, often called a smooth surface.

EXAMPLE (circle). Let $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. It is covered by four neighborhoods which are half circles, each corresponds to points $(x, y) \in S^1$ such that $x > 0$, $x < 0$, $y > 0$ and $y < 0$. Each of these neighborhoods is diffeomorphic to $(-1, 1)$. For example consider the projection from $U = \{(x, y) \in S^1 \mid x > 0\} \to$

$(-1,1)$ given by $(x,y) \mapsto y$. The map $(x,y) \mapsto y$ is smooth on $\mathbb{R}^2$, so it is smooth on U. The inverse map $y \mapsto (\sqrt{1-y^2}, y)$ is smooth on $(-1,1)$. Therefore the projection is a diffeomorphism.

REMARK. We are discussing smooth manifolds embedded in Euclidean spaces. There is a notion of abstract smooth manifold, but we do not discuss it now.

DEFINITION. If M and N are two smooth manifolds in $\mathbb{R}^k$ and $M \subset N$ then we say that M is a *submanifold* of N.

Problems.

18.2. From our definition, a smooth function f defined on $D \subset \mathbb{R}^k$ does not necessarily have partial derivatives defined at boundary points of D. However, show that if D is the closure of an open subspace of $\mathbb{R}^k$ then the partial derivatives of f are defined and are continuous on D. For example, $f : [a,b] \to \mathbb{R}$ is smooth if and only if f has right-derivative at a and left-derivative at b, or equivalently, f is smooth on an open interval (c,d) containing $[a,b]$.

18.3. If X and Y are diffeomorphic and X is an m-dimensional manifold then so is Y.

18.4. The sphere $S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$ is a smooth manifold of dimension n, covered by the hemispheres.

There is a another way to see that S^n as a manifold, by using two stereographic projections, one from the North Pole and one from the South Pole.

18.5. Show that the hyperboloid $x^2 + y^2 - z^2 = 1$ is a manifold. Is the surface $x^2 + y^2 - z^2 = 0$ a manifold?

18.6. The torus can be obtained by rotating around the z axis a circle on the xOz plane not intersecting the z axis. Show that the torus is a smooth manifold.

18.7. Consider the union of the curve $y = \sin \frac{1}{x}$, $x > 0$ and the segment $\{(0,y) | -1 \leq y \leq 1\}$ (the Topologist's sine curve, see Section 4.4). Is it a manifold?

18.8. Consider the union of the curve $y = x^3 \sin \frac{1}{x}$, $x \neq 0$ and the point $(0,0)$. Is it a smooth manifold?

18.9. Is the trace of the path $\gamma(t) = (\frac{1}{2}\sin(2t), \cos(t))$, $t \in (0, 2\pi)$ (the figure 8) a smooth manifold?

18.10. A simple closed regular path is a map $\gamma : [a,b] \to \mathbb{R}^m$ such that γ is injective on $[a,b)$, γ is smooth, $\gamma^{(k)}(a) = \gamma^{(k)}(b)$ for all integer $k \geq 0$, and $\gamma'(t) \neq 0$ for all $t \in [a,b]$. Show that the trace of a simple closed regular path is a smooth 1-dimensional manifold.

18.11. The trace of the path$((2+\cos(1.5t))\cos t, (2+\cos(1.5t))\sin t, \sin(1.5t))$, $0 \leq t \leq 4\pi$ is often called the trefoil knot. Draw it (using computer). Show that the trefoil knot is a smooth 1-dimensional manifold (in fact it is diffeomorphic to the circle S^1, but this is more difficult).

18.12. Show that any open subset of a manifold is a manifold.

18.13. Show that a connected manifold is also path-connected.

18.14. Show that any diffeomorphism from S^{n-1} onto S^{n-1} can be extended to a diffeomorphism from $D^n = B'(0,1)$ onto D^n.

18.15. * Show that our definition of smooth manifold coincides with the definitions in [**Spi65**, p. 109] and [**Mun91**, p. 196].

19. Tangent spaces and derivatives

Derivatives of maps on $\mathbb{R}^n$. We summarize here several results about derivatives of functions defined on open sets in $\mathbb{R}^n$. See for instance **[Spi65]** or **[Lan97]** for more details.

Let U be an open set in $\mathbb{R}^k$ and V be an open set in $\mathbb{R}^l$. Let $f : U \to V$ be smooth. We define the derivative of f at $x \in U$ to be the linear map df_x such that

$$\begin{aligned} df_x : \mathbb{R}^k &\to \mathbb{R}^l \\ h &\mapsto df_x(h) = \lim_{h\to 0} \frac{f(x+th)-f(x)}{t}. \end{aligned}$$

Thus $df_x(h)$ is the directional derivative of f at x in the direction of h, measuring rate of change in the direction of h at x. The derivative df_x is a linear approximation of f at x.

Because we assumed that all the first order partial derivatives of f exist and are continuous, the derivative of f exists. In the canonical coordinate system of $\mathbb{R}^n$ the derivative map df_x is represented by an $l \times k$-matrix $Jf_x = [\frac{\partial f_i}{\partial x_j}(x)]_{1\le i\le l,\, 1\le j\le k}$, called the Jacobian of f at x, thus $df_x(h) = Jf_x \cdot h$.

PROPOSITION (chain rule). *Let U, V, W be open subsets of $\mathbb{R}^k$, $\mathbb{R}^l$, $\mathbb{R}^p$ respectively, let $f : U \to V$ and $g : V \to W$ be smooth maps, and let $y = f(x)$. Then*

$$d(g \circ f)_x = dg_y \circ df_x.$$

In other words, the following commutative diagram

$$\begin{tikzcd} & V \arrow[dr, "g"] & \\ U \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & W \end{tikzcd}$$

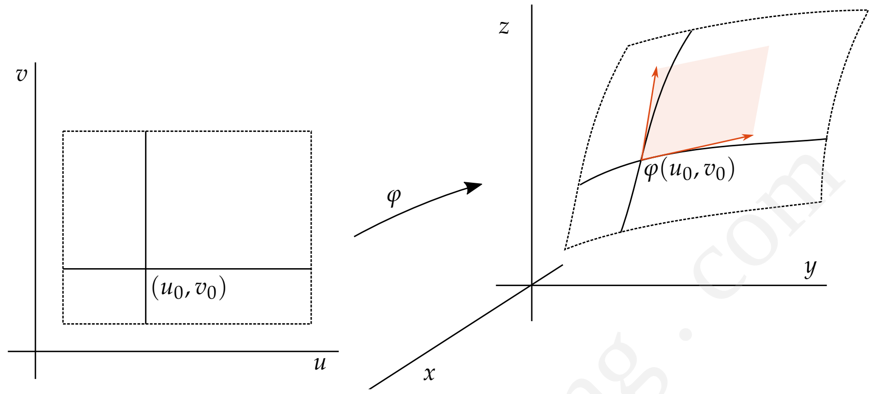
induces the commutative diagram

$$\begin{tikzcd} & TV_y \arrow[dr, "dg_y"] & \\ TU_x \arrow[ur, "df_x"] \arrow[rr, "d(g\circ f)_x"'] & & TW_{g(y)} \end{tikzcd}$$

PROPOSITION. *Let U and V be open subsets of $\mathbb{R}^k$ and $\mathbb{R}^l$ respectively. If $f : U \to V$ is a diffeomorphism then the derivative df_x is a linear isomorphism, and $k = l$.*

As a corollary, $\mathbb{R}^k$ and $\mathbb{R}^l$ are not diffeomorphic if $k \ne l$.

Tangent spaces of manifolds. To motivate the definition of tangent spaces of manifolds we recall the notion of tangent spaces of surfaces. Consider a parametrized surface in $\mathbb{R}^3$ given by $\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$. Consider a point $\varphi(u_0,v_0)$ on the surface. Near to (u_0,v_0) if we fix $v = v_0$ and only allow u to change then we get a parametrized path $\varphi(u,v_0)$ passing through $\varphi(u_0,v_0)$. The velocity vector of the path $\varphi(u, v_0)$ is a "tangent vector" to the path at the point $\varphi(u_0, v_0)$, and is given by the partial derivative with respect to u, that is, $\frac{\partial \varphi}{\partial u}(u_0, v_0)$. Similarly we have another "tangent vector" $\frac{\partial \varphi}{\partial v}(u_0, v_0)$. Then the "tangent space" of the surface at $\varphi(u_0, v_0)$ is the plane spanned by the above two tangent vectors (under some further conditions for this notion to be well-defined).



We can think of a manifold as a multi-dimensional surface. Therefore our definition of tangent space of manifold is a natural generalization.

DEFINITION. Let M be an m-dimensional manifold in $\mathbb{R}^k$. Let $x \in M$ and let $\varphi : U \to M$, where U is an open set in $\mathbb{R}^m$, be a parametrization of a neighborhood of x, and $x = \varphi(u)$. We define the *tangent space* of M at x, denoted by TM_x (or $T_x M$, or $TM(x)$), to be the vector space spanned by $\frac{\partial \varphi}{\partial u_i}(u_0)$, $1 \leq i \leq m$. Each element of the tangent space is called a *tangent vector*.

Since $\frac{\partial \varphi}{\partial u_i}(u_0) = d\varphi(u_0)(e_i)$, we get $TM_x = d\varphi_u(TU_u) = d\varphi_u(\mathbb{R}^m)$: the derivative of the local parametrization brings the Euclidean space onto the tangent space.

EXAMPLE. Consider a surface $z = f(x, y)$. Then the tangent plane at $(x, y, f(x, y))$ consists of the linear combinations of the vectors $(1, 0, f_x(x, y))$ and $(0, 1, f_y(x, y))$.

EXAMPLE. Consider the circle S^1. Let $(x(t), y(t))$ be any path on S^1. The tangent space of S^1 at (x, y) is spanned by the velocity vector $(x'(t), y'(t))$ if this vector is not 0. Since $x(t)^2 + y(t)^2 = 1$, differentiating both sides with respect to t we get $x(t)x'(t) + y(t)y'(t) = 0$, or in other words $(x'(t), y'(t))$ is perpendicular to $(x(t), y(t))$. Thus the tangent space is perpendicular to the radius.

PROPOSITION. *If M is an m-dimensional manifold then the tangent space TM_x is an m-dimensional linear space.*

PROOF. Since a parametrization φ is a diffeomorphism, there is a smooth map F from an open set in $\mathbb{R}^k$ to $\mathbb{R}^m$ such that $F \circ \varphi = \mathrm{Id}$. So $dF_{\varphi(0)} \circ d\varphi_0 = \mathrm{Id}_{\mathbb{R}^m}$. This implies that $d\varphi_0$ is an injective linear map, thus the image of $d\varphi_0$ is an m-dimensional vector space. $\square$

PROPOSITION. *The tangent space does not depend on the choice of parametrization.*

PROOF. Consider the following diagram, where U, U' are open, φ and φ' are parametrizations of open neighborhood of $x \in M$.

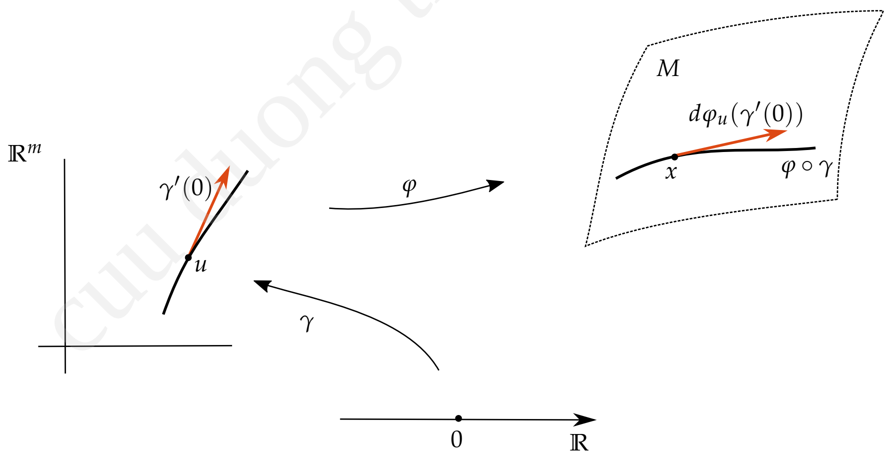
$$\begin{tikzcd} & M & \\ U \arrow[ur, "\varphi"] \arrow[rr, "\varphi'^{-1}\circ\varphi"'] & & U' \arrow[ul, "\varphi'"'] \end{tikzcd}$$

Notice that the map $\varphi'^{-1} \circ \varphi$ is to be understood as follows. We have that $\varphi(U) \cap \varphi'(U')$ is a neighborhood of $x \in M$. Restricting to $\varphi^{-1}(\varphi(U) \cap \varphi'(U'))$, the map $\varphi'^{-1} \circ \varphi$ is well-defined, and is a diffeomorphism. The above diagram gives us, with any $v \in \mathbb{R}^m$:

$$d\varphi_u(v) = d\varphi'_{\varphi'^{-1}\circ\varphi(u)}\big(d(\varphi'^{-1} \circ \varphi)_u(v)\big).$$

Thus any tangent vector with respect to the parametrization φ is also a tangent vector with respect to the parametrization φ'. We conclude that the tangent space does not depend on the choice of parametrization. $\square$

Here is another interpretation of tangent vector. Each vector in $\mathbb{R}^m$ is the velocity vector of a path in $\mathbb{R}^m$. The local parametrization brings this path to M. The velocity vector of this path is a tangent vector of M.



Derivatives of maps on manifolds. Let $M \subset \mathbb{R}^k$ and $N \subset \mathbb{R}^l$ be manifolds of dimensions m and n respectively. Let $f : M \to N$ be smooth. Let $x \in M$. There is a neighborhood W of x in $\mathbb{R}^k$ and a smooth extension F of f to W. The derivative of f is defined to be the restriction of the derivative of F. Precisely:

DEFINITION. The *derivative* of f at x is defined to be $df_x = dF_x|_{TM_x}$, i.e.

$$\begin{aligned} df_x : TM_x &\to TN_{f(x)} \\ h &\mapsto df_x(h) = dF_x(h). \end{aligned}$$

We need to show that the derivative is well-defined:

PROPOSITION. *$df_x(h) \in TN_{f(x)}$ and does not depend on the choice of F.*

PROOF. We have a commutative diagram

$$\begin{tikzcd}
W \arrow[r, "F"] & N \\
U \arrow[u, "\varphi"] \arrow[ur, "f\circ\varphi"] \arrow[r, "\psi^{-1}\circ f\circ\varphi"'] & V \arrow[u, "\psi"']
\end{tikzcd}$$

Let us explain this diagram. Assume that $\varphi(u) = x$, $\psi(v) = f(x)$, $h = d\varphi_u(w)$. Take a parametrization $\psi(V)$ of a neighborhood of $f(x)$. Then $f^{-1}(\psi(V))$ is an open neighborhood of x in M. From the definition we can find an open set W in $\mathbb{R}^k$ such that $W \cap M$ is an open neighborhood of x in M parametrized by $\varphi(U)$, and f has an extension to a function F defined on W which is smooth.

The diagram induces that $df_x(d\varphi_u(w)) = dF_x\big(d\varphi_u(w)\big) = d\psi_v\big(d(\psi^{-1}\circ f \circ \varphi)_u(w)\big)$ for every $w \in \mathbb{R}^m$. From this identity we get the desired conclusion. $\square$

Thus, although as noted in the previous section a smooth map defined on a general subset of $\mathbb{R}^k$ may not have derivatives, on a manifold the derivative can be defined, in a natural manner as the restriction of the derivative of the extension map to the tangent space of the manifold.

PROPOSITION (chain rule). *If $f : M \to N$ and $g : N \to P$ are smooth functions between manifolds, then*

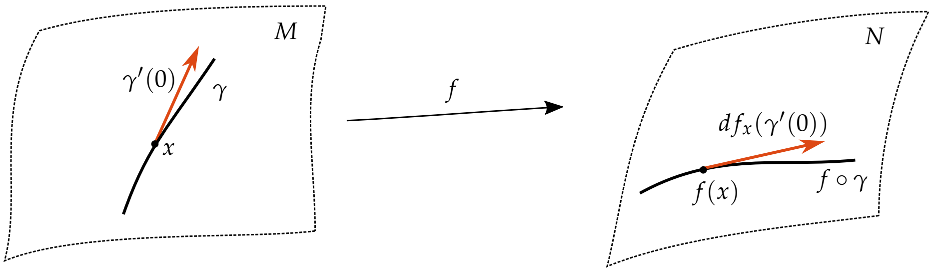
$$d(g\circ f)_x = dg_{f(x)} \circ df_x.$$

PROOF. There is an open neighborhood V of y in $\mathbb{R}^l$ and a smooth extension G of g to V. There is an open neighborhood U of x in $\mathbb{R}^k$ such that $U \subset F^{-1}(V)$ and there is a smooth extension F of f U. Then $d(g\circ f)_x = d(G\circ F)_x|_{TM_x} = (dG_y \circ dF_x)|_{TM_x} = dG_y|_{TN_y} \circ dF_x|_{TM_x} = dg_y \circ df_x$. $\square$

The derivative brings a tangent vector of M to a tangent vector of N in a natural manner. Indeed, a tangent vector at $x \in M$ is the velocity $\gamma'(0)$ of a path $\gamma(t)$ with $\gamma(0) = x$. The map f brings this path to N, giving the path $f\circ\gamma$. The velocity vector of this new path at $f(x)$ is $(f\circ\gamma)'(0) = df_x(\gamma'(0))$, which is the image under the derivative of f of the velocity of γ at x.

PROPOSITION. *If $f : M \to N$ is a diffeomorphism then $df_x : TM_x \to TN_{f(x)}$ is a linear isomorphism. In particular the dimensions of the two manifolds are same.*

PROOF. Let $m = \dim M$ and $n = \dim N$. Since $df_x \circ df^{-1}_{f(x)} = \mathrm{Id}_{TN_{f(x)}}$ and $df^{-1}_{f(x)} \circ df_x = \mathrm{Id}_{TM_x}$ we deduce, via the rank of df_x that $m \geq n$. Doing the same with $df^{-1}_{f(x)}$ we get $m \leq n$, hence $m = n$. From that df_x must be a linear isomorphism. $\square$



Problems.

19.1. Calculate the tangent spaces of S^n.

19.2. Calculate the tangent spaces of the hyperboloid $x^2 + y^2 - z^2 = 1$.

19.3. Show that if $\text{Id} : M \to M$ is the identify map then $d(\text{Id})_x$ is $\text{Id} : TM_x \to TM_x$.

19.4. Show that if M is a submanifold of N then TM_x is a subspace of TN_x.

19.5. A path on a manifold $M \subset \mathbb{R}^k$ is a smooth map γ from an open interval of $\mathbb{R}$ to M. Suppose $\gamma(t_0) = x$. The vector $\gamma'(t_0) \in \mathbb{R}^k$ is called the velocity vector of the path at $t = t_0$. Show that any tangent vector of M at x is the velocity vector of a path in M going through x.

19.6. Show that if M and N are manifolds and $M \subset N$ then $TM_x \subset TN_x$.

19.7 (Cartesian products of manifolds). If $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^l$ are manifolds then $X \times Y \subset \mathbb{R}^{k+l}$ is also a manifold. Furthermore $T(X \times Y)_{(x,y)} = TX_x \times TY_y$.

19.8. Calculate the derivative of the maps:

(a) $f : (0, 2\pi) \to S^1$, $f(t) = (\cos t, \sin t)$.
(b) $f : S^1 \to \mathbb{R}$, $f(x, y) = e^y$.

20. Regular values

Let $f : M \to N$ be smooth. A point in M is called a *critical point* (điểm dừng, điểm tới hạn) of f if the derivative of f at that point is not surjective. Otherwise the point is called a *regular point* (điểm thường, điểm chính qui) of f.

EXAMPLE. Let U be an open set in $\mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be smooth. Then $x \in U$ is a critical point of f if and only if $\nabla f(x) = 0$.

A point in N is called a *critical value* of f if it is the value of f at a critical point. Otherwise the point is called a *regular value* of f.

Thus y is a critical value of f if and only if $f^{-1}(y)$ contains a critical point. In particular, if $f^{-1}(y) = \emptyset$ then y is considered a regular value in this convention.

EXAMPLE. If $f : M \to N$ where $\dim(M) < \dim(N)$ then every $x \in M$ is a critical point and every $y \in N$ is a critical value of f.

The Inverse function theorem and the Implicit function theorem. First we state the Inverse function theorem in Multivariables Calculus.

THEOREM 20.1 (Inverse function theorem). *Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be smooth. If df_x is bijective then f is locally a diffeomorphism.*

More concisely, if $\det(Jf_x) \neq 0$ then there is an open neighborhood U of x and an open neighborhood V of $f(x)$ such that $f|_U : U \to V$ is a diffeomorphism.

For a proof, see for instance [**Spi65**]. Usually the result is stated for continuously differentiable function (i.e. C^1), but the result for smooth functions follows, since the Jacobian matrix of the inverse map is the inverse matrix of the Jacobian of the original map, and the entries of an inverse matrix can be obtained from the entries of the original matrix via smooth operations, namely $A^{-1} = \frac{1}{\det A} A^*$, where $A^*_{i,j} = (-1)^{i+j} \det(A^{j,i})$, and $A^{j,i}$ is obtained from A by omitting the ith row and jth column.

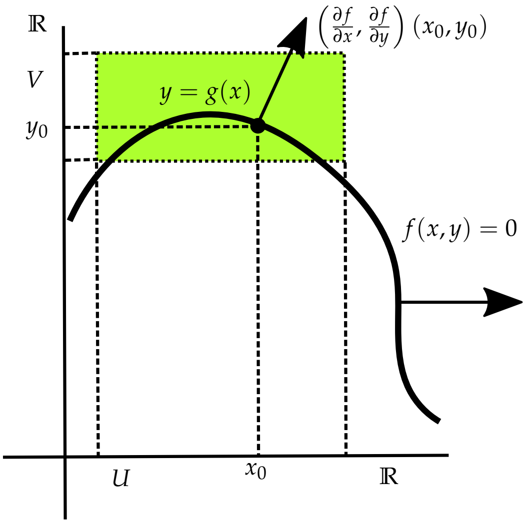
THEOREM 20.2 (Implicit function theorem). *Suppose that*

$$\begin{aligned} f : \mathbb{R}^m \times \mathbb{R}^n &\to \mathbb{R}^n \\ (x, y) &\mapsto f(x, y) \end{aligned}$$

is smooth and the $n \times n$-matrix $\frac{\partial f}{\partial y}(x_0, y_0)$ is non-singular then there is a neighborhood $U \times V$ of (x_0, y_0) such that for each $x \in U$ there is a unique $g(x) \in V$ satisfying $f(x, g(x)) = 0$, and the function g is smooth.

The conclusion of the theorem implies that on $U \times V$ the implicit equation $f(x, y) = 0$ has a unique solution $y = g(x)$. It also implies that $\{(x, y) \in U \times V \mid f(x, y) = 0\} = \{(x, g(x)) \mid x \in U\}$. In other words, on the open neighborhood $U \times V$ of (x_0, y_0) the level set $f^{-1}(0)$ is the graph of a smooth function.

The Implicit function theorem can be obtained by setting $F(x, y) = (x, f(x, y))$ and applying the Inverse function theorem to F.



COROLLARY 20.3. *Let $f : \mathbb{R}^{m+n} \to \mathbb{R}^n$ be smooth and $f(x) = y$. If df_x is surjective then there is a an open neighborhood of x in the level set $f^{-1}(y)$ which is diffeomorphic to an open subset of $\mathbb{R}^m$.*

PROOF. All we need is a permutation of variables to bring the function to the form used the the Implicit function theorem. Since df_x is onto, the Jacobian matrix $Jf(x)$ is of rank n, therefore it has n independent columns, corresponding to n variables $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$. Let σ be any permutation of $\{1, 2, \ldots, (m+n)\}$ such that $\sigma(m+1) = i_1, \sigma(m+2) = i_2, \ldots, \sigma(m+n) = i_n$. Let $\varphi : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ be the map that permute the variables as $(x_1, x_2, \ldots, x_{m+n}) \mapsto (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(m+n)})$. Consider the function $f \circ \varphi$. By the chain rule $J(f \circ \varphi)(\varphi^{-1}(x)) = Jf(x) \cdot J\varphi(\varphi^{-1}(x))$. Then it can be verified that the last n columns of the matrix $J(f \circ \varphi)(\varphi^{-1}(x))$ are the previous n independent columns of $Jf(x)$.

Now we can apply the Implicit function theorem to $f \circ \varphi$. Write $(u_0, v_0) = \varphi^{-1}(x) \in \mathbb{R}^m \times \mathbb{R}^n$. There is an open neighborhood U of u_0 in $\mathbb{R}^m$ and an open neighborhood V of v_0 in $\mathbb{R}^n$ such that

$$(U \times V) \cap (f \circ \varphi)^{-1}(y) = \{(u, h(u)) \mid u \in U\}.$$

The latter set is the graph of the smooth function h on U, therefore it is diffeomorphic to U (18.1). On the other hand the former set is $(U \times V) \cap \left(\varphi^{-1}(f^{-1}(y))\right) = \varphi^{-1}\left(\varphi(U \times V) \cap f^{-1}(y)\right)$. Since φ is a diffeomorphism, we can conclude that $\varphi(U \times V) \cap f^{-1}(y)$ is diffeomorphic to U. $\square$

Preimage of a regular value.

PROPOSITION 20.4. *If $\dim(M) = \dim(N)$ and y is a regular value of f then $f^{-1}(y)$ is a discrete set. In other words, $f^{-1}(y)$ is a zero dimensional manifold. Furthermore if M is compact then $f^{-1}(y)$ is a finite set.*

PROOF. If $x \in f^{-1}(y)$ then there is a neighborhood of x on which f is a bijection (a consequence of the Inverse function theorem, 20.19). That neighborhood contains no other point in $f^{-1}(y)$. Thus $f^{-1}(y)$ is a discrete set, i.e. each point has a neighborhood containing no other points.

If M is compact then the set $f^{-1}(y)$ is compact. A discrete compact space must be finite (6.4). $\square$

The following theorem is the Implicit function theorem for manifolds.

THEOREM 20.5 (level set at a regular value is a manifold). *If y is a regular value of $f : M \to N$ then $f^{-1}(y)$ is a manifold of dimension* $\dim(M) - \dim(N)$.

PROOF. Let $m = \dim(M)$ and $n = \dim(N)$. The case $m = n$ is already considered in 20.4. Now we assume $m > n$. Let $x_0 \in f^{-1}(y_0)$. Consider the diagram

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ O \arrow[u, "\varphi"] \arrow[r, "g"'] & W \arrow[u, "\psi"'] \end{tikzcd}$$

where $g = \psi^{-1} \circ f \circ \varphi$ and $\psi(w_0) = y_0$. Since df_{x_0} is onto, $dg_{\varphi^{-1}(x_0)}$ is also onto. Denote $\varphi^{-1}(x_0) = (u_0, v_0) \in \mathbb{R}^{m-n} \times \mathbb{R}^n$. Applying 20.3 to g, there is an open neighborhood U of u_0 in $\mathbb{R}^{m-n}$ and an open neighborhood V of v_0 in $\mathbb{R}^n$ such that $U \times V$ is contained in O and $(U \times V) \cap g^{-1}(w_0)$ is diffeomorphic to U. Now

$$\begin{aligned} \varphi\Big((U \times V) \cap g^{-1}(w_0)\Big) &= \varphi(U \times V) \cap \Big(\varphi(g^{-1}(w_0))\Big) \\ &= \varphi(U \times V) \cap \Big(\varphi(\varphi^{-1}(f^{-1}(\psi(w_0))))\Big) \\ &= \varphi(U \times V) \cap f^{-1}(y). \end{aligned}$$

Thus $\varphi(U \times V) \cap f^{-1}(y)$, which is an open neighborhood of x_0 in $f^{-1}(y)$, is diffeomorphic to U. This ends the proof. $\square$

EXAMPLE. To be able to follow the proof more easily the reader can try to work it out for an example, such as the case where M is the graph of the function $z = x^2 + y^2$, and f is the height function $f((x, y, z)) = z$ defined on M.

EXAMPLE. The n-sphere S^n is a subset of $\mathbb{R}^{n+1}$ determined by the implicit equation $\sum_{i=1}^{n+1} x_i^2 = 1$. Since 1 is a regular value of the function $f(x_1, x_2, \dots, x_{n+1}) = \sum_{i=1}^{n+1} x_i^2$ we conclude that S^n is a manifold of dimension n.

Lie groups. The set $M_n(\mathbb{R})$ of $n \times n$ matrices over $\mathbb{R}$ can be identified with the Euclidean manifold $\mathbb{R}^{n^2}$.

Consider the map $\det : M_n(\mathbb{R}) \to \mathbb{R}$. Let $A = [a_{i,j}] \in M_n(\mathbb{R})$. Since $\det(A) = \sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)} = \sum_j (-1)^{i+j} a_{i,j} \det(A^{i,j})$, we can see that det is a smooth function.

Let us find the critical points of det. A critical point is a matrix $A = [a_{i,j}]$ at which $\frac{\partial \det}{\partial a_{i,j}}(A) = (-1)^{i+j} \det(A^{i,j}) = 0$ for all i, j. In particular, $\det(A) = 0$. So 0 is the only critical value of det.

Therefore $\mathrm{SL}_n(\mathbb{R}) = \det^{-1}(1)$ is a manifold of dimension $n^2 - 1$.

Furthermore we note that the group multiplication in $\mathrm{SL}_n(\mathbb{R})$ is a smooth map from $\mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R})$ to $\mathrm{SL}_n(\mathbb{R})$. The inverse operation is a smooth map from $\mathrm{SL}_n(\mathbb{R})$ to itself. We then say that $\mathrm{SL}_n(\mathbb{R})$ is a Lie group.

DEFINITION. A *Lie group* is a smooth manifold which is also a group, for which the group operations are compatible with the smooth structure, namely the group multiplication and inversion are smooth.

Let $O(n)$ be the group of orthogonal $n \times n$ matrices, the group of linear transformation of $\mathbb{R}^n$ that preserves distances.

PROPOSITION. *The orthogonal group $O(n)$ is a Lie group.*

PROOF. Let $S(n)$ be the set of symmetric $n \times n$ matrices. This is clearly a manifold of dimension $\frac{n^2+n}{2}$.

Consider the smooth map $f : M(n) \to S(n)$, $f(A) = AA^t$. We have $O(n) = f^{-1}(I)$. We will show that I is a regular value of f.

We compute the derivative of f at $A \in f^{-1}(I)$:

$$df_A(B) = \lim_{t \to 0} \frac{f(A + tB) - f(A)}{t} = BA^t + AB^t.$$

We note that the tangents spaces of $M(n)$ and $S(n)$ are themselves. To check whether df_A is onto for $A \in O(n)$, we need to check that given $C \in S(n)$ there is a $B \in M(n)$ such that $C = BA^t + AB^t$. We can write $C = \frac{1}{2}C + \frac{1}{2}C$, and the equation $\frac{1}{2}C = BA^t$ will give a solution $B = \frac{1}{2}CA$, which is indeed a solution to the original equation. □

Problems.

20.6. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 - y^2$. Show that if $a \neq 0$ then $f^{-1}(a)$ is a 1-dimensional manifold, but $f^{-1}(0)$ is not. Show that if a and b are both positive or both negative then $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic.

20.7. Let $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y) = x^2 + y^2 - z^2$. Show that if $a \neq 0$ then $f^{-1}(a)$ is a 2-dimensional manifold, but $f^{-1}(0)$ is not. Show that if a and b are both positive or both negative then $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic.

20.8. Show that the equation $x^5 + y^4 + z^3 = 1$ determine a manifold in $\mathbb{R}^3$.

20.9. Is the intersection of the two surfaces $z = x^2 + y^2$ and $z = 1 - x^2 - y$ a manifold?

20.10. Show that the height function $(x, y, z) \mapsto z$ on the sphere S^2 has exactly two critical points.

20.11. Show that if f achieves local extremum at x then x is a critical point of f.

20.12. Show that a smooth function on a compact manifold must have at least two critical points.

20.13. Let $\dim(M) = \dim(N)$, M be compact and S be the set of all regular values of $f : M \to N$. For $y \in S$, let $|f^{-1}(y)|$ be the number of elements of $f^{-1}(y)$. Show that the map

$$\begin{array}{ccc} S & \to & \mathbb{N} \\ y & \mapsto & |f^{-1}(y)|. \end{array}$$

is locally constant. In other words, each regular value has a neighborhood where the number of preimages of regular values is constant.

20.14. Let M be a compact manifold and let $f : M \to \mathbb{R}$ be smooth. Show that the set of regular values of f is open.

20.15. Use regular value to show that the torus T^2 is a manifold.

20.16. Find the regular values of the function $f(x, y, z) = [4x^2(1 - x^2) - y^2]^2 + z^2 - \frac{1}{4}$ (and draw a corresponding level set).

20.17. Find a counter-example to show that 20.4 is not correct if regular value is replaced by critical value.

20.18. $\surd$ If $f : M \to N$ is smooth, y is a regular of f, and $x \in f^{-1}(y)$, then $\ker df_x = Tf^{-1}(y)_x$.

20.19 (inverse function theorem for manifolds). Let M and N be two manifolds of the same dimensions, and let $f : M \to N$ be smooth. Show that if x is a regular point of f then there is a neighborhood in M of x on which f is a diffeomorphism onto its image.

20.20. Show that S^1 is a Lie group.

20.21. Show that the set of all invertible $n \times n$-matrices $\mathrm{GL}(n; \mathbb{R})$ is a Lie group and find its dimension.

20.22. In this problem we find the tangent spaces of $\mathrm{SL}_n(\mathbb{R})$.

(a) Check that the derivative of the determinant map $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is represented by a gradient vector whose (i, j)-entry is $(-1)^{i+j} \det(A^{i,j})$.
(b) Determine the tangent space of $\mathrm{SL}_n(\mathbb{R})$ at $A \in \mathrm{SL}_n(\mathbb{R})$.
(c) Show that the tangent space of $\mathrm{SL}_n(\mathbb{R})$ at the identity matrix is the set of all $n \times n$ matrices with zero traces.

21. Critical points and the Morse lemma

Partial derivatives. Let $f : M \to \mathbb{R}$. Let U be an open neighborhood in M parametrized by φ. For each $x = \varphi(u)$ we define the *first partial derivatives*:

$$\left(\frac{\partial}{\partial x_i} f\right)(x) = \frac{\partial}{\partial u_i}(f \circ \varphi)(u).$$

In other words, $\left(\frac{\partial}{\partial x_i} f\right)(\varphi(u)) = \frac{\partial}{\partial u_i}(f \circ \varphi)(u)$. Of course this definition depends on local coordinates.

If f is defined on $\mathbb{R}^m$ then this is the usual partial derivative.

To understand $\left(\frac{\partial}{\partial x_i} f\right)(x)$ better, we can think that the parametrization φ brings the coordinate system of $\mathbb{R}^m$ to the neighborhood U, then $\left(\frac{\partial}{\partial x_i} f\right)(x)$ is the rate of change of $f(x)$ when the variable x changes along the path in U which is the composition of the standard path te_i along the ith axis of $\mathbb{R}^m$ with φ.

We can write

$$\begin{aligned}\left(\frac{\partial}{\partial x_i} f\right)(x) &= \frac{\partial}{\partial u_i}(f \circ \varphi)(u) = d(f \circ \varphi)(u)(e_i) \\ &= \left(df(x) \circ d\varphi(u)\right)(e_i) = df(x)(d\varphi(u)(e_i)).\end{aligned}$$

Thus $\left(\frac{\partial}{\partial x_i} f\right)(x)$ is the value of the derivative map $df(x)$ at the image of the unit vector e_i of $\mathbb{R}^m$.

Gradient vector. The tangent space TM_x inherits the Euclidean inner product from the ambient space $\mathbb{R}^k$. In this inner product space the linear map $df_x : TM_x \to \mathbb{R}$ is represented by a vector in TM_x which we called the *gradient vector* $\nabla f(x)$. This vector is determined by the property $\langle \nabla f(x), v \rangle = df_x(v)$ for any $v \in TM_x$. Notice that the gradient vector $\nabla f(x)$ is defined on the manifold, *not depending on local coordinates*.

In a local parametrization the vectors $d\varphi(u)(e_i) = \frac{\partial \varphi}{\partial u_i}(u)$, $1 \le i \le m$ constitutes a vector basis for TM_x. In this basis the coordinates of $\nabla f(x)$ are

$$\langle \nabla f(x), d\varphi(u)(e_i) \rangle = df_x(d\varphi(u)(e_i)) = \frac{\partial f}{\partial x_i}(x).$$

In other words, in that basis we have the familiar formula $\nabla f = \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_m} \rangle$. This formula depends on local coordinates. It implies that $\nabla f : M \to \mathbb{R}^m$ is a smooth function.

We have several simple observations:

PROPOSITION. *A point is a critical point if and only if the gradient vector at that point is zero.*

PROPOSITION. *At a local extremum point the gradient vector must be zero.*

Second derivatives. Since $\frac{\partial}{\partial x_i} f$ is a smooth function on U, we can take its partial derivatives. Thus we define the *second partial derivatives*:

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j} f\right)(x).$$

In other words,

$$\begin{aligned}\frac{\partial^2 f}{\partial x_i \partial x_j}(\varphi(u)) &= \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j} f\right)(\varphi(u)) = \frac{\partial}{\partial u_i}\left(\frac{\partial f}{\partial x_j} \circ \varphi\right)(u) \\ &= \frac{\partial}{\partial u_i}\left(\frac{\partial}{\partial u_j}(f \circ \varphi)\right)(u) = \frac{\partial^2}{\partial u_i \partial u_j}(f \circ \varphi)(u).\end{aligned}$$

Non-degenerate critical points. Consider the Hessian matrix of second partial derivatives:

$$Hf(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right)_{1 \le i,j \le m}.$$

If this matrix is non-degenerate, then we say that x is a *non-degenerate critical point* of f.

LEMMA 21.1. *The non-degeneracy of a critical point does not depend on choices of local coordinates.*

PROOF. We can see that the problem is reduced to the case of functions on $\mathbb{R}^m$. If $f : \mathbb{R}^m \to \mathbb{R}$ and φ is a change of variables (i.e. a diffeomorphism) of $\mathbb{R}^m$ then we have

$$\frac{\partial}{\partial u_i}(f \circ \varphi)(u) = \sum_k \frac{\partial f}{\partial x_k}(x) \cdot \frac{\partial \varphi_k}{\partial u_i}(u).$$

Then

$$\begin{aligned}\frac{\partial^2}{\partial u_j \partial u_i}(f \circ \varphi)(u) &= \sum_k \left[\left(\sum_l \frac{\partial^2 f}{\partial x_l \partial x_k}(x) \cdot \frac{\partial \varphi_l}{\partial u_j}(u)\right) \cdot \frac{\partial \varphi_k}{\partial u_i}(u) + \frac{\partial f}{\partial x_k}(x) \cdot \frac{\partial^2 \varphi}{\partial u_j \partial u_i}(u)\right] \\ &= \sum_{k,l} \frac{\partial^2 f}{\partial x_l \partial x_k}(x) \cdot \frac{\partial \varphi_l}{\partial u_j}(u) \cdot \frac{\partial \varphi_k}{\partial u_i}(u).\end{aligned}$$

In other words: $H(f \circ \varphi)(u) = J\varphi(u)^t [Hf(\varphi(u))] J\varphi(u)$. This formula immediately gives us the conclusion. □

Morse lemma.

THEOREM (Morse's lemma). *Suppose that $f : M \to \mathbb{R}$ is smooth and p is a non-degenerate critical point of f. There is a local coordinate φ in a neighborhood of p such that $\varphi(p) = 0$ and in that neighborhood*

$$f(x) = f(p) - \varphi(x)_1^2 - \varphi(x)_2^2 - \cdots - \varphi(x)_k^2 + \varphi(x)_{k+1}^2 + \varphi(x)_{k+2}^2 + \cdots + \varphi(x)_m^2.$$

In other words, in a neighborhood of 0,

$$(f \circ \varphi^{-1})(u) = (f \circ \varphi^{-1})(0) - u_1^2 - u_2^2 - \cdots - u_k^2 + u_{k+1}^2 + u_{k+2}^2 + \cdots + u_m^2.$$

If we abuse notations by using local coordinates and write x_i for $u_i = \varphi(x)_i$ then we can write

$$f(x) = f(p) - x_1^2 - x_2^2 - \cdots - x_k^2 + x_{k+1}^2 + x_{k+2}^2 + \cdots + x_m^2.$$

The number k does not depend on the choice of such local coordinates and is called the index of the non-degenerate critical point p.

EXAMPLE. Non-degenerate critical points of index 0 are local minima, and the ones with maximum indexes are local maxima.

PROOF. Since we only need to prove the formula for $f \circ \varphi^{-1}$, we only need to work in $\mathbb{R}^m$.

First, we write

$$\begin{aligned} f(x) &= f(0) + \int_0^1 \frac{d}{dt} f(tx)\, dt \\ &= f(0) + \sum_{i=1}^m \int_0^1 \left(\frac{\partial f}{\partial x_i}(tx)\right) x_i\, dt \\ &= f(0) + \sum_{i=1}^m x_i \int_0^1 \left(\frac{\partial f}{\partial x_i}(tx)\right) dt. \end{aligned}$$

A result of Analysis (see for example [**Lan97**, p. 276]) tells us that the functions $g_i(x) = \int_0^1 \left(\frac{\partial}{\partial x_i} f(tx)\right) dt$ are smooths. Notice that $g_i(0) = \frac{\partial f}{\partial x_i}(0) = 0$. Furthermore

$$\frac{\partial g_i}{\partial x_j}(x) = \int_0^1 \frac{\partial^2 f}{\partial x_j \partial x_i}(tx) t\, dt,$$

therefore $\frac{\partial g_i}{\partial x_j}(0) = \frac{1}{2}\frac{\partial^2 f}{\partial x_j \partial x_i}(0)$.

Apply this construction once again to g_i we obtain smooth functions $g_{i,j}$ such that $g_{i,j}(0) = \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(0)$ and

$$f(x) = f(0) + \sum_{i,j=1}^m x_i x_j g_{i,j}(x).$$

Set $h_{i,j} = (g_{i,j} + g_{j,i})/2$ then $h_{i,j} = h_{j,i}$, $h_{i,j}(0) = \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(0)$, and

$$f(x) = f(0) + \sum_{i,j=1}^m x_i x_j h_{i,j}(x).$$

The rest of the proof is a simple completing the square. Since the matrix $\big(h_{i,j}(0)\big)$ is non-degenerate by a permutation of variables if necessary, we can assume that $h_{1,1}(0) \neq 0$. Then there is a neighborhood of 0 such that $h_{1,1}(x)$ does not change its sign. In that neighborhood, if $h_{1,1}(0) > 0$ then

$$
\begin{aligned}
f(x) &= f(0) + \quad h_{1,1}(x)x_1^2 + \sum_{1<j}(h_{1,j}(x) + h_{j,1}(x))x_1x_j + \sum_{1<i,j} h_{i,j}(x)x_ix_j \\
&= f(0) + \quad h_{1,1}(x)x_1^2 + 2\sum_{1<j} h_{1,j}(x)x_1x_j + \sum_{1<i,j} h_{i,j}(x)x_ix_j \\
&= f(0) + \quad \left(\sqrt{h_{1,1}(x)}x_1\right)^2 + 2\sqrt{h_{1,1}(x)}x_1 \frac{\sum_{1<j} h_{1,j}(x)x_j}{\sqrt{h_{1,1}(x)}} + \sum_{1<i,j} h_{i,j}(x)x_ix_j \\
&= f(0) + \quad \left[\sqrt{h_{1,1}(x)}x_1 + \sum_{1<j} \frac{h_{1,j}(x)}{\sqrt{h_{1,1}(x)}}x_j\right]^2 \\
&\qquad - \left(\sum_{1<j} \frac{h_{1,j}(x)}{\sqrt{h_{1,1}(x)}}x_j\right)^2 + \sum_{1<i,j} h_{i,j}(x)x_ix_j.
\end{aligned}
$$

Similarly, if $h_{1,1}(0) < 0$ then

$$
\begin{aligned}
f(x) &= f(0) - \quad \left[\sqrt{-h_{1,1}(x)}x_1 - \sum_{1<j} \frac{h_{1,j}(x)}{\sqrt{-h_{1,1}(x)}}x_j\right]^2 \\
&\qquad + \left(\sum_{1<j} \frac{h_{1,j}(x)}{\sqrt{-h_{1,1}(x)}}x_j\right)^2 + \sum_{1<i,j} h_{i,j}(x)x_ix_j.
\end{aligned}
$$

Combining both cases, we define the new variables:

$$
\begin{aligned}
v_1 &= \sqrt{|h_{1,1}(x)|}x_1 + \operatorname{sign}(h_{1,1}(0)) \sum_{1<j} \frac{h_{1,j}(x)}{\sqrt{|h_{1,1}(x)|}}x_j, \\
v_i &= x_i,\ i > 1.
\end{aligned}
$$

Since

$$
\frac{\partial v_1}{\partial x_1}(0) = \sqrt{|h_{1,1}(0)|} \neq 0
$$

the Jacobian matrix $\left(\frac{\partial v_i}{\partial x_j}(0)\right)$ is non-singular. By the Inverse function theorem, there is a neighborhood of 0 where the correspondence $x \mapsto v$ is a diffeomorphism, that is, a change of variables. With the new variables we have

$$
f(v) = f(0) + \operatorname{sign}(h_{1,1}(0))v_1^2 + \sum_{1<i,j} h'_{i,j}(v)v_iv_j.
$$

By a direct calculation, we can check that in these variables

$$
Hf(0) = \begin{pmatrix} \operatorname{sign}(h_{1,1}(0)) & 0 \\ 0 & (2h'_{i,j}(0))_{1<i,j\leq m} \end{pmatrix}.
$$

Using 21.1 we conclude that the matrix $(h'_{i,j}(0))_{1<i,j\leq m}$ must be non-singular. Thus the induction process can be carried out. Finally we can permute the variables such that in the final form of f the negative signs are in front.

$\square$

Problems.

21.2. Show that the gradient vector is always normal to level surfaces.

21.3. Give a generalization of the method of Lagrange multipliers to manifolds.

21.4. For the specific case of $f(x) = \sum_{1\leq i,j\leq m} a_{i,j}x_ix_j$ where $a_{i,j}$ are real numbers, to prove the Morse's lemma we can use a diagonalization of a quadratic form or a symmetric matrix, considered in Linear Algebra. The change of variables corresponds to using a new vector basis consisting of eigenvectors of the matrix.

21.5. Recover the classification of critical points from Calculus.

22. Flows

Vector fields.

DEFINITION. A smooth tangent *vector field* on a manifold $M \subset \mathbb{R}^k$ is a smooth map $V : M \to \mathbb{R}^k$ such that $V(x) \in TM_x$ for each $x \in M$.

EXAMPLE. If $f : M \to \mathbb{R}$ is smooth then the gradient ∇f is a smooth vector field on M.

An *integral curve* at a point $x \in M$ with respect to the vector field V is a smooth path $\gamma : (a,b) \to M$ such that $0 \in (a,b)$, $\gamma(0) = x$, and $\gamma'(t) = V(\gamma(t))$ for all $t \in (a,b)$. It is a path going through x and at every moment takes the vector of the given vector field as its velocity vector. An integral curve of a vector field is tangent to the vector field. Integral curves are also called *solution curves*, *trajectories*, or *flow lines*.

In a local coordinate around x, a vector field on that neighborhood corresponds to a vector field on $\mathbb{R}^m$, and an integral curve in that neighborhood corresponds to an integral curve on $\mathbb{R}^m$. Thus, by using local coordinate, we can consider a local integral curve as a solution to the differential equation $\gamma'(t) = V(\gamma(t))$ in $\mathbb{R}^m$ subjected to the initial condition $\gamma(0) = x$, or in more common notations:

$$\begin{cases} \frac{dx}{dt} = V(x), \\ x(0) = x_0. \end{cases}$$

Flows. For each $x \in M$, let $\phi(t,x)$, or $\phi_t(x)$, be an integral curve at x, with t belongs to an interval $J(x)$. We have a map

$$\begin{aligned} \phi : D = \{(t,x) \mid x \in M,\ t \in J(x)\} \subset \mathbb{R} \times M &\to M \\ (t,x) &\mapsto \phi_t(x), \end{aligned}$$

with the properties $\phi_0(x) = x$, and $\frac{d}{dt}(\phi)(t,x) = V(\phi(t,x))$. This map ϕ is called a *flow* (dòng) generated by the vector field V.

THEOREM. *For each smooth vector field there exists a unique smooth flow, in the sense that any two integral curves at the same point must agree on the intersection of their domains. The domain of this flow can be taken to be an open set.*

This theorem is just an interpretation of the theorem in Differential Equations on the existence, uniqueness, and dependence on initial conditions of solutions to differential equations, see for example **[HS74]**, **[Lan97]**.

THEOREM (Group law). *Given $x \in M$, if $\phi_t(x)$ is defined on $(-\epsilon, \epsilon)$ then for $s \in (-\epsilon, \epsilon)$ such that $s + t \in (-\epsilon, \epsilon)$ we have*

$$\phi_{t+s}(x) = \phi_t(\phi_s(x)).$$

PROOF. Define $\gamma(t) = \phi_{t+s}(x)$. Then $\gamma(0) = \phi_s(x)$, and $\gamma'(t) = \frac{d}{dt}(\phi)(t+s,x) = V(\phi(t+s,x)) = V(\gamma(t))$. Thus $\gamma(t)$ is an integral curve at $\phi_s(x)$. But

$\phi_t(\phi_s(x))$ is another integral curve at $\phi_s(x)$. By uniqueness of integral curves, $\gamma(t)$ must agree with $\phi_t(\phi_s(x))$ on their common domains. $\square$

THEOREM. *If $\phi_t : M \to M$ is defined for $t \in (-\epsilon, \epsilon)$ then it is a diffeomorphism.*

PROOF. Since the flow ϕ is smooth the map ϕ_t is smooth. Its inverse map ϕ_{-t} is also smooth. $\square$

When every integral curve can be extended without bound in both directions, in other words, for all x the map $\phi_t(x)$ is defined for all $t \in \mathbb{R}$, we say that the flow is *complete*. If the flow is complete then ϕ_t is defined for all $t \in \mathbb{R}$, we can think of ϕ_t as moving every point along integral curves for an amount of time t.

THEOREM. *On a compact manifold any flow is complete.*

PROOF. Although a priori each integral curve has its own domain, first we will show that for a compact manifold all integral curves can have same domains. Since the domain D of the flow can be taken to be an open subset of $\mathbb{R} \times M$, each $x \in M$ has an open neighborhood U_x and a corresponding interval $(-\epsilon_x, \epsilon_x)$ such that $(-\epsilon_x, \epsilon_x) \times U_x$ is contained in D. The collection $\{U_x \mid x \in M\}$ is an open cover of M therefore there is a finite subcover. That implies that there is a positive real number ϵ such that for every $x \in M$ the integral curve $\phi_t(x)$ is defined on $(-\epsilon, \epsilon)$.

Now $\phi_t(x)$ can be extended inductively by intervals of length $\epsilon/2$ to be defined on $\mathbb{R}$. For example, if $t > 0$ then there is $n \in \mathbb{N}$ such that $n\frac{\epsilon}{2} \le t < (n+1)\frac{\epsilon}{2}$, if $n \ge 2$ then define

$$\phi_t(x) = \phi_{t-n\frac{\epsilon}{2}}\left(\phi_{n\frac{\epsilon}{2}}(x)\right),$$

where $\phi_{n\frac{\epsilon}{2}}(x) = \phi_{\frac{\epsilon}{2}}\left(\phi_{(n-1)\frac{\epsilon}{2}}(x)\right)$. $\square$

THEOREM. *Let M be a compact smooth manifold and $f : M \to \mathbb{R}$ be smooth. If the interval $[a, b]$ only contains regular values of f then the level sets $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic.*

PROOF. The idea of the proof is to construct a diffeomorphism from $f^{-1}(a)$ to $f^{-1}(b)$ by pushing along the flow lines of the gradient vector field of f. However since $\nabla f(x)$ can be zero outside of $f^{-1}([a, b])$ we need a modification to ∇f.

Suppose that $a < b$. By 20.14 there are intervals $[a, b] \subset (c, d) \subset [c, d] \subset (h, k)$ such that (h, k) contains only regular values of f. Thus on $f^{-1}((h, k))$ the vector $\nabla f(x)$ never vanish.

By 22.4 there is a smooth function ψ that is 1 on $f^{-1}([c, d])$ and is 0 outside $f^{-1}((h, k))$. Let $F = \psi\frac{\nabla f}{||\nabla f||^2}$, then F is a well-defined smooth vector field on M. Notice that F is basically a rescale of ∇f.

Let ϕ be the flow generated by F. We have:

$$\frac{d}{dt} f(\phi_t(x)) = df_{\phi_t(x)}\left(\frac{d}{dt}\phi_t(x)\right) = \langle \nabla f(\phi_t(x)), F(\phi_t(x))\rangle = \psi(\phi_t(x)).$$

Fix $x \in f^{-1}(a)$. Since $\phi_t(x)$ is continuous with respect to t and $\phi_0(x) = x$, there is an $\epsilon > 0$ such that $\phi_t(x) \in f^{-1}((c, d))$ for $t \in [0, \epsilon)$. Let ϵ_0 be the supremum (or

∞) of the set of such ϵ. Then for $t \in [0, \epsilon_0)$ we have $\phi_t(x) \in f^{-1}((c, d))$, and so

$$\frac{d}{dt} f(\phi_t(x)) = \psi(\phi_t(x)) = 1.$$

This means the flow line is going at constant speed 1. We get $f(\phi_t(x)) = t + a$ for $t \in [0, \epsilon_0)$. If $\epsilon_0 \leq b - a$ then by continuity $f(\phi_{\epsilon_0}(x)) = \epsilon_0 + a \leq b < d$. This implies there is $\epsilon' > \epsilon_0$ such that $f(\phi_t(x)) < d$ for $t \in [\epsilon_0, \epsilon')$, a contradiction. Thus $\epsilon_0 > b - a$. We now observe that $f(\phi_{b-a}(x)) = b$. Thus ϕ_{b-a} maps $f^{-1}(a)$ to $f^{-1}(b)$, so it is the desired diffeomorphism. □

THEOREM 22.1 (Homogeneity of manifolds). *On a connected manifold there is a self diffeomorphism that brings any given point to any given point.*

PROOF. First we can locally bring any point to a given point without outside disturbance. That translates to a problem on $\mathbb{R}^n$: we will show that for any $c \in B(0,1)$ there is a diffeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h|_{\mathbb{R}^n \setminus B(0,1)} = 0$ and $h(0) = c$.

By 22.2 there is a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f|_{B'(0,||c||)} = 1$ and $f|_{\mathbb{R}^n \setminus B(0,1)} = 0$. Consider the vector field $F : \mathbb{R}^n \to \mathbb{R}^n$, $F(x) = f(x)c$. This is a smooth vector field with compact support. The flow generated by this vector field is a smooth map $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\begin{aligned} \phi_0(x) &= x, \\ \frac{d}{dt}\phi_t(x) &= F(\phi_t(x)). \end{aligned}$$

□

Problems.

22.2. √ The following is a common smooth function:

$$f(x) = \begin{cases} e^{-1/x}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0. \end{cases}$$

(a) Show that $f(x)$ is smooth.
(b) Let $a < b$ and let $g(x) = f(x-a)f(b-x)$. Then g is smooth, $g(x)$ is positive on (a, b) and is zero everywhere else.
(c) Let
$$h(x) = \frac{\int_{-\infty}^{x} g(x)\,dx}{\int_{-\infty}^{\infty} g(x)\,dx}.$$
Then $h(x)$ is smooth, $h(x) = 0$ if $x \leq a$, $0 < h(x) < 1$ if $a < x < b$, and $h(x) = 1$ if $x \geq b$.
(d) The function
$$k(x) = \frac{f(x-a)}{f(x-a) + f(b-x)}$$
also has the above properties of $h(x)$.
(e) In $\mathbb{R}^n$, construct a smooth function whose value is 0 outside of the ball of radius b, 1 inside the ball of radius a, where $0 < a < b$, and between 0 and 1 in between the two balls.

22.3 (Smooth Urysohn lemma). Let $A \subset U \subset \mathbb{R}^n$ where A is compact and U is open. Show that there exists a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $0 \leq \varphi(x) \leq 1$, $\varphi|_A = 1$, $\varphi|_{\mathbb{R}^n \setminus U} = 0$.

22.4 (Smooth Urysohn lemma for manifolds). Let M be a smooth manifold, $A \subset U \subset M$ where A is compact and U is open in M. Show that there is a smooth function $\varphi : M \to \mathbb{R}$ such that $0 \leq \varphi(x) \leq 1$, $\varphi|_A = 1$, $\varphi|_{M \setminus U} = 0$.

23. Manifolds with boundaries

The closed half-space $\mathbb{H}^m = \{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \mid x_m \geq 0\} \subset \mathbb{R}^m$ whose topological boundary is $\partial\mathbb{H}^m = \{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \mid x_m = 0\}$ is our model for a manifold with boundary.

DEFINITION. A subspace M of $\mathbb{R}^k$ is called a *manifold with boundary* of dimension m if each point in M has a neighborhood diffeomorphic to either $\mathbb{R}^m$ or $\mathbb{H}^m$, where in the second case the point is sent to $\partial\mathbb{H}^m$. The set of all points of the first type is called the *interior* of M. The set of all points of the second type is called the *boundary* of M, denoted by ∂M.

A point belongs to either the interior or the boundary, not both, because of the following:

LEMMA. *$\mathbb{H}^m$ is not diffeomorphic to $\mathbb{R}^m$.*

PROOF. Suppose $f : \mathbb{R}^m \to \mathbb{H}^m$ is a diffeomorphism. For any $x \in \mathbb{R}^m$, df_x is non-singular, therefore by the Inverse function theorem f is a diffeomorphism from an open ball containing x onto an open ball containing $f(x)$. Thus $f(x)$ must be an interior point (in topological sense) of $\mathbb{H}^m$. This implies that f cannot be onto $\mathbb{H}^m$, a contradiction.

Alternatively we can use Invariance of dimension 9.26. □

REMARK. The boundary of a manifold is generally not the same as its topological boundary.

REMARK. On convention, when we talk about a manifold we still mean a manifold as earlier defined, that is, with no boundary. A manifold with boundary can have empty boundary, in which case it is a manifold.

PROPOSITION. *The interior of an m-manifold with boundary is an m-manifold without boundary. The boundary of an m-manifold with boundary is an $(m-1)$-manifold without boundary.*

PROOF. The part about the interior is clear. Let us consider the part about the boundary.

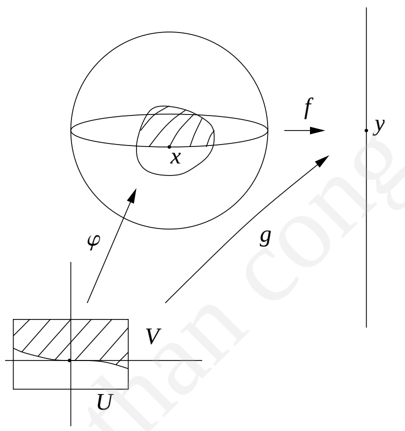
Let M be an m-manifold and let $x \in \partial M$. Let φ be a diffeomorphism from a neighborhood U of x in M to $\mathbb{H}^m$. We can check that if $y \in U$ then $\varphi(y) \in \partial\mathbb{H}^m$ if and only if $y \in \partial M$. Thus the restriction $\varphi|_{U \cap \partial M}$ is a diffeomorphism from a neighborhood of x in ∂M to $\partial\mathbb{H}^m$, which is diffeomorphic to $\mathbb{R}^{m-1}$. □

The *tangent space of a manifold with boundary* M is defined as follows. It x is an interior point of M then TM_x is defined as before. If x is a boundary point then there is a parametrization $\varphi : \mathbb{H}^m \to M$, where $\varphi(0) = x$. Notice that by continuity φ has well-defined partial derivatives at 0. This implies that the derivative $d\varphi_0 : \mathbb{R}^m \to \mathbb{R}^k$ is well-defined. Then TM_x is still defined as $d\varphi_0(\mathbb{R}^m)$. The Chain rule still holds. The notion of critical point is defined exactly as for manifolds.

THEOREM 23.1. *Let M be an m-dimensional manifold without boundary. Let $f : M \to \mathbb{R}$ be smooth and let y be a regular value of f. Then the set $f^{-1}([y,\infty))$ is an m-dimensional manifold with boundary $f^{-1}(y)$.*

PROOF. Let $N = f^{-1}([y,\infty))$. Since $f^{-1}((y,\infty))$ is an open subspace of M, it is an m-manifold without boundary.

The crucial case is when $x \in f^{-1}(y)$. Let φ be a parametrization of a neighborhood of x in M, with $\varphi(0) = x$. Let $g = f \circ \varphi$. As in the proof of 20.5, by the Implicit function theorem, there is an open ball U in $\mathbb{R}^{m-1}$ containing 0 and an open interval V in $\mathbb{R}$ containing 0 such that in $U \times V$ the set $g^{-1}(y)$ is a graph $\{(u, h(u)) \mid u \in U\}$ where h is smooth.



Since $(U \times V) \setminus g^{-1}(y)$ consists of two connected components, exactly one of the two is mapped via g to (y,∞), otherwise x will be a local extremum point of f, and so $df_x = 0$, violating the assumption. In order to be definitive, let us assume that $W = \{(u,v) \mid v \geq h(u)\}$ is mapped by g to $[y,\infty)$. Then $\varphi(W) = \varphi(U \times V) \cap f^{-1}([y,\infty))$ is a neighborhood of x in N parametrized by $\varphi|_W$. On the other hand W is diffeomorphic to an open neighborhood of 0 in $\mathbb{H}^m$. To show this, consider the map $\psi(u,v) = (u, v - h(u))$ on $U \times V$. Then ψ is a smooth bijection on open subspaces of $\mathbb{R}^m$, whose Jacobian is non-singular, therefore is a diffeomorphism. The restriction $\psi|_W$ is a diffeomorphism to $\psi(U \times V) \cap \mathbb{H}^m$. Thus x is a boundary point of N. □

EXAMPLE. Let f be the height function on S^2 and let y be a regular value. Then the set $f^{-1}((-\infty, y])$ is a disk with the circle $f^{-1}(y)$ as the boundary.

EXAMPLE. If y is a regular value of the height function on D^2 then $f^{-1}(y)$ is a 1-dimensional manifold with boundary on ∂D^2.

EXAMPLE. The closed disk D^n is an n-manifold with boundary.

THEOREM 23.2. *Let M be an m-dimensional manifold with boundary, let N be an n-manifold with or without boundary. Let $f : M \to N$ be smooth. Suppose that $y \in N$ is a regular value of both f and $f|_{\partial M}$. Then $f^{-1}(y)$ is an $(m-n)$-manifold with boundary $\partial M \cap f^{-1}(y)$.*

PROOF. That $f^{-1}(y) \setminus \partial M$ is an $(m-n)$-manifold without boundary is already proved in 20.5.

We consider the crucial case of $x \in \partial M \cap f^{-1}(y)$.

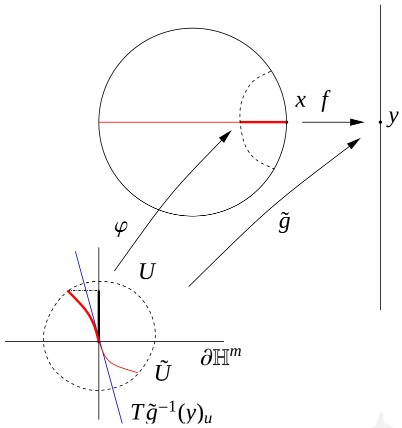


FIGURE 23.3.

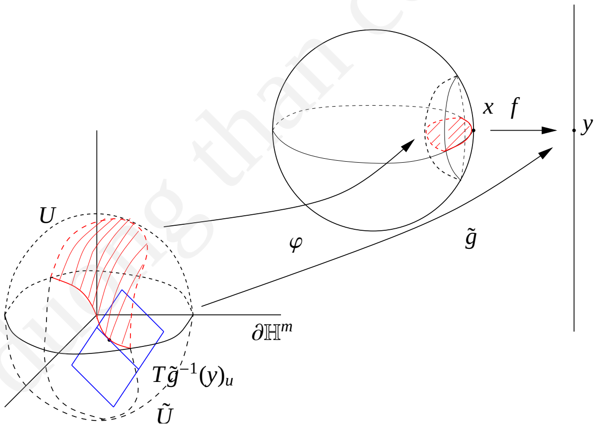


FIGURE 23.4.

The map g can be extended to $\tilde{g}$ defined on an open neighborhood $\tilde{U}$ of 0 in $\mathbb{R}^m$. As before, $\tilde{g}^{-1}(y)$ is a graph of a function of $(m-n)$ variables so it is an $(m-n)$-manifold without boundary.

Let $p : \tilde{g}^{-1}(y) \to \mathbb{R}$ be the projection to the last coordinate (the height function). We have $g^{-1}(y) = p^{-1}([0,\infty))$ therefore if we can show that 0 is a regular value of p then the desired result follows from 23.1 applied to $\tilde{g}^{-1}(y)$ and p. For that we need to show that the tangent space $T\tilde{g}^{-1}(y)_u$ at $u \in p^{-1}(0)$ is not contained in $\partial\mathbb{H}^m$. Note that since $u \in p^{-1}(0)$ we have $u \in \partial\mathbb{H}^m$.

Since $\tilde{g}$ is regular at u, the null space of $d\tilde{g}_u$ on $T\tilde{U}_u = \mathbb{R}^m$ is exactly $T\tilde{g}^{-1}(y)_u$, of dimension $m-n$. On the other hand, $\tilde{g}|_{\partial\mathbb{H}^m}$ is regular at u, which implies that the null space of $d\tilde{g}_u$ restricted to $T(\partial\mathbb{H}^m)_u = \partial\mathbb{H}^m$ has dimension $(m-1)-n$. Thus $T\tilde{g}^{-1}(y)_u$ is not contained in $\partial\mathbb{H}^m$. $\square$

Problems.

23.5. Check that $\mathbb{R}^m$ cannot be diffeomorphic to $\mathbb{H}^m$.

23.6. Show that the subspace $\{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \mid x_m > 0\}$ is diffeomorphic to $\mathbb{R}^m$.

23.7. A simple regular path is a map $\gamma : [a, b] \to \mathbb{R}^m$ such that γ is injective, smooth, and $\gamma^{(k)}(t) \neq 0$ for all $t \in [a, b]$. Show that the trace of a simple closed regular path is a smooth 1-dimensional manifold with boundary.

23.8. Suppose that M is an n-manifold without boundary. Show that $M \times [0, 1]$ is an $(n+1)$-manifold with boundary. Show that the boundary of $M \times [0, 1]$ consists of two connected components, each of which is diffeomorphic to M.

23.9. Let M be a compact smooth manifold and $f : M \to \mathbb{R}$ be smooth. Show that if the interval $[a, b]$ only contains regular values of f then the sublevel sets $f^{-1}((-\infty, a])$ and $f^{-1}((-\infty, b])$ are diffeomorphic.

24. Sard theorem

Sard theorem. We use the following result from Analysis:

THEOREM (Sard Theorem). *The set of critical values of a smooth map from* $\mathbb{R}^m$ *to* $\mathbb{R}^n$ *is of Lebesgue measure zero.*

For a proof see for instance **[Mil97]**. Sard theorem also holds for smooth functions from $\mathbb{H}^m$ to $\mathbb{R}^n$. This is left as a problem.

Since a set of measure zero must have empty interior, we have:

COROLLARY. *The set of regular values of a smooth map from* $\mathbb{R}^m$ *to* $\mathbb{R}^n$ *is dense in* $\mathbb{R}^n$*.*

An application of Sard theorem for manifolds is the following:

THEOREM. *If* M *and* N *are two manifolds with boundary and* $f : M \to N$ *is smooth then the set of all regular values of* f *is dense in* N*. In particular* f *has a regular value in* N*.*

PROOF. Consider any open subset V of N parametrized by $\psi : V' \to V$. Then $f^{-1}(V)$ is an open submanifold of M. We only need to prove that $f|_{f^{-1}(V)}$ has a regular value in V. Let C be the set of all critical points of $f|_{f^{-1}(V)}$.

We can cover $f^{-1}(V)$ (or any manifold) by a *countable* collection I of parametrized open neighborhoods. This is possible because a Euclidean space has a countable topological basis (see 2.17).

For each $U \in I$ we have a commutative diagram:

$$\begin{tikzcd} U \arrow[r, "f"] & V \\ U' \arrow[u, "\varphi_U"] \arrow[r, "g_U"] & V' \arrow[u, "\psi"'] \end{tikzcd}$$

where U' is an open subset of $\mathbb{H}^m$ and V' is an open subset of $\mathbb{H}^n$. From this diagram, x is a critical point of f in U if and only if $\varphi_U^{-1}(x)$ is a critical point of g_U. Thus the set of critical points of g_U is $\varphi_U^{-1}(C \cap U)$.

Now we write

$$f(C) = \bigcup_{U \in I} f(C \cap U) = \bigcup_{U \in I} \psi(g_U(\varphi^{-1}(C \cap U))) = \psi\left(\bigcup_{U \in I} g_U(\varphi^{-1}(C \cap U))\right).$$

By Sard Theorem the set $g_U(\varphi_U^{-1}(C \cap U))$ is of measure zero. This implies that the set $D = \bigcup_{U \in I} g_U(\varphi_U^{-1}(C \cap U))$ is of measure zero, since a countable union of sets of measure zero is a set of measure zero. As a consequence D must have empty topological interior.

Since ψ is a homeomorphism, $\psi(D) = f(C)$ must also have empty topological interior. Thus $f(C) \subsetneq V$, so there must be a regular value of f in V. $\square$

If $N \subset M$ and $f : M \to N$ such that $f|_N = \mathrm{id}_N$ then f is called a *retraction* from M to N and N is a *retract* of M.

LEMMA 24.1. *Let M be a compact manifold with boundary. There is no smooth map $f : M \to \partial M$ such that $f|_{\partial M} = id_{\partial M}$. In other words there is no smooth retraction from M to its boundary.*

PROOF. Suppose that there is such a map f. Let y be a regular value of f. Since $f|_{\partial M}$ is the identity map, y is also a regular value of $f|_{\partial M}$. By Theorem 23.2 the inverse image $f^{-1}(y)$ is a 1-manifold with boundary $f^{-1}(y) \cap \partial M = \{y\}$. But a 1-manifold cannot have boundary consisting of exactly one point. This result is contained in the classification of compact one-dimensional manifolds. □

THEOREM 24.2 (Classification of compact one-dimensional manifolds). *A smooth compact connected one-dimensional manifold is diffeomorphic to either a circle, in which case it has no boundary, or an arc, in which case its boundary consists of two points.*

See **[Mil97]** for a proof.

Repeating of the proof for the continuous Brouwer fixed point theorem as in 16.3, we get:

COROLLARY 24.3 (smooth Brouwer fixed point theorem). *A smooth map from the disk D^n to itself has a fixed point.*

PROOF. Suppose that f does not have a fixed point, i.e. $f(x) \neq x$ for all $x \in D^n$. The straight line from $f(x)$ to x will intersect the boundary ∂D^n at a point $g(x)$. Then $g : D^n \to \partial D^n$ is a smooth function which is the identity on ∂D^n. That is impossible, by 24.1. □

A proof for the continuous version of the theorem using the smooth version can be found in **[Mil97]**.

Problems.

24.4. Show that Sard theorem also holds for smooth functions from $\mathbb{H}^m$ to $\mathbb{R}^n$.

24.5. Show that a smooth loop on S^2 (i.e. a smooth map from S^1 to S^2) cannot cover S^2. Similarly, there is no smooth surjective maps from $\mathbb{R}$ to $\mathbb{R}^n$ with $n > 1$. In other words, there is no smooth space filling curves, in contrast to the continuous case (compare 9).

24.6. Check that the function g in the proof of 24.3 is smooth.

24.7. Let A be an $n \times n$ matrix whose entries are all nonnegative real numbers. We will derive the Frobenius theorem which says that A must have a real nonnegative eigenvalue.

(a) Suppose that A is not singular. Check that the map $v \mapsto \frac{Av}{||Av||}$ brings $Q = \{(x_1, x_2, \ldots, x_n) \in S^{n-1} \mid x_i \geq 0, 1 \leq i \leq n\}$ to itself.
(b) Prove that Q is homeomorphic to the closed ball D^{n-1}.
(c) Use the continuous Brouwer fixed point theorem to prove that A has a real nonnegative eigenvalue.

25. Orientation

Orientation on vector spaces. On a finite dimensional real vector space, two vector bases are said to determine the same orientation of the space if the change of bases matrix has positive determinant. Being of the same orientation is an equivalence relation on the set of all bases. With this equivalence relation the set of all bases is divided into two equivalence classes. If we choose one of the two classes as the preferred one, then we say the vector space is *oriented* and the chosen equivalence class is called the *orientation* (or the positive orientation).

Thus any finite dimensional real vector space is *orientable* (i.e. can be oriented) with two possible orientations.

EXAMPLE. The standard positive orientation of $\mathbb{R}^n$ is represented by the basis

$$\{e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)\}.$$

Unless stated otherwise, $\mathbb{R}^n$ is always oriented this way.

Let T be an isomorphism from an oriented finite dimensional real vector space V to an oriented finite dimensional real vector space W. Then T brings a basis of V to a basis of W. There are only two possibilities. Either T brings a positive basis of V to a positive basis of W, or T brings a positive basis of V to a negative basis of W. In the first case we say that T is *orientation-preserving*, and in the second case we say that T is *orientation-reversing*.

Orientation on manifolds. Roughly, a manifold is oriented if at each point an orientation for the tangent space is chosen and this orientation should be smoothly depended on the point.

DEFINITION. A manifold M is said to be *oriented* if at each point x an orientation for the tangent space TM_x is chosen and at each point there exists a local coordinate (U, φ) such that for each x in U the derivative $d\varphi_x : TM_x \to \mathbb{R}^m$ is orientation-preserving.

Thus in this local coordinate the orientation of TM_x is given by the basis $\left\{\frac{\partial \varphi}{\partial x_1}(x), \frac{\partial \varphi}{\partial x_2}(x), \ldots, \frac{\partial \varphi}{\partial x_m}(x)\right\}$ where $\frac{\partial \varphi}{\partial x_1}(x) = d\varphi^{-1}_{\varphi(x)}(e_i)$. Roughly, the local coordinate brings the orientation of $\mathbb{R}^m$ to the manifold.

If a manifold is oriented then the set of orientations of its tangent spaces is called an *orientation* of the manifold and the the manifold is said to be *orientable*.

Another approach to orientation of manifold is to orient each parametrized neighborhood then require that the orientations on overlapping neighborhoods agree. Concisely, suppose that $\varphi : U \to M$ is a parametrization of a neighborhood in M. At each point, the orientation on TM_x is given by the image of the standard basis of $\mathbb{R}^n$ via $d\varphi_u$, i.e. it is given by the basis $\left\{\frac{\partial \varphi}{\partial u_1}(u), \frac{\partial \varphi}{\partial u_2}(u), \ldots, \frac{\partial \varphi}{\partial u_n}(u)\right\}$ where $\frac{\partial \varphi}{\partial u_i}(u) = d\varphi_u(e_i)$. Suppose that $\psi : V \to M$ parametrizes an overlapping neighborhood. Since $d\psi_v = d(\psi \circ \varphi^{-1})_v \circ d\varphi_u$, the consistency requirement is that the map $d(\psi \circ \varphi^{-1})_v$ must be orientation preserving on $\mathbb{R}^m$. In other words, we can say that the change of coordinates must be orientation preserving.

EXAMPLE. If a manifold is parametrized by one parametrization, that is, it is covered by one local coordinate, then it is orientable, since we can take the unique parametrization to bring an orientation of $\mathbb{R}^m$ to the entire manifold. In particular, any open subset of $\mathbb{R}^k$ is an orientable manifold.

EXAMPLE. The graph of a smooth function $f : D \to \mathbb{R}^l$, where $D \subset \mathbb{R}^k$ is an open set, is an orientable manifold, since this graph can be parametrized by a single parametrization, namely $x \mapsto (x, f(x))$.

PROPOSITION. *If $f : \mathbb{R}^k \to \mathbb{R}$ is smooth and a is a regular value of f then $f^{-1}(a)$ is an orientable manifold.*

PROOF. Let $M = f^{-1}(a)$. If $x \in M$ then $\ker df_x = TM_x$, so the gradient vector $\nabla f(x)$ is perpendicular to TM_x. In other words the gradient vector is always perpendicular to the level set. In particular, $\nabla f(x)$ does not belong to TM_x. Choose the orientation on TM_x represented by a basis $b(x) = \{b_1(x), \ldots, b_{k-1}(x)\}$ such that the ordered set $\{b_1(x), \ldots, b_{k-1}(x), \nabla f(x)\}$ is a positive basis in the standard orientation of $\mathbb{R}^k$. That means $\det(b_1(x), \ldots, b_{k-1}(x), \nabla f(x)) > 0$.

We check that this orientation is smoothly depended on the point. Let $\varphi : \mathbb{R}^{k-1} \to U \subset M$ be a local parametrization of a neighborhood U of x, with $\varphi(0) = x$. We can assume that basis $\left\{\frac{\partial \varphi}{\partial u_1}(0), \frac{\partial \varphi}{\partial u_2}(0), \ldots, \frac{\partial \varphi}{\partial u_{k-1}}(0)\right\}$ is in the same orientation as $b(x)$, if that is not the case we can interchange two variables of φ. We can check that $\left\{\frac{\partial \varphi}{\partial u_1}(u), \frac{\partial \varphi}{\partial u_2}(u), \ldots, \frac{\partial \varphi}{\partial u_{k-1}}(u)\right\}$ is in the same orientation as $b(\varphi(u))$ for all $u \in \mathbb{R}^{k-1}$. Indeed, consider $\det\left(\frac{\partial \varphi}{\partial u_1}(u), \frac{\partial \varphi}{\partial u_2}(u), \ldots, \frac{\partial \varphi}{\partial u_{k-1}}(u), \nabla f(\varphi(u))\right)$. This is a continuous real function on $u \in \mathbb{R}^{k-1}$ whose value at 0 is positive, therefore its value is always positive. $\square$

EXAMPLE. The sphere is orientable.

EXAMPLE. The torus is orientable.

PROPOSITION. *A connected orientable manifold has exactly two orientations.*

PROOF. Suppose the manifold M is orientable. There is an orientation o on M. Then $-o$ is a different orientation on M. Suppose that o_1 is an orientation on M, we show that o_1 is either o or $-o$.

If two orientations agrees at a point they must agree locally around that point. Indeed, from the definition there is a neighborhood V of x and a local coordinates $\varphi : V \to \mathbb{R}^m$ that brings the orientation o_1 to the standard orientation of $\mathbb{R}^m$, and a local coordinates $\psi : V \to \mathbb{R}^k$ that brings the orientation o to the standard orientation of $\mathbb{R}^m$. Assuming $\varphi(x) = \psi(x) = 0$, then $\det J(\psi^{-1} \circ \varphi)$ is smooth on $\mathbb{R}^m$ and is positive at 0, therefore it is always positive. That implies o_1 and o agree on V.

Let U be the set of all points x in M such that the orientation of TM_x with respect to o_1 is the same with the orientation of TM_x with respect to o. Then U is open in M. Similarly the complement $M \setminus U$ is also open. Since M is connected, either $U = M$ or $U = \emptyset$. $\square$

Orientable surfaces. A two dimensional smooth manifold in $\mathbb{R}^3$ is called a (smooth) *surface*. A surface is *two-sided* if there is a smooth way to choose a unit normal vector $N(p)$ at each point $p \in S$. That is, there is a smooth map $N : S \to \mathbb{R}^3$ such that at each $p \in S$ the vector $N(p)$ has length 1 and is perpendicular to TS_p.

PROPOSITION. *A surface is orientable if and only if it is two-sided.*

PROOF. If the surface S is orientable then its tangent spaces could be oriented smoothly. That means at each point $p \in S$ there is a local parametrization $r(u,v)$ such that $\{r_u(u,v), r_v(u,v)\}$ gives the orientation of TS_p. Then the unit normal vector $\dfrac{r_u(u,v) \times r_v(u,v)}{||r_u(u,v) \times r_v(u,v)||}$ is defined smoothly on the surface.

Conversely, if there is a smooth unit normal vector N on the surface then we orient each tangent plane TS_p by a basis $\{v_1, v_2\}$ such that $\{v_1, v_2, N(p)\}$ is in the same orientation as the standard orientation of $\mathbb{R}^3$. For each point p take a local parametrization $r : \mathbb{R}^2 \to S$, $r(0,0) = p$, such that $\{r_u(0,0), r_v(0,0)\}$ is in the orientation of TS_p (take any local parametrization, if it gives the opposite orientation at p then just switch the variables). Since $\langle r_u(u,v) \times r_v(u,v), N(r(u,v))\rangle$ is smooth, its sign does not change, and since the sign at $(0,0)$ is positive, the sign is always positive. Thus $\{r_u(u,v), r_v(u,v)\}$ is in the orientation of $TS_{r(u,v)}$. That means the orientation is smooth. $\square$

Now we are able to prove a famous fact, that the Mobius band (see 9.4) is not orientable.

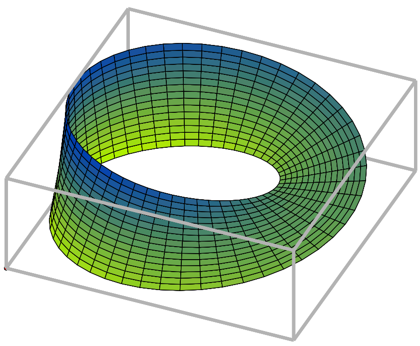


FIGURE 25.1. The Mobius band is not orientable and is not two-sided.

Visually, if we pick a normal vector to the surface at a point in the center of the Mobius band, then move that normal vector smoothly along the center circle of the band. When we come back at the initial point after one loop, we realize that the normal vector is now in the opposite direction. That demonstrate that the

Mobius surface is not two-sided. Similarly if we choose an orientation at a point then move that orientation continuously along the band then when we comeback the orientation has been switched.

We can write this argument rigorously below.

THEOREM 25.2. *The Mobius band is not orientable.*

PROOF. Recall a parametrization of a neighborhood of the (open) Mobius band M from Figure 9.6:

$$\begin{aligned} \varphi_1 : (0, 2\pi) \times (-1, 1) &\to M \\ (s, t) &\mapsto \left((2 + t\cos\frac{s}{2})\cos s, (2 + t\cos\frac{s}{2})\sin s, t\sin\frac{s}{2} \right). \end{aligned}$$

This parametrization misses a subset of M, namely the interval $[1, 3]$ on the x-axis. So we need one more parametrization to cover this part. We can take

$$\begin{aligned} \varphi_2 : (-\pi, \pi) \times (-1, 1) &\to M \\ (s, t) &\mapsto \left((2 + t\cos\frac{s}{2})\cos s, (2 + t\cos\frac{s}{2})\sin s, t\sin\frac{s}{2} \right). \end{aligned}$$

Thus φ_2 is given by the same formula as φ_1, but on a different domain. This parametrization misses the subset $\{-2\} \times \{0\} \times [-1, 1]$ of M.

Suppose that M is orientable. Take an orientation for M. Then either φ_1 agrees with this orientation or disagrees with this orientation over the entire connected domain of φ_1. The same is true for φ_2. That implies that φ_1 and φ_2 either induce the same orientations over their entire domains, or they induces the opposite orientations over their domains.

Calculating directly, we get the normal vector given by φ_1 at the point $\varphi_1(s, 0)$ on the center circle is:

$$(\varphi_1)_s \times (\varphi_1)_t(s, 0) = \left(2\cos s \sin\frac{s}{2}, 2\sin s \sin\frac{s}{2}, -2\cos\frac{s}{2} \right).$$

The normal vector given by φ_2 at the point $\varphi_2(s, 0)$ on the center circle is by the same formula:

$$(\varphi_2)_s \times (\varphi_2)_t(s, 0) = \left(2\cos s \sin\frac{s}{2}, 2\sin s \sin\frac{s}{2}, -2\cos\frac{s}{2} \right).$$

At the point $(0, 2, 0) = \varphi_1(\frac{\pi}{2}, 0) = \varphi_2(\frac{\pi}{2}, 0)$ the two normal vectors agree, but at $(0, -2, 0) = \varphi_1(\frac{3\pi}{2}, 0) = \varphi_2(-\frac{\pi}{2}, 0)$ they are opposite. Thus φ_1 and φ_2 do not give the same orientation, a contradiction. □

Orientation on the boundary of an oriented manifold. Suppose that M is a manifold with boundary and the interior of M is oriented. We orient the boundary of M as follows. Suppose that under an orientation-preserving parametrization φ the point $\varphi(x)$ is on the boundary ∂M of M. Then the orientation $\{b_2, b_3, \dots, b_n\}$ of $\partial \mathbb{H}^n$ such that the ordered set $\{-e_n, b_2, b_3, \dots, b_n\}$ is a positive basis of $\mathbb{R}^n$ will induce the positive orientation for $T\partial M_{\varphi(x)}$ through $d\varphi(x)$. This is called *the outer normal first orientation of the boundary*.

Problems.

25.3. Show that two diffeomorphic manifolds are are either both orientable or both unorientable.

25.4. Suppose that $f : M \to N$ is a diffeomorphism of connected oriented manifolds with boundary. Show that if there is an x such that $df_x : TM_x \to TN_{f(x)}$ is orientation-preserving then f is orientation-preserving.

25.5. Let $f : \mathbb{R}^k \to \mathbb{R}^l$ be smooth and let a be a regular value of f. Show that $f^{-1}(a)$ is an orientable manifold.

25.6. Consider the map $-\mathrm{id} : S^n \to S^n$ with $x \mapsto -x$. Show that $-\mathrm{id}$ is orientation-preserving if and only if n is odd.

26. Topological degrees of maps

Let M and N be boundaryless, oriented manifolds of the same dimensions m. Further suppose that M is compact.

Let $f : M \to N$ be smooth. Suppose that x is a regular point of f. Then df_x is an isomorphism from TM_x to $TN_{f(x)}$. Let $\text{sign}(df_x) = 1$ if df_x preserves orientations, and $\text{sign}(df_x) = -1$ otherwise.

For any regular value y of f, let

$$\deg(f, y) = \sum_{x \in f^{-1}(y)} \text{sign}(df_x).$$

Notice that the set $f^{-1}(y)$ is finite because M is compact (see 20.4).

This number $\deg(f, y)$ is called the *Brouwer degree* (bậc Brouwer) [15] or *topological degree* of the map f with respect to the regular value y.

From the Inverse Function Theorem 20.19, each regular value y has a neighborhood V and each preimage x of y has a neighborhood U_x on which f is a diffeomorphism onto V, either preserving or reversing orientation. Therefore we can interpret that $\deg(f, y)$ *counts the algebraic number of times the function f covers the value y*.

EXAMPLE. Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Then $\deg(f, 1) = 0$. This could be explained geometrically from the graph of f, as f covers the value 1 twice in opposite directions at $x = -1$ and $x = 1$.

EXAMPLE. Consider $f(x) = x^3 - x$ with the regular value 0. From the graph of f we see that f covers the value 0 three times in positive direction at $x = -1$ and $x = 1$ and negative direction at $x = 0$, therefore we see right away that $\deg(f, 0) = 1$.

On the other hand, if we consider the regular value -1 then f covers this value only once in positive direction, thus $\deg(f, 1) = 1$.

Homotopy invariance. In this section we will show that the Brouwer degree does not depend on the choice of regular values and is invariant under smooth homotopy.

LEMMA. *Let M be the boundary of a compact oriented manifold X, oriented as the boundary of X. If $f : M \to N$ extends to a smooth map $F : X \to N$ then $\deg(f, y) = 0$ for every regular value y.*

PROOF. (a) Assume that y is a regular value of F. Then $F^{-1}(y)$ is a 1-dimensional manifold of dimension 1 whose boundary is $F^{-1}(y) \cap M = f^{-1}(y)$, by Theorem 23.1.

By the Classification of one-dimensional manifolds, $F^{-1}(y)$ is the disjoint union of arcs and circles. Let A be a component that intersects M. Then A is an arc with boundary $\{a, b\} \subset M$.

[15] L. E. J. Brouwer (1881–1966) is a Dutch mathematician. He had many important contributions in the early development of topology, and founded Intuitionism.

We will show that $\operatorname{sign}(\det(df_a)) = -\operatorname{sign}(\det(df_b))$. Taking sum over all arc components of $F^{-1}(y)$ would give us $\deg(f,y) = 0$.

An orientation on A. Let $x \in A$. Recall that TA_x is the kernel of $dF_x : TX_x \to TN_y$. We will choose the orientation on TA_x such that this orientation together with the pull-back of the orientation of TN_y via dF_x is the orientation of X. Let $(v_2, v_3, \ldots, v_{n+1})$ be a positive basis for TN_y. Let $v_1 \in TA_x$ such that $\{v_1, dF_x^{-1}(v_2), \ldots, dF_x^{-1}(v_{n+1})\}$ is a positive basis for TX_x. Then v_1 determine the positive orientation on TA_x.

At $x = a$ or at $x = b$ we have $df_x = dF_x|_{TM_x}$. Therefore df_x is orientation-preserving on TM_x oriented by the basis $\{df_x^{-1}(v_2), \ldots, df_x^{-1}(v_{n+1})\}$.

We claim that exactly one of the two above orientations of TM_x at $x = a$ or $x = b$ is opposite to the orientation of TM_x as the boundary of X. This would show that $\operatorname{sign}(\det(df_a)) = -\operatorname{sign}(\det(df_b))$.

Observe that if at a the orientation of TA_a is pointing outward with respect to X then b the orientation of TA_b is pointing inward, and vice versa. Indeed, since A is a smooth arc it is parametrized by a smooth map $\gamma(t)$ such that $\gamma(0) = a$ and $\gamma(1) = b$. If we assume that the orientation of $TA_{\gamma(t)}$ is given by $\gamma'(t)$ then it is clear that at a the orientation is inward and at b it is outward.

(b) Suppose now that y is not a regular value of F. There is a neighborhood of y in the set of all regular values of f such that $\deg(f,z)$ does not change in this neighborhood. Let z be a regular value of F in this neighborhood, then $\deg(f,z) = \deg(F,z) = 0$ by (a), and $\deg(f,z) = \deg(f,y)$. Thus $\deg(f,y) = 0$. □

LEMMA. *If f is smoothly homotopic to g then* $\deg(f,y) = \deg(g,y)$ *for any common regular value y.*

PROOF. Let $I = [0,1]$ and $X = M \times I$. Since f be homotopic to g there is a smooth map $F : X \to N$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$.

The boundary of X is $(M \times \{0\}) \sqcup (M \times \{1\})$. Then F is an extension of the pair f, g from ∂X to X, thus $\deg(F|_{\partial X}, y) = 0$ by the above lemma.

Note that one of the two orientations of $M \times \{0\}$ or $M \times \{1\}$ as the boundary of X is opposite to the orientation of M (this is essentially for the same reason as in the proof of the above lemma). Therefore $\deg(F|_{\partial X}, y) = \pm(\deg(f,y) - \deg(g,y)) = 0$, so $\deg(f,y) = \deg(g,y)$. □

LEMMA 26.1 (Homogeneity of manifold). *Let N be a connected boundaryless manifold and let y and z be points of N. Then there is a self diffeomorphism $h : N \to N$ that is smoothly isotopic to the identity and carries y to z.*

We do not present a proof for this lemma. The reader can find a proof in [**Mil97**, p. 22].

THEOREM 26.2. *Let M and N be boundaryless, oriented manifolds of the same dimensions. Further suppose that M is compact and N is connected. The Brouwer degree of a map from M to N does not depend on the choice of regular values and is invariant under smooth homotopy.*

Therefore from now on we will write $\deg(f)$ instead of $\deg(f, y)$.

PROOF. We have already shown that degree is invariant under homotopy.

Let y and z be two regular values for $f : M \to N$. Choose a diffeomorphism h from N to N that is isotopic to the identity and carries y to z.

Note that h preserves orientation. Indeed, there is a smooth isotopy $F : N \times [0,1] \to N$ such that $F_0 = h$ and $F_1 = \mathrm{id}$. Let $x \in N$, and let $\varphi : \mathbb{R}^m \to N$ be an orientation-preserving parametrization of a neighborhood of x with $\varphi(0) = x$. Since $dF_t(x) \circ d\varphi_0 : \mathbb{R}^m \times \mathbb{R}$ is smooth with respect to t, the sign of $dF_t(x)$ does not change with t.

As a consequence, $\deg(f, y) = \deg(h \circ f, h(y))$.

Finally since $h \circ f$ is homotopic to $id \circ f$, we have $\deg(h \circ f, h(y)) = \deg(\mathrm{id} \circ f, h(y)) = \deg(f, h(y)) = \deg(f, z)$. □

EXAMPLE. Let M be a compact, oriented and boundaryless manifold. Then the degree of the identity map on M is 1. On the other hand the degree of a constant map on M is 0. Therefore the identity map is not homotopic to a constant map.

EXAMPLE 26.3 (Proof of the Brouwer fixed point theorem via the Brouwer degree). We can prove that D^{n+1} cannot retract to its boundary (this is 24.1 for the case of D^{n+1}) as follows. Suppose that there is such a retraction, a smooth map $f : D^{n+1} \to S^n$ that is the identity on S^n. Define $F : [0,1] \times S^n$ by $F(t, x) = f(tx)$. Then F is a smooth homotopy from a constant map to the identity map on the sphere. But these two maps have different degrees.

THEOREM (The fundamental theorem of Algebra). *Any non-constant polynomial with real coefficients has at least one complex root.*

PROOF. Let $p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n$, with $a_i \in \mathbb{R}$, $1 \le i \le n$. Suppose that p has no root, that is, $p(z) \neq 0$ for all $z \in \mathbb{C}$. As a consequence, $a_n \neq 0$.

For $t \in [0,1]$, let

$$q_t(z) = (1-t)^n z^n + a_1 (1-t)^{n-1} t z^{n-1} + \cdots + a_{n-1} (1-t) t^{n-1} z + a_n t^n.$$

Then $q_t(z)$ is continuous with respect to the pair (t, z). Notice that if $t \neq 0$ then $q_t(z) = t^n p((1-t)t^{-1} z)$, and $q_0(z) = z^n$ while $q_1(z) = a_n$.

If we restrict z to the set $\{z \in \mathbb{C} \mid |z| = 1\} = S^1$ then $q_t(z)$ has no roots, so $\frac{q_t(z)}{|q_t(z)|}$ is a continuous homotopy of maps from S^1 to itself, starting with the polynomial z^n and ending with the constant polynomial $\frac{a_n}{|a_n|}$. But these two polynomials have different degrees, a contradiction. □

EXAMPLE. Let $v : S^1 \to \mathbb{R}^2$, $v((x, y)) = (-y, x)$, then it is a nonzero (not zero anywhere) tangent vector field on S^1.

Similarly we can find a nonzero tangent vector field on S^n with odd n.

THEOREM 26.4 (The Hairy Ball Theorem). *If n is even then every smooth tangent vector field on S^n has a zero.*

PROOF. Suppose that v is a nonzero tangent smooth vector field on S^n. Let $w(x) = \frac{v(x)}{||v(x)||}$, then w is a unit smooth tangent vector field on S^n.

Notice that $w(x)$ is perpendicular to x. On the plane spanned by x and $w(x)$ we can easily rotate vector x to vector $-x$. Precisely, let $F_t(x) = \cos(t) \cdot x + \sin(t) \cdot w(x)$ with $0 \le t \le \pi$, then F is a homotopy on S^n from x to $-x$. But the degrees of these two maps are different, see 26.14. □

Problems.

26.5. Find the topological degree of a polynomial on $\mathbb{R}$. Notice that although the domain $\mathbb{R}$ is not compact, the topological degree is well-defined for polynomial.

26.6. Let $f : S^1 \to S^1$, $f(z) = z^n$, where $n \in \mathbb{Z}$. We can also consider f as a vector-valued function $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (f_1(x,y), f_2(x,y))$. Then $f = f_1 + if_2$.

(a) Recalling the notion of complex derivative and the Cauchy-Riemann condition, check that $\det(Jf_z) = |f'(z)|^2$.
(b) Check that all values of f are regular.
(c) Check that $\deg(f,y) = n$ for all $y \in S^1$.

26.7. Show that $\deg(f,y)$ is locally constant on the subspace of all regular values of f.

26.8. What happens if we drop the condition that N is connected in Theorem 26.2? Where do we use this condition?

26.9. Let M and N be oriented boundaryless manifolds, M is compact and N is connected. Let $f : M \to N$. Show that if $\deg(f) \neq 0$ then f is onto, i.e. the equation $f(x) = y$ always has a solution.

26.10. Let $r_i : S^n \to S^n$ be the reflection map

$$r_i((x_1, x_2, \dots, x_i, \dots, x_{n+1})) = (x_1, x_2, \dots, -x_i, \dots, x_{n+1}).$$

Compute $\deg(r_i)$.

26.11. Let $f : S^n \to S^n$ be the map that interchanges two coordinates:

$$f((x_1, x_2, \dots, x_i, \dots, x_j, \dots, x_{n+1})) = (x_1, x_2, \dots, x_j, \dots, x_i, \dots, x_{n+1}).$$

Compute $\deg(f)$.

26.12. Suppose that M, N, P are compact, oriented, connected, boundaryless m-manifolds. Let $M \xrightarrow{f} N \xrightarrow{g} P$. Then $\deg(g \circ f) = \deg(f)\deg(g)$.

26.13. Let M be a compact connected smooth manifold. Let $f : M \to M$ be smooth.

(a) Show that if f is bijective then $\deg f = \pm 1$.
(b) Let $f^2 = f \circ f$. Show that $\deg(f^2) \ge 0$.

26.14. Let $r : S^n \to S^n$ be the antipodal map

$$r((x_1, x_2, \dots, x_{n+1})) = (-x_1, -x_2, \dots, -x_{n+1}).$$

Compute $\deg(r)$.

26.15. Let $f : S^4 \to S^4$, $f((x_1, x_2, x_3, x_4, x_5)) = (x_2, x_4, -x_1, x_5, -x_3)$. Find $\deg(f)$.

26.16. Find a map from S^2 to itself of any given degree.

26.17. If $f, g : S^n \to S^n$ be smooth such that $f(x) \neq -g(x)$ for all $x \in S^n$ then f is smoothly homotopic to g.

26.18. Let $f : M \to S^n$ be smooth. Show that if $\dim(M) < n$ then f is homotopic to a constant map.

26.19 (Brouwer fixed point theorem for the sphere). Let $f : S^n \to S^n$ be smooth. If $\deg(f) \neq (-1)^{n+1}$ then f has a fixed point.

26.20. Show that any map of from S^n to S^n of odd degree carries a certain pair of antipodal points to a pair of antipodal point.

Guide for further reading

We have closely followed John Milnor's masterpiece [**Mil97**]. Another excellent text is [**GP74**]. There are not many textbooks such as these two books, presenting differential topology to undergraduate students.

The book [**Hir76**] is a technical reference for some advanced topics. The book [**DFN85**] is a masterful presentation of modern topology and geometry, with some enlightening explanations, but it sometimes requires knowledge of many topics. The book [**Bre93**] is rather similar in aim, but is more like a traditional textbook.

An excellent textbook for differential geometry of surfaces is [**dC76**].

Suggestions for some problems

1.7: There is an infinitely countable subset of B.

1.13: $\bigcup_{n=1}^{\infty}[n, n+1] = [1, \infty)$.

1.17: Use the idea of the Cantor diagonal argument in the proof of 1.3. In this case the issue of different presentations of same real numbers does not appear.

1.18: Proof by contradiction.

2.18: Show that each ball in one metric contains a ball in the other metric with the same center.

3.16: Consider a map from the unit ball to the space, such as: $x \mapsto \dfrac{1}{1 - ||x||}x$.

3.26: Compare the subinterval $[1, 2\pi)$ and its image via φ.

4.17: Use the characterization of connected subspaces of the Euclidean line.

4.24: Let A be countable and $x \in \mathbb{R}^2 \setminus A$. There is a line passing through x that does not intersect A (by an argument involving countability of sets).

4.30: Use 3.15 to modify each letter part by part.

5.11: Consider the set of all irrational numbers.

5.21: Let C be a countable subset of $[0, \Omega)$. The set $\bigcup_{c \in C}[0, c)$ is countable while the set $[0, \Omega)$ is uncountable. This implies C is bounded from above.

6.9: Use Lebesgue's number.

6.12: See the proof of 6.1.

6.14: Use 6.12.

6.15: Use 6.14.

6.17: Let X be a compact metric space, and let I be an open cover of X. For each $x \in X$ there is an open set $U_x \in I$ containing x. There is a number $\epsilon_x > 0$ such that the ball $B(x, 2\epsilon_x)$ is contained in U_x. The collection $\{B(x, \epsilon_x) \mid x \in X\}$ is an open cover of X, therefore there is a finite subcover $\{B(x_i, \epsilon_i) \mid 1 \leq i \leq n\}$. Let $\epsilon = \min\{\epsilon_i \mid 1 \leq i \leq n\}$. Suppose that $y \in B(x, \epsilon)$. There is an i_0, $1 \leq i_0 \leq n$, such that $x \in B(x_{i_0}, \epsilon_{i_0})$. We have $d(y, x_{i_0}) \leq d(y, x) + d(x, x_{i_0}) < \epsilon + \epsilon_{i_0} \leq 2\epsilon$. This implies y belongs to the element $U_{x_{i_0}}$ of I, and so $B(x, \epsilon)$ is contained in $U_{x_{i_0}}$.

6.20: Use6.19.

6.27: Use 6.12

6.28: Use 6.27.

6.30: ($\Leftarrow$) Use 6.15 and 5.8.

6.31: Show that if U is an open set of X containing Y then U contains X_n for some n. Use 6.15. Look at the Topologist's sine curve.

7.6: Look at their bases.

7.12: Only need to show that the projection of an element of the basis is open.

7.14: Use 7.2 to prove that the inclusion map is continuous.

7.15: Use 7.14.

7.16: Let (x_i) and (y_i) be in $\prod_{i\in I} X_i$. Let $\gamma_i(t)$ be a continuous path from x_i to y_i. Let $\gamma(t) = (\gamma_i(t))$.

7.17: (b) Use 7.14. (c) Fix a point $x \in \prod_{i\in I} X_i$. Use (b) to show that the set A_x of points that differs from x at at most finitely many coordinates is connected. Furthermore A_x is dense in $\prod_{i\in I} X_i$.

7.18: Use 7.14. It is enough to prove for the case an open cover of $X \times Y$ by open sets of the form a product of an open set in X with an open set in Y. For each "slice" $\{x\} \times Y$ there is finite subcover $\{U_{x,i} \times V_{x,i} \mid 1 \le i \le n_x\}$. Take $U_x = \bigcap_{i=1}^{n_x} U_{x,i}$. The collection $\{U_x \mid x \in X\}$ covers X so there is a subcover $\{U_{x_j} \mid 1 \le j \le n\}$. The collection $\{U_{x_j,i} \times V_{x_j,i} \mid 1 \le i \le n_{x_j}, 1 \le j \le n\}$ is a finite subcover of $X \times Y$.

7.24: ($\Leftarrow$) Use 6.15 and the Urysohn lemma 8.1.

8.5: ($\Leftarrow$) Use 6.15 and the Urysohn lemma 8.1.

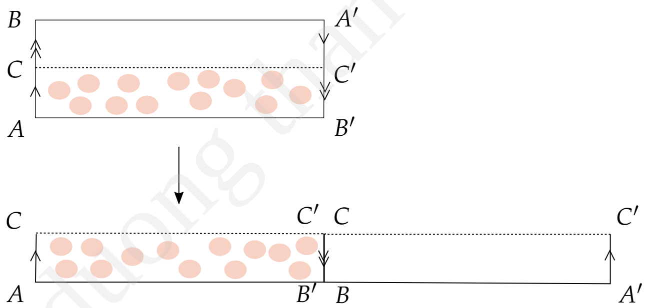
8.6: Use 6.28 and 6.13.

8.10: Use 8.9.

8.11: See 7.1 and 7.3.

8.13: Use 6.27. Use a technique similar to the one in 7.18.

9.15: To give a rigorous argument we can simply describe the figure below.



The map from $X = ([0,1] \times [0,1]) \setminus \left([0,1] \times \{\frac{1}{2}\}\right)$ to $Y = [0,2] \times [0, \frac{1}{2})$ given by

$$(x,y) \mapsto \begin{cases} (x,y), & y < \frac{1}{2}, \\ (x+1, 1-y), & y > \frac{1}{2}, \end{cases}$$

is bijective and is continuous. The induced map to $Y/(0,y) \sim (2,y)$ is surjective and is continuous. Then its induced map on $X/(0,y) \sim (1, 1-y)$ is bijective and is continuous, hence is a homeomorphism between $X/(0,y) \sim (1,1-y)$ and $Y/(0,y) \sim (2,y)$.

9.19: The idea is easy to be visualized in the cases $n = 1$ and $n = 2$. Let $S^+ = \{x = (x_1, x_2, \ldots, x_{n+1}) \in S^n \mid x_1 \ge 0\}$, the upper hemisphere. Let $S^0 = \{x = (x_1, x_2, \ldots, x_{n+1}) \in S^n \mid x_1 = 0\}$, the equator. Let $f : S^n \to S^+$ be given by $f(x) = x$ if $x \in S^+$ and $f(x) = -x$ otherwise. Then the following diagram is commutative:

$$\begin{tikzcd}
S^n \arrow[r, "f"] \arrow[d] \arrow[dr, "p\circ f"] & S^+ \arrow[d, "p"] \\
S^n/x \sim -x \arrow[r, "\tilde{f}"] & S^+/x \sim -x,\ x \in S^0
\end{tikzcd}$$

Then it is not difficult to show that $S^+/x \sim -x,\ x \in S^0$ is homeomorphic to $\mathbb{RP}^n = D^n/x \sim -x,\ x \in \partial D^n$.

9.25: Examine the following diagram, and use 9.1 to check that the maps are continuous.

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X/R_1 \arrow[r] \arrow[dl] & (X/R_1)/\tilde{R}_2 \arrow[dll, bend left] \\
X/(R_1 \cup R_2) \arrow[urr] & &
\end{tikzcd}$$

10.15: One approach is using 3.20.

11.22: Deleting an open disk is the same as deleting the interior of a triangle.

11.25: Count edges from of the set of triangles. Count vertices from the set of triangles. Count edges from the set of vertices, notice that each vertex belongs to at most $(v-1)$ edges.

15.6: Consider the dihedral group D_3. In particular consider the subgroup of the group of invertible 2×2 matrices generated by $\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ (a rotation of an angle $\frac{2\pi}{3}$) and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (a reflection).

15.2: See [**Hat01**, p. 52].

16.10: Use Mayer-Vietoris sequence.

16.7: First take a deformation retraction to a sphere.

16.8: Show that $\mathbb{R}^3 \setminus S^1$ is homotopic to Y which is a closed ball minus a circle inside. Show that $Y = S^1 \vee S^2$, [**Hat01**, p. 46]. Or write Y as a union of two halves, each of which is a closed ball minus a straight line, and use the Van Kampen theorem.

16.16: Let S be a convex compact subset of $\mathbb{R}^n$. Suppose that $p_0 \in \operatorname{Int} S$. For $x \in S^{n-1}$, let p_x be the intersection of the ray from p_0 in the direction of x with the boundary of S. Rigorously let $t_x = \sup\{t \in \mathbb{R}^+ \mid p_0 + tx \in S\}$ and let $p_x = p_0 + t_x x$. Check that the map $D^n \to S$ sending the straight segment $[0, x]$ linearly to $[p_0, p_x]$ is well-defined and is bijective. To prove that it is a homemorphism it is easier to consider the inverse map.

17.5: Let X be the simplical complex, A be the set of vertices path-connected to the vertex v_0 by edges, U be the union of simplexes containing vertices in A. Let B be the set of vertices not path-connected to the vertex v_0 by edges, V be the union of simplexes containing vertices in B. Then U and V are closed and disjoint.

17.6: Use problem 17.5.

18.6: The torus is given by the equation $(\sqrt{x^2+y^2} - b)^2 + z^2 = a^2$ where $0 < a < b$.

18.7: Consider a neighborhood of a point on the y-axis. Can it be homeomorphic to an open neighborhood in $\mathbb{R}$?

18.13: See 4.3.

18.14: See 3.25 and 22.2.

18.15: Use the Implicit function theorem.

20.19: Consider

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ U \arrow[u, "\varphi"] \arrow[r, "\psi^{-1}\circ f\circ\varphi"] & V \arrow[u, "\psi"'] \end{tikzcd}$$

Since df_x is surjective, it is bijective. Then $d(\psi^{-1} \circ f \circ \varphi)_u = d\psi^{-1}_{f(x)} \circ df_x \circ d\varphi_u$ is an isomorphism. The Inverse function theorem can be applied to $\psi^{-1} \circ f \circ \varphi$.

20.13: Each $x \in f^{-1}(y)$ has a neighborhood U_x on which f is a diffeomorphism. Let $V = [\bigcap_{x \in f^{-1}(y)} f(U_x)] \setminus f(M \setminus \bigcup_{x \in f^{-1}(y)} U_x)$. Consider $V \cap S$.

20.22: Use Problem 20.18.

22.2: Show that $f^{(n)}(x) = e^{-1/x} P_n(1/x)$ where $P_n(x)$ is a polynomial.

22.3: Cover A by finitely many balls $B_i \subset U$. For each i there is a smooth function φ_i which is positive in B_i and is zero outside of B_i.

26.19: If f does not have a fixed point then f will be homotopic to the reflection map.

26.17: Note that $f(x)$ and $g(x)$ will not be antipodal points. Use the homotopy

$$F_t(x) = \frac{(1-t)f(x) + tg(x)}{||(1-t)f(x) + tg(x)||}.$$

26.18: Using Sard Theorem show that f cannot be onto.

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You may say I'm a dreamer
But I'm not the only one
I hope someday you'll join us ...
John Lennon, Imagine.