Gorenstein homological algebra

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© 2004 Henrik Holm (according to the Danish legislation) with exception of the following three papers (which are included in the thesis):

- *Gorenstein homological dimensions*, © 2003 Elsevier B. V. All rights reserved.
- *Gorenstein derived functors* and *Rings with finite Gorenstein injective dimension*, © 2003 American Mathematical Society.

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Dear reader,

The present manuscript constitutes my Ph.D. thesis which is written at the University of Copenhagen under the supervision of Hans-Bjørn Foxby, during the period June 2001 – May 2004.

The thesis consists of six papers, of which three are joint work with co-authors. It is a pleasure to express my gratitude to my advisor Hans-Bjørn Foxby and to my co-authors Lars Winther Christensen, Anders Frankild and Peter Jørgensen for many interesting discussions and good times.

Finally, I am grateful for the many excellent papers by Edgar E. Enochs and Overtoun M. G. Jenda. Their work has inspired me much during the last three years.

Henrik Holm
Copenhagen, May 2004

Contents

Introduction

This introduction, page numbered with small Roman numerals i–xx, has three sections:

1. The structure of this thesis.
2. Some background material on Gorenstein dimensions.
3. An introduction to the papers in this thesis.

We begin with:

1. The structure of this thesis

This thesis consists of six parts which are numbered with capital Roman numerals I—VI. Each part contains a paper, and has been given a headline identical to the title of the paper it contains. The six papers are:

[I] H. Holm, *Gorenstein homological dimensions*, Journal of Pure and Applied Algebra **189** (2004), 167–193.

[II] H. Holm, *Gorenstein derived functors*, Proceedings of the American Mathematical Society **132** (2004), no. 7, 1913–1923.

[III] H. Holm, *Rings with finite Gorenstein injective dimension*, Proceedings of the American Mathematical Society **132** (2004), no. 5, 1279–1283.

[IV] L. W. Christensen, A. Frankild and H. Holm, *On Gorenstein projective, injective and flat dimensions — a functorial description with applications*, preprint 2003 (submitted), 39 pages, available from `http://arXiv.org/abs/math.AC/0403156`.

[V] H. Holm and P. Jørgensen, *Cohen-Macaulay injective, projective, and flat dimension*, preprint (2004), 17 (+5) pages.

[VI] H. Holm and P. Jørgensen, *Semi-dualizing modules and related Gorenstein homological dimensions*, preprint (2004), 25 pages.

The six papers are arranged chronologically, that is, the order of which the papers are presented here reflects the order in which they were produced.

The paper in each part I—VI has its own local page numbering, using Arabic numerals. For example, part IV is page numbered IV.1, IV.2, IV.3, ...

Each paper also has its own references. In this introduction, references [I] – [VI] refer to the papers listed above, whereas other references refer to the bibliography on page xix.

Finally, as could be expected, the six papers have somewhat different layouts and styles, making the typographical (but hopefully not the mathematical) look of this thesis slightly less homogeneous than one could wish for. This should have little or no influence on the readability.

2. Some background material on Gorenstein dimensions

The purpose of this section is to give (to the best of the author's understanding) some historical background information about Gorenstein dimensions.

In 1966/67, Auslander introduces the *G–class*, $G(A)$, for any commutative and noetherian ring A. By [1, Définition p. 55] it consists of all finitely generated A–modules M satisfying the two conditions:

(1) $\mathrm{Ext}^i_A(M,A) = \mathrm{Ext}^i_A(\mathrm{Hom}_A(M,A),A) = 0$ for all $i > 0$,
(2) The natural biduality homomorphism $M \longrightarrow \mathrm{Hom}_A(\mathrm{Hom}_A(M,A),A)$ is an isomorphism.

It is easy to see that every finitely generated projective A–module belongs to $G(A)$.

Using resolutions of modules from the class $G(A)$, Auslander [1, Définition p. 60] defines the *G–dimension*, $G\text{–}\dim_A M$, for any finitely generated module M. Among other results, he proves the following nice properties for this new dimension:

Theorem A (measure formula). *Let A be a commutative and noetherian ring, and let M be a finitely generated A–module. If $G\text{–}\dim_A M < \infty$, then:*

$$G\text{–}\dim_A M \;=\; \sup\left\{t \,\middle|\, \mathrm{Ext}^t_A(M,A) \neq 0\right\}.$$

Theorem B (refinement inequality). *If A is a commutative and noetherian ring, and M is a finitely generated A–module, then there is an inequality:*

$$G\text{–}\dim_A M \;\leqslant\; \mathrm{pd}_A M.$$

If $\mathrm{pd}_A M < \infty$, then equality holds.

Theorem C (Auslander–Bridger formula). *If $(A,\mathfrak{m},k)$ is a commutative, noetherian and local ring, and M is a finitely generated A–module such that $G\text{–}\dim_A M < \infty$, then:*

$$G\text{–}\dim_A M + \mathrm{depth}_A M \;=\; \mathrm{depth}\, A.$$

Theorem D (characterization of Gorenstein rings). *For a commutative, noetherian and local ring $(A,\mathfrak{m},k)$, the following conditions are equivalent:*

(i) *A is Gorenstein,*
(ii) *$G\text{–}\dim_A M < \infty$ for all finitely generated A–modules M,*
(iii) *$G\text{–}\dim_A k < \infty$.*

Theorems A, C and D are [1, Corollaire p. 66], [1, Théorème 2 p. 60] and [1, Théorème 3 and Remarque p. 64], respectively. Auslander does not formulate Theorem B explicitly, but it follows from e.g. Theorem A. In the case where A is local and Gorenstein, he mentions the result on [1, p. 52].

Auslander's ideas where developed even further in his work with Bridger [2], where the base ring is only assumed to be associative and two-sided noetherian. In this paper, the appropriate reference for Theorem A is [2, Remarks after (3.7)].

In the 1990's, Christensen [6, Chapter 2] and Yassemi [23] studied the G–dimension for complexes of A–modules, and developed a satisfactory theory.

It became clear that $G\text{–}\dim_A(-)$ was an interesting numerical invariant, but some natural questions arose:

- If we call the modules in Auslander's G–class for *finitely generated Gorenstein projective modules*, then how does one define non-finite (or general) Gorenstein projective modules?
- Is there also a notion of Gorenstein injective and Gorenstein flat modules?

Concerning the next result it is, as far as the author can tell, not possible to find an exact reference; but reading the proof of the last claim in [1, Proposition 8 p. 67], or of the equivalence (b) $\Leftrightarrow$ (c) in [2, Proposition (4.11)], we get:

Theorem E. *If A is an associative and two-sided noetherian ring, and M is a finitely generated left A–module, then the following conditions are equivalent:*

(i) *M belongs to $G(A)$, or equivalently, $G\text{–}\dim_A M = 0$,*
(ii) *There exists an exact sequence $\cdots \to L_1 \to L_0 \to L_{-1} \to \cdots$ of finitely generated free left A–modules such that $M = \mathrm{Im}(L_0 \to L_{-1})$ and such that $\mathrm{Hom}_A(-, A)$ leaves this complex exact.*

With Theorem E in mind, Enochs-Jenda [9] defined in 1995 a notion of Gorenstein projective and Gorenstein injective modules over an arbitrary associative ring R:

Definition F (Gorenstein projective modules). *A left R–module M is said to be Gorenstein projective if and only if there is an exact sequence:*

$$\cdots \to P_1 \to P_0 \to P_{-1} \to \cdots$$

of projective left R–modules such that $M = \mathrm{Im}(P_0 \to P_{-1})$ and such that $\mathrm{Hom}_R(-, P)$ leaves the complex above exact for any projective left R–module P.

Definition G (Gorenstein injective modules). *A left R–module N is said to be Gorenstein injective if and only if there is an exact sequence:*

$$\cdots \to E^{-1} \to E^0 \to E^1 \to \cdots$$

of injective left R–modules such that $N = \operatorname{Ker}(E^0 \to E^1)$ and such that $\operatorname{Hom}_R(E, -)$ *leaves the complex above exact for any injective left R–module E.*

While Definition F is formulated very explicitly in [9, Definition 2.1], Definition G is more hidden, but it can be read between the lines.

Furthermore, Enochs-Jenda-Torrecillas [11] introduced Gorenstein flat modules, also over an arbitrary associative ring R:

Definition H (Gorenstein flat modules). *A left R–module M is said to be Gorenstein flat if and only if there is an exact sequence:*

$$\cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$$

of flat left R–modules such that $M = \operatorname{Im}(F_0 \to F_{-1})$ and such that $I \otimes_R -$ leaves the complex above exact for any injective right R–module I.

Of course, one obvious problem remained for the Gorenstein projective modules:

- Assume that the associative ring R is two-sided noetherian, and that M is a finitely generated R–module. Is it then true that M is Gorenstein projective according to Enochs' and Jenda's Definition F, if and only if, M satisfies the equivalent conditions of Theorem E?

At the end of the 1990's, this problem was answered affirmatively by Avramov-Buchweitz-Martsinkovsky-Reiten [3]. However, as far as this author is informed, [3] was regrettably never published. Fortunately, Christensen included (with proper credits, of course) the result of Avramov-Buchweitz-Martsinkovsky-Reiten in [6, Theorem (4.2.6)]. Note that even though the ring is assumed to be commutative in [6], the proof of [6, Theorem (4.2.6)] works, with obvious modifications, in the associative case as well.

Building resolutions of the Gorenstein projective modules from Definition F, one can introduce a new homological dimension:

Definition I (Gorenstein projective dimension). If

$$M \;=\; \cdots \longrightarrow M_{i+1} \xrightarrow{\partial^M_{i+1}} M_i \xrightarrow{\partial^M_i} M_{i-1} \longrightarrow \cdots$$

(differentials are going to the right) is a complex of left R–modules which is homologically right-bounded, then the *Gorenstein projective dimension* of M is defined as

$$\operatorname{Gpd}_R M = \inf\left\{ \sup\{\ell \in \mathbb{Z} \,|\, G_\ell \neq 0\} \;\middle|\; \begin{array}{l} G \text{ is a genuinely right-bounded complex} \\ \text{of left } R\text{–modules with } G \simeq M \text{ in } \mathsf{D}(R), \\ \text{and every } G_\ell \text{ is Gorenstein projective} \end{array} \right\}.$$

Here $\mathsf{D}(R)$ is the derived category of the abelian category of R–modules, and $\simeq$ denotes isomorphism in this category. If M is a left R–module, then the formula specializes to:

$$\operatorname{Gpd}_R M = \inf \left\{ n \in \mathbb{N}_0 \;\middle|\; \begin{array}{l} 0 \to G_n \to \cdots \to G_0 \to M \to 0 \text{ is an exact sequence of} \\ \text{left } R\text{–modules, and every } G_\ell \text{ is Gorenstein projective} \end{array} \right\}.$$

There are, of course, similar definitions of the *Gorenstein injective dimension*, $\operatorname{Gid}_R(-)$, and of the *Gorenstein flat dimension*, $\operatorname{Gfd}_R(-)$.

It is remarkable that Enochs and Jenda, as far as this author can tell, never wrote down the definitions of $\operatorname{Gpd}_R(-)$, $\operatorname{Gid}_R(-)$ and $\operatorname{Gfd}_R(-)$. The first time these definitions are encountered in writing is in [6, Definitions (4.4.2), (6.2.2) and (5.2.3)].

The reason why one impose the homological boundedness condition on M in Definition I, is merely to ensure the existence of a projective resolution. This point of view may be considered a bit "old fashioned", and e.g. Veliche [21] introduces $\operatorname{Gpd}_R(-)$ without any boundedness conditions on the homology (using that every complex has a semi-projective resolution).

With Definitions F, G, H and I at hand, homological algebra was ready for at study of Gorenstein projective, Gorenstein injective and Gorenstein flat modules and their related dimensions; a "Gorenstein homological algebra", one could say.

Not surprisingly, it has turned out that the this new Gorenstein homological algebra is very much related to "classical homological algebra", in fact, there seems to exist the following meta-theorem:

- Every result in classical homological algebra has a counterpart in Gorenstein homological algebra.

Actually, one could say that the study of Gorenstein homological algebra boils down to proving this meta-theorem (in sufficiently many cases).

Many authors have contributed to the theory of Gorenstein homological algebra: Auslander, Avramov, Bridger, Buchweitz, Christensen, Enochs, Foxby, Frankild, Gerko, Golod, Iyengar, Jenda, Jørgensen, Khatami, Martsinkovsky, Reiten, Sather-Wagstaff, Takahashi, Veliche, Yassemi, Yoshino and Xu, just to mention a few.

If someone feels offended by this list, because they unintentionally have been left out, the mistake only reflects this authors forgetfulness or ignorance; and the affected parties have his sincerest apologies.

The purpose of the following last section in this introduction is to describe the contributions of [I] – [VI].

3. An introduction to the papers in this thesis

Each paper in this thesis, of course, has its own introduction, giving an overview of its contents and main results. However, for the reader's convenience, and to outline the main thread of this manuscript, we have chosen to give a joint and overall summary of all the papers [I] – [VI].

In describing the work in [I] – [VI], we will sometimes need to refer to other papers for which this author can take no credit; such references will be made very clear.

We work in the following setup:

(3.1) **Setup.** Throughout this section, R is a fixed associative ring with unit, and all R–modules are *left* R–modules.

In some of the theorems to follow, the ring is assumed to be commutative and noetherian; and in the formulation of such results, we like to use A, rather than R, to denote the base ring.

We often work within the derived category $\mathsf{D}(R)$ of the category of R–modules; cf. e.g. [17, Chapter I] and [22, Chapter 10]; complexes $M \in \mathsf{D}(R)$ have differentials going to the right:

$$M = \ \cdots \longrightarrow M_{i+1} \xrightarrow{\partial^M_{i+1}} M_i \xrightarrow{\partial^M_i} M_{i-1} \longrightarrow \cdots$$

We use $\mathsf{D}(R)$ with subscripts "$\sqsupset$", "$\sqsubset$", or "$\square$" to indicate that the homology is bounded to the right, left, or in both directions; and with superscript "f" to indicate that all the homology modules are finitely generated.

We consistently use the hyper-homological notation from [6, Appendix], in particular we use $\mathbf{R}\mathrm{Hom}_R(-,-)$ for the right derived Hom functor, and $- \otimes^{\mathbf{L}}_R -$ for the left derived tensor product functor.

$\sim \circ \sim$

Let us start out by establishing some fundamental results about the classes of Gorenstein projective, Gorenstein injective and Gorenstein flat modules. In [I, Theorem (2.5)] we prove the following:

(3.2) **Theorem.** *The Gorenstein projective R–modules have the following properties:*

1. *If $\{M_i\}_{i\in I}$ is a family of R–modules, then $\coprod_{i\in I} M_i$ is Gorenstein projective if and only if every M_i is Gorenstein projective.*
2. *If $0 \to M \to M' \to M'' \to 0$ is a short exact sequence of R–modules, and M'' is Gorenstein projective, then M is Gorenstein projective if and only if M' is Gorenstein projective.*

There are of course similar results for the classes of Gorenstein injective and Gorenstein flat modules; cf. [I, Theorems (2.6) and (3.7)].

By [I, Theorem (2.6)], the class of Gorenstein injective modules is always closed under arbitrary direct products; but when R is commutative and noetherian admitting a dualizing complex (please see (3.7) below), then [IV, Theorem (6.10)] gives even more:

(3.3) **Theorem.** *Assume that R is commutative and noetherian admitting a dualizing complex. Then a filtered direct limit of Gorenstein injective R–modules is again Gorenstein injective. In particular, the class of Gorenstein injective modules is closed under arbitrary direct sums.*

By [I, Theorem (3.7)], the class of Gorenstein flat modules is closed under countable filtered direct limits. By work of Enochs and Lópes-Ramos [13, Theorem 2.4] the class of Gorenstein flat modules is, in fact, closed under arbitrary filtered direct limits.

Having brought up this theorem by Enochs and Lópes-Ramos, it is worth mentioning that we generalize exactly that result in [V, Lemma (5.10)], so that it also holds for the class of *C–Gorenstein flat* modules, where C is any semi-dualizing R–module (see also Definition (3.29) below).

∼ ◦ ∼

The above mentioned fundamental properties which the classes of Gorenstein projective, injective and flat R–modules possess, are key ingredients in describing the related Gorenstein dimensions:

By [2, (3.11)], the class of finitely generated R–modules M with $G\text{–}\dim_R M = 0$ (here R is two-sided noetherian) has similar properties to the ones in Theorem (3.2).

Using techniques from standard homological algebra and from [2], we are able to get the following characterization [I, Theorem (2.20)] of the Gorenstein projective dimension:

(3.4) **Theorem.** *Let M be an R–module with finite Gorenstein projective dimension, and let n be an integer. Then the following conditions are equivalent.*

- *(i)* $\mathrm{Gpd}_R M \leqslant n$.
- *(ii)* $\mathrm{Ext}^i(M,L) = 0$ *for all $i > n$, and all R–modules L with finite* $\mathrm{pd}_R L$.
- *(iii)* $\mathrm{Ext}^i(M,Q) = 0$ *for all $i > n$, and all projective R–modules Q.*
- *(iv) For every exact sequence $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$ of R–modules, where $G_0, \dots, G_{n-1}$ are Gorenstein projective, then also K_n is Gorenstein projective.*

Consequently, the Gorenstein projective dimension of M is determined by the formulas:

$$\begin{aligned}\operatorname{Gpd}_R M &= \sup\{\, i \in \mathbb{N}_0 \mid \exists\, L\colon\ \operatorname{pd}_R L \text{ is finite and } \operatorname{Ext}^i(M,L) \neq 0 \,\}\\ &= \sup\{\, i \in \mathbb{N}_0 \mid \exists\, Q\colon\ Q \text{ is projective and } \operatorname{Ext}^i(M,Q) \neq 0 \,\}.\end{aligned}$$

Combining Theorem (3.4) with the techniques in [6], we take in [III, Theorem (2.2)] the step into the world of R–complexes:

(3.5) **Theorem.** *Let $M \in \mathsf{D}_{\sqsupset}(R)$ be a complex of finite Gorenstein projective dimension. For $n \in \mathbb{Z}$ the following are equivalent:*

- *(i)* $\operatorname{Gpd}_R M \leqslant n$.
- *(ii)* $n \geqslant \inf U - \inf \mathbf{R}\mathrm{Hom}(M,U)$ *for all* $U \in \mathsf{D}(R)$ *of finite projective or finite injective dimension with* $\mathrm{H}(U) \neq 0$.
- *(iii)* $n \geqslant -\inf \mathbf{R}\mathrm{Hom}(M,Q)$ *for all projective R–modules Q.*
- *(iv)* $n \geqslant \sup M$ *and the cokernel* $\mathrm{C}_n^A = \operatorname{Coker}(A_{n+1} \to A_n)$ *is a Gorenstein projective module for any genuinely right-bounded complex $A \simeq M$ of Gorenstein projective modules.*

Moreover, the following hold:

$$\begin{aligned}\operatorname{Gpd}_R M &= \sup\left\{\, \inf U - \inf \mathbf{R}\mathrm{Hom}(M,U) \;\middle|\; \operatorname{pd}_R U < \infty \ \text{ and } \ \mathrm{H}(U) \neq 0 \right\}\\ &= \sup\left\{\, -\inf \mathbf{R}\mathrm{Hom}(M,Q) \;\middle|\; Q \ \text{ is projective} \right\}\\ &\leqslant \ \operatorname{FPD}(R) + \sup M.\end{aligned}$$

Here $\operatorname{FPD}(R)$ is the (left) finitistic projective dimension of R. In reality, $\operatorname{FPD}(R)$ should be replaced by the (left) finitistic Gorenstein projective dimension of R:

$$\operatorname{FGPD}(R) = \sup\left\{ \operatorname{Gpd}_R M \;\middle|\; \begin{array}{l} M \text{ is a left } R\text{–module with finite} \\ \text{Gorenstein projective dimension} \end{array} \right\},$$

but this agrees with $\operatorname{FPD}(R)$ by [I, Theorem (2.28)]:

(3.6) **Theorem.** *There is an equality,* $\operatorname{FGPD}(R) = \operatorname{FPD}(R)$.

There are also Gorenstein injective and Gorenstein flat versions of Theorems (3.4), (3.5) and (3.6) (in the Gorenstein flat case, R has to be right coherent).

From the previous theorems we see a distinguished feature of all three Gorenstein dimensions: in order to determine the dimension of a given module or complex, we have to know in advance that this dimension is finite.

So, how does one determine if a given complex has finite, say, Gorenstein injective dimension? One approach to this problem is suggested in [IV], but before going into that, it is necessary to introduce some notation:

(3.7) **Dualizing complexes.** When A is commutative and noetherian, a complex $D \in \mathsf{D}(A)$ is said to be *dualizing* for A if it fulfills the requirements:

(1) D has finite homology, that is, $D \in \mathsf{D}^{\mathrm{f}}_{\Box}(A)$.
(2) D has finite injective dimension.
(3) The canonical (homothety) morphism $A \longrightarrow \mathbf{R}\mathrm{Hom}_A(D,D)$ is an isomorphism in $\mathsf{D}(A)$.

If A is local, this definition coincides with the classical one [17, Chapter V, §2], but we use definition (3.7) for local and non-local rings alike.

(3.8) **Auslander and Bass classes.** If C is a semi-dualizing complex; cf. [7, Definition (2.1)], for a commutative and noetherian ring A, then we can consider the adjoint pair of functors,

$$\mathsf{D}(A) \underset{\mathbf{R}\mathrm{Hom}_A(C,-)}{\overset{C\otimes^{\mathbf{L}}_A -}{\rightleftarrows}} \mathsf{D}(A).$$

As usual, let η denote the unit and ε the counit of the adjoint pair, cf. [19, Chapter 4].

The *Auslander* and *Bass classes* with respect to the semi-dualizing complex C are defined in terms of η and ε being isomorphisms. To be precise, the definition [7, Definition (4.1)] of the Auslander class reads:

$$\mathsf{A}_C(A) = \left\{ M \in \mathsf{D}_{\Box}(A) \;\middle|\; \begin{array}{l} \eta_M \colon M \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_A(C, C \otimes^{\mathbf{L}}_A M) \text{ is an} \\ \text{isomorphism, and } C \otimes^{\mathbf{L}}_A M \text{ is bounded} \end{array} \right\},$$

while the definition of the Bass class reads:

$$\mathsf{B}_C(A) = \left\{ N \in \mathsf{D}_{\Box}(A) \;\middle|\; \begin{array}{l} \varepsilon_N \colon C \otimes^{\mathbf{L}}_A \mathbf{R}\mathrm{Hom}_A(C,N) \xrightarrow{\simeq} N \text{ is an} \\ \text{isomorphism, and } \mathbf{R}\mathrm{Hom}_A(C,N) \text{ is bounded} \end{array} \right\}.$$

The Auslander and Bass classes are full triangulated subcategories of $\mathsf{D}(A)$, and the adjoint pair $(C \otimes^{\mathbf{L}}_A -, \mathbf{R}\mathrm{Hom}_A(C,-))$ provides quasi-inverse equivalences between the Auslander and Bass classes,

$$\mathsf{A}_C(A) \underset{\mathbf{R}\mathrm{Hom}_A(C,-)}{\overset{C\otimes^{\mathbf{L}}_A -}{\rightleftarrows}} \mathsf{B}_C(A).$$

This equivalence, introduced in [4], has come to be called *Foxby equivalence*.

By [7, Proposition (4.4)], all complexes of finite flat dimension belong to $\mathsf{A}_C(A)$, while complexes of finite injective dimension belong to $\mathsf{B}_C(A)$.

Having introduced Auslander and Bass classes, we are ready to state the two important results from [IV, Theorems (4.3) and (4.5)]:

(3.9) **Theorem.** *Let A be a commutative and noetherian ring. If A admits a dualizing complex D, then for any $M \in \mathsf{D}_{\sqsupset}(A)$, the following conditions are equivalent:*

(*i*) *M belongs to the Auslander class, $M \in \mathsf{A}_D(A)$.*
(*ii*) *M has finite Gorenstein projective dimension, $\mathrm{Gpd}_A M < \infty$.*
(*iii*) *M has finite Gorenstein flat dimension, $\mathrm{Gfd}_A M < \infty$.*

(3.10) **Theorem.** *Let A be a commutative and noetherian ring. If A admits a dualizing complex D, then for any $N \in \mathsf{D}_{\sqsubset}(A)$, the following conditions are equivalent:*

(*i*) *N belongs to the Bass class, $N \in \mathsf{B}_D(A)$.*
(*ii*) *N has finite Gorenstein injective dimension, $\mathrm{Gid}_A N < \infty$.*

When $(A, \mathfrak{m}, k)$ is commutative, noetherian, local and Cohen-Macaulay admitting a dualizing module, Theorems (3.9) and (3.10) were proved by Enochs-Jenda-Xu [12].

$\sim \circ \sim$

From these general descriptions of the Gorenstein dimensions, we move on to more specific formulas. The following dimension was introduce by Foxby [14]:

(3.11) **Large restricted flat dimension.** For a complex $M \in \mathsf{D}_{\square}(R)$, the *large restricted flat dimension* is defined by:

$$\mathsf{Rfd}_R M \;=\; \sup\left\{ \sup(T \otimes^{\mathbf{L}}_R M) \,\middle|\, T \text{ is a (right) } R\text{–module with } \mathrm{fd}_R T < \infty \right\}.$$

When M is an R–module the definition reads:

$$\mathsf{Rfd}_R M \;=\; \sup\left\{ i \geqslant 0 \,\middle|\, \begin{array}{l} \mathrm{Tor}^R_i(T, M) \neq 0 \text{ for some (right)} \\ R\text{–module } T \text{ with } \mathrm{fd}_R T < \infty \end{array} \right\}.$$

In [6, Theorem (5.3.6)] it is proved that when A is commutative and noetherian, then $\mathsf{Rfd}_A(-)$ satisfies a Chouinard formula for any $X \in \mathsf{D}_{\square}(A)$:

$$\mathsf{Rfd}_A X \;=\; \sup\left\{ \operatorname{depth} A_{\mathfrak{p}} - \operatorname{depth}_{A_{\mathfrak{p}}} X_{\mathfrak{p}} \,\middle|\, \mathfrak{p} \in \operatorname{Spec} A \right\}.$$

The large restricted flat dimension and other related dimensions are studied further in [8].

In [I, Theorem (3.19)] we prove the following result for modules:

(3.12) **Theorem.** *For any R–module M, we have two inequalities,*

$$\mathsf{Rfd}_R M \;\leqslant\; \mathrm{Gfd}_R M \;\leqslant\; \mathrm{fd}_R M.$$

Now assume that A is a commutative and noetherian ring. If $\mathrm{Gfd}_A M$ is finite, then:

$$\mathsf{Rfd}_A M \;=\; \mathrm{Gfd}_A M.$$

If $\mathrm{fd}_A M$ *is finite, then we have two equalities:*

$$\mathsf{R}\mathrm{fd}_A M \;=\; \mathrm{Gfd}_A M \;=\; \mathrm{fd}_A M.$$

(3.13) **Remark.** For the specific implication $\mathrm{fd}_R M < \infty \;\Rightarrow\; \mathrm{Gfd}_R M \;=\; \mathrm{fd}_R M$, the ring R only needs to be associative (and not commutative and noetherian).

Using Theorem (3.12) above, we go even further in [IV, Theorem (6.11)] as we prove:

(3.14) **Theorem.** *Assume that* A *is commutative and noetherian. For a complex* $M \in \mathsf{D}_{\square}(A)$ *of finite Gorenstein flat dimension, the next equality holds:*

$$\mathrm{Gfd}_A M \;=\; \sup\left\{\operatorname{depth} A_{\mathfrak{p}} - \operatorname{depth}_{A_{\mathfrak{p}}} M_{\mathfrak{p}} \,\middle|\, \mathfrak{p} \in \operatorname{Spec} A\right\}.$$

This result has the Gorenstein injective counterpart [IV, Theorem (6.9)]:

(3.15) **Theorem.** *Assume that* A *is commutative and noetherian admitting a dualizing complex. For a complex* $N \in \mathsf{D}_{\square}(A)$ *of finite Gorenstein injective dimension, the next equality holds:*

$$\mathrm{Gid}_A N \;=\; \sup\left\{\operatorname{depth} A_{\mathfrak{p}} - \operatorname{width}_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \,\middle|\, \mathfrak{p} \in \operatorname{Spec} A\right\}.$$

It is also worth to emphasize the Gorenstein Bass formula [IV, Theorem (6.4)]:

(3.16) **Theorem.** *Assume that* $(A, \mathfrak{m}, k)$ *is commutative, noetherian and local, admitting a dualizing complex. For any complex* $N \in \mathsf{D}^{\mathrm{f}}_{\square}(A)$ *of finite Gorenstein injective dimension, the next equation holds,*

$$\mathrm{Gid}_A N \;=\; \operatorname{depth} A - \inf N.$$

In particular, if N *is a finitely generated* A*–module of finite Gorenstein injective dimension, then*

$$\mathrm{Gid}_A N \;=\; \operatorname{depth} A.$$

Other results in this family may be found in e.g. [IV, Corollaries (6.8) and (6.14)]:

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The paper [II], although connected with [I], stands a little bit out, and thus requires a somewhat independent introduction:

A definition which is essential in order to understand that paper is:

(3.17) **Proper resolutions.** Let $\mathcal{X}$ be a class of R–modules, and let M be any R–module. We then define two types of resolutions:

(a) An (augmented) *proper left $\mathcal{X}$–resolution* of M is a complex,

$$\cdots \to X_1 \to X_0 \to M \to 0$$

(not necessarily exact) where $X_0, X_1, \ldots \in \mathcal{X}$ and such that

$$\cdots \to \mathrm{Hom}_R(X, X_1) \to \mathrm{Hom}_R(X, X_0) \to \mathrm{Hom}_R(X, M) \to 0$$

is exact for every $X \in \mathcal{X}$.

(b) Similarly, an (augmented) *proper right $\mathcal{X}$–resolution* of M is a complex,

$$0 \to M \to X^0 \to X^1 \to \cdots$$

(not necessarily exact) where $X^0, X^1, \ldots \in \mathcal{X}$ and such that

$$\cdots \to \mathrm{Hom}_R(X^1, X) \to \mathrm{Hom}_R(X^0, X) \to \mathrm{Hom}_R(M, X) \to 0$$

is exact for every $X \in \mathcal{X}$.

We refer the reader to e.g. [10, Chapter 8.1–8.2] for more information about these types of resolutions.

For a given class $\mathcal{X}$, the question about existence of proper (left or right) $\mathcal{X}$–resolutions is in general very difficult. However, in a special case we are able to say something in [I, Theorem (2.10)]:

(3.18) **Theorem.** *If M is an R–module such that $n = \mathrm{Gpd}_R M$ is finite, then M admits a proper left Gorenstein projective resolution of the form:*

$$0 \to P_n \to \cdots \to P_1 \to G_0 \to M \to 0,$$

where G_0 is Gorenstein projective and $P_1, \ldots, P_n$ are projective.

In [I, Theorems (2.15) and (3.23)] there are similar results about existence of proper right Gorenstein injective resolutions and proper left Gorenstein flat resolutions.

Later, Enochs and López-Ramos [13, Corollaries 2.7 and 2.11] have proved that when R is left noetherian, there always exist proper right Gorenstein injective resolutions; and if R is right coherent, there always exist proper left Gorenstein flat resolutions.

In [VI, Theorems 5.6 and 5.11] we generalize these results to encompass proper right C–Gorenstein injective and proper left C–Gorenstein flat resolution, where C is any semi-dualizing R–module; see also Theorem (3.38) below.

Furthermore, Jørgensen [18, Theorem 3.2] have proved that if A is commutative, noetherian and admits a dualizing complex, then every A–module has a proper left Gorenstein projective resolution.

The nice thing about proper resolutions is the following: If M and M' both have, say, a proper right $\mathcal{X}$–resolution, then every homomorphism $M \to M'$ lifts (uniquely up to homotopy) to a chain map of the given $\mathcal{X}$–resolutions; see also [I, Proposition (1.8)].

Consequently, proper resolutions, provided that they exist, can be used to (left or right) derive additive (covariant or contravariant) functors on the category of modules; cf. [II, Section 2].

And deriving functors is what we do in [II], as we introduce:

(3.19) **Definition.** Fix an R–module U. For any integer n we introduce the n'th Gorenstein (projective) right derived of $\mathrm{Hom}_R(-,U)$, and the n'th Gorenstein (injective) right derived of $\mathrm{Hom}_R(U,-)$; which are denoted

$$\mathrm{Ext}^n_{\mathcal{GP}}(-,U) \quad \text{and} \quad \mathrm{Ext}^n_{\mathcal{GI}}(U,-),$$

respectively, in the following way:

For any R–module M which admits a (non-augmented) proper left Gorenstein projective resolution, $\boldsymbol{G} = \cdots \to G_1 \to G_0 \to 0$, we define:

$$\mathrm{Ext}^n_{\mathcal{GP}}(M,U) := \mathrm{H}^n\mathrm{Hom}_R(\boldsymbol{G},U),$$

and for any R–module N which admits a (non-augmented) proper right Gorenstein injective resolution, $\boldsymbol{H} = 0 \to H^0 \to H^1 \to \cdots$, we define

$$\mathrm{Ext}^n_{\mathcal{GI}}(U,N) := \mathrm{H}^n\mathrm{Hom}_R(U,\boldsymbol{H}).$$

For the next result [II, Theorem (3.6) and Definition (3.7)], it is, as far as this author can prove, unfortunately not enough to know that proper left Gorenstein projective and proper right Gorenstein injective resolutions exist; they have to exist in the special form of Theorem (3.18) and [I, Theorem (2.15)].

(3.20) **Theorem.** *For all R–modules M and N such that $\mathrm{Gpd}_R M$ and $\mathrm{Gid}_R N$ are finite, we have isomorphisms of abelian groups:*

$$(*) \qquad \mathrm{Ext}^n_{\mathcal{GP}}(M,N) \cong \mathrm{Ext}^n_{\mathcal{GI}}(M,N),$$

which are functorial in M and N. In this situation, we write $\mathrm{GExt}^n_R(M,N)$ for the group $(*)$.

Naturally we want to compare GExt with the classical Ext, and as might be expected we get [II, Theorem (3.8)(iii)]:

(3.21) **Theorem.** *Let M and N be R–modules such that $\mathrm{Gpd}_R M$ and $\mathrm{Gid}_R N$ are finite. If either $\mathrm{pd}_R M$ or $\mathrm{id}_R N$ is finite, then there are isomorphisms:*

$$\mathrm{GExt}^n_R(M,N) \cong \mathrm{Ext}^n_R(M,N),$$

which are functorial in M and N.

We can also left derive the tensor product functor with proper left Gorenstein flat or proper left Gorenstein projective resolutions, giving us $g\mathrm{Tor}^R_n(-,-)$ and $\mathrm{GTor}^R_n(-,-)$, respectively. One can establish a theory for these functors which is similar to that of $\mathrm{GExt}^n_R(-,-)$, and this is the aim of [II, Section 4].

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The are three central results in the short paper [III]; namely [III, Theorems (2.1), (2.2) and (2.6)]. The paper deals with associative rings but let us here just state those three results for commutative rings:

(3.22) **Theorem.** *Let A be a commutative ring, and let M be any A–module. Then the following conclusions hold:*

(1) *If* $\mathrm{pd}_A M < \infty$, *then* $\mathrm{Gid}_A M = \mathrm{id}_A M$.
(2) *If* $\mathrm{id}_A M < \infty$, *then* $\mathrm{Gpd}_A M = \mathrm{pd}_A M$.
(3) *If A is noetherian with* $\dim A < \infty$, *then* $\mathrm{id}_A M < \infty$ *implies* $\mathrm{Gfd}_A M = \mathrm{fd}_A M$.

As a consequence of these three results, we derive the main result [III, Corollary (3.3)] of that paper:

(3.23) **Theorem.** *A commutative, noetherian and local ring $(A, \mathfrak{m}, k)$ is Gorenstein if (and only if) there exists an A–module M with* $\mathrm{depth}_A M < \infty$ *satisfying one of the following two conditions:*

(1) $\mathrm{fd}_A M < \infty$ *and* $\mathrm{Gid}_A M < \infty$, *or*
(2) $\mathrm{id}_A M < \infty$ *and* $\mathrm{Gfd}_A M < \infty$.

This result is generalized in [VI, Theorem 2.18] to encompass complexes as well:

(3.24) **Theorem.** *If $(A, \mathfrak{m}, k)$ is commutative, noetherian and local, then the following conditions are equivalent:*

(i) *A is Gorenstein.*
(ii) *There exists an A–complex M such that all three numbers* $\mathrm{fd}_A M$, $\mathrm{Gid}_A M$ *and* $\mathrm{width}_A M$ *are finite.*
(iii) *There exists an A–complex N such that all three numbers* $\mathrm{id}_A N$, $\mathrm{Gpd}_A N$ *and* $\mathrm{depth}_A N$ *are finite.*
(iv) *There exists an A–complex N such that all three numbers* $\mathrm{id}_A N$, $\mathrm{Gfd}_A N$ *and* $\mathrm{depth}_A N$ *are finite.*

In the case where A admits a dualizing complex, [6, (3.3.5)] compared with [VI, Theorems (4.3) and (4.5)] give Theorem (3.24) above.

(3.25) **Remark.** In connection with Theorem (3.24) above, it will be useful to mention a few extra conditions, all of which are well-known to be equivalent with $(A, \mathfrak{m}, k)$ being Gorenstein:

(ii') $\mathrm{Gid}_A M$ is finite for all $M \in \mathsf{D}_\Box(A)$.
(ii'') $\mathrm{Gid}_A k$ is finite.

(iii') $\mathrm{Gpd}_A M$ is finite for all $M \in \mathsf{D}_\square(A)$.
(iii'') $\mathrm{Gpd}_A k$ is finite.
(iv') $\mathrm{Gfd}_A M$ is finite for all $M \in \mathsf{D}_\square(A)$.
(iv'') $\mathrm{Gfd}_A k$ is finite.

The are many ways to see this; but let us give one short argument: If $(A, \mathfrak{m}, k)$ is Gorenstein, then A is a dualizing module for A with

$$\mathsf{A}_A(A) = \mathsf{B}_A(A) = \mathsf{D}_\square(A).$$

Thus [6, Corollaries (6.2.5), (4.4.5) and (5.2.6)] (which are complex-versions of the main results in [12]) imply that (ii'), (iii') and (iv') hold. Of course, the primed statements imply the corresponding double primed ones. Note that we always have

$$\mathrm{Gpd}_A k = \mathrm{Gfd}_A k = \mathrm{Gid}_A k.$$

The first equality is by [6, Theorems (4.2.6) and (5.1.11)], as k is finitely generated, and the second equality is by [6, Theorem (6.4.2)], since $\mathrm{Hom}_A(k, E_A(k)) \cong k$ by e.g. [5, Proposition 3.2.12(a)]. As already pointed out, finiteness of $\mathrm{Gpd}_A k$ implies that $(A, \mathfrak{m}, k)$ is Gorenstein by [1, Théorème 3 and Remarque p. 64].

However, Theorem (3.24) is definitely not where [VI] has its main emphasis, and we will shortly return to its real focus.

Theorem (3.24) and Remark (3.25) characterize Gorenstein local rings in terms of the three Gorenstein dimensions $\mathrm{Gid}_A(-)$, $\mathrm{Gpd}_A(-)$ and $\mathrm{Gfd}_A(-)$. One could ask and hope for a similar characterization of Cohen-Macaulay local rings, and this is exactly what we do in [V], as we introduce [V, Definition 2.2]:

(3.26) **Definition.** Let A be a commutative and noetherian ring. For any *semi-dualizing* A–module C; cf. [16] where the term *suitable* is used, we may consider the *trivial extension* $A \ltimes C$; cf. [20, p. 2], of A by C (which, of course, is defined for any module C).

Then, for any (appropriately homologically bounded) A–complex M we introduce three numbers:

$$\begin{aligned}
\mathrm{CMid}_A M &= \inf \left\{ \mathrm{Gid}_{A\ltimes C} M \,\middle|\, C \text{ is a semi-dualizing } A\text{–module} \right\}, \\
\mathrm{CMpd}_A M &= \inf \left\{ \mathrm{Gpd}_{A\ltimes C} M \,\middle|\, C \text{ is a semi-dualizing } A\text{–module} \right\}, \\
\mathrm{CMfd}_A M &= \inf \left\{ \mathrm{Gfd}_{A\ltimes C} M \,\middle|\, C \text{ is a semi-dualizing } A\text{–module} \right\},
\end{aligned}$$

called the *Cohen-Macaulay injective*, *Cohen-Macaulay projective* and *Cohen-Macaulay flat dimension* of M, respectively.

In [V, Theorem 5.1] we prove that the three Cohen-Macaulay dimensions characterize Cohen-Macaulay local rings (admitting a dualizing module) in the same way as the three Gorenstein dimensions characterize Gorenstein local rings in Theorem (3.24) and Remark (3.25) above:

(3.27) **Theorem.** *Let $(A, \mathfrak{m}, k)$ be a commutative, noetherian and local ring. Then the following conditions are equivalent.*

(1) *A is Cohen-Macaulay and admits a dualizing module.*
(2) $\mathrm{CMid}_A M$ *is finite for all* $M \in \mathsf{D}_{\square}(A)$.
(3) *There exists an A–complex M such that all three numbers* $\mathrm{fd}_A M$, $\mathrm{CMid}_A M$ *and* $\mathrm{width}_A M$ *are finite.*
(4) $\mathrm{CMid}_A k$ *is finite.*
(5) $\mathrm{CMpd}_A M$ *is finite for all* $M \in \mathsf{D}_{\square}(A)$.
(6) *There exists an A–complex M such that all three numbers* $\mathrm{id}_A M$, $\mathrm{CMpd}_A M$ *and* $\mathrm{depth}_A M$ *are finite.*
(7) $\mathrm{CMpd}_A k$ *is finite.*
(8) $\mathrm{CMfd}_A M$ *is finite for all* $M \in \mathsf{D}_{\square}(A)$.
(9) *There exists an A–complex M such that all three numbers* $\mathrm{id}_A M$, $\mathrm{CMfd}_A M$ *and* $\mathrm{depth}_A M$ *are finite.*
(10) $\mathrm{CMfd}_A k$ *is finite.*

Gerko [15, Definition 3.2] has also introduced a Cohen-Macaulay dimension, denoted $\text{CM-dim}_A(-)$, defined for finitely generated A–modules. In [V, Theorem 5.4] we show that Gerko's CM-dimension is a refinement of our Cohen-Macaulay projective dimension:

(3.28) **Theorem.** *Let $(A, \mathfrak{m}, k)$ be a commutative, noetherian and local ring, and let M be a finitely generated A–module. Then there are inequalities:*

$$\text{CM-dim}_A M \leqslant \mathrm{CMpd}_A M \leqslant \text{G-dim}_A M,$$

and if one of these numbers is finite then the inequalities to its left are equalities.

Given an A–complex M and a semi-dualizing A–module C, we have just demonstrated how useful it is to change rings from A to $A \ltimes C$, and then consider the three "ring changed" Gorenstein dimensions:

$$\mathrm{Gid}_{A \ltimes C} M \quad , \quad \mathrm{Gpd}_{A \ltimes C} M \quad \text{and} \quad \mathrm{Gfd}_{A \ltimes C} M.$$

In [VI] we take these ideas further, as we give different interpretations of the "ring changed" Gorenstein dimensions above. In the following we will mainly deal with the, say, Gorenstein injective case.

Over a commutative and noetherian ring A, we introduce [VI, Definition 2.7] a new class of modules:

(3.29) **Definition.** Let A be commutative and noetherian, and let C be a semi-dualizing A–module. An A–module M is called *C–Gorenstein injective* if:

(I1) $\mathrm{Ext}_A^{\geqslant 1}(\mathrm{Hom}_A(C, I), M) = 0$ for all injective A–modules I.

(I2) There exist injective A–modules $I_0, I_1, \ldots$ together with an exact sequence:
$$\cdots \to \mathrm{Hom}_A(C, I_1) \to \mathrm{Hom}_A(C, I_0) \to M \to 0,$$
and also, this sequence stays exact when we apply to it the functor $\mathrm{Hom}_A(\mathrm{Hom}_A(C, J), -)$ for any injective A–module J.

It turns out [VI, Example 2.8] that if I is an injective A–module, then I (together with $\mathrm{Hom}_A(C, I)$) is C–Gorenstein injective. Thus, we have existence of C–Gorenstein injective resolutions. In analogy with Definition I in Section 2 we may therefore define [VI, Definition 2.9]:

(3.30) **Definition (C–Gorenstein injective dimension).** Let A be commutative and noetherian, and let C be a semi-dualizing A–module. For $M \in \mathsf{D}_{\sqsubset}(A)$, we define the *$C$–Gorenstein injective dimension* as:
$$C\text{-}\mathrm{Gid}_R M = \inf\left\{ \sup\{\ell \in \mathbb{Z} \,|\, E_{-\ell} \neq 0\} \;\middle|\; \begin{array}{l} E \text{ is a genuinely left-bounded complex} \\ \text{of } A\text{–modules with } M \simeq E \text{ in } \mathsf{D}(A)\text{, and} \\ \text{every } E_\ell \text{ is } C\text{–Gorenstein injective} \end{array} \right\}.$$
If M is an A–module, then the formula specializes to:
$$C\text{-}\mathrm{Gid}_R M = \inf\left\{ n \in \mathbb{N}_0 \;\middle|\; \begin{array}{l} 0 \to M \to E^0 \to \cdots \to E^n \to 0 \text{ is an exact sequence} \\ \text{of } A\text{–modules, and every } E^\ell \text{ is } C\text{–Gorenstein injective} \end{array} \right\}.$$

(3.31) **Remark.** Note that if $C = A$, then $C\text{-}\mathrm{Gid}_A(-)$ becomes the normal $\mathrm{Gid}_A(-)$.

There is of course also a C–Gorenstein projective dimension, $C\text{-}\mathrm{Gpd}_A(-)$. Recall that Christensen [7, Definition (3.11)] has introduced a G–dimension, $G\text{-}\dim_C Z$, for any semi-dualizing A–complex C and any complex $Z \in \mathsf{D}^{\mathrm{f}}_{\square}(A)$. We compare this dimension with our $C\text{-}\mathrm{Gpd}_A(-)$ in [VI, Proposition 3.1]:

(3.32) **Proposition.** *Let A be commutative and noetherian, and let C be a semi-dualizing A–module. For any complex $M \in \mathsf{D}^{\mathrm{f}}_{\square}(A)$ we have:*
$$C\text{-}\mathrm{Gpd}_A M \;=\; G\text{-}\dim_C M.$$

The connection between the C–Gorenstein injective dimension and the "ring changed" Gorenstein injective dimension, $\mathrm{Gid}_{A \ltimes C}(-)$ is settled in [VI, Theorem 2.16]:

(3.33) **Theorem.** *Let A be commutative and noetherian, and let C be a semi-dualizing A–module. For any $M \in \mathsf{D}_{\sqsubset}(A)$, we have:*
$$C\text{-}\mathrm{Gid}_A M \;=\; \mathrm{Gid}_{A \ltimes C} M.$$

As a corollary [VI, Corollary 2.17] we derive:

(3.34) **Corollary.** *Let A be commutative and noetherian. For any $M \in \mathsf{D}_{\sqsubset}(A)$, we have:*

$$\mathrm{Gid}_{A \ltimes A} M \;=\; \mathrm{Gid}_{A[x]/(x^2)} M \;=\; \mathrm{Gid}_A M.$$

Using [IV, Theorems (4.3) and (4.5)] (which are Theorems (3.9) and (3.10) in this introduction), we prove the generalization [VI, Theorem 4.6]:

(3.35) **Theorem.** *Let A be commutative and noetherian, admitting a dualizing complex D. For a semi-dualizing A–module C we consider the semi-dualizing A–complex:*

$$C^{\dagger} \;=\; \mathbf{R}\mathrm{Hom}_A(C, D).$$

Then, for any $M \in \mathsf{D}_{\sqsupset}(A)$ and $N \in \mathsf{D}_{\sqsubset}(A)$ we have:

1. $M \in \mathsf{A}_{C^{\dagger}}(A) \iff C\text{-}\mathrm{Gpd}_A M < \infty \iff C\text{-}\mathrm{Gfd}_A M < \infty.$
2. $N \in \mathsf{B}_{C^{\dagger}}(A) \iff C\text{-}\mathrm{Gid}_A N < \infty.$

In the Theorem above, $\mathsf{A}_{C^{\dagger}}(A)$ and $\mathsf{B}_{C^{\dagger}}(A)$ are the Auslander and Bass classes with respect to $C^{\dagger}$; cf. (3.8). The Auslander class with respect to C (and not $C^{\dagger}$) is interesting for the following reason [VI, Theorem 4.2]:

(3.36) **Theorem.** *Let A be commutative and noetherian with a semi-dualizing module C. For any complex $M \in \mathsf{A}_C(A)$ we have an equality:*

$$C\text{-}\mathrm{Gid}_A M \;=\; \mathrm{Gid}_A(C \otimes^{\mathbf{L}}_A M).$$

Finally, we introduce [VI, Definitions 5.1, 5.2 and 5.3] a *proper* variant of $C\text{-}\mathrm{Gid}_A(-)$ for modules:

(3.37) **Definition.** Let A be commutative and noetherian with a semi-dualizing module C. A *proper right C–Gorenstein injective resolution* of an A–module M is an exact sequence:

$$(\dagger) \qquad 0 \to M \to E^0 \to E^1 \to \cdots,$$

where $E^0, E^1, \ldots$ are C–Gorenstein injective, and such that $(\dagger)$ stays exact when we apply to it the functor $\mathrm{Hom}_A(-, E)$ for every C–Gorenstein injective module E; cf. Definition (3.17)(b).

If M has a proper right C–Gorenstein injective resolution, then we define the *proper right C–Gorenstein dimension dimension* of M by:

$$C\text{-}\mathsf{Gid}_A M \;=\; \inf\left\{ n \in \mathbb{N}_0 \;\middle|\; \begin{array}{l} 0 \to M \to E^0 \to \cdots \to E^n \to 0 \text{ is a proper} \\ \text{right } C\text{–Gorenstein injective resolution of } M \end{array} \right\}.$$

In [VI, Theorem 5.6] we show:

(3.38) **Theorem.** *Let A be commutative and noetherian with a semi-dualizing module C. Then every A–module M has a proper right C–Gorenstein injective resolution, and there is an equality:*

$$C\text{-}\mathsf{Gid}_A M \;=\; C\text{-}\mathrm{Gid}_A M.$$

Hopefully, this section gave the reader some idea of what the papers in this thesis are about. For further details, one has of course to read on.

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Part I

Gorenstein homological dimensions

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GORENSTEIN HOMOLOGICAL DIMENSIONS

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Abstract. In basic homological algebra, the projective, injective and flat dimensions of modules play an important and fundamental role. In this paper, the closely related Gorenstein projective, Gorenstein injective and Gorenstein flat dimensions are studied.

There is a variety of nice results about Gorenstein dimensions over special commutative noetherian rings; very often local Cohen–Macaulay rings with a dualizing module. These results are done by Avramov, Christensen, Enochs, Foxby, Jenda, Martsinkovsky and Xu among others. The aim of this paper is to generalize these results, and to give homological descriptions of the Gorenstein dimensions over *arbitrary* associative rings.

Introduction

Throughout this paper, R denotes a non-trivial associative ring. All modules are—if not specified otherwise—*left* R–modules.

When R is two-sided and noetherian, Auslander and Bridger [2] introduced in 1969 the G-dimension, $\text{G-dim}_R M$, for every *finite*, that is, finitely generated, R–module M (see also [1] from 1966/67). They proved the inequality $\text{G-dim}_R M \leqslant \text{pd}_R M$, with equality $\text{G-dim}_R M = \text{pd}_R M$ when $\text{pd}_R M$ is finite. Furthermore they showed the generalized Auslander–Buchsbaum formula (sometimes known as the Auslander-Bridger formula) for the G-dimension.

Over a general ring R, Enochs and Jenda defined in [9] a homological dimension, namely the *Gorenstein projective dimension*, $\text{Gpd}_R(-)$, for arbitrary (non–finite) modules. It is defined via resolutions with (the so-called) *Gorenstein projective modules*. Avramov, Buchweitz, Martsinkovsky and Reiten prove that a *finite* module over a noetherian ring is Gorenstein projective if and only if $\text{G-dim}_R M = 0$ (see the remark following [7, Theorem (4.2.6)]).

Section 2 deals with this Gorenstein projective dimension, $\text{Gpd}_R(-)$. First we establish the following fundamental.

Theorem. *The class of all Gorenstein projective modules is resolving, in the sense that if* $0 \to M' \to M \to M'' \to 0$ *is a short exact sequence of R–modules, where M'' is Gorenstein projective, then M' is Gorenstein projective if and only if M is Gorenstein projective.*

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Key words and phrases. G-dimension, Gorenstein projective dimension, Gorenstein injective dimension, Gorenstein flat dimension, (pre)covers, (pre)envelopes.

This result is a generalization of [10, Theorems 10.2.8 and 11.5.6], and of [7, Corollary (4.3.5)], which all put restrictions on either the base ring, or on the modules. The result is also the main ingredient in the following important functorial description of the Gorenstein dimension.

Theorem. *Let M be a (left) R–module with finite Gorenstein projective dimension, and let $n \geqslant 0$ be an integer. Then the following conditions are equivalent.*

(*i*) $\mathrm{Gpd}_R M \leqslant n$.

(*ii*) $\mathrm{Ext}^i_R(M,L) = 0$ *for all $i > n$, and all R–modules L with finite* $\mathrm{pd}_R L$.

(*iii*) $\mathrm{Ext}^i_R(M,Q) = 0$ *for all $i > n$, and all projective R–modules Q.*

(*iv*) *For every exact sequence $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$, if $G_0, \ldots, G_{n-1}$ are Gorenstein projective, then also K_n is Gorenstein projective.*

Note that this theorem generalizes [7, Theorem (4.4.12)], which is only proved for local noetherian Cohen–Macaulay rings admitting a dualizing module.

Next, we get the following generalization of [15, Theorem 5.5.6] (where the ring is assumed to be local, noetherian and Cohen–Macaulay with a dualizing module):

Theorem. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R–modules. If any two of the modules M', M or M'' have finite Gorenstein projective dimension, then so has the third.*

In Section 2 we also investigate *Gorenstein projective precovers*. Recall that a Gorenstein projective precover of a module M is a homomorphism of modules, $G \to M$, where G is Gorenstein projective, such that the sequence

$$\mathrm{Hom}_R(Q,G) \to \mathrm{Hom}_R(Q,M) \to 0$$

is exact for every Gorenstein projective module Q. We show that every module M with finite Gorenstein projective dimension admits a nice Gorenstein projective precover:

Theorem. *Let M be an R–module with finite Gorenstein projective dimension n. Then M admits a surjective Gorenstein projective precover $\varphi\colon G \twoheadrightarrow M$ where $K = \mathrm{Ker}\,\varphi$ satisfies $\mathrm{pd}_R K = n-1$ (if $n = 0$, this should be interpreted as $K = 0$).*

Using these precovers, we show that there is an equality between the classical (left) finitistic projective dimension, FPD(R), and the related (*left*) *finitistic Gorenstein projective dimension*, $\mathrm{FGPD}(R)$, of the base ring R. The latter is defined as:

$$\mathrm{FGPD}(R) \;=\; \sup\left\{ \mathrm{Gpd}_R M \;\middle|\; \begin{array}{l} M \text{ is a left } R\text{–module with finite} \\ \text{Gorenstein projective dimension.} \end{array} \right\}.$$

Important note. Above we have only mentioned the Gorenstein projective dimension for an R–module M. Dually one can also define the *Gorenstein injective dimension*, $\mathrm{Gid}_R M$. All the results concerning Gorenstein projective dimension (with the exception of Proposition (2.16) and Corollary (2.21)), have a Gorenstein injective counterpart.

With some exceptions, we do not state or prove these "dual" Gorenstein injective results. This is left to the reader.

Section 3 deals with *Gorenstein flat modules*, together with the *Gorenstein flat dimension*, $\mathrm{Gfd}_R(-)$, in a way much similar to how we treated Gorenstein projective modules, and the Gorenstein projective dimension in Section 2.

For right coherent rings, a (left) R–module M is Gorenstein flat if, and only if, its Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a (right) Gorenstein injective R–module (please see Theorem 3.6). Using this we can prove the next generalization of [7, Theorem (5.2.14)] and [10, Theorem 10.3.8].

Theorem. *If R is right coherent, $n \geqslant 0$ is an integer and M is a left R–module with finite Gorenstein flat dimension, then the following four conditions are equivalent.*

(*i*) $\mathrm{Gfd}_R M \leqslant n$.

(*ii*) $\mathrm{Tor}_i^R(L, M) = 0$ *for all right R–modules L with finite* $\mathrm{id}_R L$, *and all $i > n$.*

(*iii*) $\mathrm{Tor}_i^R(I, M) = 0$ *for all injective right R–modules I, and all $i > n$.*

(*iv*) *For every exact sequence $0 \to K_n \to T_{n-1} \to \cdots \to T_0 \to M \to 0$ if $T_0, \ldots, T_{n-1}$ are Gorenstein flat, then also K_n is Gorenstein flat.*

Besides the Gorenstein flat dimension of an R–module M, also the *large restricted flat dimension*, $\mathsf{Rfd}_R M$, is of interest. It is defined as follows:

$$\mathsf{Rfd}_R M \;=\; \sup\left\{ i \geqslant 0 \;\middle|\; \begin{array}{l} \mathrm{Tor}_i^R(L, M) \neq 0 \text{ for some (right)} \\ R\text{–module with finite flat dimension.} \end{array} \right\}.$$

This numerical invariant is investigated in [8, Section 2] and in [7, Chapters 5.3 – 5.4]. It is conjectured by Foxby that if $\mathrm{Gfd}_R M$ is finite, then $\mathsf{Rfd}_R M = \mathrm{Gfd}_R M$. Christensen [7, Theorem (5.4.8)] proves this for local noetherian Cohen–Macaulay rings with a dualizing module. We have the following extension.

Theorem. *For any (left) R–module M there are inequalities,*

$$\mathsf{Rfd}_R M \;\leqslant\; \mathrm{Gfd}_R M \;\leqslant\; \mathrm{fd}_R M.$$

Now assume that R is commutative and noetherian. If $\mathrm{Gfd}_R M$ is finite, then we have equality $\mathsf{Rfd}_R M = \mathrm{Gfd}_R M$. If $\mathrm{fd}_R M$ is finite, then $\mathsf{Rfd}_R M = \mathrm{Gfd}_R M = \mathrm{fd}_R M$.

Furthermore we prove that every module, M, with finite Gorenstein flat dimension admits a special *Gorenstein flat precover*, $G \twoheadrightarrow M$, and we show that the classical (left) finitistic flat dimension, $\mathrm{FFD}(R)$, is equal to the (*left*) *finitistic Gorenstein flat dimension*, $\mathrm{FGFD}(R)$ of R.

Notation. By $\mathcal{M}(R)$ we denote the class of all R–modules, and by $\mathcal{P}(R)$, $\mathcal{I}(R)$ and $\mathcal{F}(R)$ we denote the classes of all projective, injective and flat R–modules respectively. Furthermore we let $\overline{\mathcal{P}}(R)$, $\overline{\mathcal{I}}(R)$ and $\overline{\mathcal{F}}(R)$ denote the classes of all R–modules with finite projective, injective and flat dimensions respectively.

(Note that in the related paper [5] by Avramov and Martsinkovsky, studying finite modules, the symbol $\mathcal{F}(R)$ denotes the class of finite modules, $\mathcal{P}(R)$ the class of finite projective modules, and $\widetilde{\mathcal{P}}(R)$ the class of finite modules with finite projective dimension).

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1. Resolving classes

This section contains some general remarks about resolving classes, which will be important in our treatment of Gorenstein projective modules in the next section.

(1.1) **Resolving classes.** Inspired by Auslander-Bridger's result [2, (3.11)], we define the following terms for any class, $\mathcal{X}$, of R–modules.

- (a) We call $\mathcal{X}$ *projectively resolving* if $\mathcal{P}(R) \subseteq \mathcal{X}$, and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathcal{X}$ the conditions $X' \in \mathcal{X}$ and $X \in \mathcal{X}$ are equivalent.
- (b) We call $\mathcal{X}$ *injectively resolving* if $\mathcal{I}(R) \subseteq \mathcal{X}$, and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathcal{X}$ the conditions $X \in \mathcal{X}$ and $X'' \in \mathcal{X}$ are equivalent.

Note that we do not require that a projectively/injectively class is closed under direct summands, as in [2, (3.11)]. The reason for this will become clear in Proposition (1.4) below.

(1.2) **Orthogonal classes.** For any class, $\mathcal{X}$, of R–modules, we define the associated *left orthogonal*, respectively, *right orthogonal*, class by:

$$^{\perp}\mathcal{X} \;=\; \{\, M \in \mathcal{M}(R) \mid \operatorname{Ext}^i_R(M, X) = 0 \text{ for all } X \in \mathcal{X}, \text{ and all } i > 0\,\},$$

respectively,

$$\mathcal{X}^{\perp} \;=\; \{\, N \in \mathcal{M}(R) \mid \operatorname{Ext}^i_R(X, N) = 0 \text{ for all } X \in \mathcal{X}, \text{ and all } i > 0\,\}.$$

(1.3) **Example.** It is well-known that $\mathcal{P}(R) = {}^{\perp}\mathcal{M}(R)$, and that $\mathcal{P}(R)$ and $\mathcal{F}(R)$ both are projectively resolving classes, whereas $\mathcal{I}(R) = \mathcal{M}(R)^{\perp}$ is an injectively resolving class. Furthermore, it is easy to see the equalities,

$${}^{\perp}\overline{\mathcal{P}}(R) = {}^{\perp}\mathcal{P}(R) \qquad \text{and} \qquad \overline{\mathcal{I}}(R)^{\perp} = \mathcal{I}(R)^{\perp}.$$

In general, the class ${}^{\perp}\mathcal{X}$ is projectively resolving, and closed under arbitrary direct sums. Similarly, the class $\mathcal{X}^{\perp}$ is injectively resolving, and closed under arbitrary direct products.

The next result is based on a technique of Eilenberg.

(1.4) **Proposition (Eilenberg's swindle).** *Let X be a class of R–modules which is either projectively resolving, or injectively resolving. If $\mathcal{X}$ is closed under countable direct sums, or closed under countable direct products, then $\mathcal{X}$ is also closed under direct summands.*

Proof. Assume that Y is a direct summand of $X \in \mathcal{X}$. We wish to show that $Y \in \mathcal{X}$. Write $X = Y \oplus Z$ for some module Z. If $\mathcal{X}$ is closed under countable direct sums, then we define $W = Y \oplus Z \oplus Y \oplus Z \oplus \cdots$ (direct sum), and note that $W \cong X \oplus X \oplus \cdots \in \mathcal{X}$. If $\mathcal{X}$ is closed under countable direct products, then we put $W = Y \times Z \times Y \times Z \times \cdots$ (direct product), and note that $W \cong X \times X \times \cdots \in \mathcal{X}$. In either case we have $W \cong Y \oplus W$, so in particular the sum $Y \oplus W$ belongs to $\mathcal{X}$. If $\mathcal{X}$ is projectively resolving, then we consider the split exact sequence $0 \to Y \to Y \oplus W \to W \to 0$, and if $\mathcal{X}$ is injectively resolving, then we consider $0 \to W \to W \oplus Y \to Y \to 0$. In either case we conclude that $Y \in \mathcal{X}$. $\square$

(1.5) **Resolutions.** For any R–module M we define two types of resolutions.

- (a) A *left $\mathcal{X}$–resolution* of M is an exact sequence $\boldsymbol{X} = \cdots \to X_1 \to X_0 \to M \to 0$ with $X_n \in \mathcal{X}$ for all $n \geqslant 0$.
- (b) A *right $\mathcal{X}$–resolution* of M is an exact sequence $\boldsymbol{X} = 0 \to M \to X^0 \to X^1 \to \cdots$ with $X^n \in \mathcal{X}$ for all $n \geqslant 0$.

Now let $\boldsymbol{X}$ be any (left or right) $\mathcal{X}$–resolution of M. We say that $\boldsymbol{X}$ is *proper* (respectively, *co–proper*) if the sequence $\mathrm{Hom}_R(Y, \boldsymbol{X})$ (respectively, $\mathrm{Hom}_R(\boldsymbol{X}, Y)$) is exact for all $Y \in \mathcal{X}$.

In this paper we *only* consider proper left $\mathcal{X}$–resolutions, and co–proper right $\mathcal{X}$–resolutions (and *never* proper right $\mathcal{X}$–resolutions, or co–proper left $\mathcal{X}$–resolutions).

It is straightforward to show the next result.

(1.6) **Proposition.** *Let $\mathcal{X}$ be a class of R–modules, and let $\{M_i\}_{i \in I}$ be a family of R-modules. Then the following hold.*

- (i) *If $\mathcal{X}$ is closed under arbitrary direct products, and if each of the modules M_i admits a (proper) left $\mathcal{X}$–resolution, then so does the product $\prod M_i$.*
- (ii) *If $\mathcal{X}$ is closed under arbitrary direct sums, and if each of the modules M_i admits a (co–proper) right $\mathcal{X}$–resolution, then so does the sum $\coprod M_i$.*

(1.7) **Horseshoe lemma.** *Let $\mathcal{X}$ be a class of R–modules. Assume that $\mathcal{X}$ is closed under finite direct sums, and consider an exact sequence $0 \to M' \to M \to M'' \to 0$ of R–modules, such that*

$$0 \longrightarrow \mathrm{Hom}_R(M'', Y) \longrightarrow \mathrm{Hom}_R(M, Y) \longrightarrow \mathrm{Hom}_R(M', Y) \longrightarrow 0$$

is exact for every $Y \in \mathcal{X}$. If both M' and M'' admits (co–proper) right $\mathcal{X}$-resolutions, then so does M.

Proof. Dualizing the proof of [10, Lemma 8.2.1], we can construct the (co–proper) resolution of M as the degreewise sum of the two given (co–proper) resolutions for M' and M''. $\square$

(1.8) **Proposition.** *Let $f\colon M \to \widetilde{M}$ be a homomorphism of modules, and consider the diagram,*

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "f"] & X^0 \arrow[r] & X^1 \arrow[r] & X^2 \arrow[r] & \cdots \\ 0 \arrow[r] & \widetilde{M} \arrow[r] & \widetilde{X}^0 \arrow[r] & \widetilde{X}^1 \arrow[r] & \widetilde{X}^2 \arrow[r] & \cdots \end{tikzcd}$$

where the upper row is a co–proper right $\mathcal{X}$–resolution of M, and the lower row is a right $\mathcal{X}$–resolution of $\widetilde{M}$. Then $f\colon M \to \widetilde{M}$ induces a chain map of complexes,

(1)
$$\begin{tikzcd} 0 \arrow[r] & X^0 \arrow[r] \arrow[d, dotted, "f^0"] & X^1 \arrow[r] \arrow[d, dotted, "f^1"] & X^2 \arrow[r] \arrow[d, dotted, "f^2"] & \cdots \\ 0 \arrow[r] & \widetilde{X}^0 \arrow[r] & \widetilde{X}^1 \arrow[r] & \widetilde{X}^2 \arrow[r] & \cdots \end{tikzcd}$$

with the property that the square,

$$\begin{tikzcd} M \arrow[r] \arrow[d, "f"'] & X^0 \arrow[d, "f^0"] \\ \widetilde{M} \arrow[r] & \widetilde{X}^0 \end{tikzcd}$$

commutes. Furthermore, the chain map (1) *is uniquely determined upto homotopy by this property.*

Proof. Please see [10, Exercise 2, p. 169], or simply "dualize" the argument following [10, Proposition 8.1.3]. □

2. Gorenstein projective and Gorenstein injective modules

In this section we give a detailed treatment of Gorenstein projective modules. The main purpose is to give functorial descriptions of the Gorenstein projective dimension.

(2.1) **Definition.** A *complete projective resolution* is an exact sequence of projective modules, $\boldsymbol{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$, such that $\mathrm{Hom}_R(\boldsymbol{P}, Q)$ is exact for every projective R–module Q.

An R–module M is called *Gorenstein projective* (G-projective for short), if there exists a complete projective resolution $\boldsymbol{P}$ with $M \cong \mathrm{Im}(P_0 \to P^0)$. The class of all Gorenstein projective R–modules is denoted $\mathcal{GP}(R)$.

Gorenstein injective (G-injective for short) modules are defined dually, and the class of all such modules is denoted $\mathcal{GI}(R)$.

(2.2) **Observation.** If $\boldsymbol{P}$ is a complete projective resolution, then by symmetry, all the images, and hence also all the kernels, and cokernels of $\boldsymbol{P}$ are Gorenstein projective modules. Furthermore, every projective module is Gorenstein projective.

Using the definitions, we immediately get the following characterization of Gorenstein projective modules.

(2.3) **Proposition.** *An R–module M is Gorenstein projective if, and only if, M belongs to the left orthogonal class ${}^{\perp}\mathcal{P}(R)$, and admits a co–proper right $\mathcal{P}(R)$–resolution.*

Furthermore, if $\boldsymbol{P}$ is a complete projective resolution, then $\mathrm{Hom}_R(\boldsymbol{P}, L)$ is exact for all R–modules L with finite projective dimension. Consequently, when M is Gorenstein projective, then $\mathrm{Ext}^i_R(M, L) = 0$ for all $i > 0$ and all R–modules L with finite projective dimension. □

As the next result shows, we can always assume that the modules in a complete projective resolution are free.

(2.4) **Proposition.** *If M is a Gorenstein projective module, then there is a complete projective resolution, $\boldsymbol{F} = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$, consisting of free modules F_n and F^n such that $M \cong \mathrm{Im}(F_0 \to F^0)$.*

Proof. Only the construction of the "right half", $0 \to M \to F^0 \to F^1 \to \cdots$ of $\boldsymbol{F}$ is of interest. By Proposition (2.3), M admits a co–proper right $\mathcal{P}(R)$–resolution, say

$$0 \to M \to Q^0 \to Q^1 \to \cdots .$$

We successively pick projective modules $P^0, P^1, P^2, \ldots$, such that all of the modules

$$F^0 = Q^0 \oplus P^0 \qquad \text{and} \qquad F^n = Q^n \oplus P^{n-1} \oplus P^n \quad \text{for} \quad n > 0,$$

are free. By adding $0 \longrightarrow P^i \xrightarrow{=} P^i \longrightarrow 0$ to the co–proper right $\mathcal{P}(R)$–resolution above in degrees i and $i+1$, we obtain the desired sequence. □

Next we set out to investigate how Gorenstein projective modules behave in short exact sequences. The following theorem is due to Foxby and Martsinkovsky, but the proof presented here differs somewhat from their original ideas. Also note that Enochs and Jenda in [10, Theorems 10.2.8 and 11.5.6], have proved special cases of the result.

(2.5) **Theorem.** *The class $\mathcal{GP}(R)$ of all Gorenstein projective R–modules is projectively resolving. Furthermore $\mathcal{GP}(R)$ is closed under arbitrary direct sums and under direct summands.*

Proof. The left orthogonal class ${}^{\perp}\mathcal{P}(R)$ is closed under arbitrary direct sums, by Example (1.3), and so is the class of modules which admit a co–proper right $\mathcal{P}(R)$—resolution, by Proposition (1.6)(*ii*). Consequently, the class $\mathcal{GP}(R)$ is also closed under arbitrary direct sums, by Proposition (2.3).

To prove that $\mathcal{GP}(R)$ is projectively resolving, we consider any short exact sequence of R–modules, $0 \to M' \to M \to M'' \to 0$, where M'' is Gorenstein projective.

First assume that M' is Gorenstein projective. Again, using the characterization in Proposition (2.3), we conclude that M is Gorenstein projective, by the Horseshoe lemma (1.7), and by Example (1.3), which shows that the left orthogonal class ${}^{\perp}\mathcal{P}(R)$ is projectively resolving.

Next assume that M is Gorenstein projective. Since ${}^{\perp}\mathcal{P}(R)$ is projectively resolving, we get that M' belongs to ${}^{\perp}\mathcal{P}(R)$. Thus, to show that M' is Gorenstein projective, we only have to prove that M' admits a co–proper right $\mathcal{P}(R)$–resolution. By assumption, there exists co–proper right $\mathcal{P}(R)$–resolutions,

$$\boldsymbol{M} = 0 \to M \to P^0 \to P^1 \to \cdots \quad \text{and} \quad \boldsymbol{M''} = 0 \to M'' \to P''^0 \to P''^1 \to \cdots .$$

Proposition (1.8) gives a chain map $\boldsymbol{M} \to \boldsymbol{M''}$, lifting the homomorphism $M \to M''$. We let $\boldsymbol{C}$ denote the mapping cone of $\boldsymbol{M} \to \boldsymbol{M''}$, and we note the following properties:

Since $\boldsymbol{M} \to \boldsymbol{M''}$ is a quasi–isomorphism (both $\boldsymbol{M}$ and $\boldsymbol{M''}$ are exact), the long exact sequence of homology for the mapping cone shows that $\boldsymbol{C}$ is exact. Furthermore, if Q is any projective module, then $\mathrm{Hom}_R(\boldsymbol{C}, Q)$ is isomorphic to (a shift of) the mapping cone of the quasi–isomorphism,

$$\mathrm{Hom}_R(\boldsymbol{M''}, Q) \to \mathrm{Hom}_R(\boldsymbol{M}, Q),$$

and thus, also $\mathrm{Hom}_R(\boldsymbol{C}, Q)$ is exact. Next note that we have a short exact sequence of complexes,

(2)
$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & P''^0 \oplus P^1 \arrow[r, equal] \arrow[u] & P''^0 \oplus P^1 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P^0 \arrow[r] \arrow[u] & M'' \oplus P^0 \arrow[r] \arrow[u] & M'' \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & M' \arrow[r] \arrow[u] & M \arrow[r] \arrow[u] & M'' \arrow[r] \arrow[u, equal] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & \\
 & \parallel & \parallel & \parallel & \\
0 \arrow[r] & \boldsymbol{M'} \arrow[r] & \boldsymbol{C} \arrow[r] & \boldsymbol{D} \arrow[r] & 0
\end{tikzcd}$$

We claim that the first column, $\boldsymbol{M'}$, is a co–proper right $\mathcal{P}(R)$–resolution of M'. Since both $\boldsymbol{C}$ and $\boldsymbol{D}$ are exact, the long exact sequence in homology shows that $\boldsymbol{M'}$ is exact as well. Thus $\boldsymbol{M'}$ is a right $\mathcal{P}(R)$-resolution of M'.

To see that it is co–proper, we let Q be any projective module. Applying $\mathrm{Hom}_R(-, Q)$ to (2) we obtain another exact sequence of complexes,

$$0 \to \mathrm{Hom}_R(\boldsymbol{D}, Q) \to \mathrm{Hom}_R(\boldsymbol{C}, Q) \to \mathrm{Hom}_R(\boldsymbol{M'}, Q) \to 0.$$

For the first row,

$$0 \to \mathrm{Hom}_R(M'', Q) \to \mathrm{Hom}_R(M, Q) \to \mathrm{Hom}_R(M', Q) \to 0,$$

exactness follows from Proposition (2.3), since M'' is Gorenstein projective, and for the remaining rows exactness is obvious. As already noticed, $\mathrm{Hom}_R(\boldsymbol{C}, Q)$ is exact, and obviously, so is $\mathrm{Hom}_R(\boldsymbol{D}, Q)$. Thus, another application of the long exact sequence for homology shows that $\mathrm{Hom}_R(\boldsymbol{M'}, Q)$ is exact as well. Hence $\boldsymbol{M'}$ is co–proper.

Finally we have to show that the class $\mathcal{GP}(R)$ is closed under direct summands. Since $\mathcal{GP}(R)$ is projectively resolving, and closed under arbitrary direct sums, the desired conclusion follows from Proposition (1.4). $\square$

Here is the first exception to the "Important note" on page 2. We state, but do not prove, the Gorenstein injective version of Theorem (2.20) above (as we will need it in Section 3, when we deal with Gorenstein flat modules).

(2.6) **Theorem.** *The class $\mathcal{GI}(R)$ of all Gorenstein injective R–modules is injectively resolving. Furthermore $\mathcal{GI}(R)$ is closed under arbitrary direct products and under direct summands.*

(2.7) **Proposition.** *Let M be any R–module and consider two exact sequences,*

$$0 \longrightarrow K_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0\,,$$

$$0 \longrightarrow \widetilde{K}_n \longrightarrow \widetilde{G}_{n-1} \longrightarrow \cdots \longrightarrow \widetilde{G}_0 \longrightarrow M \longrightarrow 0\,,$$

where $G_0, \ldots, G_{n-1}$ and $\widetilde{G}_0, \ldots, \widetilde{G}_{n-1}$ are Gorenstein projective modules. Then K_n is Gorenstein projective if and only if $\widetilde{K}_n$ is Gorenstein projective.

Proof. Since the class of Gorenstein projective modules is projectively resolving and closed under arbitrary sums, and under direct summands, by Theorem (2.5), the stated result is a direct consequence of [2, Lemma (3.12)]. $\square$

At this point we introduce the Gorenstein projective dimension:

(2.8) **Definition.** The *Gorenstein projective dimension*, $\mathrm{Gpd}_R M$, of an R–module M is defined by declaring that $\mathrm{Gpd}_R M \leqslant n$ ($n \in \mathbb{N}_0$) if, and only if, M has a Gorenstein projective resolution of length n. We use $\overline{\mathcal{GP}}(R)$ to denote the class of all R–modules with *finite* Gorenstein projective dimension.

Similarly, one defines the *Gorenstein injective dimension*, $\mathrm{Gid}_R M$, of M, and we use $\overline{\mathcal{GI}}(R)$ to denote the class of all R–modules with finite Gorenstein injective dimension.

Hereafter, we immediately deal with Gorenstein projective precovers, and proper left $\mathcal{GP}(R)$–resolutions. We begin with a definition of precovers.

(2.9) **Precovers.** Let $\mathcal{X}$ be any class of R–modules, and let M be an R–module. An *$\mathcal{X}$–precover* of M is an R–homomorphism $\varphi\colon X \to M$, where $X \in \mathcal{X}$, and such that the sequence,

$$\mathrm{Hom}_R(X', X) \xrightarrow{\mathrm{Hom}_R(X',\varphi)} \mathrm{Hom}_R(X', M) \longrightarrow 0$$

is exact for every $X' \in \mathcal{X}$. ($\mathcal{X}$–preenvelopes of M are defined "dually".)

For more details about precovers (and preenvelopes), the reader may consult [10, Chapters 5 and 6] or [15, Chapter 1]. Instead of saying $\mathcal{GP}(R)$-precover, we shall use the term *Gorenstein projective precover*.

In the case where $(R, \mathfrak{m}, k)$ is a local noetherian Cohen–Macaulay ring admitting a dualizing module, special cases of the results below can be found in [12, Theorem 2.9 and 2.10].

(2.10) **Theorem.** *Let M be an R–module with finite Gorenstein projective dimension n. Then M admits a surjective Gorenstein projective precover, $\varphi\colon G \twoheadrightarrow M$, where $K = \operatorname{Ker}\varphi$ satisfies $\operatorname{pd}_R K = n-1$ (if $n=0$, this should be interpreted as $K=0$).*

In particular, M admits a proper left Gorenstein projective resolution (that is, a proper left $\mathcal{GP}(R)$-resolution) of length n.

Proof. Pick an exact sequence, $0 \to K' \to P_{n-1} \to \cdots \to P_0 \to M \to 0$, where $P_0, \dots, P_{n-1}$ are projectives. Then K' is Gorenstein projective by Proposition (2.7). Hence there is an exact sequence $0 \to K' \to Q^0 \to \cdots \to Q^{n-1} \to G \to 0$, where $Q^0, \dots, Q^{n-1}$ are projectives, G is Gorenstein projective, and such that the functor $\operatorname{Hom}_R(-, Q)$ leaves this sequence exact, whenever Q is projective.

Thus there exist homomorphisms, $Q^i \to P_{n-1-i}$ for $i = 0, \dots, n-1$, and $G \to M$, such that the following diagram is commutative.

(3)
$$\begin{tikzcd} 0 \arrow[r] & K' \arrow[r] \arrow[d, equal] & Q^0 \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & Q^{n-1} \arrow[r] \arrow[d, dotted] & G \arrow[r] \arrow[d, dotted] & 0 \\ 0 \arrow[r] & K' \arrow[r] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

This diagram gives a chain map between complexes,

$$\begin{tikzcd} 0 \arrow[r] & Q^0 \arrow[r] \arrow[d] & \cdots \arrow[r] & Q^{n-1} \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

which induces an isomorphism in homology. Its mapping cone is exact, and all the modules in it, except for $P_0 \oplus G$ (which is Gorenstein projective), are projective. Hence the kernel K of $\varphi\colon P_0 \oplus G \twoheadrightarrow M$ satisfies $\operatorname{pd}_R K \leqslant n-1$ (and then necessarily $\operatorname{pd}_R K = n-1$).

Since K has finite projective dimension, we have $\operatorname{Ext}^1_R(G', K) = 0$ for any Gorenstein projective module G', by Proposition (2.3), and thus the homomorphism

$$\operatorname{Hom}_R(G', \varphi)\colon \operatorname{Hom}_R(G', P_0 \oplus G) \to \operatorname{Hom}_R(G', M)$$

is surjective. Hence $\varphi\colon P_0 \oplus G \twoheadrightarrow M$ is the desired precover of M. $\square$

(2.11) **Corollary.** *Let $0 \to G' \to G \to M \to 0$ be a short exact sequence where G and G' are Gorenstein projective modules, and where $\operatorname{Ext}^1_R(M, Q) = 0$ for all projective modules Q. Then M is Gorenstein projective.*

Proof. Since $\operatorname{Gpd}_R M \leqslant 1$, Theorem (2.10) above gives the existence of an exact sequence $0 \to Q \to \widetilde{G} \to M \to 0$, where Q is projective, and $\widetilde{G}$ is Gorenstein projective. By our assumption $\operatorname{Ext}^1_R(M, Q) = 0$, this sequence splits, and hence M is Gorenstein projective by Theorem (2.5). $\square$

(2.12) **Remark.** If R is left noetherian and M is finite, then all the modules appearing in the proof of Theorem (2.10) can be chosen to be finite. Consequently, the module G in the Gorenstein projective precover $\varphi\colon G \twoheadrightarrow M$ of Theorem (2.10) (and hence also K) can be chosen to be finite. Let us write it out:

(2.13) **Corollary.** *Every finite R-module N with finite Gorenstein projective dimension has a finite surjective Gorenstein projective precover, $0 \to K \to G \to N \to 0$, such that the kernel K has finite projective dimension.*

(2.14) **Observation.** Over a local noetherian ring $(R, \mathfrak{m}, k)$ admitting a dualizing module, Auslander and Buchweitz introduces in their paper [3]

(i) a *maximal Cohen-Macaulay approximation*, $0 \to I_N \to M_N \to N \to 0$, and

(ii) a *hull of finite injective dimension*, $0 \to N \to I^N \to M^N \to 0$

for every finite R-module N. Here M_N and M^N are finite maximal Cohen-Macaulay modules, and I_N, I^N have finite injective dimension.

Note how the sequence $0 \to K \to G \to N \to 0$ from Corollary (2.13) resembles their maximal Cohen-Macaulay approximation.

(2.15) **Theorem.** *Let N be an R–module with finite Gorenstein injective dimension n. Then N admits an injective Gorenstein injective preenvelope, $\varphi\colon N \hookrightarrow H$, where $C = \operatorname{Coker}\varphi$ satisfies $\operatorname{id}_R C = n-1$ (if $n = 0$, this should be interpreted as $C = 0$).*

In particular, N admits a co–proper right Gorenstein injective resolution (that is, a co–proper right $\mathcal{GI}(R)$-resolution) of length n. □

Using completely different methods, Enochs and Jenda proved in [9, Theorem 2.13] the Gorenstein injective dual version of Proposition (2.11) above. However, Proposition (2.11) itself is only proved for (left) coherent rings and *finitely presented* (right) modules in [10, Theorem 10.2.8].

We now wish to prove how the Gorenstein projective dimension, which is defined in terms of resolutions, can be mesured by the Ext-functors in a way much similar to how these functors mesures the ordinary projective dimension.

(2.16) **Proposition.** *Assume that R is left noetherian, and that M is a finite (left) R–module with Gorenstein projective dimension m (possibly $m = \infty$). Then M has a Gorenstein projective resolution of length m, consisting of finite Gorenstein projective modules.*

Proof. Simply apply Proposition (2.7) to a resolution of M by finite projective modules. □

Using Propositions (2.3) and (2.7) together with standard arguments, we immediately obtain the next two results.

(2.17) **Lemma.** *Consider an exact sequence $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$ where $G_0, \ldots, G_{n-1}$ are Gorenstein projective modules. Then*

$$\operatorname{Ext}^i_R(K_n, L) \cong \operatorname{Ext}^{i+n}_R(M, L)$$

for all R–modules L with finite projective dimension, and all integers $i > 0$. □

(2.18) **Proposition.** *Let $0 \to K \to G \to M \to 0$ be an exact sequence of R–modules where G is Gorenstein projective. If M is Gorenstein projective, then so is K. Otherwise we get $\mathrm{Gpd}_R K = \mathrm{Gpd}_R M - 1 \geqslant 0$.* □

(2.19) **Proposition.** *If $(M_\lambda)_{\lambda\in\Lambda}$ is any family of R–modules, then we have an equality*

$$\mathrm{Gpd}_R(\textstyle\coprod M_\lambda) \;=\; \sup\{\, \mathrm{Gpd}_R M_\lambda \mid \lambda \in \Lambda \,\}.$$

Proof. The inequality '$\leqslant$' is clear since $\mathcal{GP}(R)$ is closed under direct sums by Theorem (2.5). For the converse inequality '$\geqslant$', it suffices to show that if M' is any direct summand of an R–module M, then $\mathrm{Gpd}_R M' \leqslant \mathrm{Gpd}_R M$. Naturally we may assume that $\mathrm{Gpd}_R M = n$ is finite, and then proceed by induction on n.

The induction start is clear, because if M is Gorenstein projective, then so is M', by Theorem (2.5). If $n > 0$, we write $M = M' \oplus M''$ for some module M''. Pick exact sequences $0 \to K' \to G' \to M' \to 0$ and $0 \to K'' \to G'' \to M'' \to 0$, where G' and G'' are Gorenstein projectives. We get a commutative diagram with split-exact rows,

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & M' \arrow[r] \arrow[u] & M \arrow[r] \arrow[u] & M'' \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & G' \arrow[r] \arrow[u] & G' \oplus G'' \arrow[r] \arrow[u] & G'' \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & K' \arrow[r] \arrow[u] & K' \oplus K'' \arrow[r] \arrow[u] & K'' \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Applying Proposition (2.18) to the middle column in this diagram, we get that $\mathrm{Gpd}_R(K' \oplus K'') = n - 1$. Hence the induction hypothesis yields that $\mathrm{Gpd}_R K' \leqslant n - 1$, and thus the short exact sequence $0 \to K' \to G' \to M' \to 0$ shows that $\mathrm{Gpd}_R M' \leqslant n$, as desired. □

(2.20) **Theorem.** *Let M be an R–module with finite Gorenstein projective dimension, and let n be an integer. Then the following conditions are equivalent.*

- (*i*) $\mathrm{Gpd}_R M \leqslant n$.
- (*ii*) $\mathrm{Ext}^i_R(M, L) = 0$ *for all $i > n$, and all R–modules L with finite* $\mathrm{pd}_R L$.
- (*iii*) $\mathrm{Ext}^i_R(M, Q) = 0$ *for all $i > n$, and all projective R–modules Q.*
- (*iv*) *For every exact sequence $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$ where $G_0, \ldots, G_{n-1}$ are Gorenstein projectives, then also K_n is Gorenstein projective.*

Consequently, the Gorenstein projective dimension of M is determined by the formulas:

$$\begin{aligned}\mathrm{Gpd}_R M &= \sup\{\, i \in \mathbb{N}_0 \mid \exists\, L \in \overline{\mathcal{P}}(R)\colon\ \mathrm{Ext}^i_R(M,L) \neq 0\,\}\\ &= \sup\{\, i \in \mathbb{N}_0 \mid \exists\, Q \in \mathcal{P}(R)\colon\ \mathrm{Ext}^i_R(M,Q) \neq 0\,\}.\end{aligned}$$

Proof. The proof is 'cyclic'. Obviously $(ii) \Rightarrow (iii)$ and $(iv) \Rightarrow (i)$, so we only have to prove the last two implications.

To prove $(i) \Rightarrow (ii)$, we assume that $\mathrm{Gpd}_R M \leqslant n$. By definition there is an exact sequence, $0 \to G_n \to \cdots \to G_0 \to M \to 0$, where $G_0, \ldots, G_n$ are Gorenstein projectives. By Lemma (2.17) and Proposition (2.3), we conclude the equalities $\mathrm{Ext}^i_R(M,L) \cong \mathrm{Ext}^{i-n}_R(G_n,L) = 0$ whenever $i > n$, and L has finite projective dimension, as desired.

To prove $(iii) \Rightarrow (iv)$, we consider an exact sequence,

$$0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0, \tag{4}$$

where $G_0, \ldots, G_{n-1}$ are Gorenstein projectives. Applying Lemma (2.17) to this sequence, and using the assumption, we get that $\mathrm{Ext}^i_R(K_n,Q) \cong \mathrm{Ext}^{i+n}_R(M,Q) = 0$ for every integer $i > 0$, and every projective module Q. Decomposing (4) into short exact sequences, and applying Proposition (2.18) successively n times, we see that $\mathrm{Gpd}_R K_n < \infty$, since $\mathrm{Gpd}_R M < \infty$. Hence there is an exact sequence,

$$0 \to G'_m \to \cdots \to G'_0 \to K_n \to 0,$$

where $G'_0, \ldots, G'_m$ are Gorenstein projectives. We decompose it into short exact sequences, $0 \to C'_j \to G'_{j-1} \to C'_{j-1} \to 0$, for $j = 1, \ldots, m$, where $C'_m = G'_m$ and $C'_0 = K_n$. Now another use of Lemma (2.17) gives that

$$\mathrm{Ext}^1_R(C'_{j-1},Q) \;\cong\; \mathrm{Ext}^j_R(K_n,Q) \;=\; 0$$

for all $j = 1, \ldots, m$, and all projective modules Q. Thus Proposition (2.11) can be applied successively to conclude that $C'_m, \ldots, C'_0$ (in that order) are Gorenstein projectives. In particular $K_n = C'_0$ is Gorenstein projective.

The last formulas in the theorem for determination of $\mathrm{Gpd}_R M$ are a direct consequence of the equivalence between $(i) - (iii)$. □

(2.21) **Corollary.** *If R is left noetherian, and M is a finite (left) module with finite Gorenstein projective dimension, then*

$$\mathrm{Gpd}_R M = \sup\{\, i \in \mathbb{N}_0 \mid \mathrm{Ext}^i_R(M,R) \neq 0\,\}.$$

Proof. By Theorem (2.20), it suffices to show that if $\mathrm{Ext}^i_R(M,Q) \neq 0$ for some projective module Q, then also $\mathrm{Ext}^i_R(M,R) \neq 0$. We simply pick another module P, such that $Q \oplus P \cong R^{(\Lambda)}$ for some index set Λ, and then note that $\mathrm{Ext}^n_R(M,R)^{(\Lambda)} \cong \mathrm{Ext}^n_R(M,Q) \oplus \mathrm{Ext}^n_R(M,P) \neq 0$. □

(2.22) **Theorem.** *Let N be an R–module with finite Gorenstein injective dimension, and let n be an integer. Then the following conditions are equivalent.*

(i) $\mathrm{Gid}_R N \leqslant n$.

(*ii*) $\mathrm{Ext}^i_R(L,N)=0$ *for all* $i>n$, *and all* R*–modules* L *with finite* $\mathrm{id}_R L$.
(*iii*) $\mathrm{Ext}^i_R(I,N)=0$ *for all* $i>n$, *and all injective* R*–modules* I.
(*iv*) *For every exact sequence* $0\to N\to H^0\to\cdots\to H^{n-1}\to C^n\to 0$ *where* $H^0,\ldots,H^{n-1}$ *are Gorenstein injective, then also* C^n *is Gorenstein injective.*

Consequently, the Gorenstein injective dimension of N *is determined by the formulas:*

$$\begin{aligned}\mathrm{Gid}_R N &= \sup\{\, i\in\mathbb{N}_0 \mid \exists\, L\in\overline{\mathcal{I}}(R)\colon\ \mathrm{Ext}^i_R(L,N)\neq 0\,\}\\ &= \sup\{\, i\in\mathbb{N}_0 \mid \exists\, I\in\mathcal{I}(R)\colon\ \mathrm{Ext}^i_R(I,N)\neq 0\,\}.\end{aligned}$$

Comparing Theorem (2.22) above with Matlis' Structure Theorem on injective modules we get the next result.

(2.23) **Corollary.** *If* R *is commutative and noetherian, and* N *is a module with finite Gorenstein injective dimension, then*

$$\mathrm{Gid}_R N=\sup\{\, i\in\mathbb{N}_0 \mid \exists\,\mathfrak{p}\in\operatorname{Spec}R:\ \mathrm{Ext}^i_R(E_R(R/\mathfrak{p}),N)\neq 0\,\}.$$

Here $E_R(R/\mathfrak{p})$ *denotes the injective hull of* $R/\mathfrak{p}$. □

(2.24) **Theorem.** *Let* $0\to M'\to M\to M''\to 0$ *be a short exact sequence of* R*–modules. If any two of the modules* M, M', *or* M'' *have finite Gorenstein projective dimension, then so has the third.*

Proof. The proof of [5, Proposition 3.4] shows that this theorem is a formal consequence of Proposition (2.7). □

(2.25) **Theorem.** *Let* $0\to N'\to N\to N''\to 0$ *be a short exact sequence of* R*–modules. If any two of the modules* N, N', *or* N'' *have finite Gorenstein injective dimension, then so has the third.* □

(2.26) **Remark.** The theory of Gorenstein projective modules is particularly nice when the ring $(R,\mathfrak{m},k)$ is local, noetherian, Cohen–Macaulay and has a dualizing module. In that case we can consider the Auslander class $\mathcal{A}(R)$, defined in [4, (3.1)]. See also [15, Definition 5.5.1].

From [12, Corollary 2.4], the following implications are known for any R–module M:

$$M\in\mathcal{A}(R)\iff \mathrm{Gpd}_R M<\infty \iff \mathrm{Gpd}_R M\leqslant \dim R.$$

In this case the previous Theorem (2.24) is trivial, as it is easy to see that $\mathcal{A}(R)$ is closed under short exact sequences (this can be found in e.g [15, Theorem 5.5.6]).

Similar remarks are to be said about the Bass class $\mathcal{B}(R)$ and the Gorenstein injective dimension.

It is only natural to investigate how much the usual projective dimension differs from the Gorenstein projective one. The answer follows easily from Theorem (2.20).

(2.27) **Proposition.** *If M is an R–module with finite projective dimension, then* $\mathrm{Gpd}_R M = \mathrm{pd}_R M$. *In particular there is an equality of classes* $\mathcal{GP}(R) \cap \overline{\mathcal{P}}(R) = \mathcal{P}(R)$.

Proof. Assume that $n = \mathrm{pd}_R M$ is finite. By definition, there is always an inequality $\mathrm{Gpd}_R M \leqslant \mathrm{pd}_R M$, and consequently, we also have $\mathrm{Gpd}_R M \leqslant n < \infty$. In order to show that $\mathrm{Gpd}_R M = n$, we need, by Theorem (2.20), the existence of a projective module P, such that $\mathrm{Ext}^n_R(M, P) \neq 0$.

Since $\mathrm{pd}_R M = n$, there is some module, N, with $\mathrm{Ext}^n_R(M, N) \neq 0$. Let P be any projective module which surjects onto N. From the long exact homology sequence, it now follows that also $\mathrm{Ext}^n_R(M, P) \neq 0$, as desired. □

Using relative homological algebra, Enochs and Jenda have shown similar results to Proposition (2.27) above in [10, Propositions 10.1.2 and 10.2.3].

We end this section with an applications of Gorenstein projective precovers. We compare the (*left*) *finitistic Gorenstein projective dimension* of the base ring R,

$$\mathrm{FGPD}(R) \;=\; \sup \left\{ \mathrm{Gpd}_R M \;\middle|\; \begin{array}{l} M \text{ is a (left) } R\text{–module with finite} \\ \text{Gorenstein projective dimension.} \end{array} \right\},$$

with the usual, and well-investigated, (left) finitistic projective dimension, $\mathrm{FPD}(R)$.

(2.28) **Theorem.** *For any ring R there is an equality* $\mathrm{FGPD}(R) = \mathrm{FPD}(R)$.

Proof. Clearly $\mathrm{FPD}(R) \leqslant \mathrm{FGPD}(R)$ by Proposition (2.27). Note that if M is a module with $0 < \mathrm{Gpd}_R M < \infty$, then Theorem (2.10) in particular gives the existence of a module K with $\mathrm{pd}_R K = \mathrm{Gpd}_R M - 1$, and hence we get $\mathrm{FGPD}(R) \leqslant \mathrm{FPD}(R) + 1$. Proving the inequality $\mathrm{FGPD}(R) \leqslant \mathrm{FPD}(R)$, we may therefore assume that

$$0 < \mathrm{FGPD}(R) = m < \infty.$$

Pick a module M with $\mathrm{Gpd}_R M = m$. We wish to find a module L with $\mathrm{pd}_R L = m$. By Theorem (2.10) there is an exact sequence $0 \to K \to G \to M \to 0$ where G is Gorenstein projective, and $\mathrm{pd}_R K = m - 1$. Since G is Gorenstein projective, there exists a projective module Q with $G \subseteq Q$, and since also $K \subseteq G$, we can consider the quotient $L = Q/K$. Note that $M \cong G/K$ is a submodule of L, and thus we get a short exact sequence $0 \to M \to L \to L/M \to 0$.

If L is Gorenstein projective, then Proposition (2.18) will imply that $\mathrm{Gpd}_R(L/M) = m + 1$, since $\mathrm{Gpd}_R M = m > 0$. But this contradict the fact that $m = \mathrm{FGPD}(R) < \infty$. Hence L is *not* Gorenstein projective, in particular, L is not projective. Therefore the short exact sequence $0 \to K \to Q \to L \to 0$ shows that $\mathrm{pd}_R L = \mathrm{pd}_R K + 1 = m$. □

For the (left) finitistic Gorenstein injective dimension, $\mathrm{FGID}(R)$, and the usual (left) finitistic injective dimension, $\mathrm{FID}(R)$, we of course also have:

(2.29) **Theorem.** *For any ring R there is an equality* $\mathrm{FGID}(R) = \mathrm{FID}(R)$. □

3. Gorenstein flat modules

The treatment of Gorenstein flat R–modules is different from the way we handled Gorenstein projective modules. This is because Gorenstein flat modules are defined by the tensor product functor $- \otimes_R -$ and not by $\mathrm{Hom}_R(-,-)$. However, over a right coherent ring there is a connection between Gorenstein flat left modules and Gorenstein injective right modules, and this allow us to get good results.

(3.1) **Definition.** A *complete flat resolution* is an exact sequence of flat (left) R–modules, $\boldsymbol{F} = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$, such that $I \otimes_R \boldsymbol{F}$ is exact for every injective right R–module I.

An R–module M is called *Gorenstein flat* (*G-flat* for short), if there exists a complete flat resolution $\boldsymbol{F}$ with $M \cong \mathrm{Im}(F_0 \to F^0)$. The class of all Gorenstein flat R–modules is denoted $\mathcal{GF}(R)$.

There is a nice connection between Gorenstein flat and Gorenstein injective modules, and this enable us to prove that the class of Gorenstein flat modules is projectively resolving. We begin with:

(3.2) **Proposition.** *The class $\mathcal{GF}(R)$ is closed under arbitrary direct sums.*

Proof. Simply note that a (degreewise) sum of complete flat resolutions again is a complete flat resolution (as tensor products commutes with sums). $\square$

(3.3) **Remark.** From Bass [6, Corollary 5.5], and Gruson–Raynaud [14, Seconde partie, Theorem (3.2.6)], we have that $\mathrm{FPD}(R) = \dim R$, when R is commutative and noetherian.

(3.4) **Proposition.** *If R is right coherent with finite left finitistic projective dimension, then every Gorenstein projective (left) R–module is also Gorenstein flat.*

Proof. It suffices to prove that if $\boldsymbol{P}$ is a complete projective resolution, then $I \otimes_R \boldsymbol{P}$ is exact for all injective right modules I. Since R is right coherent, $F = \mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ is a flat (left) R–module by [15, Lemma 3.1.4]. Since $\mathrm{FPD}(R)$ is finite, Jensen [13, Proposition 6] implies that F has finite projective dimension, and consequently $\mathrm{Hom}_R(\boldsymbol{P}, F)$ is exact by Proposition (2.3). By adjointness,

$$\mathrm{Hom}_{\mathbb{Z}}(I \otimes_R \boldsymbol{P}, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_R(\boldsymbol{P}, F),$$

and the desired result follows. $\square$

(3.5) **Example.** Let R be any integral domain which is not a field, and let K denote the field of fractions of R. Then K is a flat (and hence Gorenstein flat) R–module which is not contained in any free R–module, in particular, K cannot be Gorenstein projective.

(3.6) **Theorem.** *For any (left) R–module M, we consider the following conditions.*

(*i*) *M is a Gorenstein flat (left) R–module.*

(ii) *The Pontryagin dual* $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ *is a Gorenstein injective right* R*–module.*

(iii) M *admits a co–proper right flat resolution (that is, a co–proper right* $\mathcal{F}(R)$*–resolution), and* $\mathrm{Tor}_i^R(I, M) = 0$ *for all injective right* R*–modules* I*, and all integers* $i > 0$*.*

Then $(i) \Rightarrow (ii)$*. If* R *is right coherent, then also* $(ii) \Rightarrow (iii) \Rightarrow (i)$*, and hence all three conditions are equivalent.*

Proof. As the theorem is stated, it is an extended non-commutative version of [7, Theorem (6.4.2)], which deals with commutative, noetherian rings. However, a careful reading of the proof, compared with basic facts about the Pontryagin dual, gives this stronger version. □

(3.7) **Theorem.** *If* R *is right coherent, then the class* $\mathcal{GF}(R)$ *of Gorenstein flat* R*–modules is projectively resolving and closed under direct summands.*

Furthermore, if $M_0 \to M_1 \to M_2 \to \cdots$ *is a sequence of Gorenstein flat modules, then the direct limit* $\varinjlim M_n$ *is again Gorenstein flat.*

Proof. Using Theorem (2.6) together with the equivalence $(i) \Leftrightarrow (ii)$ in Theorem (3.6) above, we see that $\mathcal{GF}(R)$ is projectively resolving. Now, comparing Proposition (3.2) with Proposition (1.4), we get that $\mathcal{GF}(R)$ is closed under direct summands.

Concerning the last statement, we pick for each n a co–proper right flat resolution $\boldsymbol{G}_n$ of M_n (which is possible by Theorem (3.6)*(iii)*), as illustrated in the next diagram.

(5)
$$\begin{tikzcd}
\boldsymbol{G}_0 \arrow[d, dotted] & = & 0 \arrow[r] & M_0 \arrow[r] \arrow[d] & G_0^0 \arrow[r] \arrow[d, dotted] & G_0^1 \arrow[r] \arrow[d, dotted] & G_0^2 \arrow[r] \arrow[d, dotted] & \cdots \\
\boldsymbol{G}_1 \arrow[d, dotted] & = & 0 \arrow[r] & M_1 \arrow[r] \arrow[d] & G_1^0 \arrow[r] \arrow[d, dotted] & G_1^1 \arrow[r] \arrow[d, dotted] & G_1^2 \arrow[r] \arrow[d, dotted] & \cdots \\
 & \vdots & & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

By Proposition (1.8), each map $M_n \to M_{n+1}$ can be lifted to a chain map $\boldsymbol{G}_n \to \boldsymbol{G}_{n+1}$ of complexes. Since we are dealing with *sequences* (and not arbitrary direct systems), each column in (5) is again a direct system. Thus it makes sense to apply the exact functor $\varinjlim$ to (5), and doing so, we obtain an exact complex,

$$\boldsymbol{G} \;=\; \varinjlim \boldsymbol{G}_n \;=\; 0 \to \varinjlim M_n \to \varinjlim G_n^0 \to \varinjlim G_n^1 \to \cdots\,,$$

where each module $G^k = \varinjlim G_n^k$, $k = 0, 1, 2, \ldots$ is flat. When I is injective right R–module, then $I \otimes_R \boldsymbol{G}_n$ is exact because: since $F = \mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ is a flat (left) R–module (recall that R is right coherent), we get exactness of

$$\mathrm{Hom}_R(\boldsymbol{G}_n, F) \;=\; \mathrm{Hom}_R(\boldsymbol{G}_n, \mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})) \;\cong\; \mathrm{Hom}_{\mathbb{Z}}(I \otimes_R \boldsymbol{G}_n, \mathbb{Q}/\mathbb{Z}),$$

and hence of $I \otimes_R \boldsymbol{G}_n$, since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$–module. Since $\varinjlim$ commutes with the homology functor, we also get exactness of

$$I \otimes_R \boldsymbol{G} \;\cong\; \varinjlim(I \otimes_R \boldsymbol{G}_n).$$

Thus we have constructed the "right half", $\boldsymbol{G}$, of a complete flat resolution for $\varinjlim M_n$. Since M_n is Gorenstein flat, we also have

$$\mathrm{Tor}_i^R(I, \varinjlim M_n) \;\cong\; \varinjlim \mathrm{Tor}_i^R(I, M_n) \;=\; 0$$

for $i > 0$, and all injective right modules I. Thus $\varinjlim M_n$ is Gorenstein flat. $\square$

(3.8) **Proposition.** *Assume that R is right coherent, and consider a short exact sequence of (left) R–modules $0 \to G' \to G \to M \to 0$, where G and G' are Gorenstein flats. If $\mathrm{Tor}_1^R(I, M) = 0$ for all injective right modules I, then M is G-flat.*

Proof. Define $H = \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$ and $H' = \mathrm{Hom}_{\mathbb{Z}}(G', \mathbb{Q}/\mathbb{Z})$, which are Gorenstein injective by the general implication $(i) \Rightarrow (ii)$ in Theorem (3.6). Applying the dual of Proposition (2.11) (about Gorenstein injective modules) to the exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \to H \to H' \to 0,$$

and noting that we have an isomorphism,

$$\mathrm{Ext}_R^1(I, \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \;\cong\; \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}_1^R(I, M), \mathbb{Q}/\mathbb{Z}) = 0$$

for all injective right modules I, we see that $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is Gorenstein injective. Since R is right coherent, we conclude that M is Gorenstein flat. $\square$

Next we introduce the Gorenstein flat dimension via resolutions, and show how the Tor-functors can be used to measure this dimension when R is right coherent.

(3.9) **Gorenstein flat dimension.** As done in [11] (and similar to the Gorenstein projective case), we define the *Gorenstein flat dimension*, $\mathrm{Gfd}_R M$, of a module M by declaring that $\mathrm{Gfd}_R M \leqslant n$ if, and only if, M has a resolution by Gorenstein flat modules of length n. We let $\overline{\mathcal{GF}}(R)$ denote the class of all R–modules with finite Gorenstein flat dimension.

(3.10) **Proposition (Flat base change).** *Consider a flat homomorphism of commutative rings $R \to S$ (that is, S is flat as an R–module). Then for any (left) R–module M we have an inequality,*

$$\mathrm{Gfd}_S(S \otimes_R M) \leqslant \mathrm{Gfd}_R M.$$

Proof. If $\boldsymbol{F}$ is a complete flat resolution of R–modules, then $S \otimes_R \boldsymbol{F}$ is an exact (since S is R–flat) sequence of flat S–modules. If I is an injective S–module, then, since S is R–flat, I is also an injective R–module. Thus we have exactness of

$$I \otimes_S (S \otimes_R \boldsymbol{F}) \;\cong\; (I \otimes_S S) \otimes_R \boldsymbol{F} \;\cong\; I \otimes_R \boldsymbol{F},$$

and hence $S \otimes_R \boldsymbol{F}$ is a complete flat resolution of S–modules. $\square$

(3.11) **Proposition.** *For any (left) R–module M there is an inequality,*

$$\mathrm{Gid}_R \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \leqslant \mathrm{Gfd}_R M.$$

If R is right coherent, then we have the equality,

$$\mathrm{Gid}_R \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = \mathrm{Gfd}_R M.$$

Proof. The inequality follows directly from the implication $(i) \Rightarrow (ii)$ in Theorem (3.6). Now assume that R is right coherent. For the converse inequality, we may assume that $\mathrm{Gid}_R \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = m$ is finite. Pick an exact sequence,

$$0 \to K_m \to G_{m-1} \to \cdots \to G_0 \to M \to 0,$$

where $G_0, \ldots, G_{m-1}$ are Gorenstein flats. Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ to this sequence, we get exactness of

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \to H^0 \to \cdots \to H^{m-1} \to C^m \to 0,$$

where we have defined $H^i = \mathrm{Hom}_{\mathbb{Z}}(G_i, \mathbb{Q}/\mathbb{Z})$ for $i = 0, \ldots, m-1$, together with $C^m = \mathrm{Hom}_{\mathbb{Z}}(K_m, \mathbb{Q}/\mathbb{Z})$. Since $H^0, \ldots, H^{m-1}$ are Gorenstein injective, Theorem (2.22) implies that $C^m = \mathrm{Hom}_{\mathbb{Z}}(K_m, \mathbb{Q}/\mathbb{Z})$ is Gorenstein injective. Now another application of Theorem (3.6) gives that K_m is Gorenstein flat (since R is right coherent), and consequently $\mathrm{Gfd}_R M \leqslant m = \mathrm{Gid}_R \mathrm{Hom}_R(M, \mathbb{Q}/\mathbb{Z})$. □

Using the connection between Gorenstein flat and Gorenstein injective dimension, which Proposition (3.11) establishes, together the Gorenstein injective versions of Propositions (2.18) and (2.19)), we get the next two results.

(3.12) **Proposition.** *Assume that R is right coherent. Let $0 \to K \to G \to M \to 0$ be a short exact sequence of R–modules where G is Gorenstein flat, and define $n = \mathrm{Gfd}_R M$. If M is Gorenstein flat, then so is K. If otherwise $n > 0$, then $\mathrm{Gfd}_R K = n - 1$.*

(3.13) **Proposition.** *Assume that R is right coherent. If $(M_\lambda)_{\lambda \in \Lambda}$ is any family of (left) R–modules, then we have an equality,*

$$\mathrm{Gfd}_R(\textstyle\coprod M_\lambda) = \sup\{\, \mathrm{Gfd}_R M_\lambda \mid \lambda \in \Lambda \,\}.$$

The next theorem is a generalization of [7, Theorem (5.2.14)], which is proved only for (commutative) local, noetherian Cohen–Macaulay rings with a dualizing module.

(3.14) **Theorem.** *Assume that R is right coherent. Let M be a (left) R-module with finite Gorenstein flat dimension, and let $n \geqslant 0$ be an integer. Then the following four conditions are equivalent.*

- (i) $\mathrm{Gfd}_R M \leqslant n$.
- (ii) $\mathrm{Tor}_i^R(L, M) = 0$ *for all right R–modules L with finite* $\mathrm{id}_R L$, *and all $i > n$.*
- (iii) $\mathrm{Tor}_i^R(I, M) = 0$ *for all injective right R–modules I, and all $i > n$.*
- (iv) *For every exact sequence $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$, where $G_0, \ldots, G_{n-1}$ are Gorenstein flats, then also K_n is Gorenstein flat.*

Consequently, the Gorenstein flat dimension of M is determined by the formulas:

$$\begin{aligned}\mathrm{Gfd}_R M &= \sup\{\, i \in \mathbb{N}_0 \mid \exists\, L \in \overline{\mathcal{I}}(R)\colon\ \mathrm{Tor}_i^R(L,M) \neq 0 \,\} \\ &= \sup\{\, i \in \mathbb{N}_0 \mid \exists\, I \in \mathcal{I}(R)\colon\ \mathrm{Tor}_i^R(I,M) \neq 0 \,\}.\end{aligned}$$

Proof. Combine the adjointness isomorphism,

$$\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}_i^R(L,M),\mathbb{Q}/\mathbb{Z}) \cong \mathrm{Ext}_R^i(L,\mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z}))$$

for right R–modules L, together with the identity from Proposition (3.11),

$$\mathrm{Gid}_R \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z}) = \mathrm{Gfd}_R M,$$

and use Theorem (2.22). □

(3.15) **Theorem.** *Assume that R is right coherent. If any two of the modules M, M' or M'' in a short exact sequence $0 \to M' \to M \to M'' \to 0$ have finite Gorenstein flat dimension, then so has the third.*

Proof. Consider $0 \to \mathrm{Hom}_{\mathbb{Z}}(M'',\mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(M',\mathbb{Q}/\mathbb{Z}) \to 0$. Using Proposition (3.11) together with Theorem (2.25), the desired conclusion easily follows. □

Next, we examine the large restricted flat dimension, and relate it to the usual flat dimension, and to the Gorenstein flat dimension.

(3.16) **Large restricted flat dimension.** For a R–module M, we consider the *large restricted flat dimension*, which is defined by

$$\mathsf{Rfd}_R M = \sup\left\{ i \geqslant 0 \;\middle|\; \begin{array}{l}\mathrm{Tor}_i^R(L,M) \neq 0 \text{ for some (right)} \\ R\text{–module with finite flat dimension.}\end{array} \right\}.$$

(3.17) **Lemma.** *Assume that R is right coherent. Let M be any R–module with finite Gorenstein flat dimension n. Then there exists a short exact sequence $0 \to K \to G \to M \to 0$ where G is Gorenstein flat, and where $\mathrm{fd}_R K = n-1$ (if $n = 0$, this should be interpreted as $K = 0$).*

Proof. We may assume that $n > 0$. We start by taking an exact sequence,

$$0 \to K' \to F_{n-1} \to \cdots \to F_0 \to M \to 0,$$

where $F_0, \ldots, F_{n-1}$ are flats. Then K' is Gorenstein flat by Theorem (3.14), and hence Theorem (3.6)(*iii*) gives an exact sequence $0 \to K' \to G^0 \to \cdots \to G^{n-1} \to G' \to 0$, where $G^0, \ldots, G^{n-1}$ are flats, G' is Gorenstein flat, and such that the functor $\mathrm{Hom}_R(-,F)$ leaves this sequence exact whenever F is a flat R–module. Consequently, we get homomorphisms, $G^i \to F_{n-1-i}$, $i = 0, \ldots, n-1$, and $G' \to M$, giving a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r] \arrow[d, equal] & G^0 \arrow[r] \arrow[d, dotted] & G^1 \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & G^{n-1} \arrow[r] \arrow[d, dotted] & G' \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & K' \arrow[r] & F_{n-1} \arrow[r] & F_{n-2} \arrow[r] & \cdots \arrow[r] & F_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

The argument following diagram (3) in the proof of Theorem (2.10) finishes the proof. □

(3.18) **Remark.** As noticed in the proof of Theorem (2.10), the homomorphism $G \twoheadrightarrow M$ in a short exact sequence $0 \to K \to G \to M \to 0$ where $\mathrm{pd}_R K$ is finite, is necessarily a Gorenstein projective precover of M.

But the homomorphism $G \twoheadrightarrow M$ in the exact sequence $0 \to K \to G \to M \to 0$ established above in Lemma (3.17), where $\mathrm{fd}_R K$ is finite, is not necessarily a Gorenstein flat cover of M, since it is not true that $\mathrm{Ext}^1_R(T, K) = 0$ whenever T is Gorenstein flat and $\mathrm{fd}_R K$ is finite.

We make up for this loss in Theorem (3.23). Meanwhile, we have the application below of the simpler Lemma (3.17).

The large restricted flat dimension was investigated in [8, Section 2] and in [7, Chapters 5.3 – 5.4]. It is conjectured by Foxby that if $\mathrm{Gfd}_R M$ is finite, then $\mathsf{Rfd}_R M = \mathrm{Gfd}_R M$. Christensen [7, Theorem (5.4.8)] proves this for local noetherian Cohen–Macaulay rings with a dualizing module. We have the following extension:

(3.19) **Theorem.** *For any R–module M, we have two inequalities,*

$$\mathsf{Rfd}_R M \leqslant \mathrm{Gfd}_R M \leqslant \mathrm{fd}_R M.$$

Now assume that R is commutative and noetherian. If $\mathrm{Gfd}_R M$ is finite, then:

$$\mathsf{Rfd}_R M = \mathrm{Gfd}_R M.$$

If $\mathrm{fd}_R M$ is finite, then we have two equalities:

$$\mathsf{Rfd}_R M = \mathrm{Gfd}_R M = \mathrm{fd}_R M.$$

Proof. The last inequality $\mathrm{Gfd}_R M \leqslant \mathrm{fd}_R M$ is clear. Concerning $\mathsf{Rfd}_R M \leqslant \mathrm{Gfd}_R M$, we may assume that $n = \mathrm{Gfd}_R M$ is finite, and then proceed by induction on $n \geqslant 0$.

If $n = 0$, then M is Gorenstein flat. We wish to prove that $\mathrm{Tor}^R_i(L, M) = 0$ for all $i > 0$, and all right modules L with finite flat dimension. Therefore assume that $\ell = \mathrm{fd}_R L$ is finite. Since M is Gorenstein flat, there exists an exact sequence,

$$0 \to M \to G^0 \to \cdots \to G^{\ell-1} \to T \to 0,$$

where $G^0, \ldots, G^{\ell-1}$ are flats (and T is Gorenstein flat). By this sequence we conclude that $\mathrm{Tor}^R_i(-, M) \cong \mathrm{Tor}^R_{i+\ell}(-, T)$ for all $i > 0$, in particular we get that $\mathrm{Tor}^R_i(L, M) \cong \mathrm{Tor}^R_{i+\ell}(L, T) = 0$ for all $i > 0$, since $i + \ell > \mathrm{fd}_R L$.

Next we assume that $n > 0$. Pick a short exact sequence $0 \to K \to T \to M \to 0$ where T is Gorenstein flat, and $\mathrm{Gfd}_R K = n - 1$. By induction hypothesis we have

$$\mathsf{Rfd}_R K \leqslant \mathrm{Gfd}_R K = n - 1,$$

and hence $\mathrm{Tor}^R_j(L, K) = 0$ for all $j > n - 1$, and all (right) R–modules L with finite flat dimension. For such an L, and an integer $i > n$, we use the long exact sequence,

$$0 = \mathrm{Tor}^R_i(L, T) \to \mathrm{Tor}^R_i(L, M) \to \mathrm{Tor}^R_{i-1}(L, K) = 0,$$

to conclude that $\mathrm{Tor}^R_i(L, M) = 0$. Therefore $\mathsf{Rfd}_R M \leqslant n = \mathrm{Gfd}_R M$.

Now assume that R is commutative and noetherian. If $\mathrm{fd}_R M$ is finite, then [7, Proposition (5.4.2)] implies that $\mathsf{Rfd}_R M = \mathrm{fd}_R M$, and hence also $\mathsf{Rfd}_R M = \mathrm{Gfd}_R M = \mathrm{fd}_R M$.

Next assume that $\mathrm{Gfd}_R M = n$ is finite. We have to prove that $\mathsf{Rfd}_R M \geqslant n$. Naturally we may assume that $n > 0$. By Lemma (3.17) there exists a short exact sequence, say $0 \to K \to T \to M \to 0$, where T is Gorenstein flat and $\mathrm{fd}_R K = n-1$. Since T is Gorenstein flat, we have a short exact sequence $0 \to T \to G \to T' \to 0$ where G is flat and T' is Gorenstein flat. Since $K \subseteq T \subseteq G$, we can consider the residue class module $Q = G/K$.

Because G is flat and $\mathrm{fd}_R K = n - 1$, exactness of $0 \to K \to G \to Q \to 0$ shows that $\mathrm{fd}_R Q \leqslant n$. Note that $M \cong T/K$ is a submodule of $Q = G/K$ with $Q/M \cong (G/K)/(T/K) \cong G/T \cong T'$, and thus we get a short exact sequence $0 \to M \to Q \to T' \to 0$. Since $\mathrm{Gfd}_R M = n$, Theorem (3.14) gives an injective module I with $\mathrm{Tor}_n^R(I, M) \neq 0$. Applying $I \otimes_R -$ to $0 \to M \to Q \to T' \to 0$, we get

$$0 = \mathrm{Tor}_{n+1}^R(I, T') \to \mathrm{Tor}_n^R(I, M) \to \mathrm{Tor}_n^R(I, Q),$$

showing that $\mathrm{Tor}_n^R(I, Q) \neq 0$. Since $\mathrm{Gfd}_R Q \leqslant \mathrm{fd}_R Q \leqslant n < \infty$, Theorem (3.14) gives that $\mathrm{Gfd}_R Q \geqslant n$. Therefore $\mathrm{fd}_R Q = n$, and consequently $\mathsf{Rfd}_R Q = \mathrm{fd}_R Q = n$.

Thus we get the existence of an R–module L with finite flat dimension, such that $\mathrm{Tor}_n^R(L, Q) \neq 0$. Since T' is Gorenstein flat, then $\mathsf{Rfd}_R T' \leqslant 0$, and so the exactness of $\mathrm{Tor}_n^R(L, M) \to \mathrm{Tor}_n^R(L, Q) \to \mathrm{Tor}_n^R(L, T') = 0$ proves that also $\mathrm{Tor}_n^R(L, M) \neq 0$. Hence $\mathsf{Rfd}_R M \geqslant n$, as desired. □

Our next goal is to prove that over a right coherent ring, every (left) module M with finite $\mathrm{Gfd}_R M$, admits a Gorenstein flat precover.

This result can be found in [12, Theorem 3.5] for local noetherian Cohen–Macaulay rings $(R, \mathfrak{m}, k)$, admitting a dualizing module. Actually the proof presented there almost works in the general case, when we use as input the strong results about Gorenstein flat modules from this sections.

(3.20) **Cotorsion modules.** Xu [15, Definition 3.1.1], calls an R–module K for *cotorsion*, if $\mathrm{Ext}_R^1(F, K) = 0$ for all flat R–modules F. In [15, Lemma 2.1.1] it is proved that if $\varphi\colon F \to M$ is a flat cover of any module M, then the kernel $K = \mathrm{Ker}\,\varphi$ is cotorsion. Furthermore, if R is right coherent, and M is a left R–module with finite flat dimension, then M *has* a flat cover by [15, Theorem 3.1.11].

(3.21) **Pure injective modules.** Recall that a short exact sequence,

$$0 \to A \to B \to C \to 0,$$

of (left) modules is called *pure exact* if $0 \to X \otimes A \to X \otimes B \to X \otimes C \to 0$ is exact for every (right) module X. In this case we also say that A is a *pure submodule* of B. A module H is called *pure injective* if the sequence

$$0 \longrightarrow \mathrm{Hom}_R(C, H) \longrightarrow \mathrm{Hom}_R(B, H) \longrightarrow \mathrm{Hom}_R(A, H) \longrightarrow 0$$

is exact for every pure exact sequence $0 \to A \to B \to C \to 0$. By [15, Theorem 2.3.8], every R–module M has a pure injective envelope, denoted $PE(M)$, such that $M \subseteq PE(M)$. If R is right coherent, and F is flat, then both $PE(F)$ and $PE(F)/F$ are flat too, by [15, Lemma 3.1.6]. Also note that every pure injective module is cotorsion.

(3.22) **Proposition.** *Assume that R is right coherent. If T is a Gorenstein flat R–module, then $\mathrm{Ext}_R^i(T,K)=0$ for all integers $i>0$, and all cotorsion R–modules K with finite flat dimension.*

Proof. We use induction on the finite number $\mathrm{fd}_R K = n$. If $n=0$, then K is flat. Consider the Pontryagin duals $K^* = \mathrm{Hom}_{\mathbb{Z}}(K,\mathbb{Q}/\mathbb{Z})$, and $K^{**} = \mathrm{Hom}_{\mathbb{Z}}(K^*,\mathbb{Q}/\mathbb{Z})$. Since R is right coherent, and K^* is injective, then K^{**} is flat, by [15, Lemma 3.1.4]. By [15, Proposition 2.3.5], K is a pure submodule of K^{**}, and hence K^{**}/K is flat. Since K is cotorsion, $\mathrm{Ext}_R^1(K^{**}/K,K)=0$, and consequently,

$$0 \to K \to K^{**} \to K^{**}/K \to 0$$

is split exact. Therefore, K is a direct summand of K^{**}, which implies that $\mathrm{Ext}_R^i(T,K)$ is a direct summand of

$$\mathrm{Ext}_R^i(T,K^{**}) \;\cong\; \mathrm{Ext}_R^i(T,\mathrm{Hom}_{\mathbb{Z}}(K^*,\mathbb{Q}/\mathbb{Z})) \;\cong\; \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}_i^R(K^*,T),\mathbb{Q}/\mathbb{Z}) \;=\; 0,$$

where $\mathrm{Tor}_i^R(K^*,T)=0$, since T is Gorenstein flat, and K^* is injective.

Now assume that $n=\mathrm{fd}_R K>0$. By the remarks (3.20) above, we can pick a short exact sequence $0\to K'\to F\to K\to 0$, where $F\to K$ is a flat cover of K, and K' is cotorsion with $\mathrm{fd}_R K' = n-1$. Since both K' and K are cotorsion, then so is F, by [15, Proposition 3.1.2]. Applying the induction hypothesis, the long exact sequence,

$$0=\mathrm{Ext}_R^i(T,F)\to \mathrm{Ext}_R^i(T,K)\to \mathrm{Ext}_R^{i+1}(T,K')=0,$$

gives the desired conclusion. □

(3.23) **Theorem.** *Assume that R is right coherent ring R, and that M is an R–module with finite Gorenstein flat dimension n. Then M admits a surjective Gorenstein flat precover $\varphi\colon T \twoheadrightarrow M$, where $K=\mathrm{Ker}\,\varphi$ satisfies $\mathrm{fd}_R K = n-1$ (if $n=0$, this should be interpreted as $K=0$).*

In particular, M admits a proper left Gorenstein flat resolution (that is, a proper left $\mathcal{GF}(R)$–resolution) of length n.

Proof. We may assume that $n>0$. By Proposition (3.22), it suffices to construct an exact sequence $0\to K\to T\to M\to 0$ where K is cotorsion with $\mathrm{fd}_R K = n-1$. By Lemma (3.17) there exists a short exact sequence $0\to K'\to T'\to M\to 0$ where T' is Gorenstein flat and $\mathrm{fd}_R K' = n-1$. Since $\mathrm{fd}_R K'$ is finite, then K' has a flat cover by the remarks in (3.20), say $\psi\colon F\twoheadrightarrow K'$, and the kernel $C=\mathrm{Ker}\,\psi$ is cotorsion. Now consider the pushout diagram,

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d] & K' \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & C \arrow[r, dotted] & PE(F) \arrow[r, dotted] \arrow[d] & K \arrow[r] \arrow[d, dotted] & 0 \\
 & & PE(F)/F \arrow[r, equal] \arrow[d] & PE(F)/F \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

In the sequence $0 \to C \to PE(F) \to K \to 0$, both C and $PE(F)$ are cotorsion, and hence also K is cotorsion by [15, Proposition 3.1.2]. Furthermore, since $PE(F)/F$ is flat, the short exact sequence $0 \to K' \to K \to PE(F)/F \to 0$ shows that

$$\mathrm{fd}_R K \;=\; \mathrm{fd}_R K' \;=\; n-1.$$

Finally we consider the pushout diagram,

(6)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K' \arrow[r] \arrow[d] & T' \arrow[r] \arrow[d, dotted] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r, dotted] \arrow[d] & T \arrow[r, dotted] \arrow[d, dotted] & M \arrow[r] & 0 \\
 & PE(F)/F \arrow[r, equal] \arrow[d] & PE(F)/F \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

In the second column in (6), both T' and $PE(F)/F$ are Gorenstein flat, and hence also T is Gorenstein flat, since the class $\mathcal{GF}(R)$ is projectively resolving by Theorem (3.7). Therefore the lower row in diagram (6), $0 \to K \to T \to M \to 0$, is the desired sequence. $\square$

Finally we may compare the (*left*) *finitistic Gorenstein flat dimension* of the base ring R, defined by

$$\mathrm{FGFD}(R) \;=\; \sup\left\{ \mathrm{Gpd}_R M \;\middle|\; \begin{array}{l} M \text{ is a (left) } R\text{–module with} \\ \text{finite Gorenstein flat dimension.} \end{array} \right\},$$

with the usual (left) finitistic flat dimension, $\mathrm{FFD}(R)$.

(3.24) **Theorem.** *If R is right coherent, then* $\mathrm{FGFD}(R) = \mathrm{FFD}(R)$.

Proof. Analogous to the proof of Theorem (2.28), using Proposition (3.12) instead of (2.18), and Theorem (3.23) above instead of (2.10). □

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Part II

Gorenstein derived functors

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GORENSTEIN DERIVED FUNCTORS

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ABSTRACT. Over any associative ring R it is standard to derive $\mathrm{Hom}_R(-,-)$ using projective resolutions in the first variable, or injective resolutions in the second variable, and doing this, one obtains $\mathrm{Ext}^n_R(-,-)$ in both cases. We examine the situation, where projective and injective modules are replaced by Gorenstein projective and Gorenstein injective ones, respectively. Furthermore we derive the tensor product $-\otimes_R-$ using Gorenstein flat modules.

1. INTRODUCTION

When R is a two-sided Noetherian ring, Auslander and Bridger [2] introduced in 1969 the G-dimension, $\mathrm{G\text{-}dim}_R M$, for every *finite*, that is, finitely generated R-module M. They proved the inequality $\mathrm{G\text{-}dim}_R M \leqslant \mathrm{pd}_R M$, with equality $\mathrm{G\text{-}dim}_R M = \mathrm{pd}_R M$ when $\mathrm{pd}_R M < \infty$, along with a generalized Auslander-Buchsbaum formula (sometimes known as the Auslander-Bridger formula) for the G-dimension.

The (finite) modules with G-dimension zero are called *Gorenstein projectives*. Over a general ring R, Enochs and Jenda defined in [6] Gorenstein projective modules. Avramov, Buchweitz, Martsinkovsky and Reiten prove that if R is two-sided Noetherian, and G is a finite Gorenstein projective module, then the new definition agrees with that of Auslander and Bridger, see the remark following [4, Theorem (4.2.6)]. Using Gorenstein projective modules, one can introduce the Gorenstein projective dimension for arbitrary R-modules. At this point we need to introduce:

(1.1) **Notation.** Throughout this paper, we use the following notation:

- R is an associative ring. All modules are—if not specified otherwise—*left* R-modules, and the category of all R-modules is denoted $\mathcal{M}$. We use $\mathcal{A}$ for the category of abelian groups (that is, $\mathbb{Z}$-modules).

- We use $\mathcal{GP}$, $\mathcal{GI}$ and $\mathcal{GF}$ for the categories of *Gorenstein projective*, *Gorenstein injective* and *Gorenstein flat* R-modules; please see [6] and [8], or Definition (2.7) below.

- Furthermore, for each R-module M we write $\mathrm{Gpd}_R M$, $\mathrm{Gid}_R M$ and $\mathrm{Gfd}_R M$ for the Gorenstein projective, Gorenstein injective, and Gorenstein flat dimension of M, respectively.

Now, given our base ring R, the usual right derived functors $\mathrm{Ext}^n_R(-,-)$ of $\mathrm{Hom}_R(-,-)$ are important in homological studies of R. The material presented here deals with the Gorenstein right derived functors, $\mathrm{Ext}^n_{\mathcal{GP}}(-,-)$ and $\mathrm{Ext}^n_{\mathcal{GI}}(-,-)$, of $\mathrm{Hom}_R(-,-)$.

More precisely, let N be a fixed R-module. For an R-module M which has a *proper left $\mathcal{GP}$-resolution* $\boldsymbol{G} = \cdots \to G_1 \to G_0 \to 0$ (please see (2.1) below for the definition of proper resolutions), we define
$$\mathrm{Ext}^n_{\mathcal{GP}}(M,N) \;:=\; \mathrm{H}^n(\mathrm{Hom}_R(\boldsymbol{G},N)).$$

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From (2.4) it will follow that $\mathrm{Ext}^n_{\mathcal{GP}}(-,N)$ is a well-defined contravariant functor, defined on the full subcategory, $\mathsf{LeftRes}_{\mathcal{M}}(\mathcal{GP})$, of $\mathcal{M}$, consisting of all R-modules which have a proper left $\mathcal{GP}$-resolution.

For fixed R-module M', there is a similar definition of the functor $\mathrm{Ext}^n_{\mathcal{GI}}(M',-)$, which is defined on the full subcategory, $\mathsf{RightRes}_{\mathcal{M}}(\mathcal{GI})$, of $\mathcal{M}$, consisting of all R-modules which have a proper right $\mathcal{GI}$-resolution. Now, the best one could *hope* for is the existence of isomorphisms,

$$\mathrm{Ext}^n_{\mathcal{GP}}(M,N) \;\cong\; \mathrm{Ext}^n_{\mathcal{GI}}(M,N),$$

which are functorial in each variable $M \in \mathsf{LeftRes}_{\mathcal{M}}(\mathcal{GP})$ and $N \in \mathsf{RightRes}_{\mathcal{M}}(\mathcal{GI})$. The aim of this paper, is to show a slightly weaker result.

When R is n-Gorenstein (meaning that R is both left and right Noetherian, with self-injective dimension $\leqslant n$ from both sides), Enochs and Jenda [9, Theorem 12.1.4] have proved the exististence of such functorial isomorphisms $\mathrm{Ext}^n_{\mathcal{GP}}(M,N) \cong \mathrm{Ext}^n_{\mathcal{GI}}(M,N)$, for all R-modules M and N.

It is important to note that for an n-Gorenstein ring R, we have $\mathrm{Gpd}_R M < \infty$, $\mathrm{Gid}_R M < \infty$, and also $\mathrm{Gfd}_R M < \infty$ for all R-modules M; please see [9, Theorems 11.2.1, 11.5.1, 11.7.6]. For any ring R, [12, Theorem 2.15] (which is restated in this paper as Proposition (3.1)) implies that the category $\mathsf{LeftRes}_{\mathcal{M}}(\mathcal{GP})$ contains all R-modules M with $\mathrm{Gpd}_R M < \infty$, that is, every R-module with finite G-projective dimension has a proper left $\mathcal{GP}$-resolution. Also every R-module with finite G-injective dimension has a proper right $\mathcal{GI}$-resolution, so $\mathsf{RightRes}_{\mathcal{M}}(\mathcal{GI})$ contains all R-modules N with $\mathrm{Gid}_R N < \infty$.

Theorem (3.6) in this text, proves that the functorial isomorphisms $\mathrm{Ext}^n_{\mathcal{GP}}(M,N) \cong \mathrm{Ext}^n_{\mathcal{GI}}(M,N)$ holds over *arbitrary* rings R, provided that $\mathrm{Gpd}_R M < \infty$ and $\mathrm{Gid}_R N < \infty$. By the remarks above, this result generalizes that of Enochs and Jenda.

Furthermore, Theorems (4.8) and (4.10) give similar results about the Gorenstein left derived of the tensor product $-\otimes_R -$, using proper left $\mathcal{GP}$-resolutions and proper left $\mathcal{GF}$-resolutions. This has also been proved by Enochs and Jenda [9, Theorem 12.2.2] in the case when R is n-Gorenstein.

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2. Preliminaries

Let $T\colon \mathcal{C} \to \mathcal{E}$ be any additive functor between abelian categories. One usually derives T using resolutions consisting of projective or injective objects (if the category $\mathcal{C}$ has enough projectives or injectives). This section is a very brief note on how to derive functors T with resolutions consisting of objects in some subcategory $\mathcal{X} \subseteq \mathcal{C}$. The general discussion presented here will enable us to give very short proofs of the main theorems in the next section.

(2.1) **Proper Resolutions.** Let $\mathcal{X} \subseteq \mathcal{C}$ be a full subcategory. A *proper left $\mathcal{X}$-resolution* of $M \in \mathcal{C}$ is a complex $\boldsymbol{X} = \cdots \to X_1 \to X_0 \to 0$ where $X_i \in \mathcal{X}$, together with a morphism $X_0 \to M$, such that $\boldsymbol{X}^+ := \cdots \to X_1 \to X_0 \to M \to 0$ is also a complex, and such that the sequence

$$\mathrm{Hom}_{\mathcal{C}}(X,\boldsymbol{X}^+) \;=\; \cdots \to \mathrm{Hom}_{\mathcal{C}}(X,X_1) \to \mathrm{Hom}_{\mathcal{C}}(X,X_0) \to \mathrm{Hom}_{\mathcal{C}}(X,M) \to 0$$

is exact for every $X \in \mathcal{X}$. We sometimes refer to $\boldsymbol{X}^+ = \cdots \to X_1 \to X_0 \to M \to 0$ as an *augmented* proper left $\mathcal{X}$-resolution. We do not require that $\boldsymbol{X}^+$ itself is exact. Furthermore, we use

$$\mathsf{LeftRes}_{\mathcal{C}}(\mathcal{X})$$

to denote the full subcategory of $\mathcal{C}$, consisting of those objects which have a proper left $\mathcal{X}$-resolution. Note that $\mathcal{X}$ is a subcategory of $\mathsf{LeftRes}_{\mathcal{C}}(\mathcal{X})$.

Proper right $\mathcal{X}$-resolutions are defined dually, and the full subcategory of $\mathcal{C}$, consisting of those objects which have a proper right $\mathcal{X}$-resolution, is denoted $\mathsf{RightRes}_{\mathcal{C}}(\mathcal{X})$.

The importance of working with *proper* resolutions comes from the following:

(2.2) **Proposition.** *Let $f\colon M \to M'$ be a morphism in $\mathcal{C}$, and consider the diagram*

$$\begin{tikzcd} \cdots \arrow[r] & X_2 \arrow[r] \arrow[d, dotted, "f_2"] & X_1 \arrow[r] \arrow[d, dotted, "f_1"] & X_0 \arrow[r] \arrow[d, dotted, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\ \cdots \arrow[r] & X_2' \arrow[r] & X_1' \arrow[r] & X_0' \arrow[r] & M' \arrow[r] & 0 \end{tikzcd}$$

where the upper row is a complex with $X_n \in \mathcal{X}$ for all $n \geqslant 0$, and the lower row is an augmented proper left $\mathcal{X}$-resolution of M'. Then the following conclusions hold:

(*i*) *There exists morphisms $f_n\colon X_n \to X_n'$ for all $n \geqslant 0$, making the diagram above commutative. The chain map $\{f_n\}_{n\geqslant 0}$ is called a lift of f.*

(*ii*) *If $\{f_n'\}_{n\geqslant 0}$ is another lift of f, then the chain maps $\{f_n\}_{n\geqslant 0}$ and $\{f_n'\}_{n\geqslant 0}$ are homotopic.*

Proof. The proof is an exercise, please see [9, Exercise 8.1.2]. □

(2.3) **Remark.** A few comments are in order:

- In our applications, the class $\mathcal{X}$ contains all projectives. Consequently, all the augmented proper left $\mathcal{X}$-resolutions occuring in this paper will be exact. Also all augmented proper right $\mathcal{Y}$-resolutions will be exact, when $\mathcal{Y}$ is a class of R-modules containing all injectives.

- Recall (see [15, Definition 1.2.2]) that an *$\mathcal{X}$-precover* of $M \in \mathcal{C}$ is a morphism $\varphi\colon X \to M$, where $X \in \mathcal{X}$, such that the sequence

$$\mathrm{Hom}_{\mathcal{C}}(X', X) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(X',\varphi)} \mathrm{Hom}_{\mathcal{C}}(X', M) \longrightarrow 0$$

  is exact for every $X' \in \mathcal{X}$. Hence, in an augmented proper left $\mathcal{X}$-resolution $\boldsymbol{X}^+$ of M, the morphisms $X_{i+1} \to \mathrm{Ker}(X_i \to X_{i-1})$, $i > 0$, and $X_0 \to M$ are $\mathcal{X}$-precovers.

- What we have called *proper* $\mathcal{X}$-resolutions, Enochs and Jenda [9, Definition 8.1.2] simply denote *$\mathcal{X}$-resolutions*. We have adopted the terminology *proper* from [3, Section 4].

(2.4) **Derived Functors.** Consider an additive functor $T\colon \mathcal{C} \to \mathcal{E}$ between abelian categories. Let us assume that T is covariant, say. Then (as usual) we can define the n^{th} left derived functor

$$\mathrm{L}_n^{\mathcal{X}}T\colon \mathsf{LeftRes}_{\mathcal{C}}(\mathcal{X}) \to \mathcal{E}$$

of T, with respect to the class $\mathcal{X}$, by setting $\mathrm{L}_n^{\mathcal{X}}T(M) = \mathrm{H}_n(T(\boldsymbol{X}))$, where $\boldsymbol{X}$ is any proper left $\mathcal{X}$-resolution of $M \in \mathsf{LeftRes}_{\mathcal{C}}(\mathcal{X})$. Similarly, the n^{th} right derived functor

$$\mathrm{R}^n_{\mathcal{X}}T\colon \mathsf{RightRes}_{\mathcal{C}}(\mathcal{X}) \to \mathcal{E}$$

of T with respect to $\mathcal{X}$, is defined by $\mathrm{R}^n_{\mathcal{X}}T(N) = \mathrm{H}_n(T(\boldsymbol{Y}))$, where $\boldsymbol{Y}$ is any proper right $\mathcal{X}$-resolution of $N \in \mathsf{RightRes}_{\mathcal{C}}(\mathcal{X})$. These constructions are well-defined and functorial in the arguments M and N by Proposition (2.2).

The situation where T is contravariant is handled similarly. We refer to [9, Section 8.2] for a more detailed discussion on this matter.

(2.5) **Balanced Functors.** Next we consider yet another abelian category $\mathcal{D}$, together with a full subcategory $\mathcal{Y} \subseteq \mathcal{D}$ and an additive functor $F\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ in *two* variables. We will assume that F is contravariant in the first variable, and covariant in the second variable.

Actually, the variance of the variables of F is not important, and the definitions and results below can easily be modified to fit the situation, where F is covariant in both variables, say.

For fixed $M \in \mathcal{C}$ and $N \in \mathcal{D}$ we can then consider the two right derived functors as in (2.4):

$$\mathrm{R}^n_{\mathcal{X}}F(-,N)\colon \mathsf{LeftRes}_{\mathcal{C}}(\mathcal{X}) \to \mathcal{E} \qquad \text{and} \qquad \mathrm{R}^n_{\mathcal{Y}}F(M,-)\colon \mathsf{RightRes}_{\mathcal{D}}(\mathcal{Y}) \to \mathcal{E}.$$

If furthermore $M \in \mathsf{LeftRes}_{\mathcal{C}}(\mathcal{X})$ and $N \in \mathsf{RightRes}_{\mathcal{D}}(\mathcal{Y})$, we can ask for a sufficient condition to ensure that

$$\mathrm{R}^n_{\mathcal{X}}F(M,N) \;\cong\; \mathrm{R}^n_{\mathcal{Y}}F(M,N),$$

functorial in M and N. Here we have written $\mathrm{R}^n_{\mathcal{X}}F(M,N)$ for the functor $\mathrm{R}^n_{\mathcal{X}}F(-,N)$ applied to M. Another, and perhaps better, notation could be

$$\mathrm{R}^n_{\mathcal{X}}F(-,N)[M].$$

Enochs and Jenda have in [5] developed a machinery for answering such questions. They operate with the term *left/right balanced functor* (hence the headline), which we will not define here (but the reader might consult [5, Definition 2.1]). Instead we shall focus on the following result:

(2.6) **Theorem.** *Consider the functor $F\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which is contravariant in the first variable and covariant in the second variable, together with the full subcategories $\mathcal{X} \subseteq \mathcal{C}$ and $\mathcal{Y} \subseteq \mathcal{D}$. Assume that we have full subcategories $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{Y}}$ of $\mathsf{LeftRes}_{\mathcal{C}}(\mathcal{X})$ and $\mathsf{RightRes}_{\mathcal{D}}(\mathcal{Y})$, respectively, satisfying:*

- *(i)* $\mathcal{X} \subseteq \widetilde{\mathcal{X}}$ *and* $\mathcal{Y} \subseteq \widetilde{\mathcal{Y}}$.
- *(ii) Every $M \in \widetilde{\mathcal{X}}$ has an augmented proper left $\mathcal{X}$-resolution $\cdots \to X_1 \to X_0 \to M \to 0$, such that $0 \to F(M,Y) \to F(X_0,Y) \to F(X_1,Y) \to \cdots$ is exact for all $Y \in \mathcal{Y}$.*
- *(iii) Every $N \in \widetilde{\mathcal{Y}}$ has an augmented proper right $\mathcal{Y}$-resolution $0 \to N \to Y^0 \to Y^1 \to \cdots$, such that $0 \to F(X,N) \to F(X,Y^0) \to F(X,Y^1) \to \cdots$ is exact for all $X \in \mathcal{X}$.*

Then we have functorial isomorphisms

$$\mathrm{R}^n_{\mathcal{X}}F(M,N) \;\cong\; \mathrm{R}^n_{\mathcal{Y}}F(M,N),$$

for all $M \in \widetilde{\mathcal{X}}$ and $N \in \widetilde{\mathcal{Y}}$.

Proof. Please see [5, Proposition 2.3]. That the isomorphisms are functorial follows from the construction. The functoriality becomes more clear if one consults the proof of [9, Proposition 8.2.14], or the proofs of [14, Theorems 2.7.2 and 2.7.6]. $\square$

In the next paragraphs we apply the results above to special categories $\mathcal{X}$, $\widetilde{\mathcal{X}}$, $\mathcal{C}$ and $\mathcal{Y}$, $\widetilde{\mathcal{Y}}$, $\mathcal{D}$, including the categories mentioned in (1.1). For completeness we include a definition of Gorenstein projective, Gorenstein injective and Gorenstein flat modules:

(2.7) **Definition.** A *complete projective resolution* is an exact sequence of projective modules,

$$\boldsymbol{P} \;=\; \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots\,,$$

such that $\mathrm{Hom}_R(\boldsymbol{P},Q)$ is exact for every projective R-module Q. An R-module M is called *Gorenstein projective* (*G-projective* for short), if there exists a complete projective resolution $\boldsymbol{P}$ with $M \cong \mathrm{Im}(P_0 \to P_{-1})$. *Gorenstein injective* (*G-injective* for short) modules are defined dually.

A *complete flat resolution* is an exact sequence of flat (left) R-modules,

$$\boldsymbol{F} \;=\; \cdots \to F_1 \to F_0 \to F_{-1} \to \cdots\,,$$

such that $I \otimes_R \boldsymbol{F}$ is exact for every injective right R-module I. An R-module M is called *Gorenstein flat* (*G-flat* for short), if there exists a complete flat resolution $\boldsymbol{F}$ with $M \cong \mathrm{Im}(F_0 \to F_{-1})$.

3. Gorenstein Deriving $\mathrm{Hom}_R(-,-)$

We now return to categories of *modules.* We use $\widetilde{\mathcal{GP}}$, $\widetilde{\mathcal{GI}}$ and $\widetilde{\mathcal{GF}}$ to denote the class of R-modules with finite Gorenstein projective dimension, finite Gorenstein injective dimension, and finite Gorenstein flat dimension, respectively.

Recall that every projective module is Gorenstein projective. Consequently, $\mathcal{GP}$-precovers are always surjective, and $\widetilde{\mathcal{GP}}$ contains all modules with finite projective dimension.

We now consider the functor $\mathrm{Hom}_R(-,-)\colon \mathcal{M} \times \mathcal{M} \to \mathcal{A}$, together with the categories

$$\mathcal{X} \;=\; \mathcal{GP}\;,\; \widetilde{\mathcal{X}} \;=\; \widetilde{\mathcal{GP}} \qquad \text{and} \qquad \mathcal{Y} \;=\; \mathcal{GI}\;,\; \widetilde{\mathcal{Y}} \;=\; \widetilde{\mathcal{GI}}.$$

In this case we define, in the sense of Derived Functors (2.4),

$$\mathrm{Ext}^n_{\mathcal{GP}}(-,N) \;=\; \mathrm{R}^n_{\mathcal{GP}}\mathrm{Hom}_R(-,N) \qquad \text{and} \qquad \mathrm{Ext}^n_{\mathcal{GI}}(M,-) \;=\; \mathrm{R}^n_{\mathcal{GI}}\mathrm{Hom}_R(M,-),$$

for fixed R-modules M and N. We wish, of course, to apply Theorem (2.6) to this situation. Note that by [12, Theorem 2.15], we have:

(3.1) **Proposition.** *If M is an R-module with $\mathrm{Gpd}_R M < \infty$, then there exists a short exact sequence $0 \to K \to G \to M \to 0$, where $G \to M$ is a $\mathcal{GP}$-precover of M (please see Remark (2.3)), and $\mathrm{pd}_R K = \mathrm{Gpd}_R M - 1$ (in the case where M is Gorenstein projective, this should be interpreted as $K = 0$).*

Consequently, every R-module with finite Gorenstein projective dimension has a proper left $\mathcal{GP}$-resolution (that is, there is an inclusion $\widetilde{\mathcal{GP}} \subseteq \mathsf{LeftRes}_{\mathcal{M}}(\mathcal{GP})$).

Furthermore we will need the following from [12, Theorem 2.22]:

(3.2) **Theorem.** *Let M be any R-module with $\mathrm{Gpd}_R M < \infty$. Then*

$$\mathrm{Gpd}_R M \;=\; \{\; n \geqslant 0 \mid \mathrm{Ext}^n_R(M,L) \neq 0 \textit{ for some } R\textit{-module } L \textit{ with } \mathrm{pd}_R L < \infty \;\}.$$

(3.3) **Remark.** I may be useful to compare Theorem (3.2) to the classical projective dimension, which for an R-module M is given by:

$$\mathrm{pd}_R M \;=\; \{\; n \geqslant 0 \mid \mathrm{Ext}^n_R(M,L) \neq 0 \text{ for some } R\text{-module } L \;\}.$$

It also follows that if $\mathrm{pd}_R M < \infty$, then every projective resolution of M is actually a proper left $\mathcal{GP}$-resolution of M.

(3.4) **Lemma.** *Assume that M is an R-module with finite Gorenstein projective dimension, and let $\boldsymbol{G}^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $\mathcal{GP}$-resolution of M (which exists by Proposition (3.1)). Then $\mathrm{Hom}_R(\boldsymbol{G}^+, H)$ is exact for all Gorenstein injective modules H.*

Proof. We split the proper resolution $\boldsymbol{G}^+$ into short exact sequences. Hence it suffices to show exactness of $\mathrm{Hom}_R(\boldsymbol{S}, H)$ for all Gorenstein injective modules H, and all short exact sequences

$$\boldsymbol{S} = 0 \to K \to G \to M \to 0\;,$$

where $G \to M$ is a $\mathcal{GP}$-precover of some module M with $\mathrm{Gpd}_R M < \infty$ (recall that $\mathcal{GP}$-precovers are always surjective). By Proposition (3.1), there is a special short exact sequence,

$$\boldsymbol{S}' = \ 0 \longrightarrow K' \xrightarrow{\ \iota\ } G' \xrightarrow{\ \pi\ } M \longrightarrow 0\,,$$

where $\pi\colon G' \to M$ is a $\mathcal{GP}$-precover and $\mathrm{pd}_R K' < \infty$.

It is easy to see (as in Proposition (2.2)) that the complexes $\boldsymbol{S}$ and $\boldsymbol{S}'$ are homotopy equivalent, and thus so are the complexes $\mathrm{Hom}_R(\boldsymbol{S}, H)$ and $\mathrm{Hom}_R(\boldsymbol{S}', H)$ for every (Gorenstein injective) module H. Hence it suffices to show the exactness of $\mathrm{Hom}_R(\boldsymbol{S}', H)$, whenever H is Gorenstein injective.

Now let H be any Gorenstein injective module. We need to prove the exactness of

$$\mathrm{Hom}_R(G', H) \xrightarrow{\mathrm{Hom}_R(\iota, H)} \mathrm{Hom}_R(K', H) \longrightarrow 0\,.$$

To see this, let $\alpha\colon K' \to H$ be any homomorphism. We wish to find $\varrho\colon G' \to H$ such that $\varrho\iota = \alpha$. Now pick an exact sequence

$$0 \longrightarrow \widetilde{H} \longrightarrow E \xrightarrow{\ g\ } H \longrightarrow 0\,,$$

where E is injective, and $\widetilde{H}$ is Gorenstein injective (the sequence in question is just a part of the complete injective resolution which defines H). Since $\widetilde{H}$ is Gorenstein injective and $\mathrm{pd}_R K' < \infty$, we get $\mathrm{Ext}^1_R(K', \widetilde{H}) = 0$ by [7, Lemma 1.3], and thus a lifting $\varepsilon\colon K' \to E$ with $g\varepsilon = \alpha$.

$$\begin{tikzcd} & K' \arrow[r, "\iota"] \arrow[dl, "\alpha"'] \arrow[d, dotted, "\varepsilon"] & G' \arrow[dl, dotted, "\widetilde{\varepsilon}"] \\ H & E \arrow[l, "g"] & \end{tikzcd}$$

Next, injectivity of E gives $\widetilde{\varepsilon}\colon G' \to E$ with $\widetilde{\varepsilon}\iota = \varepsilon$. Now $\varrho = g\widetilde{\varepsilon}\colon G' \to H$ is the desired map. $\square$

With a similar proof we get:

(3.5) **Lemma.** *Assume that N is an R-module with finite Gorenstein injective dimension, and let $\boldsymbol{H}^+ = 0 \to N \to H^0 \to H^1 \to \cdots$ be an augmented proper right $\mathcal{GI}$-resolution of N (which exists by the dual of Proposition (3.1)). Then $\mathrm{Hom}_R(G, \boldsymbol{H}^+)$ is exact for all Gorenstein projective modules G.* $\square$

Comparing Proposition (3.4) and (3.5) with Theorem (2.6), we obtain:

(3.6) **Theorem.** *For all R-modules M and N with $\mathrm{Gpd}_R M < \infty$ and $\mathrm{Gid}_R N < \infty$, we have isomorphisms*

$$\mathrm{Ext}^n_{\mathcal{GP}}(M, N) \ \cong\ \mathrm{Ext}^n_{\mathcal{GI}}(M, N),$$

which are functorial in M and N. $\square$

(3.7) **Definition of GExt.** Let M and N be R-modules with $\mathrm{Gpd}_R M < \infty$ and $\mathrm{Gid}_R N < \infty$. Then we write

$$\mathrm{GExt}^n_R(M, N) \ :=\ \mathrm{Ext}^n_{\mathcal{GP}}(M, N) \ \cong\ \mathrm{Ext}^n_{\mathcal{GI}}(M, N)$$

for the isomorphic abelian groups in Theorem (3.6) above.

Naturally we want to compare GExt with the classical Ext. This is done in:

(3.8) **Theorem.** *Let M and N be any R-modules. Then the following conclusions hold:*

(*i*) *There are natural isomorphisms* $\mathrm{Ext}^n_{\mathcal{GP}}(M,N) \cong \mathrm{Ext}^n_R(M,N)$ *under each of the conditions*

$$(\dagger)\ \ \mathrm{pd}_R M < \infty \qquad \text{or} \qquad (\ddagger)\ \ M \in \mathsf{LeftRes}_{\mathcal{M}}(\mathcal{GP}) \ \text{ and } \ \mathrm{id}_R N < \infty.$$

(*ii*) *There are natural isomorphisms* $\mathrm{Ext}^n_{\mathcal{GI}}(M,N) \cong \mathrm{Ext}^n_R(M,N)$ *under each of the conditions*

$$(\dagger)\ \ \mathrm{id}_R N < \infty \qquad \text{or} \qquad (\ddagger)\ \ N \in \mathsf{RightRes}_{\mathcal{M}}(\mathcal{GI}) \ \text{ and } \ \mathrm{pd}_R M < \infty.$$

(*iii*) *Assume that* $\mathrm{Gpd}_R M < \infty$ *and* $\mathrm{Gid}_R N < \infty$*. If either* $\mathrm{pd}_R M < \infty$ *or* $\mathrm{id}_R N < \infty$*, then*

$$\mathrm{GExt}^n_R(M,N) \cong \mathrm{Ext}^n_R(M,N)$$

functorial in M *and* N*.*

Proof. (*i*) Assume that $\mathrm{pd}_R M < \infty$, and pick any projective resolution $\boldsymbol{P}$ of M. By Remark (3.3), $\boldsymbol{P}$ is also a proper left $\mathcal{GP}$-resolution of M, and thus

$$\mathrm{Ext}^n_{\mathcal{GP}}(M,N) \ = \ \mathrm{H}^n(\mathrm{Hom}_R(\boldsymbol{P},N)) \ = \ \mathrm{Ext}^n_R(M,N).$$

In the case where $M \in \mathsf{LeftRes}_{\mathcal{M}}(\mathcal{GP})$ and $\mathrm{id}_R N = m < \infty$, we see that Gorenstein projective modules are acyclic for the functor $\mathrm{Hom}_R(-,N)$, that is, $\mathrm{Ext}^i_R(G,N) = 0$ (the usual Ext) for every Gorenstein projective module G, and every integer $i > 0$.

This is because, if G is a Gorenstein projective module, and $i > 0$ is an integer, then there exists an exact sequence $0 \to G \to Q^0 \to \cdots \to Q^{m-1} \to C \to 0$, where $Q^0, \ldots, Q^{m-1}$ are projective modules. Breaking this exact sequence into short exact ones, and applying $\mathrm{Hom}_R(-,N)$, we get $\mathrm{Ext}^i_R(G,N) \cong \mathrm{Ext}^{m+i}_R(C,N) = 0$, as claimed.

Therefore [11, Proposition 1.2A] implies that $\mathrm{Ext}^n_R(-,N)$ can be computed using (proper) left Gorenstein projective resolutions of the argument in the first variable, as desired.

The proof of (*ii*) is similar. The claim (*iii*) is a direct consequence of (*i*) and (*ii*), together with the Definition (3.7) of $\mathrm{GExt}^n_R(-,-)$. $\square$

4. Gorenstein Deriving $- \otimes_R -$

Dealing with the tensor product we need of course both left and right R-modules. Thus the following addition to Notation (1.1) is needed:

If $\mathcal{C}$ *is any of the categories in Notation (1.1) (*$\mathcal{M}$*,* $\mathcal{GP}$*, etc.), we write* ${}_R\mathcal{C}$*, respectively,* $\mathcal{C}_R$*, for the category of left, respectively, right,* R*-modules with the property describing the modules in* $\mathcal{C}$*.*

Now we consider the functor $- \otimes_R -\colon \mathcal{M}_R \times {}_R\mathcal{M} \to \mathcal{A}$. For fixed $M \in \mathcal{M}_R$ and $N \in {}_R\mathcal{M}$ we define, in the sense of Derived Functors (2.4):

$$\mathrm{Tor}^{\mathcal{GP}_R}_n(-,N) \ := \ \mathrm{L}^{\mathcal{GP}_R}_n(- \otimes_R N) \qquad \text{and} \qquad \mathrm{Tor}^{{}_R\mathcal{GP}}_n(M,-) \ := \ \mathrm{L}^{{}_R\mathcal{GP}}_n(M \otimes_R -),$$

together with

$$\mathrm{Tor}^{\mathcal{GF}_R}_n(-,N) \ := \ \mathrm{L}^{\mathcal{GF}_R}_n(- \otimes_R N) \qquad \text{and} \qquad \mathrm{Tor}^{{}_R\mathcal{GF}}_n(M,-) \ := \ \mathrm{L}^{{}_R\mathcal{GF}}_n(M \otimes_R -).$$

The first two Tor*s* uses proper left Gorestein *projective* resolutions, and the last two Tor*s* uses proper left Gorenstein *flat* resolutions. In order to compare these different Tor*s*, we wish, of course, to apply (a version of) Theorem (2.6) to different combinations of

$$(\mathcal{X}\ ,\ \widetilde{\mathcal{X}}) \ = \ (\mathcal{GP}_R\ ,\ \widetilde{\mathcal{GP}}_R) \quad \text{or} \quad (\mathcal{GF}_R\ ,\ \widetilde{\mathcal{GF}}_R),$$

and

$$(\mathcal{Y}\ ,\ \widetilde{\mathcal{Y}}) \ = \ ({}_R\mathcal{GP}\ ,\ {}_R\widetilde{\mathcal{GP}}) \quad \text{or} \quad ({}_R\mathcal{GF}\ ,\ {}_R\widetilde{\mathcal{GF}}).$$

Namely the covariant-covariant version of Theorem (2.6), instead of the stated contravariant-covariant version. We will need the classical notion:

(4.1) **Definition.** The *left finitistic projective dimension* $\mathsf{LeftFPD}(R)$ of R is defined as

$$\mathsf{LeftFPD}(R) = \sup\{ \operatorname{pd}_R M \mid M \text{ is a } \textit{left } R\text{-module with } \operatorname{pd}_R M < \infty \}.$$

The right finitistic projective dimension $\mathsf{RightFPD}(R)$ of R is defined similarly.

(4.2) **Remark.** When R is commutative and Noetherian, $\mathsf{LeftFPD}(R)$ and $\mathsf{RightFPD}(R)$ equals the Krull dimension of R, by [10, Théorème (3.2.6) (Seconde partie)].

We will need the following three results, [12, Proposition 3.4], [12, Theorems 3.6 and 3.23], respectively:

(4.3) **Proposition.** *If R is right coherent with finite $\mathsf{LeftFPD}(R)$, then every Gorenstein projective left R-module is also Gorenstein flat. That is, there is an inclusion ${}_R\mathcal{GP} \subseteq {}_R\mathcal{GF}$.* □

(4.4) **Theorem.** *For any left R-module M, we consider the following three conditions:*

- (i) *The left R-module M is G-flat.*
- (ii) *The Pontryagin dual $\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ (which is a right R-module) is G-injective.*
- (iii) *M has an augmented proper right resolution $0 \to M \to F^0 \to F^1 \to \cdots$ consisting of flat left R-modules, and $\operatorname{Tor}_i^R(I, M) = 0$ for all injective right R-modules I, and all $i > 0$.*

The implication $(i) \Rightarrow (ii)$ always holds. If R is right coherent, then also $(ii) \Rightarrow (iii) \Rightarrow (i)$, and hence all three conditions are equivalent. □

(4.5) **Proposition.** *Assume that R is right coherent. If M is a left R-module with $\operatorname{Gfd}_R M < \infty$, then there exists a short exact sequence $0 \to K \to G \to M \to 0$, where $G \to M$ is a ${}_R\mathcal{GF}$-precover of M, and $\operatorname{fd}_R K = \operatorname{Gfd}_R M - 1$ (in the case where M is Gorenstein flat, this should be interpreted as $K = 0$).*

In particular, every left R-module with finite Gorenstein flat dimension has a proper left ${}_R\mathcal{GF}$-resolution (that is, there is an inclusion $\widetilde{{}_R\mathcal{GF}} \subseteq \mathsf{LeftRes}_{{}_R\mathcal{M}}({}_R\mathcal{GF})$). □

Our first result is:

(4.6) **Lemma.** *Let M be a left R-module with $\operatorname{Gpd}_R M < \infty$, and let $\boldsymbol{G}^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left ${}_R\mathcal{GP}$-resolution of M (which exists by Proposition (3.1)). Then the following conclusions hold:*

- (i) *$T \otimes_R \boldsymbol{G}^+$ is exact for all Gorenstein flat right R-modules T.*
- (ii) *If R is left coherent with finite $\mathsf{RightFPD}(R)$, then $T \otimes_R \boldsymbol{G}^+$ is exact for all Gorenstein projective right R-modules T.*

Proof. (i) By Theorem (4.4) above, the Pontryagin dual $H = \operatorname{Hom}_{\mathbb{Z}}(T, \mathbb{Q}/\mathbb{Z})$ is a Gorenstein injective left R-module. Hence $\operatorname{Hom}_R(\boldsymbol{G}^+, H) \cong \operatorname{Hom}_{\mathbb{Z}}(T \otimes_R \boldsymbol{G}^+, \mathbb{Q}/\mathbb{Z})$ is exact by Proposition (3.4). Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, $T \otimes_R \boldsymbol{G}^+$ is exact too.

(ii) With the given assumptions on R, the dual of Proposition (4.3) implies that every Gorenstein projective right R-module also is Gorenstein flat. □

(4.7) **Lemma.** *Assume that R is right coherent with finite $\mathsf{LeftFPD}(R)$. Let M be a left R-module with $\operatorname{Gfd}_R M < \infty$, and let $\boldsymbol{G}^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper* *left ${}_R\mathcal{GF}$-resolution of M (which exists by Proposition (4.5), since R is right coherent). Then the following conclusion hold:*

- (i) $\mathrm{Hom}_R(\boldsymbol{G}^+, H)$ *is exact for all Gorenstein injective left R-modules H.*
- (ii) $T \otimes_R \boldsymbol{G}^+$ *is exact for all Gorenstein flat right R-modules T.*
- (iii) *If R is also left coherent with finite* $\mathsf{RightFPD}(R)$*, then $T \otimes_R \boldsymbol{G}^+$ is exact for all Gorenstein projective right R-modules T.*

Proof. (i) Since $\mathrm{Gfd}_R M < \infty$ and R is right coherent, Proposition (4.5) gives a special short exact sequence $0 \to K' \to G' \to M \to 0$, where $G' \to M$ is a ${}_R\mathcal{GF}$-precover of M, and $\mathrm{fd}_R K' < \infty$. Since R has $\mathsf{LeftFPD}(R) < \infty$, [13, Proposition 6] implies that also $\mathrm{pd}_R K' < \infty$. Now the proof of Proposition (3.4) applies.

(ii) If T is a Gorenstein flat right R-module, then the left R-module $H = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{Q}/\mathbb{Z})$ is Gorenstein injective; by (the dual of) Theorem (4.4) above. By the result (i), just proved, we have exactness of

$$\mathrm{Hom}_R(\boldsymbol{G}^+, H) \;\cong\; \mathrm{Hom}_{\mathbb{Z}}(T \otimes_R \boldsymbol{G}^+, \mathbb{Q}/\mathbb{Z}).$$

Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, we also have exactness of $T \otimes_R \boldsymbol{G}^+$, as desired.

(iii) Under the extra assumptions on R, the dual of Proposition (4.3) implies that every Gorenstein projective right R-module is also Gorenstein flat. Thus (iii) follows from (ii). □

(4.8) **Theorem.** *Assume that R is both left and right coherent, and that both* $\mathsf{LeftFPD}(R)$ *and* $\mathsf{RightFPD}(R)$ *are finite. For every right R-module M, and every left R-module N, the following conclusions hold:*

- (i) *If $\mathrm{Gfd}_R M < \infty$ and $\mathrm{Gfd}_R N < \infty$, then*
$$\mathrm{Tor}_n^{\mathcal{GF}_R}(M, N) \cong \mathrm{Tor}_n^{{}_R\mathcal{GF}}(M, N).$$
- (ii) *If $\mathrm{Gpd}_R M < \infty$ and $\mathrm{Gfd}_R N < \infty$, then*
$$\mathrm{Tor}_n^{\mathcal{GP}_R}(M, N) \cong \mathrm{Tor}_n^{\mathcal{GF}_R}(M, N) \cong \mathrm{Tor}_n^{{}_R\mathcal{GF}}(M, N).$$
- (iii) *If $\mathrm{Gfd}_R M < \infty$ and $\mathrm{Gpd}_R N < \infty$, then*
$$\mathrm{Tor}_n^{\mathcal{GF}_R}(M, N) \cong \mathrm{Tor}_n^{{}_R\mathcal{GP}}(M, N) \cong \mathrm{Tor}_n^{{}_R\mathcal{GF}}(M, N).$$
- (iv) *If $\mathrm{Gpd}_R M < \infty$ and $\mathrm{Gpd}_R N < \infty$, then*
$$\mathrm{Tor}_n^{\mathcal{GP}_R}(M, N) \cong \mathrm{Tor}_n^{\mathcal{GF}_R}(M, N) \cong \mathrm{Tor}_n^{{}_R\mathcal{GP}}(M, N) \cong \mathrm{Tor}_n^{{}_R\mathcal{GF}}(M, N).$$

All the isomorphisms are functorial in M and N.

Proof. Use Lemma (4.6) and (4.7) as input in the covariant-covariant version of Theorem (2.6). □

(4.9) **Definition of $\boldsymbol{g}$Tor and GTor.** Assume that R is both left and right coherent, and that both $\mathsf{LeftFPD}(R)$ and $\mathsf{RightFPD}(R)$ are finite. Furthermore, let M be a right R-module, and let N be a left R-module. If $\mathrm{Gfd}_R M < \infty$ and $\mathrm{Gfd}_R N < \infty$, then we write

$$g\mathrm{Tor}_n^R(M, N) \;:=\; \mathrm{Tor}_n^{\mathcal{GF}_R}(M, N) \;\cong\; \mathrm{Tor}_n^{{}_R\mathcal{GF}}(M, N)$$

for the isomorphic abelian groups in Theorem (4.8)(i). If $\mathrm{Gpd}_R M < \infty$ and $\mathrm{Gpd}_R N < \infty$, then we write

$$\mathrm{GTor}_n^R(M, N) \;:=\; \mathrm{Tor}_n^{\mathcal{GP}_R}(M, N) \;\cong\; \mathrm{Tor}_n^{{}_R\mathcal{GP}}(M, N)$$

for the isomorphic abelian groups in Theorem (4.8)(iv).

We can now reformulate some of the contents of Theorem (4.8):

(4.10) **Theorem.** *Assume that R is both left and right coherent, and that both* $\mathsf{LeftFPD}(R)$ *and* $\mathsf{RightFPD}(R)$ *are finite. For every right R-module M with* $\mathrm{Gpd}_R M < \infty$, *and for every left R-module N with* $\mathrm{Gpd}_R N < \infty$, *we have isomorphisms:*

$$g\mathrm{Tor}_n^R(M,N) \;\cong\; \mathrm{GTor}_n^R(M,N),$$

which are functorial in M and N.

Finally we compare gTor (and hence GTor) with the usual Tor.

(4.11) **Theorem.** *Assume that R is both left and right coherent, and that both* $\mathsf{LeftFPD}(R)$ *and* $\mathsf{RightFPD}(R)$ *are finite. Furthermore, let M be a right R-module with* $\mathrm{Gfd}_R M < \infty$, *and let N be a left R-module with* $\mathrm{Gfd}_R N < \infty$. *If either* $\mathrm{fd}_R M < \infty$ *or* $\mathrm{fd}_R N < \infty$, *then there are isomorphisms*

$$g\mathrm{Tor}_n^R(M,N) \cong \mathrm{Tor}_n^R(M,N),$$

which are functorial in M and N.

Proof. If $\mathrm{fd}_R M < \infty$, then also $\mathrm{pd}_R M < \infty$ by [13, Proposition 6] (since $\mathsf{RightFPD}(R) < \infty$). Let $\boldsymbol{P}$ be any projective resolution of M. As noted in the Remark (3.3), $\boldsymbol{P}$ is also a proper left $\mathcal{GP}_R$-resolution of M. Hence, Theorem (4.8)(ii) and the definitions give:

$$g\mathrm{Tor}_n^R(M,N) \;=\; \mathrm{Tor}_n^{\mathcal{GP}_R}(M,N) \;=\; \mathrm{H}_n(\boldsymbol{P} \otimes_R N) \;=\; \mathrm{Tor}_n^R(M,N),$$

as desired. □

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Part III

Rings with finite Gorenstein injective dimension

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RINGS WITH FINITE GORENSTEIN INJECTIVE DIMENSION

HENRIK HOLM

ABSTRACT. In this paper we prove that for any associative ring R, and for any left R-module M with finite projective dimension, then the Gorenstein injective dimension $\mathrm{Gid}_R M$ equals the usual injective dimension $\mathrm{id}_R M$. In particular, if $\mathrm{Gid}_R R$ is finite, then also $\mathrm{id}_R R$ is finite, and thus R is Gorenstein (provided that R is commutative and Noetherian).

1. INTRODUCTION

It is well-known that among the commutative local Noetherian rings $(R, \mathfrak{m}, k)$, the *Gorenstein rings* are characterized by the condition $\mathrm{id}_R R < \infty$. From the dual of [10, Proposition 2.27] ([6, Proposition 10.2.3] is a special case) it follows that the *Gorenstein injective dimension* $\mathrm{Gid}_R(-)$ is a *refinement* of the usual injective dimension $\mathrm{id}_R(-)$ in the following sense:

For any R-module M there is an inequality $\mathrm{Gid}_R M \leqslant \mathrm{id}_R M$, and if $\mathrm{id}_R M < \infty$, then there is an equality $\mathrm{Gid}_R M = \mathrm{id}_R M$.

Now, since the injective dimension $\mathrm{id}_R R$ of R measures Gorensteinness, it is only natural to ask what does the Gorenstein injective dimension $\mathrm{Gid}_R R$ of R measures? As a consequence of Theorem (2.1) below, it turns out that

An associative ring R with $\mathrm{Gid}_R R < \infty$ also has $\mathrm{id}_R R < \infty$ (and hence R is Gorenstein, provided that R is commutative and Noetherian).

This result is proved by Christensen [2, Theorem (6.3.2)] in the case where $(R, \mathfrak{m}, k)$ is a commutative local Noetherian Cohen-Macaulay ring with a dualizing module. The aim of this paper is to prove Theorem (2.1), together with a series of related results. Among these results is Theorem (3.2), which has the nice, and easily stated Corollary (3.3):

Assume that $(R, \mathfrak{m}, k)$ is a commutative local Noetherian ring, and let M be an R-module of finite depth, that is, $\mathrm{Ext}_R^m(k, M) \neq 0$ for some $m \in \mathbb{N}_0$ (this happens for example if $M \neq 0$ is finitely generated). If either

$$(i)\ \ \mathrm{Gfd}_R M < \infty \ \text{ and } \ \mathrm{id}_R M < \infty \qquad \text{or} \qquad (ii)\ \ \mathrm{fd}_R M < \infty \ \text{ and } \ \mathrm{Gid}_R M < \infty,$$

then R is Gorenstein.

This corollary is also proved by Christensen [2, Theorem (6.3.2)] in the case where $(R, \mathfrak{m}, k)$ is Cohen-Macaulay with a dualizing module. However, Theorem (3.2) itself (dealing not only with local rings), is a generalization of [8, Proposition 2.10] (in the module case) by Foxby from 1979.

We should *briefly* mention the history of Gorenstein injective, projective and flat modules: *Gorenstein injective* modules over an arbitrary associative ring, and the related Gorenstein injective dimension, was introduced and studied by Enochs and Jenda in [3]. The dual concept, *Gorenstein projective* modules, was already introduced by Auslander and Bridger [1] in 1969,

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but only for finitely generated modules over a two-sided Noetherian ring. *Gorenstein flat* modules was also introduced by Enochs and Jenda; please see [5].

(1.1) **Setup and notation.** Let R be any associative ring with a non-zero multiplicative identity. All modules are—if not specified otherwise—*left* R-modules. If M is any R-module, we use $\mathrm{pd}_R M$, $\mathrm{fd}_R M$, and $\mathrm{id}_R M$ to denote the usual projective, flat, and injective dimesion of M, respectively. Furthermore we write $\mathrm{Gpd}_R M$, $\mathrm{Gfd}_R M$, and $\mathrm{Gid}_R M$ for the Gorenstein projective, Gorenstein flat, and Gorenstein injective dimesion of M, respectively.

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2. Rings with finite Gorenstein injecive dimension

(2.1) **Theorem.** *If M is an R-module with $\mathrm{pd}_R M < \infty$, then $\mathrm{Gid}_R M = \mathrm{id}_R M$. In particular, if $\mathrm{Gid}_R R < \infty$, then also $\mathrm{id}_R R < \infty$ (and hence R is Gorenstein, provided that R is commutative and Noetherian).*

Proof. Since $\mathrm{Gid}_R M \leqslant \mathrm{id}_R M$ always, it suffices to prove that $\mathrm{id}_R M \leqslant \mathrm{Gid}_R M$. Naturally, we may assume that $\mathrm{Gid}_R M < \infty$.

First consider the case where M is Gorenstein injective, that is, $\mathrm{Gid}_R M = 0$. By definition, M is a kernel in a complete injective resolution. This means that there exists an exact sequence $\boldsymbol{E} = \cdots \to E_1 \to E_0 \to E_{-1} \to \cdots$ of injective R-modules, such that $\mathrm{Hom}_R(I, \boldsymbol{E})$ is exact for every injective R-module I, and such that $M \cong \mathrm{Ker}(E_1 \to E_0)$. In particular, there exists a short exact sequence $0 \to M' \to E \to M \to 0$, where E is injective, and M' is Gorenstein injective. Since M' is Gorenstein injective and $\mathrm{pd}_R M < \infty$, it follows by [4, Lemma 1.3] that $\mathrm{Ext}^1_R(M, M') = 0$. Thus $0 \to M' \to E \to M \to 0$ is split-exact, so M is a direct summand of the injective module E. Therefore M itself is injective.

Next consider the case where $\mathrm{Gid}_R M > 0$. By [10, Theorem 2.15] there exists an exact sequence $0 \to M \to H \to C \to 0$ where H is Gorenstein injective and $\mathrm{id}_R C = \mathrm{Gid}_R M - 1$. As in the previous case, since H is Gorenstein injective, there exists a short exact sequence $0 \to H' \to I \to H \to 0$ where I is injective and H' is Gorenstein injective. Now consider the pull-back diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & M \arrow[u] \arrow[r] & H \arrow[u] \arrow[r] & C \arrow[r] & 0 \\
0 \arrow[r] & P \arrow[u] \arrow[r] & I \arrow[u] \arrow[r] & C \arrow[u, equal] \arrow[r] & 0 \\
 & H' \arrow[u] \arrow[r, equal] & H' \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

Since I is injective and $\mathrm{id}_R C = \mathrm{Gid}_R - 1$ we get $\mathrm{id}_R P \leqslant \mathrm{Gid}_R M$ by the second row. Since H' is Gorenstein injective and $\mathrm{pd}_R M < \infty$, it follows (as before) by [4, Lemma 1.3] that $\mathrm{Ext}^1_R(M, H') = 0$. Consequently, the first column $0 \to H' \to P \to M \to 0$ splits. Therefore $P \cong M \oplus H'$, and hence $\mathrm{id}_R M \leqslant \mathrm{id}_R P \leqslant \mathrm{Gid}_R M$. □

The theorem above has, of course, a dual counterpart:

(2.2) **Theorem.** *If M is an R-module with $\mathrm{id}_R M < \infty$, then $\mathrm{Gpd}_R M = \mathrm{pd}_R M$.* $\square$

Theorem (2.6) below is a "flat version" of the two previous theorems. First recall the following:

(2.3) **Definition.** The *left finitistic projective dimension* $\mathsf{LeftFPD}(R)$ of R is defined as

$$\mathsf{LeftFPD}(R) = \sup\{\ \mathrm{pd}_R M \mid M \text{ is a } \textit{left } R\text{-module with } \mathrm{pd}_R M < \infty\ \}.$$

The right finitistic projective dimension $\mathsf{RightFPD}(R)$ of R is defined similarly.

(2.4) **Remark.** When R is commutative and Noetherian, $\mathsf{LeftFPD}(R)$ and $\mathsf{RightFPD}(R)$ equals the Krull dimension of R, by [9, Théorème (3.2.6) (Seconde partie)].

Furthermore, we will need the following result from [10, Proposition 3.11]:

(2.5) **Proposition.** *For any (left) R-module M there is an inequality $\mathrm{Gid}_R \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \leqslant \mathrm{Gfd}_R M$. If R is right coherent, then we have the equality $\mathrm{Gid}_R \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = \mathrm{Gfd}_R M$.* $\square$

We are now ready to state:

(2.6) **Theorem.** *For any R-module M, the following conclusions hold:*

(i) *Assume that $\mathsf{LeftFPD}(R)$ is finite. If $\mathrm{fd}_R M < \infty$, then $\mathrm{Gid}_R M = \mathrm{id}_R M$.*

(ii) *Assume that R is left and right coherent with finite $\mathsf{RightFPD}(R)$. If $\mathrm{id}_R M < \infty$, then $\mathrm{Gfd}_R M = \mathrm{fd}_R M$.*

Proof. (i) If $\mathrm{fd}_R M < \infty$, then also $\mathrm{pd}_R M < \infty$, by [11, Proposition 6] (since $\mathsf{LeftFPD}(R) < \infty$). Hence the desired conclusion follows from Theorem (2.1) above.

(ii) Since R is left coherent we have $\mathrm{fd}_R \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \leqslant \mathrm{id}_R M < \infty$, by [12, Lemma 3.1.4]. By assumption, $\mathsf{RightFPD}(R) < \infty$, and therefore also $\mathrm{pd}_R \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) < \infty$, by [11, Proposition 6]. Now Theorem (2.1) gives that $\mathrm{Gid}_R \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = \mathrm{id}_R \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. It is well-known that $\mathrm{fd}_R M = \mathrm{id}_R \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ (without assumptions on R), and by Proposition (2.5) above, we also get $\mathrm{Gfd}_R M = \mathrm{Gid}_R \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, since R is right coherent. The proof is done. $\square$

3. A theorem on Gorenstein rings by Foxby

We end this paper by generalizing a theorem [8, Proposition 2.10] on Gorenstein rings by Foxby from 1979. For completeness, we briefly recall:

(3.1) **The small support.** Assume that R is commutative and Noetherian. For an R-module M, an integer n, and a prime ideal $\mathfrak{p}$ in R, we write $\beta_n^R(\mathfrak{p}, M)$, respectively, $\mu_R^n(\mathfrak{p}, M)$, for the n^{th} Betti number, respectively, n^{th} Bass number, of M at $\mathfrak{p}$.

Foxby [8, Definition p. 157] or [7, (14.8)] defines the *small (or homological) support* of an R-module M to be the set

$$\mathrm{supp}_R M = \{\ \mathfrak{p} \in \mathrm{Spec}\, R \mid \exists n \in \mathbb{N}_0 \colon \beta_n^R(\mathfrak{p}, M) \neq 0\ \}.$$

Let us mention the most basic results about the small support, all of which can be found in [8, p. 157 – 159] and [7, Chapter 14]:

(a) The small support, $\mathrm{supp}_R M$, is contained in the usual (large) support, $\mathrm{Supp}_R M$, and $\mathrm{supp}_R M = \mathrm{Supp}_R M$ if M is finitely generated. Also, if $M \neq 0$, then $\mathrm{supp}_R M \neq \emptyset$.

(b) $\mathrm{supp}_R M = \{\ \mathfrak{p} \in \mathrm{Spec}\, R \mid \exists n \in \mathbb{N}_0 \colon \mu_R^n(\mathfrak{p}, M) \neq 0\ \}$.

(*c*) Assume that $(R, \mathfrak{m}, k)$ is local. If M is an R-module with finite depth, that is,
$$\mathrm{depth}_R M := \inf\{\ m \in \mathbb{N}_0 \mid \mathrm{Ext}_R^m(k, M) \neq 0\ \} < \infty$$
(this happens for example if $M \neq 0$ is finitely generated), then $\mathfrak{m} \in \mathrm{supp}_R M$, by (*b*) above.

Now, given these facts about the small support, and the results in the previous section, the following generalization of [8, Proposition 2.10] is immediate:

(3.2) **Theorem.** *Assume that R is commutative and Noetherian. Let M be any R-module, and assume that either of the following four conditions are satisfied:*

(*i*) $\mathrm{Gpd}_R M < \infty$ *and* $\mathrm{id}_R M < \infty$,

(*ii*) $\mathrm{pd}_R M < \infty$ *and* $\mathrm{Gid}_R M < \infty$,

(*iii*) *R has finite Krull dimension, and* $\mathrm{Gfd}_R M < \infty$ *and* $\mathrm{id}_R M < \infty$,

(*iv*) *R has finite Krull dimension, and* $\mathrm{fd}_R M < \infty$ *and* $\mathrm{Gid}_R M < \infty$.

Then $R_\mathfrak{p}$ is a Gorenstein local ring for all $\mathfrak{p} \in \mathrm{supp}_R M$. □

(3.3) **Corollary.** *Assume that $(R, \mathfrak{m}, k)$ is a commutative local Noetherian ring. If there exists an R-module M of finite depth, that is,*
$$\mathrm{depth}_R M := \inf\{\ m \in \mathbb{N}_0 \mid \mathrm{Ext}_R^m(k, M) \neq 0\ \} < \infty,$$
and which satisfies either

(*i*) $\mathrm{Gfd}_R M < \infty$ *and* $\mathrm{id}_R M < \infty$, *or*

(*ii*) $\mathrm{fd}_R M < \infty$ *and* $\mathrm{Gid}_R M < \infty$,

then R is Gorenstein. □

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Part IV

On Gorenstein projective, injective and flat dimensions
—
a functorial description with applications

ON GORENSTEIN PROJECTIVE, INJECTIVE AND FLAT DIMENSIONS — A FUNCTORIAL DESCRIPTION WITH APPLICATIONS

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Abstract. For a large class of rings, including all those encountered in algebraic geometry, we establish the conjectured Morita-like equivalence between the full subcategory of complexes of finite Gorenstein flat dimension and that of complexes of finite Gorenstein injective dimension.

This functorial description meets the expectations and delivers a series of new results, which allows us to establish a well-rounded theory for Gorenstein dimensions.

Dedicated to Professor Christian U. Jensen

Introduction

For any pair of adjoint functors,

$$\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D},$$

there is a natural way to single out two full subcategories, A of C and B of D, such that the restrictions of F and G provide a quasi-inverse equivalence of categories,

$$(\dagger) \qquad \mathsf{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{B}.$$

However, we typically study categories A and B that *a priori* do not arise from a pair of adjoint functors. This is, indeed, the situation in this paper, and the motivation for seeking an equivalence between A and B is clear: The literature abounds with evidence that new insight can be been gained from the functorial machinery (†), if one can successfully make the categories at hand fit into this setup. Examples close to the nature of this paper are provided by Morita [35], Rickard [38], Dwyer and Greenlees [13], and Avramov and Foxby [6].

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In the late 1960's, Auslander and Bridger [2, 3] introduced the G–dimension ("G" for Gorenstein) for finitely generated modules over an associative and two-sided noetherian ring; and they immediately established a powerful theory for this new dimension. Among its most celebrated features are the Gorenstein parallel [3, thm. (4.20)] of the Auslander–Buchsbaum–Serre characterization of regular local rings, and the generalized Auslander–Buchsbaum formula [3, thm. (4.13)(b)].

After that, twenty years were to elapse before any real progress was made on extending the G–dimension to non-finitely generated modules. This work was initiated by Enochs and Jenda [16, 18]; over any associative ring they introduced Gorenstein projective, Gorenstein injective, and Gorenstein flat modules. In [4] Avramov, Buchweitz, Martsinkovsky, and Reiten prove that the Gorenstein projective dimension agrees with the G–dimension for finitely generated modules over an associative and two-sided noetherian ring.

On one hand, the Gorenstein projective dimension is a natural refinement of the classical projective dimension, on the other hand the definition is highly non-functorial, and it is, in general, not possible to measure the dimension in terms of vanishing of appropriate functors. Only when the Gorenstein projective dimension of a module is known to be finite, i.e. a finite resolution exists, may we measure the dimension in terms of vanishing of certain Ext modules. The situation is the same for the Gorenstein injective and flat dimensions, and this is, in fact, the Achilles' heel of the theory for Gorenstein dimensions.

One way to remedy this problem is to realize the category of modules of finite Gorenstein projective dimension as A in a setup like (†). Partial results in this direction were established in [10, 19, 23]. In this paper we establish a firm connection between the Gorenstein dimensions and the well-understood Auslander and Bass classes studied by Avramov and Foxby [6]. In their setting, the underlying category is $\mathsf{C} = \mathsf{D} = \mathsf{D}(R)$, the derived category of a commutative, noetherian and local ring R admitting a dualizing complex D. The quasi-inverse equivalence is

$$\mathsf{A}(R) \underset{\mathbf{R}\mathrm{Hom}_R(D,-)}{\overset{D\otimes^{\mathbf{L}}_R-}{\rightleftarrows}} \mathsf{B}(R),$$

where $\mathsf{A}(R)$ is the so-called Auslander class and $\mathsf{B}(R)$ the Bass class with respect to the dualizing complex D.

The central result of this paper is the following (theorem (4.3)):

Theorem A. *Let R be a commutative and noetherian ring. If R admits a dualizing complex, then the following conditions are equivalent for any right-bounded complex X.*

(i) *X belongs to the Auslander class, $X \in \mathsf{A}(R)$.*
(ii) *X has finite Gorenstein projective dimension, $\mathrm{Gpd}_R X < \infty$.*
(iii) *X has finite Gorenstein flat dimension, $\mathrm{Gfd}_R X < \infty$.*

Theorem A has the following counterpart (theorem (4.5)):

Theorem B. *Let R be a commutative and noetherian ring. If R admits a dualizing complex, then the following conditions are equivalent for any left-bounded complex Y.*

(i) *Y belongs to the Bass class, $Y \in \mathsf{B}(R)$.*
(ii) *Y has finite Gorenstein injective dimension, $\mathrm{Gid}_R\, Y < \infty$.*

Theorems A and B are established in section 4; one immediate consequence of theorem A is a Gorenstein version of the classical Gruson–Raynaud–Jensen theorem (see [31, prop. 6] and [37, Seconde partie, thm. (3.2.6)]):

Theorem C. *Let R be a commutative and noetherian ring. If R admits a dualizing complex then, for any right-bounded complex X, there is a biimplication*

$$\mathrm{Gfd}_R\, X < \infty \quad \Leftrightarrow \quad \mathrm{Gpd}_R\, X < \infty.$$

While the Gorenstein parallel of the Auslander–Buchsbaum–Serre theorem was one of the original motivations for studying G–dimension, the Gorenstein equivalent of another classic, the Bass formula, has proved more elusive. It was first established over Gorenstein rings [17, thm. 4.3], later over local Cohen–Macaulay rings with dualizing module [10, thm. (6.2.15)], and in section 6 we now prove (theorem (6.4)):

Theorem D. *Let R be a commutative, noetherian, local ring, and assume that R admits a dualizing complex. If N is a non-trivial finitely generated R–module of finite Gorenstein injective dimension, then*

$$\mathrm{Gid}_R\, N = \operatorname{depth} R.$$

For non-finite modules, the natural generalization of the Auslander–Buchsbaum and Bass formulas are Chouinard's formulas for flat and injective dimensions. Again, a similar formula for the Gorenstein flat dimension has been around for some time, while the Gorenstein injective version (theorem (6.9)) established here is new. As an important application of this theorem we prove (theorem (6.10)):

Theorem E. *Let R be a commutative and noetherian ring, admitting a dualizing complex. A filtered, direct limit of Gorenstein injective R–modules is then Gorenstein injective.*

The functorial description in section 4 draws on some fundamental properties of Gorenstein dimensions. In section 2 we synthesize these properties in three theorems; we do it in the most general setting possible today: over associative rings with unit. In fact, this is only a minor effort, as these three theorems build on the same technical machinery as the main theorems in section 4. These technical results have been grouped together in section 3.

1. Notation and Prerequisites

In this paper, we work within the derived category of the module category over a ring R. For some technical results we do, however, need go back to the category of R–complexes. When we work in greatest generality, R is just associative with unit; our most restrictive case is when R is commutative, noetherian and local, admitting a dualizing complex. Unless otherwise explicitly stated, all modules in this paper are left modules. If R is associative with unit, R^{opp} will denote the opposite ring. Recall that a right R–module is a left R^{opp}–module.

We consistently use the notation found in the appendix of [10]. In particular, the category of R–complexes is denoted $\mathsf{C}(R)$, and we use subscripts $\sqsubset$, $\sqsupset$, and $\square$ to denote genuine boundedness conditions. Thus, $\mathsf{C}_{\sqsupset}(R)$ is the full subcategory of $\mathsf{C}(R)$ of genuine right-bounded complexes.

The derived category is denoted $\mathsf{D}(R)$, and we use subscripts $\sqsubset$, $\sqsupset$, and $\square$ to denote homological boundedness conditions. Thus, $\mathsf{D}_{\sqsupset}(R)$ denotes the full subcategory of $\mathsf{D}(R)$ of homologically right-bounded complexes. Henceforth, we shall reserve the term *bounded* to signify homological boundedness. The symbol "$\simeq$" will be used to designate isomorphisms in $\mathsf{D}(R)$ and quasi-isomorphisms in $\mathsf{C}(R)$.

We also use superscript "f" to signify that the homology modules are degreewise finitely generated. A complex X is said to have *finite homology* if and only if it is homologically bounded and all the homology modules are finitely generated, that is, $X \in \mathsf{D}^{\mathrm{f}}_{\square}(R)$.

For the derived category and derived functors, the reader is referred to the original texts, Verdier's thesis [40] and Hartshorne's notes [28], and further to excellent modern accounts: Gelfand and Manin's book [26] and Neeman's book on triangulated categories [36].

Next, we recall the definition of dualizing complexes and review some technical constructions and results for later use.

(1.1) **Definition (Dualizing Complex).** When R is commutative and noetherian, a complex $D \in \mathsf{D}(R)$ is said to be *dualizing* for R if it fulfills the requirements:

(1) D has finite homology.
(2) D has finite injective dimension.
(3) The canonical (homothety) morphism $\chi^R_D \colon R \to \mathbf{R}\mathrm{Hom}_R(D,D)$ is an isomorphism in $\mathsf{D}(R)$.

If R is local, this definition coincides with the classical one [28, chap. V, §2], and in this paper we use definition (1.1) for local and non-local rings alike.

(1.2) **Dagger duality.** If R has a dualizing complex D, we may consider the functor

$$-^{\dagger} = \mathbf{R}\mathrm{Hom}_R(-,D).$$

It is safe to say that the functor $-^{\dagger}$ has been studied extensively ever since it was conceived by Grothendieck and appeared in Hartshorne's notes [28]. Observe that $X^{\dagger}$

has finite homology whenever X has finite homology, that is, we have the diagram

$$\mathsf{D}^{\mathsf{f}}_{\square}(R) \underset{-^{\dagger}}{\overset{-^{\dagger}}{\rightleftarrows}} \mathsf{D}^{\mathsf{f}}_{\square}(R).$$

The endofunctor $-^{\dagger}$ furnishes a quasi-inverse duality on $\mathsf{D}^{\mathsf{f}}_{\square}(R)$, that is, for any $X \in \mathsf{D}^{\mathsf{f}}_{\square}(R)$ the canonical biduality morphism

$$X \xrightarrow{\simeq} X^{\dagger\dagger} = \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(X,D),D) \tag{1.2.1}$$

is an isomorphism, cf. [28, prop. V.2.1].

(1.3) **Foxby equivalence.** If D is a dualizing complex for a commutative and noetherian ring R, then we can consider the adjoint pair of functors,

$$(D \otimes^{\mathbf{L}}_R -, \mathbf{R}\mathrm{Hom}_R(D,-)).$$

As usual, let η denote the unit and ε the counit of the adjoint pair, cf. [33, chap. 4].

The *Auslander* and *Bass classes* with respect to the dualizing complex D are defined in terms of η and ε being isomorphisms. To be precise, the definition of the Auslander class reads

$$\mathsf{A}(R) = \left\{ X \in \mathsf{D}_{\square}(R) \;\middle|\; \begin{array}{l} \eta_X : X \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_R X) \text{ is an} \\ \text{isomorphism, and } D \otimes^{\mathbf{L}}_R X \text{ is bounded} \end{array} \right\},$$

while the definition of the Bass class reads

$$\mathsf{B}(R) = \left\{ Y \in \mathsf{D}_{\square}(R) \;\middle|\; \begin{array}{l} \varepsilon_Y : D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D,Y) \xrightarrow{\simeq} Y \text{ is an} \\ \text{isomorphism, and } \mathbf{R}\mathrm{Hom}_R(D,Y) \text{ is bounded} \end{array} \right\}.$$

The Auslander and Bass classes are full triangulated subcategories of $\mathsf{D}(R)$, and the adjoint pair $(D \otimes^{\mathbf{L}}_R -, \mathbf{R}\mathrm{Hom}_R(D,-))$ provides quasi-inverse equivalences between the Auslander and Bass classes,

$$\mathsf{A}(R) \underset{\mathbf{R}\mathrm{Hom}_R(D,-)}{\overset{D \otimes^{\mathbf{L}}_R -}{\rightleftarrows}} \mathsf{B}(R).$$

This equivalence, introduced in [6], has come to be called *Foxby equivalence*.

Note that all complexes of finite flat dimension belong to $\mathsf{A}(R)$, while complexes of finite injective dimension belong to $\mathsf{B}(R)$.

(1.4) **Finitistic dimensions.** We write $\mathrm{FPD}(R)$ for the (left) finitistic projective dimension of R, i.e.

$$\mathrm{FPD}(R) = \sup \left\{ \mathrm{pd}_R M \;\middle|\; \begin{array}{l} M \text{ is an } R\text{–module of} \\ \text{finite projective dimension} \end{array} \right\}.$$

Similarly, we write $\mathrm{FID}(R)$ and $\mathrm{FFD}(R)$ for the (left) finitistic injective and (left) finitistic flat dimension of R.

If R is commutative and noetherian, then it is well known, cf. [8, cor. 5.5] and [37, seconde partie, thm. (3.2.6)], that

$$\mathrm{FID}(R) = \mathrm{FFD}(R) \leq \mathrm{FPD}(R) = \dim R. \tag{1.4.1}$$

2. Measuring Gorenstein dimensions

In this section we state and prove three fundamental theorems. Mimicking the style of Cartan and Eilenberg, these theorems characterize complexes of finite Gorenstein dimension in terms of resolutions and show how to determine them in terms of vanishing of certain derived functors.

Such results have previously in [10, 15, 16, 18, 29] been established in more restrictive settings, and the purpose of this section is to present them in the most general setting possible today. Certain technical results — corollaries (3.10), (3.11), and (3.15) — are required for the proofs in this section; in the interest of readability, these have been grouped together in section 3.

Throughout this section, R is an associative ring with unit.

We start by investigating the Gorenstein projective dimension. The definitions in (2.1) go back to [10, 16].

(2.1) **Gorenstein projective dimension.** Let P be a complex of projective modules with $\mathrm{H}(P) = 0$. We say that P is a *complete projective resolution* if and only if $\mathrm{H}(\mathrm{Hom}_R(P, Q)) = 0$ for every projective R–module Q.

A module M is said to be *Gorenstein projective* if and only if there exists a complete projective resolution with a cokernel isomorphic to M.

The *Gorenstein projective dimension*, $\mathrm{Gpd}_R X$, of $X \in \mathsf{D}_{\sqsupset}(R)$ is defined as

$$\mathrm{Gpd}_R X = \inf \left\{ \sup\{\ell \in \mathbb{Z} \mid A_\ell \neq 0\} \;\middle|\; \begin{array}{l} A \in \mathsf{C}_{\sqsupset}(R) \text{ is isomorphic to } X \text{ in } \mathsf{D}(R) \\ \text{and every } A_\ell \text{ is Gorenstein projective} \end{array} \right\}.$$

(2.2) **Theorem.** *Let $X \in \mathsf{D}_{\sqsupset}(R)$ be a complex of finite Gorenstein projective dimension. For $n \in \mathbb{Z}$ the following are equivalent:*

- (i) $\mathrm{Gpd}_R X \leq n$.
- (ii) $n \geq \inf U - \inf \mathbf{R}\mathrm{Hom}_R(X, U)$ *for all $U \in \mathsf{D}(R)$ of finite projective or finite injective dimension with $\mathrm{H}(U) \neq 0$.*
- (iii) $n \geq -\inf \mathbf{R}\mathrm{Hom}_R(X, Q)$ *for all projective R–modules Q.*
- (iv) $n \geq \sup X$ *and the cokernel $\mathrm{C}_n^A = \mathrm{Coker}(A_{n+1} \to A_n)$ is a Gorenstein projective module for any genuine right-bounded complex $A \simeq X$ of Gorenstein projective modules.*

Moreover, the following hold:

$$\begin{aligned} \mathrm{Gpd}_R X &= \sup \{\inf U - \inf \mathbf{R}\mathrm{Hom}_R(X, U) \mid \mathrm{pd}_R U < \infty \text{ and } \mathrm{H}(U) \neq 0\} \\ &= \sup \{-\inf \mathbf{R}\mathrm{Hom}_R(X, Q) \mid Q \text{ is projective}\} \\ &\leq \mathrm{FPD}(R) + \sup X. \end{aligned}$$

Proof. The proof of the first part is cyclic. clearly, (ii) is stronger than (iii), and this leaves us three implications to prove.

$(i) \Rightarrow (ii)$: Choose a complex $A \in \mathsf{C}_{\sqsupset}(R)$ consisting of Gorenstein projective modules, such that $A \simeq X$ and $A_\ell = 0$ for $\ell > n$. First, let U be a complex of finite projective dimension with $\mathrm{H}(U) \neq 0$. Set $i = \inf U$ and choose a bounded complex $P \simeq U$ of projective modules with $P_\ell = 0$ for $\ell < i$. By corollary (3.10) the complex $\mathrm{Hom}_R(A, P)$ is isomorphic to $\mathbf{R}\mathrm{Hom}_R(X, U)$ in $\mathsf{D}(\mathbb{Z})$; in particular, $\inf \mathbf{R}\mathrm{Hom}_R(X, U) = \inf \mathrm{Hom}_R(A, P)$. For $\ell < i - n$ and $q \in \mathbb{Z}$, either $q > n$ or $q + \ell \leq n + \ell < i$, so the module

$$\mathrm{Hom}_R(A, P)_\ell \;=\; \prod_{q \in \mathbb{Z}} \mathrm{Hom}_R(A_q, P_{q+\ell})$$

vanishes. Hence, $\mathrm{H}_\ell(\mathrm{Hom}_R(A, P)) = 0$ for $\ell < i - n$, and $\inf \mathbf{R}\mathrm{Hom}_R(X, U) \geq i - n = \inf U - n$ as desired.

Next, let U be a complex of finite injective dimension and choose a bounded complex $I \simeq U$ of injective modules. Set $i = \inf U$ and consider the soft truncation $V = I_{i}\!\supset$. The modules in V have finite injective dimension and $U \simeq V$, whence $\mathrm{Hom}_R(A, V) \simeq \mathbf{R}\mathrm{Hom}_R(X, U)$ by corollary (3.10), and the proof continues as above.

$(iii) \Rightarrow (iv)$: This part evolves in three steps. First we establish the inequality $n \geq \sup X$, next we prove that the n'th cokernel in a genuine bounded complex $A \simeq X$ of Gorenstein projectives is again Gorenstein projective, and finally we give an argument that allows us to conclude the same for $A \in \mathsf{C}_{\sqsupset}(R)$.

To see that $n \geq \sup X$, it is sufficient to show that

$$\sup \{ -\inf \mathbf{R}\mathrm{Hom}_R(X, Q) \mid Q \text{ is projective} \} \geq \sup X. \tag{1}$$

By assumption, $g = \mathrm{Gpd}_R X$ is finite, i.e. $X \simeq A$ for some complex

$$A = 0 \to A_g \to A_{g-1} \to \cdots \to A_i \to 0$$

and it is clear from definition (2.1) that $g \geq \sup X$. For any projective module Q, the complex $\mathrm{Hom}_R(A, Q)$ is concentrated in degrees $-i$ to $-g$,

$$0 \to \mathrm{Hom}_R(A_i, Q) \to \cdots \to \mathrm{Hom}_R(A_{g-1}, Q) \xrightarrow{\mathrm{Hom}_R(\partial_g^A, Q)} \mathrm{Hom}_R(A_g, Q) \to 0,$$

and isomorphic to $\mathbf{R}\mathrm{Hom}_R(X, Q)$ in $\mathsf{D}(\mathbb{Z})$, cf. corollary (3.10). First, consider the case $g = \sup X$: The differential $\partial_g^A \colon A_g \to A_{g-1}$ is not injective, as A has homology in degree $g = \sup X = \sup A$. By the definition of Gorenstein projective modules, there exists a projective module Q and an injective homomorphism $\varphi \colon A_g \to Q$. Because ∂_g^A is not injective, $\varphi \in \mathrm{Hom}_R(A_g, Q)$ cannot have the form $\mathrm{Hom}_R(\partial_g^A, Q)(\psi) = \psi \partial_g^A$ for some $\psi \in \mathrm{Hom}_R(A_{g-1}, Q)$. That is, the differential $\mathrm{Hom}_R(\partial_g^A, Q)$ is not surjective; hence $\mathrm{Hom}_R(A, Q)$ has non-zero homology in degree $-g = -\sup X$, and (1) follows.

Next, assume that $g > \sup X = s$ and consider the exact sequence

$$0 \to A_g \to \cdots \to A_{s+1} \to A_s \to \mathrm{C}_s^A \to 0.$$

It shows that $\mathrm{Gpd}_R \mathrm{C}_s^A \leq g - s$, and it is easy to check that equality must hold; otherwise, we would have $\mathrm{Gpd}_R X < g$. A straightforward computation based on corollary (3.10), cf. [10, lem. (4.3.9)], shows that

$$\mathrm{Ext}_R^m(\mathrm{C}_n^A, Q) = \mathrm{H}_{-(m+n)}(\mathbf{R}\mathrm{Hom}_R(X, Q)), \tag{2}$$

for all $m > 0$, all $n \geq \sup X$, and all projective modules Q. By [29, thm. (2.20)] we have $\mathrm{Ext}_R^{g-s}(\mathrm{C}_s^A, Q) \neq 0$ for some projective Q, whence $\mathrm{H}_{-g}(\mathbf{R}\mathrm{Hom}_R(X, Q)) \neq 0$ and (1) follows.

By assumption, $\mathrm{Gpd}_R X$ is finite, so a bounded complex $\widetilde{A} \simeq X$ of Gorenstein projective modules does exist. Consider the cokernel $\mathrm{C}_n^{\widetilde{A}}$. Since $n \geq \sup X = \sup \widetilde{A}$, it fits in an exact sequence $0 \to \widetilde{A}_t \to \cdots \to \widetilde{A}_{n+1} \to \widetilde{A}_n \to \mathrm{C}_n^{\widetilde{A}} \to 0$, where all the $\widetilde{A}_\ell$'s are Gorenstein projective modules. By (2) and [29, thm. (2.20)] it now follows that also $\mathrm{C}_n^{\widetilde{A}}$ is Gorenstein projective.

With this, it is sufficient to prove the following:

> If $P, A \in \mathsf{C}_{\sqsupset}(R)$ are complexes of, respectively, projective and Gorenstein projective modules, and $P \simeq X \simeq A$, then the cokernel C_n^P is Gorenstein projective if and only if C_n^A is so.

Let A and P be two such complexes. As P consists of projectives, there is a quasi-isomorphism $\pi\colon P \xrightarrow{\simeq} A$, cf. [5, 1.4.P], which, in turn, induces a quasi-isomorphism $\subset_n \pi$ between the truncated complexes, $\subset_n \pi\colon \subset_n P \xrightarrow{\simeq} \subset_n A$. The mapping cone

$$\mathsf{Cone}\,(\subset_n \pi) = 0 \to \mathrm{C}_n^P \to P_{n-1} \oplus \mathrm{C}_n^A \to P_{n-2} \oplus A_{n-1} \to \cdots$$

is a bounded exact complex, in which all modules but the two left-most ones are known to be (sums of) projective and Gorenstein projective modules. It follows by the resolving properties of Gorenstein projective modules, cf. [29, thm. (2.5)], that C_n^P is Gorenstein projective if and only if $P_{n-1} \oplus \mathrm{C}_n^A$ is so, which is tantamount to C_n^A being Gorenstein projective.

(iv) $\Rightarrow$ *(i)*: Choose a projective resolution P of X; by *(iv)* the truncation $\subset_n P$ is a complex of the desired type. This concludes the cyclic part of the proof.

To show the last claim, we still assume that $\mathrm{Gpd}_R X$ is finite. The two equalities are immediate consequences of the equivalence of *(i)*, *(ii)*, and *(iii)*. Moreover, it is easy to see how a complex $A \in \mathsf{C}_{\square}(R)$ of Gorenstein projective modules, which is isomorphic to X in $\mathsf{D}(R)$, may be truncated to form a Gorenstein projective resolution of the top homology module of X. Thus, by the definition we automatically obtain the inequality $\mathrm{Gpd}_R X \leq \mathrm{FGPD}(R) + \sup X$, where

$$\mathrm{FGPD}(R) = \sup \left\{ \mathrm{Gpd}_R M \;\middle|\; \begin{array}{l} M \text{ is an } R\text{–module with finite} \\ \text{Gorenstein projective dimension} \end{array} \right\}$$

is the (left) finitistic Gorenstein projective dimension, cf. paragraph (1.4). Finally, we have $\mathrm{FGPD}(R) = \mathrm{FPD}(R)$ by [29, thm. (2.28)]. □

(2.3) **Corollary.** *Assume that R is left coherent, and let $X \in \mathsf{D}_{\sqsupset}(R)$ be a complex with finitely presented homology modules. If X has finite Gorenstein projective dimension, then*

$$\mathrm{Gpd}_R X \;=\; -\inf \mathbf{R}\mathrm{Hom}_R(X, R).$$

Proof. Under the assumptions, X admits a resolution by finitely generated projective modules, say P; and thus, $\mathrm{Hom}_R(P, -)$ commutes with arbitrary sums. The proof is now a straightforward computation. □

Next, we turn to the Gorenstein injective dimension. The definitions of Gorenstein injective modules and dimension go back to [10, 15, 16]. The proof of theorem (2.5) relies on corollary (3.11) instead of (3.10) but is otherwise similar to the proof of theorem (2.2); hence it has been omitted.

(2.4) **Gorenstein injective dimension.** The definitions of *complete injective resolutions* and *Gorenstein injective modules* are dual to the ones given in definition (2.1), cf. also [10, (6.1.1) and (6.2.2)]. The *Gorenstein injective dimension*, $\mathrm{Gid}_R Y$, of $Y \in \mathsf{D}_{\sqsubset}(R)$ is defined as

$$\mathrm{Gid}_R Y = \inf\left\{ \sup\{\ell \in \mathbb{Z} \,|\, B_{-\ell} \neq 0\} \;\middle|\; \begin{array}{l} B \in \mathsf{C}_{\sqsubset}(R) \text{ is isomorphic to } Y \text{ in } \mathsf{D}(R) \\ \text{and every } B_\ell \text{ is Gorenstein injective} \end{array} \right\}.$$

(2.5) **Theorem.** *Let $Y \in \mathsf{D}_{\sqsubset}(R)$ be a complex of finite Gorenstein injective dimension. For $n \in \mathbb{Z}$ the following are equivalent:*

(*i*) $\mathrm{Gid}_R Y \leq n$.

(*ii*) $n \geq -\sup U - \inf \mathbf{R}\mathrm{Hom}_R(U,Y)$ *for all $U \in \mathsf{D}(R)$ of finite injective or finite projective dimension with* $\mathrm{H}(U) \neq 0$.

(*iii*) $n \geq -\inf \mathbf{R}\mathrm{Hom}_R(J,Y)$ *for all injective R–modules J.*

(*iv*) $n \geq -\inf Y$ *and the kernel* $\mathrm{Z}^B_{-n} = \mathrm{Ker}(B_{-n} \to B_{-(n+1)})$ *is a Gorenstein injective module for any genuine left-bounded complex $B \simeq Y$ of Gorenstein injective modules.*

Moreover, the following hold:

$$\begin{aligned} \mathrm{Gid}_R Y &= \sup\{-\sup U - \inf \mathbf{R}\mathrm{Hom}_R(U,Y) \mid \mathrm{id}_R U < \infty \text{ and } \mathrm{H}(U) \neq 0\} \\ &= \sup\{-\inf \mathbf{R}\mathrm{Hom}_R(J,Y) \mid J \text{ is injective}\} \\ &\leq \mathrm{FID}(R) - \inf Y. \quad \square \end{aligned}$$

(2.6) **Corollary.** *Assume that R is commutative and noetherian. If $Y \in \mathsf{D}_{\sqsubset}(R)$ is a complex of finite Gorenstein injective dimension, then*

$$\mathrm{Gid}_R Y \;=\; \sup\{-\inf \mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(R/\mathfrak{p}),Y) \mid \mathfrak{p} \in \mathrm{Spec}\, R\}.$$

Proof. A straightforward application of Matlis' structure theorem. $\square$

Finally, we deal with the Gorenstein flat dimension. The definition of Gorenstein flat modules goes back to [18].

(2.7) **Gorenstein flat dimension.** A *complete flat resolution* is a complex F of flat modules with $\mathrm{H}(F) = 0$ and $\mathrm{H}(J \otimes_R F) = 0$ for every injective R^{opp}–module J.

A module M is said to be *Gorenstein flat* if and only if there exists a complete flat resolution with a cokernel isomorphic to M.

The definition of the *Gorenstein flat dimension*, $\mathrm{Gfd}_R X$, of $X \in \mathsf{D}_{\sqsupset}(R)$ is similar to that of the Gorenstein projective given in definition (2.1), see also [10, (5.2.3)].

(2.8) **Theorem.** *Assume that R is right coherent, and let $X \in \mathsf{D}_{\sqsupset}(R)$ be a complex of finite Gorenstein flat dimension. For $n \in \mathbb{Z}$ the following are equivalent:*

(*i*) $\mathrm{Gfd}_R X \leq n$.

(*ii*) $n \geq \sup(U \otimes^{\mathbf{L}}_R X) - \sup U$ *for all* $U \in \mathsf{D}(R^{\mathrm{opp}})$ *of finite injective or finite flat dimension with* $\mathrm{H}(U) \neq 0$.

(*iii*) $n \geq \sup(J \otimes^{\mathbf{L}}_R X)$ *for all injective* R^{opp}*–modules* J.

(*iv*) $n \geq \sup X$ *and the cokernel* $\mathrm{C}^A_n = \mathrm{Coker}(A_{n+1} \to A_n)$ *is Gorenstein flat for any genuine right-bounded complex* $A \simeq X$ *of Gorenstein flat modules.*

Moreover, the following hold:

$$\begin{aligned}\mathrm{Gfd}_R X &= \sup\{\sup(U \otimes^{\mathbf{L}}_R X) - \sup U \mid \mathrm{id}_{R^{\mathrm{opp}}} U < \infty \text{ and } \mathrm{H}(U) \neq 0\}\\ &= \sup\{\sup(J \otimes^{\mathbf{L}}_R X) \mid J \text{ is injective}\}\\ &\leq \mathrm{FFD}(R) + \sup X.\end{aligned}$$

Proof. The first part of the proof is cyclic. The implications $(ii) \Rightarrow (iii)$ and $(iv) \Rightarrow (i)$ are immediate, and this leaves us two implications to prove.

$(i) \Rightarrow (ii)$: Choose a complex $A \in \mathsf{C}_{\square}(R)$ consisting of Gorenstein flat modules, such that $A \simeq X$ and $A_\ell = 0$ for $\ell > n$. First, let $U \in \mathsf{D}(R^{\mathrm{opp}})$ be a complex of finite injective dimension with $\mathrm{H}(U) \neq 0$. Set $s = \sup U$ and pick a genuine bounded complex $I \simeq U$ of injective modules with $I_\ell = 0$ for $\ell > s$. By corollary (3.15) the complex $I \otimes_R A$ is isomorphic to $U \otimes^{\mathbf{L}}_R X$ in $\mathsf{D}(\mathbb{Z})$; in particular, $\sup(U \otimes^{\mathbf{L}}_R X) = \sup(I \otimes_R A)$. For $\ell > n + s$ and $q \in \mathbb{Z}$ either $q > s$ or $\ell - q \geq \ell - s > n$, so the module

$$(I \otimes_R A)_\ell \;=\; \coprod_{q \in \mathbb{Z}} I_q \otimes_R A_{\ell - q}$$

vanishes. Hence, $\mathrm{H}_\ell(I \otimes_R A) = 0$ for $\ell > n+s$, forcing $\sup(U \otimes^{\mathbf{L}}_R X) \leq n+s = n+\sup U$ as desired.

Next, let $U \in \mathsf{D}(R^{\mathrm{opp}})$ be a complex of finite flat dimension and choose a bounded complex $F \simeq U$ of flat modules. Set $s = \sup U$ and consider the soft truncation $V = \subset_s F$. The modules in V have finite flat dimension and $U \simeq V$, hence $V \otimes_R A \simeq U \otimes^{\mathbf{L}}_R X$ by corollary (3.15), and the proof continues as above.

$(iii) \Rightarrow (iv)$: By assumption, $\mathrm{Gfd}_R X$ is finite, so a bounded complex $A \simeq X$ of Gorenstein flat modules does exist. For any injective R^{opp}–module J, we have $J \otimes^{\mathbf{L}}_R X \simeq J \otimes_R A$ by corollary (3.15), so

$$\begin{aligned}\sup(J \otimes^{\mathbf{L}}_R X) &= \sup(J \otimes_R A)\\ &= -\inf \mathrm{Hom}_{\mathbb{Z}}(J \otimes_R A, \mathbb{Z}/\mathbb{Q})\\ &= -\inf \mathrm{Hom}_{R^{\mathrm{opp}}}(J, \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z}/\mathbb{Q}))\\ &= -\inf \mathbf{R}\mathrm{Hom}_{R^{\mathrm{opp}}}(J, \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z}/\mathbb{Q})),\end{aligned}$$

where the last equality follows from corollary (3.11), as $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z}/\mathbb{Q})$ is a complex of Gorenstein injective modules by [29, thm. (3.6)]. As desired, we now have:

$$\begin{aligned}n &\geq \sup\{\sup(J \otimes^{\mathbf{L}}_R X) \mid J \text{ is injective}\}\\ &= \sup\{-\inf \mathbf{R}\mathrm{Hom}_{R^{\mathrm{opp}}}(J, \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z}/\mathbb{Q})) \mid J \text{ is injective}\}\\ &\geq -\inf \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z}/\mathbb{Q})\\ &= \sup A = \sup X,\end{aligned}$$

where the inequality follows from (2.5) (applied to R^{opp}). The rest of the argument is similar to the one given in the proof of theorem (2.2), as also the class of Gorenstein flat modules is resolving by [29, thm. (3.7)].

For the second part, we can argue, as we did in the proof of theorem (2.2), to see that $\operatorname{Gfd}_R X \leq \operatorname{FGFD}(R) + \sup X$, where

$$\operatorname{FGFD}(R) = \sup \left\{ \operatorname{Gfd}_R M \;\middle|\; \begin{array}{l} M \text{ is an } R\text{–module with finite} \\ \text{Gorenstein flat dimension} \end{array} \right\}$$

is the (left) finitistic Gorenstein flat dimension, cf. paragraph (1.4). By [29, thm. (3.24)] we have $\operatorname{FGFD}(R) = \operatorname{FFD}(R)$, and this concludes the proof. $\square$

(2.9) **Corollary.** *Assume that R is commutative and noetherian. If $X \in \mathsf{D}_{\sqsupset}(R)$ is a complex of finite Gorenstein flat dimension, then*

$$\operatorname{Gfd}_R X \;=\; \sup\{\sup(\mathrm{E}_R(R/\mathfrak{p}) \otimes^{\mathbf{L}}_R X) \mid \mathfrak{p} \in \operatorname{Spec} R\}.$$

Proof. A straightforward application of Matlis' structure theorem. $\square$

The next three results uncover some relations between the Gorenstein projective, injective and flat dimensions. All three are Gorenstein versions of well-established properties of the classical homological dimensions.

(2.10) **Theorem.** *For any complex $X \in \mathsf{D}_{\sqsupset}(R)$ there is an inequality,*

$$\operatorname{Gid}_{R^{\mathrm{opp}}} \operatorname{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z}) \leq \operatorname{Gfd}_R X,$$

and equality holds if R is right coherent.

Proof. If $X \simeq A$, where $A \in \mathsf{C}_{\sqsupset}(R)$ consists of Gorenstein flat modules, then $Y = \operatorname{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})$ is isomorphic in $\mathsf{D}(R^{\mathrm{opp}})$ to $B = \operatorname{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$; by [29, thm. (3.6)] B is a complex of Gorenstein injective modules. This proves the inequality.

Assume that R is right coherent, and let A, Y and B be as above. We are required to show $\operatorname{Gid}_{R^{\mathrm{opp}}} Y \geq \operatorname{Gfd}_R X$. We may assume that $n = \operatorname{Gid}_{R^{\mathrm{opp}}} Y$ is finite; note that $n \geq -\inf Y = \sup X$. By theorem (2.5) the kernel

$$\mathrm{Z}^B_{-n} \;=\; \mathrm{Z}^{\operatorname{Hom}_{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z})}_{-n} \;=\; \operatorname{Hom}_{\mathbb{Z}}(\mathrm{C}^A_n, \mathbb{Q}/\mathbb{Z})$$

is Gorenstein injective, hence [29, thm. (3.6)] informs us that the cokernel C^A_n is Gorenstein flat. Since $n \geq \sup X = \sup A$, it follows that $\subset_n A \simeq A \simeq X$, whence $\operatorname{Gfd}_R X \leq n$. $\square$

(2.11) **Proposition.** *Assume that R is right coherent and $\operatorname{FPD}(R) < \infty$. For every $X \in \mathsf{D}_{\sqsupset}(R)$ the next inequality holds*

$$\operatorname{Gfd}_R X \leq \operatorname{Gpd}_R X.$$

Proof. Under the assumptions, it follows by [29, prop. (3.4)] that every Gorenstein projective R–module also is Gorenstein flat. $\square$

We now compare the Gorenstein projective and Gorenstein flat dimension to Auslander and Bridger's G–dimension. In [3] Auslander and Bridger introduce the G–dimension, $\mathrm{G\text{–}dim}_R(-)$, for finitely generated modules over an associative ring R, which is both left and right noetherian.

The G–dimension is defined via resolutions consisting of modules from the so-called G–class, $\mathrm{G}(R)$. The G–class consists exactly of the finite R–modules M with $\mathrm{G\text{–}dim}_R M = 0$ (together with the zero-module). The basic properties are cataloged in [3, prop. (3.8)(c)].

When R is commutative and noetherian, [10, sec. 2.3] introduces a G–dimension, also denoted $\mathrm{G\text{–}dim}_R(-)$, for complexes in $\mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$. For modules it agrees with Auslander and Bridger's G–dimension. However, the definition given in [10, sec. 2.3] makes perfect sense over any associative and two-sided noetherian ring.

(2.12) **Theorem.** *Assume that R is left and right coherent. For a complex $X \in \mathsf{D}_{\sqsupset}(R)$ with finitely presented homology modules, the following hold.*

(a) *If $\mathrm{FPD}(R) < \infty$, then*

$$\mathrm{Gpd}_R X = \mathrm{Gfd}_R X.$$

(b) *If R is both left and right noetherian, then*

$$\mathrm{Gpd}_R X = \mathrm{G\text{–}dim}_R X.$$

Proof. Since R is right coherent with $\mathrm{FPD}(R) < \infty$, proposition (2.11) implies that $\mathrm{Gfd}_R X \leq \mathrm{Gpd}_R X$. To prove the opposite inequality in (a) we may assume that $n = \mathrm{Gfd}_R X$ is finite. Since R is left coherent, and since the homology modules of X are finitely presented, we can pick a projective resolution P of X, where each P_ℓ is finitely generated. The cokernel C^P_n is finitely presented, and by theorem (2.8) it is Gorenstein flat.

Reading the proof of [10, thm. (5.1.11)] (which deals with commutative, noetherian rings and is propelled by Lazard's result [32, lem. 1.1]) it is easy, but tedious, to check that over an associative and left coherent ring, any finitely presented Gorenstein flat module is also Gorenstein projective. Therefore, C^P_n is actually Gorenstein projective, which shows that $\mathrm{Gpd}_R X \leq n$ as desired.

Next, we turn to (b). By the "if" part of [10, thm. (4.2.6)], every module in the G–class is Gorenstein projective in the sense of definition (2.1). (Actually, [10, thm. (4.2.6)] is formulated under the assumption that R is commutative and noetherian, but reading the proof we see that it is valid over associative and two-sided noetherian rings as well.) It follows immediately that $\mathrm{Gpd}_R X \leq \mathrm{G\text{–}dim}_R X$.

In order to prove the opposite inequality, we may assume that $n = \mathrm{Gpd}_R X$ is finite. Let P be any projective resolution of X by finitely generated modules, and consider cokernel the C^P_n. Of course, C^P_n is finitely generated, and by theorem (2.2) it is also Gorenstein projective. Now the "only if" part of (the already mentioned "associative version" of) [10, thm. (4.2.6)] gives that C^P_n belongs to the G–class. Hence, $\subset_n P$ is a resolution of X consisting of modules from the G–class and, thus, $\mathrm{G\text{–}dim}_R X \leq n$. $\square$

The Gorenstein dimensions refine the classical homological dimensions. On the other hand, the next three lemmas show that a module of finite Gorenstein projective/injective/flat dimension can be approximated by a module, for which the corresponding classical homological dimension is also finite.

(2.13) **Lemma.** *Let M be an R–module of finite Gorenstein projective dimension. There is then an exact sequence,*

$$0 \to M \to H \to A \to 0,$$

where A is Gorenstein projective and $\mathrm{pd}_R\, H = \mathrm{Gpd}_R\, M$.

Proof. If M is Gorenstein projective, we simply take $0 \to M \to H \to A \to 0$ to be the first short exact sequence in the "right half" of a complete projective resolution of M.

We may now assume that $\mathrm{Gpd}_R\, M = n > 0$. By [29, thm. (2.10)] there exists an exact sequence,

$$0 \to K \to A' \to M \to 0,$$

where A' is Gorenstein projective, and $\mathrm{pd}_R\, K = n-1$. Since A' is Gorenstein projective, there exists (as above) a short exact sequence,

$$0 \to A' \to Q \to A \to 0,$$

where Q is projective, and A is Gorenstein projective. Consider the push-out diagram,

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & & A \arrow[u] \arrow[r, equal] & A \arrow[u] & \\
0 \arrow[r] & K \arrow[r] & Q \arrow[u] \arrow[r] & H \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[u, equal] \arrow[r] & A' \arrow[u] \arrow[r] & M \arrow[u] \arrow[r] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

The class of Gorenstein projective modules is resolving, so if H was projective, exactness of the second column would imply that $\mathrm{Gpd}_R\, M = 0$. The first row therefore shows that $\mathrm{pd}_R\, H = n$, and the second column is the desired sequence. $\square$

The next two lemmas have similar proofs.

(2.14) **Lemma.** *Let N be an R–module of finite Gorenstein injective dimension. There is then an exact sequence,*

$$0 \to B \to H \to N \to 0,$$

where B is Gorenstein injective and $\mathrm{id}_R\, H = \mathrm{Gid}_R\, N$. $\square$

(2.15) **Lemma.** *Assume that R is right coherent, and let M be an R–module of finite Gorenstein flat dimension. There is then an exact sequence,*

$$0 \to M \to H \to A \to 0,$$

where A is Gorenstein flat and $\mathrm{fd}_R H = \mathrm{Gfd}_R M$. □

(2.16) **Remarks on distinguished triangles.** It is natural to ask if finiteness of Gorenstein dimensions is closed under distinguished triangles. That is, whenever we encounter a distinguished triangle,

$$X \to Y \to Z \to \Sigma X,$$

where two of the three complexes X, Y and Z have finite, say, Gorenstein projective dimension, is then also the third complex of finite Gorenstein projective dimension?

Of course, once we have established our main theorems, (4.3) and (4.5), it follows that over a commutative and noetherian ring admitting a dualizing complex, finiteness of each of the three Gorenstein dimensions is closed under distinguished triangles. This conclusion is immediate, as the Auslander and Bass classes are full triangulated subcategories of $\mathsf{D}(R)$. But from the definitions and results of this section, it is not clear that the Gorenstein dimensions possess this property over general associative rings.

However, in [39] Veliche introduces a Gorenstein projective dimension for unbounded complexes. By [39, thm. 3.2.8(1)] the finiteness of this dimension is closed under distinguished triangles; by [39, thm. 3.3.6] it coincides, for right-bounded complexes, with the Gorenstein projective dimension studied in this paper.

3. Ubiquity of quasi-isomorphisms

In this section we establish some important, technical results on preservation of quasi-isomorphisms. It is, e.g., a crucial ingredient in the proof of the main theorem (4.3) that the functor $-\otimes_R A$ preserves certain quasi-isomorphisms, when A is a Gorenstein flat module. This is established in theorem (3.14) below. An immediate consequence of this result is that Gorenstein flat modules may sometimes substitute for real flat modules in representations of derived tensor products. This corollary, (3.15), plays an important part in the proof of theorem (2.8). The proofs in this section do not depend logically on the previous section.

Similar results on representations of the derived Hom functor are used in the proofs of theorems (2.2) and (2.5). These are also established below.

Throughout this section, R is an associative ring with unit.

We start by deriving some immediate lemmas from the definitions of Gorenstein projective, injective, and flat modules.

(3.1) **Lemma.** *If M is a Gorenstein projective R–module, then* $\mathrm{Ext}_R^m(M,T) = 0$ *for all $m > 0$ and all modules T of finite projective or finite injective dimension.*

Proof. For a module T of finite projective dimension, the vanishing of $\mathrm{Ext}_R^m(M,T)$ is an immediate consequence of the definition of Gorenstein projective modules.

Assume that $\mathrm{id}_R T = n < \infty$. Since M is Gorenstein projective, we have an exact sequence,

$$0 \to M \to P_0 \to P_{-1} \to \cdots \to P_{1-n} \to C \to 0,$$

where the P's are projective modules. Breaking this sequence into short exact ones, we see that $\mathrm{Ext}_R^m(M,T) = \mathrm{Ext}_R^{m+n}(C,T)$ for $m > 0$, so the Ext's vanish as desired since $\mathrm{Ext}_R^w(-,T) = 0$ for $w > n$. $\square$

Similarly one establishes the next two lemmas.

(3.2) **Lemma.** *If N is a Gorenstein injective R–module, then $\mathrm{Ext}_R^m(T,N) = 0$ for all $m > 0$ and all modules T of finite projective or finite injective dimension.* $\square$

(3.3) **Lemma.** *If M is a Gorenstein flat R–module, then $\mathrm{Tor}^R_m(T,M) = 0$ for all $m > 0$ and all R^{opp}–modules T of finite flat or finite injective dimension.* $\square$

From lemma (3.1) it is now a three step process to arrive at the desired results on preservation of quasi-isomorphisms by the Hom functor. We give proofs for the results pertaining with the covariant Hom functor; those on the contravariant functor have similar proofs.

(3.4) **Lemma.** *Assume that $X, Y \in \mathsf{C}(R)$ with either $X \in \mathsf{C}_{\sqsupset}(R)$ or $Y \in \mathsf{C}_{\sqsupset}(R)$. If $\mathrm{H}(\mathrm{Hom}_R(X_\ell, Y)) = 0$ for all $\ell \in \mathbb{Z}$, then $\mathrm{H}(\mathrm{Hom}_R(X,Y)) = 0$.*

Proof. Immediate from the proof of [20, lem. (6.7)]. $\square$

(3.5) **Lemma.** *Assume that $X, Y \in \mathsf{C}(R)$ with either $X \in \mathsf{C}_{\sqsubset}(R)$ or $Y \in \mathsf{C}_{\sqsubset}(R)$. If $\mathrm{H}(\mathrm{Hom}_R(X, Y_\ell)) = 0$ for all $\ell \in \mathbb{Z}$, then $\mathrm{H}(\mathrm{Hom}_R(X,Y)) = 0$.* $\square$

(3.6) **Proposition.** *Consider a class $\mathfrak{U}$ of R–modules, and let $\alpha\colon X \to Y$ be a morphism in $\mathsf{C}(R)$, such that*

$$\mathrm{Hom}_R(U,\alpha)\colon\ \mathrm{Hom}_R(U,X) \xrightarrow{\simeq} \mathrm{Hom}_R(U,Y)$$

is a quasi-isomorphism for every module $U \in \mathfrak{U}$.

Let $\widetilde{U} \in \mathsf{C}(R)$ be a complex consisting of modules from $\mathfrak{U}$. The induced morphism,

$$\mathrm{Hom}_R(\widetilde{U},\alpha)\colon\ \mathrm{Hom}_R(\widetilde{U},X) \longrightarrow \mathrm{Hom}_R(\widetilde{U},Y),$$

is then a quasi-isomorphism, provided that either

(a) *$\widetilde{U} \in \mathsf{C}_{\sqsupset}(R)$, or*
(b) *$X, Y \in \mathsf{C}_{\sqsupset}(R)$.*

Proof. Under either hypothesis (a) or (b) we wish to show exactness of the mapping cone

$$\mathsf{Cone}(\mathrm{Hom}_R(\widetilde{U},\alpha)) \simeq \mathrm{Hom}_R(\widetilde{U},\mathsf{Cone}(\alpha)).$$

Condition (b) implies that $\mathsf{Cone}(\alpha) \in \mathsf{C}_{\sqsupset}(R)$. In any event, lemma (3.4) informs us that it suffices to show that the complex $\mathrm{Hom}_R(\widetilde{U}_\ell, \mathsf{Cone}(\alpha))$ is exact for all $\ell \in \mathbb{Z}$. But this follows as all

$$\mathrm{Hom}_R(\widetilde{U}_\ell, \alpha)\colon\ \mathrm{Hom}_R(\widetilde{U}_\ell, X) \xrightarrow{\simeq} \mathrm{Hom}_R(\widetilde{U}_\ell, Y)$$

are assumed to be quasi-isomorphisms in $\mathsf{C}(R)$. □

(3.7) **Proposition.** *Consider a class $\mathfrak{V}$ of R–modules, and let $\alpha\colon X \to Y$ be a morphism in $\mathsf{C}(R)$, such that*

$$\mathrm{Hom}_R(\alpha, V)\colon\ \mathrm{Hom}_R(Y, V) \xrightarrow{\simeq} \mathrm{Hom}_R(X, V)$$

is a quasi-isomorphism for every module $V \in \mathfrak{V}$.

Let $\widetilde{V} \in \mathsf{C}(R)$ be a complex consisting of modules from $\mathfrak{V}$. The induced morphism,

$$\mathrm{Hom}_R(\alpha, \widetilde{V})\colon\ \mathrm{Hom}_R(Y, \widetilde{V}) \longrightarrow \mathrm{Hom}_R(X, \widetilde{V}),$$

is then a quasi-isomorphism, provided that either

(a) $\widetilde{V} \in \mathsf{C}_{\sqsubset}(R)$, *or*
(b) $X, Y \in \mathsf{C}_{\sqsubset}(R)$. □

(3.8) **Theorem.** *Let $V \xrightarrow{\simeq} W$ be a quasi-isomorphism between R–complexes, where each module in V and W has finite projective dimension or finite injective dimension. If $A \in \mathsf{C}_{\sqsupset}(R)$ is a complex of Gorenstein projective modules, then the induced morphism*

$$\mathrm{Hom}_R(A, V) \longrightarrow \mathrm{Hom}_R(A, W)$$

is a quasi-isomorphism under each of the next two conditions.

(a) $V, W \in \mathsf{C}_{\sqsubset}(R)$
(b) $V, W \in \mathsf{C}_{\sqsupset}(R)$

Proof. By proposition (3.6)(a) we may immediately reduce to the case, where A is a Gorenstein projective module. In this case we have quasi-isomorphisms $\mu\colon P \xrightarrow{\simeq} A$ and $\nu\colon A \xrightarrow{\simeq} \widetilde{P}$ in $\mathsf{C}(R)$, where $P \in \mathsf{C}_{\sqsupset}(R)$ and $\widetilde{P} \in \mathsf{C}_{\sqsubset}(R)$ are complexes of projective modules. More precisely, P and $\widetilde{P}$ are, respectively, the "left half" and "right half" of a complete projective resolution of A.

Let T be any R–module of finite projective or finite injective dimension. Lemma (3.1) implies that a complete projective resolution stays exact when the functor $\mathrm{Hom}_R(-, T)$ is applied to it. In particular, the induced morphisms

$$\mathrm{Hom}_R(\mu, T)\colon\ \mathrm{Hom}_R(A, T) \xrightarrow{\simeq} \mathrm{Hom}_R(P, T) \tag{1}$$

and

$$\mathrm{Hom}_R(\nu, T)\colon\ \mathrm{Hom}_R(\widetilde{P}, T) \xrightarrow{\simeq} \mathrm{Hom}_R(A, T) \tag{2}$$

are quasi-isomorphisms.

From (1) and proposition (3.7)(a) it follows that under assumption (a) both $\mathrm{Hom}_R(\mu, V)$ and $\mathrm{Hom}_R(\mu, W)$ are quasi-isomorphisms. In the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(A,V) \arrow[r] \arrow[d, "\mathrm{Hom}(\mu,V)"', "\simeq"] & \mathrm{Hom}_R(A,W) \arrow[d, "\simeq"', "\mathrm{Hom}(\mu,W)"] \\
\mathrm{Hom}_R(P,V) \arrow[r, "\simeq"] & \mathrm{Hom}_R(P,W)
\end{tikzcd}$$

the lower horizontal morphism is obviously a quasi-isomorphism, and this makes the induced morphism $\mathrm{Hom}_R(A, V) \longrightarrow \mathrm{Hom}_R(A, W)$ a quasi-isomorphism as well.

Under assumption (b), the induced morphism $\mathrm{Hom}_R(\widetilde{P}, V) \rightarrow \mathrm{Hom}_R(\widetilde{P}, W)$ is a quasi-isomorphism by proposition (3.6)(b). As the induced morphisms (2) are quasi-isomorphisms, it follows by proposition (3.7)(b) that so are $\mathrm{Hom}_R(\nu, V)$ and $\mathrm{Hom}_R(\nu, W)$. Since the diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(A,V) \arrow[r] & \mathrm{Hom}_R(A,W) \\
\mathrm{Hom}_R(\widetilde{P},V) \arrow[r, "\simeq"] \arrow[u, "\mathrm{Hom}(\nu,V)", "\simeq"'] & \mathrm{Hom}_R(\widetilde{P},W) \arrow[u, "\simeq", "\mathrm{Hom}(\nu,W)"']
\end{tikzcd}$$

commutes, we conclude that also its top vertical morphism is a quasi-isomorphism. $\square$

(3.9) **Theorem.** *Let $V \xrightarrow{\simeq} W$ be a quasi-isomorphism between R–complexes, where each module in V and W has finite projective dimension or finite injective dimension. If $B \in \mathsf{C}_{\sqsubset}(R)$ is a complex of Gorenstein injective modules, then the induced morphism*

$$\mathrm{Hom}_R(W, B) \longrightarrow \mathrm{Hom}_R(V, B)$$

is a quasi-isomorphism under each of the next two conditions.

(a) $V, W \in \mathsf{C}_{\sqsupset}(R)$
(b) $V, W \in \mathsf{C}_{\sqsubset}(R)$ $\square$

(3.10) **Corollary.** *Assume that $X \simeq A$, where $A \in \mathsf{C}_{\sqsupset}(R)$ is a complex of Gorenstein projective modules. If $U \simeq V$, where $V \in \mathsf{C}_{\sqsubset}(R)$ is a complex in which each module has finite projective dimension or finite injective dimension, then*

$$\mathbf{R}\mathrm{Hom}_R(X, U) \simeq \mathrm{Hom}_R(A, V).$$

Proof. Take an injective resolution $U \xrightarrow{\simeq} I$, where $I \in \mathsf{C}_{\sqsubset}(R)$ consists of injective modules, then $\mathbf{R}\mathrm{Hom}_R(X, U) \simeq \mathrm{Hom}_R(A, I)$. By, e.g., [5, 1.1.I and 1.4.I] there is also a quasi-isomorphism $V \xrightarrow{\simeq} I$. Whence, by theorem (3.8)(a) we immediately get $\mathbf{R}\mathrm{Hom}_R(X, U) \simeq \mathrm{Hom}_R(A, I) \simeq \mathrm{Hom}_R(A, V)$. $\square$

(3.11) **Corollary.** *Assume that $Y \simeq B$, where $B \in \mathsf{C}_{\sqsubset}(R)$ is a complex of Gorenstein injective modules. If $U \simeq V$, where $V \in \mathsf{C}_{\sqsupset}(R)$ is a complex in which each module has finite projective dimension or finite injective dimension, then*

$$\mathbf{R}\mathrm{Hom}_R(U, Y) \simeq \mathrm{Hom}_R(V, B). \quad \square$$

Next, we turn to tensor products and Gorenstein flat modules. The first lemma follows by applying Pontryagin duality to lemma (3.4) for R^{opp}.

(3.12) **Lemma.** *Assume that* $X \in \mathsf{C}(R^{\mathrm{opp}})$ *and* $Y \in \mathsf{C}(R)$ *with either* $X \in \mathsf{C}_{\sqsupset}(R^{\mathrm{opp}})$ *or* $Y \in \mathsf{C}_{\sqsubset}(R)$*. If* $\mathrm{H}(X_\ell \otimes_R Y) = 0$ *for all* $\ell \in \mathbb{Z}$*, then* $\mathrm{H}(X \otimes_R Y) = 0$*.* ☐

(3.13) **Proposition.** *Consider a class* $\mathfrak{W}$ *of* R^{opp}*–modules, and let* $\alpha\colon X \to Y$ *be a morphism in* $\mathsf{C}(R)$*, such that*

$$W \otimes_R \alpha\colon W \otimes_R X \xrightarrow{\simeq} W \otimes_R Y$$

is a quasi-isomorphism for every module $W \in \mathfrak{W}$*.*

Let $\widetilde{W} \in \mathsf{C}(R^{\mathrm{opp}})$ *be a complex consisting of modules from* $\mathfrak{W}$*. The induced morphism,*

$$\widetilde{W} \otimes_R \alpha\colon \widetilde{W} \otimes_R X \longrightarrow \widetilde{W} \otimes_R Y,$$

is then a quasi-isomorphism, provided that either

(a) $\widetilde{W} \in \mathsf{C}_{\sqsupset}(R^{\mathrm{opp}})$*, or*
(b) $X, Y \in \mathsf{C}_{\sqsubset}(R)$*.*

Proof. It follows immediately by lemma (3.12) that the mapping cone $\mathsf{Cone}\,(\widetilde{W} \otimes_R \alpha) \simeq \widetilde{W} \otimes_R \mathsf{Cone}\,(\alpha)$ is exact under either assumption (a) or (b). ☐

(3.14) **Theorem.** *Let* $V \xrightarrow{\simeq} W$ *be a quasi-isomorphism between complexes of* R^{opp}*–modules, where each module in* V *and* W *has finite injective dimension or finite flat dimension. If* $A \in \mathsf{C}_{\sqsupset}(R)$ *is a complex of Gorenstein flat modules, then the induced morphism*

$$V \otimes_R A \longrightarrow W \otimes_R A$$

is a quasi-isomorphism under each of the next two conditions.

(a) $V, W \in \mathsf{C}_{\sqsupset}(R)$
(b) $V, W \in \mathsf{C}_{\sqsubset}(R)$

Proof. Using proposition (3.13)(a), applied to R^{opp}, we immediately reduce to the case, where A is a Gorenstein flat module. In this case we have quasi-isomorphisms $\mu\colon F \xrightarrow{\simeq} A$ and $\nu\colon A \xrightarrow{\simeq} \widetilde{F}$ in $\mathsf{C}(R)$, where $F \in \mathsf{C}_{\sqsupset}(R)$ and $\widetilde{F} \in \mathsf{C}_{\sqsubset}(R)$ are complexes of flat modules. To be precise, F and $\widetilde{F}$ are, respectively, the "left half" and "right half" of a complete flat resolution of A. The proof now continues as the proof of theorem (3.8); only using proposition (3.13) instead of (3.6) and (3.7), and lemma (3.3) instead of (3.1). ☐

(3.15) **Corollary.** *Assume that* $X \simeq A$*, where* $A \in \mathsf{C}_{\sqsupset}(R)$ *is a complex of Gorenstein flat modules. If* $U \simeq V$*, where* $V \in \mathsf{C}_{\sqsupset}(R^{\mathrm{opp}})$ *is a complex in which each module has finite flat dimension or finite injective dimension, then*

$$U \otimes^{\mathbf{L}}_R X \simeq V \otimes_R A.$$

Proof. Choose a complex $P \in \mathsf{C}_{\sqsupset}(R^{\mathrm{opp}})$ of projective modules, such that $P \simeq U \simeq V$. There is then a quasi-isomorphism $P \xrightarrow{\simeq} V$, in $\mathsf{C}(R^{\mathrm{opp}})$, and it follows by theorem (3.14)(a) that $U \otimes^{\mathbf{L}}_R X \simeq P \otimes_R A \simeq V \otimes_R A$. ☐

4. Auslander and Bass Classes

We can now prove the conjectured characterization of finite Gorenstein dimensions: The objects in the Auslander class are exactly the complexes of finite Gorenstein projective/flat dimension. Similarly, the objects in the Bass class are the complexes of finite Gorenstein injective dimension.

In particular, this section shows that the full subcategory, of $\mathsf{D}(R)$, of complexes of finite Gorenstein projective/flat dimension is quasi-equivalent, by Foxby equivalence, to the full subcategory of complexes of finite Gorenstein injective dimension.

Throughout this section, R is a commutative and noetherian ring.

(4.1) **Proposition.** *If R admits a dualizing complex, then the Krull dimension of R is finite, and there exists an integer $S \geq 0$, such that*

(a) *For all complexes $X \in \mathsf{A}(R)$ and all modules M with $\operatorname{fd}_R M < \infty$, we have*

$$-\inf \mathbf{R}\mathrm{Hom}_R(X,M) \leq S + \sup X.$$

(b) *For all complexes $X \in \mathsf{A}(R)$ and all modules N with $\operatorname{id}_R N < \infty$, we have*

$$\sup\,(N \otimes^{\mathbf{L}}_R X) \leq S + \sup X.$$

(c) *For all complexes $Y \in \mathsf{B}(R)$ and all modules N with $\operatorname{id}_R N < \infty$, we have*

$$-\inf \mathbf{R}\mathrm{Hom}_R(N,Y) \leq S - \inf Y.$$

Proof. To see that R has finite Krull dimension, it suffices to prove that $\operatorname{FPD}(R) < \infty$, cf. (1.4.1). Let X be a non-trivial module of finite projective dimension, say p; then, in particular, $\operatorname{Ext}^p_R(X,T) \neq 0$ for some module T, and it is easy to see that also $\operatorname{Ext}^p_R(X,M) \neq 0$ for any module M which surjects onto T. Let M be a projective module with this property. It is well-known that X belongs to the Auslander class $\mathsf{A}(R)$. Thus, once we have established (a), it follows that

$$p = -\inf \mathbf{R}\mathrm{Hom}_R(X,M) \leq S,$$

forcing $\operatorname{FPD}(R) \leq S < \infty$.

To prove (a), we start by noting that the inequality is obvious if X or M is trivial; therefore, we may assume that $\mathrm{H}(X) \neq 0 \neq M$. Let D denote the dualizing complex. As $X \in \mathsf{A}(R)$, it is, by definition, bounded. By assumption, M has finite flat dimension, in particular, $M \in \mathsf{A}(R)$. This allows us to perform the computation below.

$$\begin{aligned}
-\inf \mathbf{R}\mathrm{Hom}_R(X,M) &= -\inf \mathbf{R}\mathrm{Hom}_R(X,\mathbf{R}\mathrm{Hom}_R(D,D \otimes^{\mathbf{L}}_R M)) \\
&\overset{(1)}{=} -\inf \mathbf{R}\mathrm{Hom}_R(D \otimes^{\mathbf{L}}_R X, D \otimes^{\mathbf{L}}_R M) \\
&\overset{(2)}{\leq} \sup\,(D \otimes^{\mathbf{L}}_R X) + \operatorname{id}_R(D \otimes^{\mathbf{L}}_R M) \\
&\overset{(3)}{\leq} \sup D + \sup X + \operatorname{id}_R(D \otimes^{\mathbf{L}}_R M) \\
&\overset{(4)}{\leq} \sup D + \sup X + \operatorname{id}_R D - \inf M \\
&= \sup D + \operatorname{id}_R D + \sup X.
\end{aligned}$$

Here (1) is by adjunction; (2) follows by [10, (A.5.2.1)] as $D \otimes^{\mathbf{L}}_R X$ is a bounded complex with non-trivial homology; (3) follows from [10, prop. (3.3.7)(a)], while (4) follows from [10, (A.5.8.3)]. By definition (1.1), the number $\sup D + \mathrm{id}_R D$ is finite, and the proof of (a) is complete.

Parts (b) and (c) have similar proofs, and the computations show that all three parts hold with $S = \sup D + \mathrm{id}_R D + \mathrm{amp}\, D \geq 0$. □

(4.2) **Lemma.** *Assume that R admits a dualizing complex. If M is an R–module satisfying*

(a) $M \in \mathsf{A}(R)$*, and*
(b) $\mathrm{Ext}^m_R(M, Q) = 0$ *for all integers $m > 0$, and all projective R–modules Q,*

then M is Gorenstein projective.

Proof. We are required to construct a complete projective resolution of M. For the left half of this resolution, any ordinary projective resolution of M will do, because of (b). In order to construct the right half, it suffices to construct a short exact sequence,

$$0 \to M \to P' \to M' \to 0, \tag{1}$$

where P' is a projective module and M' also satisfies (a) and (b). The construction of (1) is done in three steps.

1° First we show that M can be embedded in a module of finite flat dimension. Pick a projective resolution, P, together with a bounded injective resolution, I, of the dualizing complex D:

$$\mathsf{C}_{\sqsupset}(R) \ni P \xrightarrow{\simeq} D \xrightarrow{\simeq} I \in \mathsf{C}_{\square}(R).$$

Since $M \in \mathsf{A}(R)$, the complex $P \otimes_R M$ has bounded homology; in particular, $P \otimes_R M$ admits an injective resolution,

$$P \otimes_R M \xrightarrow{\simeq} J \in \mathsf{C}_{\sqsubset}(R).$$

Concordantly, we get quasi-isomorphisms

$$M \xrightarrow{\simeq} \mathrm{Hom}_R(P, P \otimes_R M) \xrightarrow{\simeq} \mathrm{Hom}_R(P, J) \xleftarrow{\simeq} \mathrm{Hom}_R(I, J),$$

where $F = \mathrm{Hom}_R(I, J) \in \mathsf{C}_{\sqsubset}(R)$ is a complex of flat modules. In particular, the modules M and $\mathrm{H}_0(F)$ are isomorphic, and $\mathrm{H}_\ell(F) = 0$ for all $\ell \neq 0$. Obviously, $\mathrm{H}_0(F)$ is embedded in the cokernel $\mathrm{C}^F_0 = \mathrm{Coker}(F_1 \to F_0)$, and C^F_0 has finite flat dimension since

$$\cdots \to F_1 \to F_0 \to \mathrm{C}^F_0 \to 0$$

is exact and $F \in \mathsf{C}_{\sqsubset}(R)$. This proves the first claim.

2° Next, we show that M can be embedded in a flat (actually free) module. Note that, by induction on $\mathrm{pd}_R K$, condition (b) is equivalent to

(b') $\mathrm{Ext}^m_R(M, K) = 0$ for all integers $m > 0$ and all modules K with $\mathrm{pd}_R K < \infty$.

By the already established **1°** there exists an embedding $M \hookrightarrow C$, where C is a module of finite flat dimension. Pick a short exact sequence,

$$0 \to K \to L \to C \to 0, \tag{2}$$

where L is free and, consequently, $\mathrm{fd}_R K < \infty$. From proposition (4.1) we learn that R is of finite Krull dimension; this forces $\mathrm{pd}_R K$ to be finite, and hence $\mathrm{Ext}^1_R(M,K) = 0$ by (b'). Applying $\mathrm{Hom}_R(M,-)$ to (2), we get an exact sequence,

$$\mathrm{Hom}_R(M,L) \to \mathrm{Hom}_R(M,C) \to \mathrm{Ext}^1_R(M,K) = 0,$$

which yields a factorization,

$$\begin{tikzcd} M \arrow[r] \arrow[dr, dotted] & C \\ & L \arrow[u] \end{tikzcd}$$

As $M \hookrightarrow C$ is a monomorphism, so is the map from M into the free module L.

3° Finally, we are able to construct (1). Pick a flat preenvelope $\varphi\colon M \to F$ of M, cf. [14, prop. 5.1]. By **2°** the module M can be embedded into a flat module, L, forcing also $\varphi\colon M \to F$ to be a monomorphism,

$$0 \longrightarrow M \xrightarrow{\varphi} F.$$

Now choose a projective module P' surjecting onto F, that is,

$$0 \to Z \to P' \xrightarrow{\pi} F \to 0$$

is exact. Repeating the argument above, we get a factorization

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[dr, dotted, "\partial"'] & F \\ & P' \arrow[u, "\pi"'] \end{tikzcd}$$

and because φ is injective so is ∂. Thus, we have a short exact sequence

$$0 \to M \xrightarrow{\partial} P' \to M' \to 0. \tag{3}$$

What remains to be proved is that M' has the same properties as M. The projective module P' belongs to the Auslander class, and by assumption so does M. Since $\mathsf{A}(R)$ is a full triangulated subcategory of $\mathsf{D}(R)$, also $M' \in \mathsf{A}(R)$. Let Q be projective; for $m > 0$ we have $\mathrm{Ext}^m_R(M,Q) = 0 = \mathrm{Ext}^m_R(P',Q)$, so it follows from the long exact sequence of Ext modules associated to (3) that $\mathrm{Ext}^m_R(M',Q) = 0$ for $m > 1$. To prove that $\mathrm{Ext}^1_R(M',Q) = 0$, we consider the right-exact sequence

$$\mathrm{Hom}_R(P',Q) \xrightarrow{\mathrm{Hom}_R(\partial,Q)} \mathrm{Hom}_R(M,Q) \to \mathrm{Ext}^1_R(M',Q) \to 0.$$

Since Q is flat and $\varphi\colon M \to F$ is a flat preenvelope, there exists, for each $\tau \in \mathrm{Hom}_R(M,Q)$, a homomorphism $\tau'\colon F \to Q$ such that $\tau = \tau'\varphi$; that is, $\tau = \tau'\pi\partial = \mathrm{Hom}_R(\partial,Q)(\tau'\pi)$. Thus, the induced map $\mathrm{Hom}_R(\partial,Q)$ is surjective and, therefore, $\mathrm{Ext}^1_R(M',Q) = 0$. $\square$

(4.3) **Theorem.** *Let R be a commutative and noetherian ring admitting a dualizing complex. For any $X \in \mathsf{D}_{\sqsupset}(R)$ the following conditions are equivalent:*

(*i*) $X \in \mathsf{A}(R)$.
(*ii*) $\operatorname{Gpd}_R X$ *is finite.*
(*iii*) $\operatorname{Gfd}_R X$ *is finite.*

Proof. By proposition (4.1) the Krull dimension $\dim R$ is finite, and $(ii) \Rightarrow (iii)$ therefore follows by proposition (2.11).

$(iii) \Rightarrow (i)$: If $\operatorname{Gfd}_R X$ is finite, then, by definition, X is isomorphic in $\mathsf{D}(R)$ to a complex $A \in \mathsf{C}_{\square}(R)$ of Gorenstein flat modules. Pick a genuine bounded injective resolution resolution I of the dualizing complex, D, together with a resolution, P, by finitely generated projective modules,

$$\mathsf{C}_{\sqsupset}(R) \ni P \xrightarrow{\simeq} D \xrightarrow{\simeq} I \in \mathsf{C}_{\square}(R).$$

Let $\lambda\colon P \xrightarrow{\simeq} I$ denote the composition $P \xrightarrow{\simeq} D \xrightarrow{\simeq} I$. By theorem (2.8) $D \otimes^{\mathbf{L}}_R X$ is bounded. Whence, to prove that $X \in \mathsf{A}(R)$ we only need to show that the unit evaluated on A, that is,

$$\eta_A\colon A \longrightarrow \operatorname{Hom}_R(P, P \otimes_R A)$$

is a quasi-isomorphism. To this end, consider the commutative diagram:

$$\begin{tikzcd}
A \arrow[rr, "\eta_A"] \arrow[d, "\simeq"] & & \operatorname{Hom}_R(P, P \otimes_R A) \arrow[d, "\simeq"', "\operatorname{Hom}(P,\lambda\otimes A)"] \\
R \otimes_R A \arrow[d, "\chi^R_I\otimes A"', "\simeq"] & & \operatorname{Hom}_R(P, I \otimes_R A) \arrow[d, "\simeq"', "\omega^{-1}_{PIA}"] \\
\operatorname{Hom}_R(I,I) \otimes_R A \arrow[rr, "\simeq", "\operatorname{Hom}(\lambda,I)\otimes A"'] & & \operatorname{Hom}_R(P,I) \otimes_R A
\end{tikzcd}$$

We claim that all the morphisms in this diagram marked with the symbol "$\simeq$" (the maps forming a "horse shoe") are quasi-isomorphisms, and hence, η_A must be the same. Following the horse shoe counter-clockwise from A, we argue as follows:

- The map $A \xrightarrow{\simeq} R \otimes_R A$ is a trivial isomorphism.
- By definition (1.1) the morphism $\chi^R_I\colon R \xrightarrow{\simeq} \operatorname{Hom}_R(I,I)$ is a quasi-isomorphism. Note that R and $\operatorname{Hom}_R(I,I)$ belong to $\mathsf{C}_{\sqsupset}(R)$ and consist of flat modules. Therefore, $\chi^R_I \otimes_R A$ is a quasi-isomorphism; see e.g. [10, (A.4.1)].
- Since λ is a quasi-isomorphism, so is $\operatorname{Hom}_R(\lambda, I)$. The complex $\operatorname{Hom}_R(I,I)$ is bounded and consists of flat modules, while $\operatorname{Hom}_R(P,I) \in \mathsf{C}_{\sqsubset}(R)$ consists of injective modules. As A is a bounded complex of Gorenstein flat modules, it follows by theorem (3.14)(b) that $\operatorname{Hom}_R(\lambda, I) \otimes_R A$ is a quasi-isomorphism.
- By e.g. [10, (A.2.10)] ω_{PIA} is a genuine isomorphism in $\mathsf{C}(R)$.
- It follows by theorem (3.14)(a) that the induced morphism $\lambda \otimes_R A\colon P \otimes_R A \xrightarrow{\simeq} I \otimes_R A$ is a quasi-isomorphism, and hence so is $\operatorname{Hom}_R(P, \lambda \otimes_R A)$.

$(i) \Rightarrow (ii)$: Assume that $X \in \mathsf{A}(R)$. By proposition (4.1) there exists an integer $S \geq 0$ such that

$$-\inf \mathbf{R}\mathrm{Hom}_R(X, M) \leq S + \sup X,$$

for all modules M with $\mathrm{fd}_R M < \infty$. Set $n = S + \sup X$, and note that $n \geq \sup X$. Take any complex $A \in \mathsf{C}_{\sqsupset}(R)$ of Gorenstein projective modules, such that $A \simeq X$. It suffices to show that the cokernel $\mathrm{C}_n^A = \mathrm{Coker}(A_{n+1} \to A_n)$ is a Gorenstein projective module. By lemma (4.2) it is enough to prove that

(a) $\mathrm{C}_n^A \in \mathsf{A}(R)$, and
(b) $\mathrm{Ext}_R^m(\mathrm{C}_n^A, Q) = 0$ for all integers $m > 0$, and all projective R–modules Q.

Consider the distinguish triangle,

$$\sqsubset_{n-1} A \longrightarrow \subset_n A \longrightarrow \Sigma^n \mathrm{C}_n^A \longrightarrow \Sigma(\sqsubset_{n-1} A).$$

Since $n \geq \sup A$, we have $\subset_n A \simeq A \simeq X \in \mathsf{A}(R)$. Moreover, note that

$$\mathrm{Gfd}_R(\sqsubset_{n-1} A) \leq \mathrm{Gpd}_R(\sqsubset_{n-1} A) \leq n - 1 < \infty,$$

whence applying the already established implication $(iii) \Rightarrow (i)$, we get $\sqsubset_{n-1} A \in \mathsf{A}(R)$. Evoking the fact that $\mathsf{A}(R)$ is a full triangulated subcategory of $\mathsf{D}(R)$, we conclude that $\Sigma^n \mathrm{C}_n^A$, and hence C_n^A, belongs to $\mathsf{A}(R)$. This establishes (a).

To verify (b), we let $m > 0$ be any integer, and Q be any projective R–module. It is a straightforward computation, cf. [10, lem. (4.3.9)], to see that

$$\mathrm{Ext}_R^m(\mathrm{C}_n^A, Q) = \mathrm{H}_{-(m+n)}(\mathbf{R}\mathrm{Hom}_R(X, Q)),$$

for all $m > 0$. Since $-\inf(\mathbf{R}\mathrm{Hom}_R(X, Q)) \leq n$, we conclude that $\mathrm{Ext}_R^m(\mathrm{C}_n^A, Q) = 0$. $\square$

Similarly, one proves the next two results linking complexes of finite Gorenstein injective dimension to the Bass class.

(4.4) **Lemma.** *Assume that R admits a dualizing complex. If N is an R–module satisfying*

(a) $N \in \mathsf{B}(R)$, *and*
(b) $\mathrm{Ext}_R^m(J, N) = 0$ *for all integers $m > 0$, and all injective R–modules J,*

then N is Gorenstein injective. $\square$

(4.5) **Theorem.** *Let R be a commutative and noetherian ring admitting a dualizing complex. For any $Y \in \mathsf{D}_{\sqsubset}(R)$ the following conditions are equivalent:*

(i) $Y \in \mathsf{B}(R)$.
(ii) $\mathrm{Gid}_R Y$ *is finite.* $\square$

5. Stability results

We now apply the functorial characterization from the previous section to show that finiteness of Gorenstein dimensions is preserved under a series of standard operations. Our first theorem is a Gorenstein version of Gruson, Raynaud, and Jensen's classical result.

In this section all rings are commutative and noetherian.

(5.1) **Theorem.** *If R admits a dualizing complex, then, for any complex $X \in \mathsf{D}_{\sqsupset}(R)$, there is a biimplication*

$$\operatorname{Gfd}_R X < \infty \quad \Leftrightarrow \quad \operatorname{Gpd}_R X < \infty.$$

Proof. This is just a reformulation of theorem (4.3). □

While the general result by Gruson, Raynaud, and Jensen is deep, a simple proof exists when R, in addition, admits a dualizing complex, cf. [21, thm. 2.6 and proof of cor. 3.4]. The situation may be similar for the Gorenstein version, so we ask whether theorem (5.1) also holds under the weaker assumption that R has finite Krull dimension?

(5.2) **Theorem.** *Let $R \xrightarrow{\varphi} S$ be a local homomorphism. If R admits a dualizing complex, then the following hold,*

$$\text{G–dim}\,\varphi < \infty \quad \Leftrightarrow \quad \operatorname{Gfd}_R S < \infty \quad \Leftrightarrow \quad \operatorname{Gpd}_R S < \infty.$$

Proof. As R admits a dualizing complex [6, thm. (4.3)] yields that G–dim $\varphi < \infty$ precisely when $S \in \mathsf{A}(R)$. It remains to invoke theorem (4.3). □

(5.3) **Theorem.** *Assume that R admits a dualizing complex, and let E be an injective R–module. For any $Y \in \mathsf{D}_{\sqsubset}(R)$ we have,*

$$\operatorname{Gfd}_R \operatorname{Hom}_R(Y, E) \leq \operatorname{Gid}_R Y,$$

and equality holds if E is faithful.

Proof. As in the proof of theorem (2.10) it suffices to prove that if N is Gorenstein injective, then $\operatorname{Hom}_R(N, E)$ is Gorenstein flat, and that the converse holds, when E is faithfully injective.

Write $-^{\vee} = \operatorname{Hom}_R(-, E)$ for short, and set $d = \operatorname{FFD}(R)$, which is finite by (1.4.1) and proposition (4.1). From theorem (2.8) we are informed that if C is any module with $\operatorname{Gfd}_R C < \infty$ then, in fact, $\operatorname{Gfd}_R C \leq d$.

Now assume that N is Gorenstein injective, and consider the left part of a complete injective resolution of N,

$$0 \to C_d \to I_{d-1} \to \cdots \to I_0 \to N \to 0. \tag{1}$$

The I_ℓ's are injective R–modules and C_d is Gorenstein injective. In particular, $C_d \in \mathsf{B}(R)$ by theorem (4.5) and $C_d^{\vee} \in \mathsf{A}(R)$ by [10, lem. (3.2.9)(b)], so $\operatorname{Gfd}_R C_d^{\vee} \leq d$. Applying the functor $-^{\vee}$ to (1) we obtain the following exact sequence:

$$0 \to N^{\vee} \to I_0^{\vee} \to \cdots \to I_{d-1}^{\vee} \to C_d^{\vee} \to 0,$$

where the $I_\ell^\vee$'s are flat R–modules. From theorem (2.8) we conclude that $N^\vee$ is Gorenstein flat.

Finally, we assume that E is faithfully injective and that $N^\vee$ is Gorenstein flat, in particular, $N^\vee \in \mathsf{A}(R)$. This forces $N \in \mathsf{B}(R)$, again by [10, lem. (3.2.9)(b)], that is, $\operatorname{Gid}_R N$ is finite. By lemma (2.14) there exists an exact sequence

$$0 \to B \to H \to N \to 0,$$

where B is Gorenstein injective and $\operatorname{id}_R H = \operatorname{Gid}_R N$. By the first part of the proof, $B^\vee$ is Gorenstein flat, and by assumption, so is $N^\vee$. Therefore, exactness of

$$0 \to N^\vee \to H^\vee \to B^\vee \to 0$$

forces $H^\vee$ to be Gorenstein flat by the resolving property of Gorenstein flat modules, cf. [29, thm (3.7)]. In particular, $\operatorname{Gid}_R N = \operatorname{id}_R H = \operatorname{fd}_R H^\vee = \operatorname{Gfd}_R H^\vee = 0$. □

As an immediate corollary we get:

(5.4) **Corollary.** *Assume that R admits a dualizing complex, and let F be a flat R–module. For any $Y \in \mathsf{D}_{\sqsubset}(R)$ we have,*

$$\operatorname{Gid}_R(Y \otimes^{\mathbf{L}}_R F) \leq \operatorname{Gid}_R Y,$$

and equality holds if F is faithful. □

(5.5) **Theorem.** *Let $R \xrightarrow{\varphi} S$ be a ring homomorphism, and assume that R admits a dualizing complex D. If φ is of finite flat dimension and $D \otimes^{\mathbf{L}}_R S$ is a dualizing complex for S, then the following hold for $X \in \mathsf{D}_{\sqsupset}(R)$ and $Y \in \mathsf{D}_{\sqsubset}(R)$:*

(a) $$\operatorname{Gfd}_R X < \infty \;\Rightarrow\; \operatorname{Gfd}_S(X \otimes^{\mathbf{L}}_R S) < \infty$$

(b) $$\operatorname{Gid}_R Y < \infty \;\Rightarrow\; \operatorname{Gid}_S(Y \otimes^{\mathbf{L}}_R S) < \infty$$

Both implications may be reversed under each of the next two extra conditions:

(1) *φ is faithfully flat.*

(2) *φ is local, and both X and Y have finite homology.*

Proof. We only prove the statements for the Gorenstein injective dimension, as the proof for the Gorenstein flat dimension is similar.

By assumption, $E = D \otimes^{\mathbf{L}}_R S$ is a dualizing complex for S. In view of the main theorem (4.5), we just need to see that that the base changed complex $Y \otimes^{\mathbf{L}}_R S$ belongs to $\mathsf{B}(S)$ when $Y \in \mathsf{B}(R)$. But this is just a special case of [11, prop. (5.9)], from where it also follows that the implication may be reversed when φ is faithfully flat.

Next, let φ be local, $Y \in \mathsf{D}^{\mathrm{f}}_{\square}(R)$, and assume that $Y \otimes^{\mathbf{L}}_R S \in \mathsf{B}(S)$. The aim is now to show that $Y \in \mathsf{B}(R)$. Since the completion maps $R \longrightarrow \widehat{R}$ and $S \longrightarrow \widehat{S}$ are faithfully flat, we may, by what we have already proved, reduce to the case where both R and S are complete local rings. In this setting φ admits a Cohen factorization

$$\begin{tikzcd} & R' \arrow[dr, "\varphi'"] & \\ R \arrow[ur, "\dot{\varphi}"] \arrow[rr, "\varphi"'] & & S, \end{tikzcd}$$

see [7, thm. (1.1)]. The intermediate ring R' is complete and local, so it admits a dualizing complex D'. From [6, (8.8)] it follows that the surjective homomorphism $R' \longrightarrow S$ is Gorenstein, and consequently, $E \simeq D' \otimes^{\mathbf{L}}_{R'} S$.

The homomorphism $R \longrightarrow R'$ is local and flat, hence faithful, whence it suffices to prove that $Z = Y \otimes^{\mathbf{L}}_R R'$ belongs to $\mathsf{B}(R')$. The remainder of the proof is built up around two applications of Iversen's amplitude inequality, which is now available for unbounded complexes [24, thm. 3.1]. The amplitude inequality yields

$$\operatorname{amp}(\mathbf{R}\mathrm{Hom}_{R'}(D', Z)) \leq \operatorname{amp}(\mathbf{R}\mathrm{Hom}_{R'}(D', Z) \otimes^{\mathbf{L}}_{R'} S),$$

which allows us to conclude that $\mathbf{R}\mathrm{Hom}_{R'}(D', Z)$ is bounded, as S is a finitely generated R'–module of finite projective dimension. The right-hand side is finite as

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{R'}(D', Z) \otimes^{\mathbf{L}}_{R'} S &\simeq \mathbf{R}\mathrm{Hom}_{R'}(D', Z \otimes^{\mathbf{L}}_{R'} S)\\ &\simeq \mathbf{R}\mathrm{Hom}_{R'}(D', Y \otimes^{\mathbf{L}}_{R} S)\\ &\simeq \mathbf{R}\mathrm{Hom}_{S}(E, Y \otimes^{\mathbf{L}}_{R} S)\end{aligned}$$

Here the first isomorphism is tensor-evaluation, the second uses associativity of the tensor product, and the third follows by adjointness. The complex $\mathbf{R}\mathrm{Hom}_S(E, Y \otimes^{\mathbf{L}}_R S)$ is bounded, as $Y \otimes^{\mathbf{L}}_R S \in \mathsf{B}(S)$. Finally, consider the commutative diagram

$$\begin{tikzcd}
E \otimes^{\mathbf{L}}_S \mathbf{R}\mathrm{Hom}_S(E, Z \otimes^{\mathbf{L}}_{R'} S) \arrow[d, "\varepsilon_{Z \otimes^{\mathbf{L}} S}"', "\simeq"] & D' \otimes^{\mathbf{L}}_{R'} \mathbf{R}\mathrm{Hom}_{R'}(D', Z \otimes^{\mathbf{L}}_{R'} S) \arrow[l, "\gamma_{Z \otimes^{\mathbf{L}} S}"', "\simeq"] \arrow[d, "\simeq"', "D' \otimes^{\mathbf{L}} \omega_{D'ZS}"] \\
Z \otimes^{\mathbf{L}}_{R'} S & (D' \otimes^{\mathbf{L}}_{R'} \mathbf{R}\mathrm{Hom}_{R'}(D', Z)) \otimes^{\mathbf{L}}_{R'} S, \arrow[l, "\varepsilon_Z \otimes^{\mathbf{L}} S"]
\end{tikzcd}$$

where $\gamma_{X \otimes^{\mathbf{L}}_{R'} S}$ is a natural isomorphism induced by adjointness. The diagram shows that $\varepsilon_Z \otimes^{\mathbf{L}}_{R'} S$ is an isomorphism. As $D' \otimes^{\mathbf{L}}_{R'} \mathbf{R}\mathrm{Hom}_{R'}(D', Z)$ has degreewise finite homology, we may apply [30, prop. 2.10], which uses the extended Iversen's amplitude inequality, to conclude that the counit ε_Z is an isomorphism. □

(5.6) **Localization.** Working directly with the definition of Gorenstein flat modules (see [10, lem. (5.1.3)]), it is easily verified that the inequality

(1) $$\operatorname{Gfd}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} \leq \operatorname{Gfd}_R X$$

holds for all complexes $X \in \mathsf{D}_{\sqsupset}(R)$ and all prime ideals $\mathfrak{p}$ in R.

Turning to the Gorenstein projective and injective dimensions, it is natural to ask if they also localize. To be precise, we ask if the inequalities

(2) $$\operatorname{Gpd}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} \leq \operatorname{Gpd}_R X \quad \text{and}$$

(3) $$\operatorname{Gid}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}} \leq \operatorname{Gid}_R Y.$$

hold for all complexes $X \in \mathsf{D}_{\sqsupset}(R)$ and $Y \in \mathsf{D}_{\sqsubset}(R)$.

When R is local and Cohen–Macaulay with a dualizing module, Foxby settles the question affirmatively in [23, cor. (3.5)]. Recently, Foxby proved (2) (in an unpublished note) for any commutative, noetherian ring of finite Krull dimension. Unfortunately, it is not clear how to employ the ideas of that proof to get a proof of (3). Thus, it remains an open question if (3) holds in general, but we have the following partial result:

(5.7) **Proposition.** *Assume that R admits a dualizing complex. For any complex $Y \in \mathsf{D}_{\sqsubset}(R)$ and any prime ideal $\mathfrak{p}$ in R, there is an inequality*

$$\operatorname{Gid}_{R_\mathfrak{p}} Y_\mathfrak{p} \leq \operatorname{Gid}_R Y.$$

Proof. It suffices to show that if N is a Gorenstein injective R–module, then $N_\mathfrak{p}$ is a Gorenstein injective $R_\mathfrak{p}$–module. This is proved in the exact same manner as in [23, cor. (3.5)] using main theorem (4.5). □

(5.8) **Local cohomology.** Let $\mathfrak{a}$ be an ideal in R. In this paragraph, we consider the right derived local cohomology functor with support in $\mathfrak{a}$, denoted $\mathbf{R}\Gamma_\mathfrak{a}(-)$, together with its right adjoint $\mathbf{L}\Lambda^\mathfrak{a}(-)$, the left derived local homology functor. The usual local cohomology functors are recovered as $\mathrm{H}^\ell_\mathfrak{a}(-) = \mathrm{H}_{-\ell}\,\mathbf{R}\Gamma_\mathfrak{a}(-)$.

Recall from [22, thm. 6.5], or see paragraph (5.10) below, that derived local cohomology (with support in any ideal $\mathfrak{a}$) sends complexes of finite flat dimension (respectively, finite injective dimension) to complexes of finite flat dimension (respectively, finite injective dimension). We close this section by proving a Gorenstein version of this result:

(5.9) **Theorem.** *Let $\mathfrak{a}$ be any ideal in R, and let $X, Y \in \mathsf{D}(R)$. Then*

(a) $$\operatorname{Gfd}_R X < \infty \quad \Rightarrow \quad \operatorname{Gfd}_R \mathbf{R}\Gamma_\mathfrak{a} X < \infty,$$

and, if R admits a dualizing complex, also

(b) $$\operatorname{Gid}_R Y < \infty \quad \Rightarrow \quad \operatorname{Gid}_R \mathbf{R}\Gamma_\mathfrak{a} Y < \infty.$$

When R has a dualizing complex, and $\mathfrak{a}$ is in the radical of R, then both implications may be reversed for $X, Y \in \mathsf{D}^\mathrm{f}_\square(R)$.

This theorem is used by Iyengar and Sather-Wagstaff in their proof of [30, thm. 8.7]. In section 6 we shall use local homology to prove a Gorenstein version of Chouinard's formula for injective dimension, see theorem (6.9). Before we go on with the proof of theorem (5.9), we need some preparations.

(5.10) **Representations of local (co)homology.** Local cohomology may be represented on $\mathsf{D}(R)$ as

(5.10.1) $$\mathbf{R}\Gamma_\mathfrak{a}(-) \;\simeq\; \mathbf{R}\Gamma_\mathfrak{a} R \otimes^{\mathbf{L}}_R - \,,$$

and $\mathbf{R}\Gamma_\mathfrak{a} R$ is isomorphic in $\mathsf{D}(R)$ to the so-called Čech, or stable Koszul, complex on $\mathfrak{a}$. In particular, $\mathbf{R}\Gamma_\mathfrak{a} R$ has finite flat dimension, so we immediately see that $\mathbf{R}\Gamma_\mathfrak{a}(-)$ preserves homological boundedness as well as finite flat and finite injective dimension.

Local homology may be represented on $\mathsf{D}(R)$ as

(5.10.2) $$\mathbf{L}\Lambda^\mathfrak{a}(-) \;\simeq\; \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_\mathfrak{a} R, -)\,.$$

This representation was discovered by Greenlees and May in [27] and investigated further by Alonso, Jeremías and Lipman in [1]. By the Gruson–Raynaud–Jensen theorem, also the projective dimension of $\mathbf{R}\Gamma_\mathfrak{a} R$ is finite, and hence $\mathbf{L}\Lambda^\mathfrak{a}(-)$ preserves homological boundedness together with finite flat and finite injective dimension.

(5.11) **Observation.** For all complexes $X, Y \in \mathsf{D}(R)$ and $X' \in \mathsf{D}^{\mathrm{f}}_{\square}(R)$ there are isomorphisms:

(5.11.1) $$\alpha : \mathbf{R}\Gamma_{\mathfrak{a}}(X \otimes^{\mathbf{L}}_R Y) \xrightarrow{\simeq} X \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{a}} Y,$$

(5.11.2) $$\beta : \mathbf{R}\Gamma_{\mathfrak{a}}\,\mathbf{R}\mathrm{Hom}_R(X', Y) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(X', \mathbf{R}\Gamma_{\mathfrak{a}} Y).$$

In particular, if R admits a dualizing complex D, then we have isomorphisms:

(5.11.3) $$\sigma : \ \mathbf{R}\Gamma_{\mathfrak{a}}\,\mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_R X) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{a}} X),$$

(5.11.4) $$\rho : \ \mathbf{R}\Gamma_{\mathfrak{a}}(D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, Y)) \xrightarrow{\simeq} D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, \mathbf{R}\Gamma_{\mathfrak{a}} Y).$$

First note that α is immediate by (5.10.1). In view of this representation, β is essentially the tensor evaluation morphism, which is invertible in $\mathsf{D}(R)$, as X' has finite homology and $\mathbf{R}\Gamma_{\mathfrak{a}} R$ is of finite flat dimension. See also [22, prop. 6.1]. The last two isomorphisms follow by composing α and β.

Proof of theorem (5.9). The first implication, (a), is immediate: The complex $\mathbf{R}\Gamma_{\mathfrak{a}} R$ is isomorphic in $\mathsf{D}(R)$ to the Čech complex, cf. (5.10), whence it follows by (5.10.1) and [10, thm. (6.4.5)] that $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$ preserves finite Gorenstein flat dimension.

To show (b), assume that $Y \in \mathsf{B}(R)$. By (5.11.2) the complex $\mathbf{R}\mathrm{Hom}_R(D, \mathbf{R}\Gamma_{\mathfrak{a}} Y)$ is isomorphic to $\mathbf{R}\Gamma_{\mathfrak{a}}\,\mathbf{R}\mathrm{Hom}_R(D, Y)$. And since both Y and $\mathbf{R}\mathrm{Hom}_R(D, Y)$ are bounded so are $\mathbf{R}\Gamma_{\mathfrak{a}} Y$ and $\mathbf{R}\mathrm{Hom}_R(D, \mathbf{R}\Gamma_{\mathfrak{a}} Y)$. It is easily verified that the diagram

$$\begin{tikzcd}
\mathbf{R}\Gamma_{\mathfrak{a}} Y \arrow[d, equal] & \mathbf{R}\Gamma_{\mathfrak{a}}(D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, Y)) \arrow[l, "\simeq"', "\mathbf{R}\Gamma_{\mathfrak{a}}\,\varepsilon_Y"] \arrow[d, "\rho", "\simeq"'] \\
\mathbf{R}\Gamma_{\mathfrak{a}} Y & D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, \mathbf{R}\Gamma_{\mathfrak{a}} Y). \arrow[l, "\varepsilon_{\mathbf{R}\Gamma_{\mathfrak{a}} Y}"']
\end{tikzcd}$$

is commutative, and it follows that the counit $\varepsilon_{\mathbf{R}\Gamma_{\mathfrak{a}} Y}$ is an isomorphism in $\mathsf{D}(R)$. In total, $\mathbf{R}\Gamma_{\mathfrak{a}} Y$ belongs to $\mathsf{B}(R)$. The implication (b) is now immediate by the main theorem (4.5).

The second half of the proof is propelled by two powerful isomorphisms connecting derived local cohomology with derived local homology. They read as follows: For all $X, Y \in \mathsf{D}(R)$ the next two morphisms

(1) $$\sqsupset_X : \ \mathbf{L}\Lambda^{\mathfrak{a}}\mathbf{R}\Gamma_{\mathfrak{a}} X \xrightarrow{\simeq} \mathbf{L}\Lambda^{\mathfrak{a}} X,$$

(2) $$\urcorner_Y : \ \mathbf{R}\Gamma_{\mathfrak{a}} Y \xrightarrow{\simeq} \mathbf{R}\Gamma_{\mathfrak{a}}\mathbf{L}\Lambda^{\mathfrak{a}} Y,$$

are isomorphisms, see [1, p. 6, cor., part (iii) and (iv)]. Furthermore, for $Z \in \mathsf{D}^{\mathrm{f}}_{\sqsupset}(R)$ there is an isomorphism:

(3) $$\mathbf{L}\Lambda^{\mathfrak{a}} Z \ \simeq \ Z \otimes^{\mathbf{L}}_R \widehat{R_{\mathfrak{a}}} \ \simeq \ Z \otimes_R \widehat{R_{\mathfrak{a}}},$$

where $\widehat{R_{\mathfrak{a}}}$ denotes the $\mathfrak{a}$-adic completion of R; see [25, prop. (2.7)].

Now, assume that $\mathfrak{a}$ is in the radical of R, that $Y \in \mathsf{D}^{\mathrm{f}}_{\square}(R)$, and that $\mathbf{R}\Gamma_{\mathfrak{a}} Y \in \mathsf{B}(R)$. We are required to show that also $Y \in \mathsf{B}(R)$.

First, we show that $\mathbf{R}\mathrm{Hom}_R(D, Y)$ is bounded. As $\mathbf{L}\Lambda^{\mathfrak{a}}(-)$ preserves homological boundedness we get that $\mathbf{L}\Lambda^{\mathfrak{a}}\,\mathbf{R}\mathrm{Hom}_R(D, \mathbf{R}\Gamma_{\mathfrak{a}} Y)$ is bounded since, already,

$\mathbf{R}\mathrm{Hom}_R(D, \mathbf{R}\Gamma_{\mathfrak{a}} Y)$ is. Observe that

$$\begin{aligned}\mathbf{L}\Lambda^{\mathfrak{a}}\, \mathbf{R}\mathrm{Hom}_R(D, \mathbf{R}\Gamma_{\mathfrak{a}} Y) &\simeq \mathbf{R}\mathrm{Hom}_R(D, \mathbf{L}\Lambda^{\mathfrak{a}}\mathbf{R}\Gamma_{\mathfrak{a}} Y)\\ &\simeq \mathbf{R}\mathrm{Hom}_R(D, \mathbf{L}\Lambda^{\mathfrak{a}} Y)\\ &\simeq \mathbf{R}\mathrm{Hom}_R(D, Y \otimes^{\mathbf{L}}_R \widehat{R_{\mathfrak{a}}})\\ &\simeq \mathbf{R}\mathrm{Hom}_R(D, Y) \otimes^{\mathbf{L}}_R \widehat{R_{\mathfrak{a}}}.\end{aligned}$$

Here the first isomorphism is adjointness in conjunction with (5.10.2), the second is due to (1), and the third is due to (3). As $\mathfrak{a}$ is in the radical of R, the completion $\widehat{R_{\mathfrak{a}}}$ is a faithful flat R–module by [34, thm. 8.14]. Thus, the last isomorphism follows from tensor evaluation, cf. [10, (A.4.24)], and faithful flatness allows us to conclude that the homology of $\mathbf{R}\mathrm{Hom}_R(D, Y)$ is bounded. Moreover, it follows by [6, (1.2.1) and (1.2.2)] that $D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, Y)$ belongs to $\mathsf{D}^{\mathsf{f}}_{\sqsupset}(R)$.

We proceed by considering the commutative diagram

$$\begin{tikzcd}
\mathbf{L}\Lambda^{\mathfrak{a}}\mathbf{R}\Gamma_{\mathfrak{a}} Y \arrow[d, "\sqsupset_Y"', "\simeq"] & \mathbf{L}\Lambda^{\mathfrak{a}}(D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, \mathbf{R}\Gamma_{\mathfrak{a}} Y)) \arrow[l, "\simeq"', "\mathbf{L}\Lambda^{\mathfrak{a}}\varepsilon_{\mathbf{R}\Gamma_{\mathfrak{a}} Y}"] \\
 & \mathbf{L}\Lambda^{\mathfrak{a}}\mathbf{R}\Gamma_{\mathfrak{a}}(D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, Y)) \arrow[u, "\simeq", "\mathbf{L}\Lambda^{\mathfrak{a}}\,\rho"'] \arrow[d, "\simeq"', "\sqsupset_{D\otimes^{\mathbf{L}}\mathbf{R}\mathrm{Hom}(D,Y)}"] \\
\mathbf{L}\Lambda^{\mathfrak{a}} Y & \mathbf{L}\Lambda^{\mathfrak{a}}(D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, Y)), \arrow[l, "\mathbf{L}\Lambda^{\mathfrak{a}}\varepsilon_Y"']
\end{tikzcd}$$

from which we deduce that $\mathbf{L}\Lambda^{\mathfrak{a}}\varepsilon_Y$ is an isomorphism. Now, since $D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, Y)$ belongs to $\mathsf{D}^{\mathsf{f}}_{\sqsupset}(R)$, it follows by (3) that we may identify $\mathbf{L}\Lambda^{\mathfrak{a}}\varepsilon_Y$ with $\varepsilon_Y \otimes_R \widehat{R_{\mathfrak{a}}}$. Whence, the counit ε_Y is an isomorphism by faithful flatness of $\widehat{R_{\mathfrak{a}}}$.

In the presence of a dualizing complex, a similar argument shows that also (a) may be reversed, if X belongs to $\mathsf{D}^{\mathsf{f}}_{\square}(R)$ and $\mathfrak{a}$ is in the radical of R. $\square$

6. Bass and Auslander–Buchsbaum formulas

The main theorems in section 2 give formulas for measuring Gorenstein dimensions. We close this paper by establishing a number alternative formulas that allow us to measure or even compute Gorenstein dimensions. Prime among these are Gorenstein versions of the celebrated Bass formula and Chouinard's non-finite versions of the Bass and Auslander–Buchsbaum formulas.

Throughout this section, R is a commutative and noetherian ring.

(6.1) **Width.** Recall that when $(R, \mathfrak{m}, k)$ is local, the *width* of an R–complex $X \in \mathsf{D}_{\sqsupset}(R)$ is defined as

$$\mathrm{width}_R\, X = \inf\,(k \otimes^{\mathbf{L}}_R X).$$

There is always an inequality,

$$\mathrm{width}_R\, X \geq \inf X, \tag{6.1.1}$$

and by Nakayama's lemma, equality holds for $X \in \mathsf{D}^{\mathsf{f}}_{\sqsupset}(R)$.

(6.2) **Proposition.** *Let R be local. For any $Y \in \mathsf{D}_\square(R)$ the following inequality holds,*

$$\operatorname{Gid}_R Y \geq \operatorname{depth} R - \operatorname{width}_R Y.$$

In particular,

$$\operatorname{Gid}_R Y \geq \operatorname{depth} R - \inf Y,$$

for $Y \in \mathsf{D}^{\mathrm{f}}_\square(R)$.

Proof. Set $d = \operatorname{depth} R$ and pick an R–regular sequence $\boldsymbol{x} = x_1, \dots, x_d$. Note that the module $T = R/(\boldsymbol{x})$ has $\operatorname{pd}_R T = d$. We may assume that $\operatorname{Gid}_R Y < \infty$, and the desired inequality now follows from the computation:

$$\begin{aligned}
\operatorname{Gid}_R Y &\overset{(1)}{\geq} -\inf \mathbf{R}\mathrm{Hom}_R(T, Y) \\
&\overset{(2)}{\geq} -\operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(T, Y) \\
&\overset{(3)}{=} \operatorname{pd}_R T - \operatorname{width}_R Y \\
&= \operatorname{depth} R - \operatorname{width}_R Y.
\end{aligned}$$

Here (1) follows from theorem (2.5); (2) is (6.1.1), while (3) is by [12, thm. (4.14)(b)]. $\square$

(6.3) **Observation.** Assume that R admits a dualizing complex. For $Y \in \mathsf{D}^{\mathrm{f}}_\square(R)$ it follows by theorems (2.10), (5.3), and (2.12) combined with (1.2.1) that

$$\operatorname{Gid}_R Y < \infty \quad \Leftrightarrow \quad \operatorname{Gfd}_R Y^\dagger < \infty \quad \Leftrightarrow \quad \operatorname{Gpd}_R Y^\dagger < \infty.$$

(6.4) **Theorem.** *Assume that R is local and admits a dualizing complex. For any complex $Y \in \mathsf{D}^{\mathrm{f}}_\square(R)$ of finite Gorenstein injective dimension, the next equation holds,*

$$\operatorname{Gid}_R Y = \operatorname{depth} R - \inf Y.$$

In particular,

$$\operatorname{Gid}_R N = \operatorname{depth} R$$

for any finitely generated R–module N of finite Gorenstein injective dimension.

Proof. By proposition (6.2) we only need to show the inequality

$$\operatorname{Gid}_R Y \leq \operatorname{depth} R - \inf Y.$$

By dagger duality (1.2.1) we have $\operatorname{Gid}_R Y = \operatorname{Gid}_R Y^{\dagger\dagger}$, and by theorem (2.5) there exists an injective R–module J, such that $\operatorname{Gid}_R Y^{\dagger\dagger} = -\inf \mathbf{R}\mathrm{Hom}_R(J, Y^{\dagger\dagger})$. In the computation,

$$\begin{aligned}
\operatorname{Gid}_R Y &\overset{(1)}{=} -\inf \mathbf{R}\mathrm{Hom}_R(Y^\dagger, J^\dagger) &&\overset{(2)}{\leq} \operatorname{Gpd}_R Y^\dagger - \inf J^\dagger \\
&\overset{(3)}{\leq} \operatorname{Gpd}_R Y^\dagger + \operatorname{Gid}_R D &&\overset{(4)}{\leq} \operatorname{Gpd}_R Y^\dagger + \operatorname{id}_R D,
\end{aligned}$$

(1) is by adjointness; (2) is by theorem (2.2), as $J^\dagger$ is a complex of finite flat dimension and hence of finite projective dimension; (3) follows from theorem (2.5), and (4) is trivial.

By observation (6.3) $\operatorname{Gpd}_R Y^\dagger$ is finite, and since $Y^\dagger$ has finite homology, it follows from theorem (2.12)(b) and [10, thm. (2.3.13)] that $\operatorname{Gpd}_R Y^\dagger = \operatorname{depth} R - \operatorname{depth}_R Y^\dagger$. Thus, we may continue the computation as follows:

$$\begin{aligned}\operatorname{Gid}_R Y &\leq \operatorname{depth} R - \operatorname{depth}_R Y^\dagger + \operatorname{id}_R D\\ &\overset{(5)}{=} \operatorname{depth} R - \inf Y - \operatorname{depth}_R D + \operatorname{id}_R D\\ &\overset{(6)}{=} \operatorname{depth} R - \inf Y.\end{aligned}$$

Both (5) and (6) stem from well-known properties of dualizing complexes, e.g. see [11, 3.1(a) and 3.5]. □

A dualizing complex D for R is said to be *normalized* if $\inf D = \operatorname{depth} R$, cf. [6, (2.5)]. This language is justified by formulas like: $\operatorname{depth}_R Y = \inf Y^\dagger$, which holds for all $Y \in \mathsf{D}^{\mathrm{f}}_{\square}(R)$ when the dagger dual $Y^\dagger$ is taken with respect to a normalized dualizing complex, see e.g. [11, 3.1(a) and 3.2(b)].

(6.5) **Corollary.** *If R is local and D is a normalized dualizing complex for R, then the next equalities hold for all $Y \in \mathsf{D}^{\mathrm{f}}_{\square}(R)$,*

$$\operatorname{Gid}_R Y = \operatorname{Gpd}_R Y^\dagger = \operatorname{Gfd}_R Y^\dagger,$$

where $Y^\dagger = \mathbf{R}\mathrm{Hom}_R(Y, D)$.

Proof. By observation (6.3) the three dimensions $\operatorname{Gid}_R Y$, $\operatorname{Gpd}_R Y^\dagger$, and $\operatorname{Gfd}_R Y^\dagger$ are simultaneously finite, and in this case, dagger duality (1.2.1) and theorem (6.4) give:

$$\operatorname{Gid}_R Y = \operatorname{Gid}_R Y^{\dagger\dagger} = \operatorname{depth} R - \inf Y^{\dagger\dagger} = \operatorname{depth} R - \operatorname{depth}_R Y^\dagger,$$

where the last equality uses that the dualizing complex is normalized. By [10, thm. (2.3.13)] we have $\text{G--dim}_R Y^\dagger = \operatorname{depth} R - \operatorname{depth}_R Y^\dagger$, and $\text{G--dim}_R Y^\dagger = \operatorname{Gpd}_R Y^\dagger = \operatorname{Gfd}_R Y^\dagger$ by theorem (2.12). □

In [9] Chouinard proves that

$$\operatorname{id}_R N \;=\; \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}$$

for any R–module N of finite injective dimension and, dually,

$$\operatorname{fd}_R M \;=\; \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}$$

for any module M of finite flat dimension. Later, Foxby [20] (for flat dimension) and Yassemi [41] (for injective dimension) extended these results to complexes.

We now extend these formulas to also encompass the Gorenstein injective and Gorenstein flat dimensions. However, in the Gorenstein injective case, we have to assume that the base ring admits a dualizing complex.

The first result in this direction is inspired by Iyengar and Sather-Wagstaff's [30, thm. 8.6].

(6.6) **Theorem.** *Assume that* $(R, \mathfrak{m}, k)$ *is commutative, noetherian and local, admitting a dualizing complex; and let* $\mathrm{E}_R(k)$ *denote the injective hull of the residue field. For any complex* $Y \in \mathsf{D}_\square(R)$ *of finite Gorenstein injective dimension, the next equality holds,*

$$\operatorname{width}_R Y = \operatorname{depth} R + \inf \mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(k), Y).$$

In particular, $\operatorname{width}_R Y$ *and* $\inf \mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(k), Y)$ *are simultaneously finite.*

Proof. By theorem (4.5) we have $Y \in \mathsf{B}(R)$; in particular, we may write

$$Y \simeq D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, Y).$$

Furthermore, we can assume that D is a normalized dualizing complex, in which case we have $\mathbf{R}\Gamma_{\mathfrak{m}} D \simeq \mathrm{E}_R(k)$. The conclusion is reached by computing as follows

$$\begin{aligned}
\operatorname{width}_R Y &= \operatorname{width}_R(D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, Y)) \\
&\overset{(1)}{=} \operatorname{width}_R D + \operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(D, Y) \\
&\overset{(2)}{=} \inf D + \inf \mathbf{L}\Lambda^{\mathfrak{m}} \mathbf{R}\mathrm{Hom}_R(D, Y) \\
&\overset{(3)}{=} \operatorname{depth} R + \inf \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{m}} D, Y) \\
&= \operatorname{depth} R + \inf \mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(k), Y).
\end{aligned}$$

Here (1) follows from e.g. [10, (A.6.5)]; (2) is by (6.1.1) and [25, thm. (2.11)], while (3) follows from (5.10.1) and (5.10.2). ☐

Over a local ring, proposition (6.2) provides an upper bound, $\operatorname{Gid}_R Y$, for the difference $\operatorname{depth} R - \operatorname{width}_R Y$. We can now give a lower bound.

(6.7) **Corollary.** *Assume that* $(R, \mathfrak{m}, k)$ *is commutative, noetherian and local, admitting a dualizing complex. If* $Y \in \mathsf{D}_\square(R)$ *is a complex of finite Gorenstein injective dimension and finite width, then*

$$\operatorname{depth} R - \operatorname{width}_R Y \geq -\sup Y.$$

In particular, if N *is a Gorenstein injective module of finite width, then*

$$\operatorname{width}_R N = \operatorname{depth} R.$$

Proof. Since $\operatorname{width}_R Y$ is finite, theorem (6.6) forces the complex $\mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(k), Y)$ to have non-trivial homology. Furthermore, Y itself also has non-trivial homology, so $s = \sup Y$ is finite. If we set $g = \operatorname{Gid}_R Y$, then theorem (2.5) implies the existence of complex $B \simeq Y$,

$$B \;=\; 0 \to B_s \to B_{s-1} \to \cdots \to B_{-g} \to 0,$$

in which all modules are Gorenstein injective. From corollary (3.11) we learn that $\mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(k), Y)$ is isomorphic to $\mathrm{Hom}_R(\mathrm{E}_R(k), B)$ in $\mathsf{D}(R)$; the latter complex obviously has $\inf \mathrm{Hom}_R(\mathrm{E}_R(k), B) \leq s$. Therefore, theorem (6.6) yields

$$\operatorname{depth} R - \operatorname{width}_R Y \;=\; -\inf \mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(k), Y) \;\geq\; -s \;=\; -\sup Y.$$

Finally, consider the case where $Y = N$ is a Gorenstein injective module. The inequality just established gives $\operatorname{depth} R - \operatorname{width}_R N \geq 0$, and proposition (6.2) gives $\operatorname{depth} R - \operatorname{width}_R N \leq 0$. ☐

(6.8) **Corollary.** *Assume that $(R,\mathfrak{m},k)$ is commutative, noetherian and local, admitting a dualizing complex D. If D is normalized, and $Y \in \mathsf{D}^{\mathrm{f}}_{\square}(R)$ is a complex of finite Gorenstein injective dimension, then*

$$\begin{aligned}\operatorname{Gid}_R Y \;=\; -\inf \mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(k),Y) \;&=\; -\operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(D,Y)\\ &=\; -\inf \mathbf{R}\mathrm{Hom}_R(D,Y).\end{aligned}$$

Proof. The first equality comes from the computation,

$$\operatorname{Gid}_R Y \overset{(1)}{=} \operatorname{depth} R - \inf Y \;=\; \operatorname{depth} R - \operatorname{width}_R Y \overset{(2)}{=} -\inf \mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(k),Y).$$

Here (1) is the Bass formula from theorem (6.4), while (2) is theorem (6.6). For the last equalities, we note that

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(k),Y) &\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{m}}D,Y)\\ &\simeq \mathbf{L}\Lambda^{\mathfrak{m}}\,\mathbf{R}\mathrm{Hom}_R(D,Y)\\ &\simeq \mathbf{R}\mathrm{Hom}_R(D,Y)\otimes^{\mathbf{L}}_R \widehat{R},\end{aligned}$$

where the last isomorphism hinges on the fact that $\mathbf{R}\mathrm{Hom}_R(D,Y)$ has finite homology, so we may apply (3) from page IV.28. Since $\widehat{R}$ is faithfully flat, the complexes $\mathbf{R}\mathrm{Hom}_R(\mathrm{E}_R(k),Y)$ and $\mathbf{R}\mathrm{Hom}_R(D,Y)$ must have the same infimum. ☐

(6.9) **Theorem.** *Assume that R is commutative and noetherian admitting a dualizing complex. For a complex $Y \in \mathsf{D}_{\square}(R)$ of finite Gorenstein injective dimension, the next equality holds,*

$$\operatorname{Gid}_R Y \;=\; \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}.$$

Proof. First we show the inequality "$\geq$". For any prime ideal $\mathfrak{p}$ in R, propositions (5.7) and (6.2) give the desired inequality,

$$\operatorname{Gid}_R Y \;\geq\; \operatorname{Gid}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}} \;\geq\; \operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}}.$$

For the converse inequality, "$\leq$", we may assume that $\mathrm{H}(Y) \neq 0$, i.e. the amplitude $a = \operatorname{amp} Y$ is a non-negative integer. Since

$$\operatorname{Gid}_R(\Sigma^s Y) \;=\; \operatorname{Gid}_R Y - s \quad \text{and} \quad \operatorname{width}_{R_{\mathfrak{p}}}(\Sigma^s Y)_{\mathfrak{p}} \;=\; \operatorname{width}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}} + s,$$

we may assume that $\sup Y = 0$, and hence $\inf Y = -a$.

Set $g = \operatorname{Gid}_R Y$. The proof now proceeds by induction on the amplitude $a \geq 0$.

Case $a = 0$: In this case $Y \simeq N$, where N is a non-zero module. We need to show that

$$\sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\} \;\geq\; g.$$

First, we prove that

$$\sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\} \;\geq\; 0.$$

To see this, we note that by [22, lem. 2.6] there exists a prime ideal $\mathfrak{p}$ in R, such that the homology of $k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}$ is non-trivial; in particular,

$$\operatorname{width}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} = \inf\,(k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} N_{\mathfrak{p}})$$

is finite. On the other hand, the Gorenstein injective dimension of $N_{\mathfrak{p}}$ is finite by proposition (5.7), and therefore corollary (6.7) implies that

$$\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \;\geq\; 0.$$

Thus, if $g = 0$ we are done, and we may in the following assume that $g > 0$. To conclude this part of the proof, it suffices to show that there exists a prime ideal $\mathfrak{p}$, such that $\operatorname{Tor}^{R_{\mathfrak{p}}}_{\operatorname{depth} R_{\mathfrak{p}} - g}(k(\mathfrak{p}), N_{\mathfrak{p}}) \neq 0$, in which case

$$\operatorname{width}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \;=\; \inf\{m \mid \operatorname{Tor}^{R_{\mathfrak{p}}}_m(k(\mathfrak{p}), N_{\mathfrak{p}}) \neq 0\} \;\leq\; \operatorname{depth} R_{\mathfrak{p}} - g.$$

By lemma (2.14) there is an exact sequence, $0 \to B \to H \to N \to 0$, where B is Gorenstein injective and $\operatorname{id}_R H = g$. By Chouinard's formula for the classical injective dimension, there exists a prime ideal $\mathfrak{p}$, such that

$$g \;=\; \operatorname{id}_R H \;=\; \operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} H_{\mathfrak{p}};$$

in particular,

$$\operatorname{Tor}^{R_{\mathfrak{p}}}_{\operatorname{depth} R_{\mathfrak{p}} - g}(k(\mathfrak{p}), H_{\mathfrak{p}}) \neq 0.$$

Applying $k(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} -$ to the short exact sequence $0 \to B_{\mathfrak{p}} \to H_{\mathfrak{p}} \to N_{\mathfrak{p}} \to 0$, we obtain an exact sequence

$$\begin{aligned} \cdots \longrightarrow \operatorname{Tor}^{R_{\mathfrak{p}}}_{\operatorname{depth} R_{\mathfrak{p}} - g}(k(\mathfrak{p}), B_{\mathfrak{p}}) &\longrightarrow \operatorname{Tor}^{R_{\mathfrak{p}}}_{\operatorname{depth} R_{\mathfrak{p}} - g}(k(\mathfrak{p}), H_{\mathfrak{p}}) \\ &\longrightarrow \operatorname{Tor}^{R_{\mathfrak{p}}}_{\operatorname{depth} R_{\mathfrak{p}} - g}(k(\mathfrak{p}), N_{\mathfrak{p}}) \longrightarrow \cdots. \end{aligned}$$

Whence, it suffices to show that $\operatorname{Tor}^{R_{\mathfrak{p}}}_{\operatorname{depth} R_{\mathfrak{p}} - g}(k(\mathfrak{p}), B_{\mathfrak{p}}) = 0$. By corollary (6.7) we have

$$\inf\{m \mid \operatorname{Tor}^{R_{\mathfrak{p}}}_m(k(\mathfrak{p}), B_{\mathfrak{p}}) \neq 0\} \;=\; \operatorname{width}_{R_{\mathfrak{p}}} B_{\mathfrak{p}} \;\geq\; \operatorname{depth} R_{\mathfrak{p}},$$

and since $g > 0$ this forces $\operatorname{Tor}^{R_{\mathfrak{p}}}_{\operatorname{depth} R_{\mathfrak{p}} - g}(k(\mathfrak{p}), B_{\mathfrak{p}}) = 0$.

Case $\boldsymbol{a > 0}$: By the induction hypothesis,

$$\operatorname{Gid}_R Y' \;=\; \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} Y'_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}$$

for all $Y' \in \mathsf{D}_{\square}(R)$ with finite Gorenstein injective dimension and $\operatorname{amp} Y' < a$. Since $-\inf Y = a > 0$, we also have $g = \operatorname{Gid}_R Y > 0$. By theorem (2.5) there exists a complex $B \simeq Y$,

$$B \;=\; 0 \to B_0 \to B_{-1} \to \cdots \to B_{-a} \to \cdots \to B_{-g} \to 0,$$

in which all modules are Gorenstein injective, and we may assume that $Y = B$. Consider the short exact sequence of complexes,

$$0 \longrightarrow B_0 \longrightarrow B \longrightarrow B' \longrightarrow 0,$$

where $B' = \sqsubset_{-1} B$. Obviously, $\operatorname{amp} B' < a$ and $\operatorname{Gid}_R B' < \infty$; and the induction hypothesis yields the existence of a prime ideal $\mathfrak{p}$, such that

$$\operatorname{Gid}_R B' \;=\; \operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} B'_{\mathfrak{p}}. \tag{1}$$

We claim that $\operatorname{Gid}_R B' = g$: Clearly, $\operatorname{Gid}_R B' \leq g$. To show the opposite inequality, it suffices, by theorem (2.5), to show the existence of an injective R–module J with

$$\mathrm{H}_{-g}(\mathbf{R}\mathrm{Hom}_R(J, B')) \neq 0.$$

Since $\mathrm{Gid}_R B = g$ there, indeed, exists an injective module J with $-\inf \mathbf{R}\mathrm{Hom}_R(J,B) = g$. The long exact sequence,

$$\cdots \longrightarrow \mathrm{H}_{-g}(\mathbf{R}\mathrm{Hom}_R(J,B_0)) \longrightarrow \mathrm{H}_{-g}(\mathbf{R}\mathrm{Hom}_R(J,B)) \\ \longrightarrow \mathrm{H}_{-g}(\mathbf{R}\mathrm{Hom}_R(J,B')) \longrightarrow \cdots,$$

now gives the desired conclusion, as $\mathrm{H}_{-g}(\mathbf{R}\mathrm{Hom}_R(J,B_0)) = 0$ by theorem (2.5) because $g > 0$. Now, (1) tells us that

$$\mathrm{width}_{R_\mathfrak{p}} B'_\mathfrak{p} \;=\; \inf\{m \mid \mathrm{H}_m(k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} B'_\mathfrak{p}) \neq 0\} \;=\; \mathrm{depth}\, R_\mathfrak{p} - g,$$

in particular, $\mathrm{H}_{\mathrm{depth}\, R_\mathfrak{p} - g}(k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} B'_\mathfrak{p}) \neq 0$. We need to prove that $\mathrm{width}_{R_\mathfrak{p}} B_\mathfrak{p} \leq \mathrm{depth}\, R_\mathfrak{p} - g$. By the long exact sequence,

$$\cdots \longrightarrow \mathrm{H}_{\mathrm{depth}\, R_\mathfrak{p} - g}(k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} B_\mathfrak{p}) \longrightarrow \mathrm{H}_{\mathrm{depth}\, R_\mathfrak{p} - g}(k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} B'_\mathfrak{p}) \\ \longrightarrow \mathrm{H}_{\mathrm{depth}\, R_\mathfrak{p} - g - 1}(k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} (B_0)_\mathfrak{p}) \longrightarrow \cdots,$$

it is sufficient to show that $\mathrm{H}_{\mathrm{depth}\, R_\mathfrak{p} - g - 1}(k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} (B_0)_\mathfrak{p}) = 0$. But this is clear, as $\inf(k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} (B_0)_\mathfrak{p}) = \mathrm{width}_{R_\mathfrak{p}} (B_0)_\mathfrak{p} \geq \mathrm{depth}\, R_\mathfrak{p}$ by corollary (6.7). □

The next result offers a peculiar application of the previous theorem.

(6.10) **Theorem.** *Assume that R is commutative and noetherian admitting a dualizing complex. A filtered direct limit of Gorenstein injective R–modules is then Gorenstein injective.*

Proof. Let $B_i \to B_j$ be a filtered, direct system of Gorenstein injective modules. By theorem (4.5) all the B_i's belong to $\mathsf{B}(R)$, and it is straightforward to verify that also the limit $\varinjlim B_i$ belongs to the Bass class. In particular, $\mathrm{Gid}_R \varinjlim B_i < \infty$.
Since tensor products and homology commute with direct limits, we have

$$\mathrm{width}_{R_\mathfrak{p}} \left(\varinjlim (B_i)_\mathfrak{p}\right) \geq \inf_i \left\{\mathrm{width}_{R_\mathfrak{p}} (B_i)_\mathfrak{p}\right\},$$

for each prime ideal $\mathfrak{p}$. By theorem (6.9) we now have

$$\begin{aligned}\mathrm{Gid}_R \varinjlim B_i &= \sup\{\, \mathrm{depth}\, R_\mathfrak{p} - \mathrm{width}_{R_\mathfrak{p}} (\varinjlim B_i)_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Spec}\, R\} \\ &\leq \sup\{\, \mathrm{depth}\, R_\mathfrak{p} - \inf_i\{\mathrm{width}_{R_\mathfrak{p}} (B_i)_\mathfrak{p}\} \mid \mathfrak{p} \in \mathrm{Spec}\, R\} \\ &= \sup_i \left\{\sup\{\, \mathrm{depth}\, R_\mathfrak{p} - \mathrm{width}_{R_\mathfrak{p}} (B_i)_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Spec}\, R\}\right\} \\ &= \sup_i \{\mathrm{Gid}_R B_i\} = 0. \quad \square\end{aligned}$$

Next, we turn to the Gorenstein flat parallel of theorem (6.9). It turns out that this case is much easier than the Gorenstein injective one, since several results are already available. Furthermore, we do not have to assume that the base ring admits a dualizing complex.

(6.11) **Theorem.** *Assume that R is commutative and noetherian. For a complex $X \in \mathsf{D}_\square(R)$ of finite Gorenstein flat dimension, the next equality holds,*

$$\mathrm{Gfd}_R X \;=\; \sup\{\, \mathrm{depth}\, R_\mathfrak{p} - \mathrm{depth}_{R_\mathfrak{p}} X_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Spec}\, R\}.$$

Proof. First we prove the inequality "$\geq$". Recall from [12, def. (2.1)] that the restricted flat dimension of a complex $X \in \mathsf{D}_\square(R)$ is defined as

$$\mathsf{Rfd}_R X \;=\; \sup\{\sup(T \otimes_R^{\mathbf{L}} X) \mid T \text{ is a module with } \operatorname{fd}_R T < \infty\}.$$

By [12, thm. (2.4)(b)] the restricted flat dimension always satisfies the formula,

$$\mathsf{Rfd}_R X \;=\; \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}.$$

Comparing the definition of restricted flat dimension with theorem (2.8)(*ii*), we immediately get the inequality

$$\operatorname{Gfd}_R X \;\geq\; \mathsf{Rfd}_R X.$$

Next, we turn to the opposite inequality "$\leq$". We may assume that $\mathrm{H}(X) \neq 0$, whence the amplitude $a = \operatorname{amp} X$ is a non-negative integer. We may even assume that $\inf X = 0$ and $\sup X = a$. The proof now proceeds by induction on the amplitude $a \geq 0$.

Case $a = 0$: This is exactly the content of [29, thm. (3.19)].

Case $a > 0$: Similar to the inductive step in the proof of theorem (6.9) (note that is does not require the presence of a dualizing complex). $\square$

(6.12) **Remark.** In [30, thm. 8.8] Iyengar and Sather-Wagstaff give a different proof of theorem (6.11), cf. [30, 8.3].

Their proof employs theorem (4.3) together with [30, thm. 8.7]. Note that theorem (6.6) is a parallel to [30, thm. 8.7]; using the same idea as in the proof of theorem (6.6) we now give a short proof of [30, thm. 8.7] without resorting to theorem (5.9).

(6.13) **Theorem.** *Assume that $(R, \mathfrak{m}, k)$ is commutative, noetherian, and local; and let $\mathrm{E}_R(k)$ denote the injective hull of the residue field. For a complex $X \in \mathsf{D}_\square(R)$ of finite Gorenstein flat dimension, the next equality holds,*

$$\operatorname{depth}_R X = \operatorname{depth} R - \sup(\mathrm{E}_R(k) \otimes_R^{\mathbf{L}} X).$$

In particular, $\operatorname{depth}_R X$ *and* $\sup(\mathrm{E}_R(k) \otimes_R^{\mathbf{L}} X)$ *are simultaneously finite.*

Proof. As in the first part of the proof of [30, thm. (8.6)], we may reduce to the complete case. In particular, we may assume that R admits a (normalized) dualizing complex D.

The rest of the proof is parallel to the proof of theorem (6.6); the computation goes as follows:

$$\begin{aligned}
\operatorname{depth}_R X &= \operatorname{depth}_R(\mathbf{R}\mathrm{Hom}_R(D, D \otimes_R^{\mathbf{L}} X)) \\
&\overset{(1)}{=} \operatorname{width}_R D + \operatorname{depth}_R(D \otimes_R^{\mathbf{L}} X) \\
&\overset{(2)}{=} \inf D - \sup \mathbf{R}\Gamma_{\mathfrak{m}}(D \otimes_R^{\mathbf{L}} X) \\
&= \operatorname{depth} R - \sup(\mathrm{E}_R(k) \otimes_R^{\mathbf{L}} X).
\end{aligned}$$

Here (1) follows from e.g. [10, (A.6.4)], and (2) is due to Grothendieck's well-known vanishing results for local cohomology. $\square$

(6.14) **Corollary.** *Assume that $(R, \mathfrak{m}, k)$ is commutative, noetherian, and local. If $X \in \mathsf{D}^{\mathrm{f}}_{\square}(R)$ is a complex with finite Gorenstein flat dimension, then*

$$\operatorname{Gfd}_R X \;=\; \sup\,(\mathrm{E}_R(k) \otimes^{\mathbf{L}}_R X).$$

If, in addition, D is a normalized dualizing complex for R, then

$$\operatorname{Gfd}_R X \;=\; -\operatorname{depth}_R(D \otimes^{\mathbf{L}}_R X). \quad \square$$

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Part V

Cohen-Macaulay injective, projective and flat dimension

COHEN-MACAULAY INJECTIVE, PROJECTIVE, AND FLAT DIMENSION

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ABSTRACT. We define three new homological dimensions — Cohen-Macaulay injective, projective, and flat dimension — which inhabit a theory similar to that of classical injective, projective, and flat dimension. Finiteness of the new dimensions characterizes Cohen-Macaulay rings with dualizing modules.

1. Introduction

The classical theory of injective, projective, and flat dimension has had great success in commutative algebra. In particular, it has been very useful to know that finiteness of these dimensions characterizes regular rings.

Several attempts have been made to mimic this success by constructing homological dimensions whose finiteness would characterize other rings than the regular ones. These efforts have given us complete intersection dimension, Gorenstein dimension, and Cohen-Macaulay dimension.

The normal practice has not been to mimic all three classical dimensions, but rather to focus on projective dimension for finitely generated modules. Hence complete intersection dimension and Cohen-Macaulay dimension only exist in this restricted sense, and the same used to be the case for Gorenstein dimension.

However, recent years have seen much work on the Gorenstein theory which now contains both Gorenstein injective, projective, and flat dimension. These dimensions inhabit a nice theory similar to the classical one. A good summary is in [2], although this reference is already a bit dated.

The purpose of this paper is to do something similar in the Cohen-Macaulay case. So we define Cohen-Macaulay injective, projective, and flat dimension, and show some central properties.

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Our main result is theorem 5.1 which lists a large number of finiteness conditions on the Cohen-Macaulay dimensions, and shows that they are all equivalent to the ground ring being Cohen-Macaulay with a dualizing module. As a sample of further possibilities, there is also an Auslander-Buchsbaum formula for Cohen-Macaulay projective dimension, see theorem 5.5.

As tools to define the Cohen-Macaulay dimensions, we use "ring changed" Gorenstein dimensions. If A is a ring with a dualizing module C, then we can consider the trivial extension ring $A \ltimes C$, and if M is a complex of A-modules, then we can consider M as a complex of $(A \ltimes C)$-modules and take "ring changed" Gorenstein dimensions of M over $A \ltimes C$. We shall develop the theory of these dimensions further in [8]. The present paper only refers to results from [8] at the end, in the proofs of lemma 5.3 and theorem 5.4.

The paper is organized as follows: Section 2 defines the Cohen-Macaulay dimensions. Section 3 studies the trivial extension ring $A \ltimes C$ when C is a semi-dualizing module for A. Section 4 gives some bounds on the injective dimension of C. And finally, section 5 studies the Cohen-Macaulay dimensions and proves the results we have stated.

Setup 1.1. Throughout, A is a commutative noetherian ring. Complexes of A-modules have the form

$$\cdots \longrightarrow M_{i+1} \longrightarrow M_i \longrightarrow M_{i-1} \longrightarrow \cdots,$$

and the words "right-bounded" and "left-bounded" are to be understood relative to this.

2. Homological dimensions

Definition 2.1. Let C be an A-module. The direct sum $A \oplus C$ can be equipped with the product

$$\begin{pmatrix} a \\ c \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} aa_1 \\ ac_1 + ca_1 \end{pmatrix}.$$

This turns $A \oplus C$ into a ring which is called the trivial extension of A by C and denoted $A \ltimes C$.

There are ring homomorphisms

$$\begin{array}{ccccc} A & \longrightarrow & A \ltimes C & \longrightarrow & A, \\ a & \longmapsto & \begin{pmatrix} a \\ 0 \end{pmatrix}, & & \\ & & \begin{pmatrix} a \\ c \end{pmatrix} & \longmapsto & a \end{array}$$

whose composition is the identity on A. These homomorphisms allow us to view any A-module as an $(A \ltimes C)$-module and any $(A \ltimes C)$-module as an A-module, and we shall do so freely.

In particular, if M is a suitably bounded complex of A-modules, we can consider the "ring changed" Gorenstein dimensions

$$\mathrm{Gid}_{A\ltimes C} M, \ \mathrm{Gpd}_{A\ltimes C} M, \text{ and } \mathrm{Gfd}_{A\ltimes C} M.$$

Before the next definition, recall that a semi-dualizing module C for A is a finitely generated module for which the canonical map $A \longrightarrow \mathrm{Hom}_A(C, C)$ is an isomorphism, while $\mathrm{Ext}^i_A(C, C) = 0$ for each $i \geqslant 1$. Equivalently, C is a finitely generated module so that the canonical morphism $A \longrightarrow \mathrm{RHom}_A(C, C)$ is an isomorphism in the derived category $\mathsf{D}(A)$. An example of a semi-dualizing module is A itself. The theory of semi-dualizing modules (and complexes) is developed in [3].

Definition 2.2. Let M and N be complexes of A-modules so that the homology of M is bounded to the left and the homology of N is bounded to the right.

The Cohen-Macaulay injective, projective, and flat dimensions of M and N over A are

$$\begin{aligned}\mathrm{CMid}_A M &= \inf\{\, \mathrm{Gid}_{A\ltimes C} M \mid C \text{ is a semi-dualizing module}\,\},\\ \mathrm{CMpd}_A N &= \inf\{\, \mathrm{Gpd}_{A\ltimes C} N \mid C \text{ is a semi-dualizing module}\,\},\\ \mathrm{CMfd}_A N &= \inf\{\, \mathrm{Gfd}_{A\ltimes C} N \mid C \text{ is a semi-dualizing module}\,\}.\end{aligned}$$

3. Lemmas on the trivial extension

Lemma 3.1. *Let C be an A-module.*

(1) *If I is a (faithfully) injective A-module then* $\mathrm{Hom}_A(A \ltimes C, I)$ *is a (faithfully) injective $(A \ltimes C)$-module.*

(2) *Each injective $(A\ltimes C)$-module is a direct summand in a module* $\mathrm{Hom}_A(A \ltimes C, I)$ *where I is an injective A-module.*

Proof. (1) Adjunction gives

$$\begin{aligned}\mathrm{Hom}_{A\ltimes C}(-, \mathrm{Hom}_A(A \ltimes C, I)) &\simeq \mathrm{Hom}_A((A \ltimes C) \otimes_{A\ltimes C} -, I)\\ &\simeq \mathrm{Hom}_A(-, I)\end{aligned} \tag{1}$$

on $(A\ltimes C)$-modules, making it clear that if I is a (faithfully) injective A-module, then $\mathrm{Hom}_A(A \ltimes C, I)$ is a (faithfully) injective $(A \ltimes C)$-module.

(2) To see that an injective $(A \ltimes C)$-module J is a direct summand in a module of the form $\operatorname{Hom}_A(A \ltimes C, I)$, it is enough to embed it into such a module. For this, first view J as an A-module and embed it into an injective A-module I. Then use equation (1) to convert the monomorphism of A-modules $J \hookrightarrow I$ to a monomorphism of $(A \ltimes C)$-modules $J \hookrightarrow \operatorname{Hom}_A(A \ltimes C, I)$. $\square$

Lemma 3.2. *Let C be a semi-dualizing module for A.*

(1) *There is an isomorphism*
$$\operatorname{RHom}_A(A \ltimes C, C) \cong A \ltimes C$$
in the derived category $\mathsf{D}(A \ltimes C)$.

(2) *There is a natural equivalence*
$$\operatorname{RHom}_{A \ltimes C}(-, A \ltimes C) \simeq \operatorname{RHom}_A(-, C)$$
of functors on $\mathsf{D}(A)$.

(3) *If M is in $\mathsf{D}(A)$ then the biduality morphisms*
$$M \longrightarrow \operatorname{RHom}_A(\operatorname{RHom}_A(M, C), C)$$
and
$$M \longrightarrow \operatorname{RHom}_{A \ltimes C}(\operatorname{RHom}_{A \ltimes C}(M, A \ltimes C), A \ltimes C)$$
are equal.

(4) *There is an isomorphism*
$$\operatorname{RHom}_{A \ltimes C}(A, A \ltimes C) \cong C$$
in $\mathsf{D}(A \ltimes C)$.

Proof. (1) Since C is semi-dualizing, it is clear that there is an isomorphism in $\mathsf{D}(A)$,
$$\operatorname{RHom}_A(A \oplus C, C) \cong C \oplus A.$$
It is easy to see that in fact, this isomorphism respects the action of $A \ltimes C$, so
$$\operatorname{RHom}_A(A \ltimes C, C) \cong A \ltimes C$$
in $\mathsf{D}(A \ltimes C)$.

(2) This is a computation,
$$\begin{aligned}
\operatorname{RHom}_{A \ltimes C}(-, A \ltimes C) &\overset{(a)}{\simeq} \operatorname{RHom}_{A \ltimes C}(-, \operatorname{RHom}_A(A \ltimes C, C)) \\
&\overset{(b)}{\simeq} \operatorname{RHom}_A((A \ltimes C) \otimes^{\mathrm{L}}_{A \ltimes C} -, C) \\
&\simeq \operatorname{RHom}_A(-, C),
\end{aligned}$$
where (a) is by part (1) and (b) is by adjunction.

(3) and (4) These are easy to obtain from (2). □

Lemma 3.3. *Let C be a semi-dualizing module for A and let I be an injective A-module.*

(1) *A and C are Gorenstein projective over $A \ltimes C$.*

(2) *$\operatorname{Hom}_A(A, I) \cong I$ and $\operatorname{Hom}_A(C, I)$ are Gorenstein injective over $A \ltimes C$.*

Proof. (1) Lemma 3.2(4) says $\operatorname{RHom}_{A\ltimes C}(A, A \ltimes C) \cong C$. That is, the dual of A with respect to the ring $A \ltimes C$ is C. But dualization with respect to the ring preserves the class of finitely generated Gorenstein projective modules by [2, obs. (1.1.7)], so to prove part (1) it is enough to see that A is Gorenstein projective over $A \ltimes C$.

By [2, prop. (2.2.2)], this will follow if $\operatorname{RHom}_{A\ltimes C}(A, A \ltimes C)$ is concentrated in degree zero and the biduality morphism

$$A \longrightarrow \operatorname{RHom}_{A\ltimes C}(\operatorname{RHom}_{A\ltimes C}(A, A \ltimes C), A \ltimes C)$$

is an isomorphism. The first of these conditions holds by lemma 3.2(4), and the second condition holds because the biduality morphism equals

$$A \longrightarrow \operatorname{RHom}_A(\operatorname{RHom}_A(A, C), C)$$

by lemma 3.2(3), and this is an isomorphism because it is equal to the canonical morphism $A \longrightarrow \operatorname{RHom}_A(C, C)$.

(2) We will prove that $\operatorname{Hom}_A(C, I)$ is Gorenstein injective over $A \ltimes C$, the case of $\operatorname{Hom}_A(A, I) \cong I$ being similar.

Since C is Gorenstein projective over $A \ltimes C$, by definition it has a complete projective resolution P. So P is a complex of $(A \ltimes C)$-modules which has C as one of its cycle modules, say $\mathrm{Z}_0(P) \cong C$. Moreover, P is an exact complex of projective $(A \ltimes C)$-modules, and

$$\operatorname{Hom}_{A\ltimes C}(P, Q)$$

is exact when Q is a projective $(A \ltimes C)$-module. Since C is finitely generated, we can assume that P consists of finitely generated $(A\ltimes C)$-modules by [2, thms. (4.1.4) and (4.2.6)].

The $(A \ltimes C)$-module $J = \operatorname{Hom}_A(A \ltimes C, I)$ is injective by lemma 3.1(1). Consider the complex

$$K = \operatorname{Hom}_{A\ltimes C}(P, J).$$

This is clearly an exact complex of injective $(A \ltimes C)$-modules. Moreover, if L is an injective $(A \ltimes C)$-module then

$$\begin{aligned}\operatorname{Hom}_{A\ltimes C}(L,K) &= \operatorname{Hom}_{A\ltimes C}(L,\operatorname{Hom}_{A\ltimes C}(P,J)) \\ &\cong \operatorname{Hom}_{A\ltimes C}(L\otimes_{A\ltimes C} P,J) \\ &\cong \operatorname{Hom}_{A\ltimes C}(P,\operatorname{Hom}_{A\ltimes C}(L,J)) \\ &= (*),\end{aligned}$$

where both $\cong$'s are by adjunction. Here $\operatorname{Hom}_{A\ltimes C}(L,J)$ is a flat $(A \ltimes C)$-module by [10, thm. 1.2], so it is the direct limit of projective $(A \ltimes C)$-modules,

$$\operatorname{Hom}_{A\ltimes C}(L,J) \cong \varinjlim Q_\alpha.$$

So

$$(*) \cong \operatorname{Hom}_{A\ltimes C}(P,\varinjlim Q_\alpha) \cong \varinjlim \operatorname{Hom}_{A\ltimes C}(P,Q_\alpha) = (**),$$

where the second $\cong$ holds because each module in P is finitely generated. Since each $\operatorname{Hom}_{A\ltimes C}(P,Q_\alpha)$ is exact, so is $(**)$.

This shows that K is a complete injective resolution over $A \ltimes C$, and

$$\begin{aligned}\mathrm{Z}_{-1}(K) &= \mathrm{Z}_{-1}(\operatorname{Hom}_{A\ltimes C}(P,J)) \\ &\cong \operatorname{Hom}_{A\ltimes C}(\mathrm{Z}_0(P),J) \\ &\cong \operatorname{Hom}_{A\ltimes C}(C,\operatorname{Hom}_A(A\ltimes C,I)) \\ &\overset{\text{(a)}}{\cong} \operatorname{Hom}_A(C\otimes_{A\ltimes C}(A\ltimes C),I) \\ &\cong \operatorname{Hom}_A(C,I),\end{aligned}$$

where (a) is again by adjunction. So K is a complete injective resolution of $\operatorname{Hom}_A(C,I)$ which is therefore Gorenstein injective. □

Lemma 3.4. *Let J be an injective A-module. Then there is an equivalence of functors on the category of $(A \ltimes C)$-modules,*

$$\operatorname{RHom}_{A\ltimes C}(\operatorname{Hom}_A(A\ltimes C,J),-) \simeq \operatorname{RHom}_A(\operatorname{Hom}_A(C,J),-).$$

Proof. We have

$$\begin{aligned}
&\mathrm{RHom}_{A\ltimes C}(\mathrm{RHom}_A(A\ltimes C, J), -)\\
&\quad\overset{\text{(a)}}{\simeq} \mathrm{RHom}_{A\ltimes C}(\mathrm{RHom}_A(\mathrm{RHom}_A(A\ltimes C, C), J), -)\\
&\quad\overset{\text{(b)}}{\simeq} \mathrm{RHom}_{A\ltimes C}((A\ltimes C)\otimes^{\mathrm{L}}_A \mathrm{RHom}_A(C, J), -)\\
&\quad\overset{\text{(c)}}{\simeq} \mathrm{RHom}_A(\mathrm{RHom}_A(C, J), -),
\end{aligned}$$

where (a) is by lemma 3.2(1), (b) is by the isomorphism [2, (A.4.24)], and (c) is by adjunction. ☐

Lemma 3.5. *Let A be a local ring with a non-zero finitely generated module C. Then*

$$A\ltimes C \textit{ is a Gorenstein ring} \Leftrightarrow C \textit{ is a dualizing module for } A.$$

Proof. This can be found between the lines in [4] or [13], or explicitly as a special case of [9, thm. 2.2]. ☐

4. Bounds on the injective dimension of C

Lemma 4.1. *Let C be a semi-dualizing module for A, and let M be an A-module which is Gorenstein injective over $A\ltimes C$. Then there exists a short exact sequence of A-modules,*

$$0\to M' \longrightarrow \mathrm{Hom}_A(C, I) \longrightarrow M \to 0,$$

where I is injective over A and M' is Gorenstein injective over $A\ltimes C$, which stays exact if one applies to it the functor $\mathrm{Hom}_A(\mathrm{Hom}_A(C, J), -)$ *for any injective A-module J.*

Proof. Since M is Gorenstein injective over $A\ltimes C$, it has a complete injective resolution. From this can be extracted a short exact sequence of $(A\ltimes C)$-modules,

$$0\to N\longrightarrow K\longrightarrow M\to 0,$$

where K is injective and N Gorenstein injective over $A\ltimes C$, which stays exact if one applies to it the functor $\mathrm{Hom}_{A\ltimes C}(L, -)$ for any injective $(A\ltimes C)$-module L.

In particular, the sequence stays exact if one applies to it the functor $\mathrm{Hom}_{A\ltimes C}(\mathrm{Hom}_A(A\ltimes C, J), -)$ for any injective A-module J, because $\mathrm{Hom}_A(A\ltimes C, J)$ is an injective $(A\ltimes C)$-module by lemma 3.1(1).

By lemma 3.1(2), the injective $(A\ltimes C)$-module K is a direct summand in $\mathrm{Hom}_A(A\ltimes C, I)$ for some injective A-module I. If $K\oplus K'\cong \mathrm{Hom}_A(A\ltimes C, I)$, then by adding K' to both the first and the second module in the short exact sequence, we may assume that the sequence has the form

$$0 \to N \longrightarrow \mathrm{Hom}_A(A \ltimes C, I) \xrightarrow{\eta} M \to 0.$$

The module N is still Gorenstein injective over $A \ltimes C$, and the sequence still stays exact if one applies to it the functor

$$\mathrm{Hom}_{A \ltimes C}(\mathrm{Hom}_A(A \ltimes C, J), -)$$

for any injective A-module J.

Now let us consider in detail the homomorphism η. Elements of the source $\mathrm{Hom}_A(A \ltimes C, I)$ have the form (α, γ) where $A \xrightarrow{\alpha} I$ and $C \xrightarrow{\gamma} I$ are homomorphisms of A-modules. The $(A \ltimes C)$-module structure of $\mathrm{Hom}_A(A \ltimes C, I)$ comes from the first variable, and one checks that it takes the form

$$\begin{pmatrix} a \\ c \end{pmatrix} \cdot (\alpha, \gamma) = (a\alpha + \chi_{\gamma(c)}, a\gamma),$$

where $\chi_{\gamma(c)}$ is the homomorphism $A \longrightarrow I$ given by $a \mapsto a\gamma(c)$.

The target of η is M which is an A-module. When viewed as an $(A \ltimes C)$-module, M is annihilated by the ideal $0 \ltimes C$, so

$$0 = \begin{pmatrix} 0 \\ c \end{pmatrix} \cdot \eta(\alpha, \gamma) = \eta(\begin{pmatrix} 0 \\ c \end{pmatrix} \cdot (\alpha, \gamma)) = \eta(\chi_{\gamma(c)}, 0), \tag{2}$$

where the last $=$ follows from the previous equation.

In fact, this implies

$$\eta(\alpha, 0) = 0 \tag{3}$$

for each $A \xrightarrow{\alpha} I$. To see so, note that there is a surjection $F \longrightarrow \mathrm{Hom}_A(C, I)$ with F free, and hence a surjection $C \otimes_A F \longrightarrow C \otimes_A \mathrm{Hom}_A(C, I)$. The target here is isomorphic to I by [3, prop. (4.4) and obs. (4.10)], so there is a surjection $C \otimes_A F \longrightarrow I$. As $C \otimes_A F$ is a direct sum of copies of C, this means that, given an element i in I, it is possible to find homomorphisms $\gamma_1, \ldots, \gamma_t : C \longrightarrow I$ and elements $c_1, \ldots, c_t$ in C with $i = \gamma_1(c_1) + \cdots + \gamma_t(c_t)$. Hence the homomorphism

$$A \xrightarrow{\alpha} I$$

given by $a \mapsto ai$ is equal to

$$\chi_{\gamma_1(c_1)+\cdots+\gamma_t(c_t)} = \chi_{\gamma_1(c_1)} + \cdots + \chi_{\gamma_t(c_t)},$$

and so equation (2) implies equation (3).

To make use of this, observe that the exact sequence of $(A \ltimes C)$-modules

$$0 \to C \longrightarrow A \ltimes C \longrightarrow A \to 0 \tag{4}$$

induces an exact sequence

$$0 \to \operatorname{Hom}_A(A, I) \longrightarrow \operatorname{Hom}_A(A \ltimes C, I) \longrightarrow \operatorname{Hom}_A(C, I) \to 0.$$

So equation (3) can be interpreted as saying that the homomorphism $\operatorname{Hom}_A(A \ltimes C, I) \xrightarrow{\eta} M$ factors through the surjection $\operatorname{Hom}_A(A \ltimes C, I) \longrightarrow \operatorname{Hom}_A(C, I)$. This means that we can construct a commutative diagram of $(A \ltimes C)$-modules with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & \operatorname{Hom}_A(A \ltimes C, I) \arrow[r, "\eta"] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M' \arrow[r] & \operatorname{Hom}_A(C, I) \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

We will show that if we view the lower row as a sequence of A-modules by means of the ring homomorphism $A \longrightarrow A \ltimes C$, then it is a short exact sequence with the properties claimed in the lemma.

First, I is injective over A by construction.

Secondly, applying the Snake Lemma to the above diagram embeds the vertical arrows into exact sequences. The leftmost of these is

$$0 \to \operatorname{Hom}_A(A, I) \longrightarrow N \longrightarrow M' \to 0.$$

Here the modules $\operatorname{Hom}_A(A, I) \cong I$ and N are Gorenstein injective over $A \ltimes C$ by lemma 3.3(2), respectively, by assumption. Hence M' is also Gorenstein injective over $A \ltimes C$ because the class of Gorenstein injective modules is injectively resolving by [7, thm. 2.6].

Thirdly, by construction, the upper sequence in the diagram stays exact if one applies to it the functor $\operatorname{Hom}_{A \ltimes C}(\operatorname{Hom}_A(A \ltimes C, J), -)$ for any injective A-module J. It follows that the same holds for the lower row. But taking H_0 of lemma 3.4 shows

$$\operatorname{Hom}_{A \ltimes C}(\operatorname{Hom}_A(A \ltimes C, J), -) \simeq \operatorname{Hom}_A(\operatorname{Hom}_A(C, J), -),$$

so the lower row in the diagram also stays exact if one applies to it the functor $\operatorname{Hom}_A(\operatorname{Hom}_A(C, J), -)$ for any injective A-module J. $\square$

The following lemmas use ${}_C\mathcal{A}(A)$ and ${}_C\mathcal{B}(A)$, the Auslander and Bass classes of the semi-dualizing module C, as introduced in [3, def. (4.1)]. The proof of the first of the lemmas can be found in [5] (there is also included a proof at the end of this paper).

Lemma 4.2. *Let C be a semi-dualizing module for A, let M in ${}_C\mathcal{A}(A)$ satisfy $\operatorname{Gid}_{A \ltimes C} M < \infty$, and write $s = \sup\{\, i \mid \mathrm{H}_i M \neq 0 \,\}$. Then there* *is a distinguished triangle in* $\mathsf{D}(A)$,

$$\Sigma^s H \longrightarrow Y \longrightarrow M \longrightarrow,$$

where H *is an* A*-module which is Gorenstein injective over* $A \ltimes C$*, and where*

$$\mathrm{id}_A(C \otimes^{\mathrm{L}}_A Y) \leqslant \mathrm{Gid}_{A \ltimes C} M.$$

Lemma 4.3. *Let* C *be a semi-dualizing module for* A*, let* M *be a complex of* A*-modules with homology bounded to the right and* $\mathrm{pd}_A M < \infty$*, and let* H *be an* A*-module which is Gorenstein injective over* $A \ltimes C$*. Then*

$$\mathrm{H}_{-(j+1)} \mathrm{RHom}_A(M, H) \;=\; 0$$

for $j \geqslant \sup\{\, i \mid \mathrm{H}_i M \neq 0 \,\}$.

Proof. Since M has homology bounded to the right and $\mathrm{pd}_A M < \infty$, there exists a bounded projective resolution P of M, and

$$\mathrm{H}_{-(j+1)} \mathrm{RHom}_A(M, H) \;\cong\; \mathrm{Ext}^1_A(C^P_j, H)$$

where C^P_j is the j'th cokernel of P. Since

$$\cdots \longrightarrow P_{j+1} \longrightarrow P_j \longrightarrow C^P_j \to 0$$

is a projective resolution of C^P_j and since P is bounded, we have $\mathrm{pd}_A C^P_j < \infty$. Hence it is enough to show

$$\mathrm{Ext}^1_A(M, H) = 0$$

for each A-module M with $\mathrm{pd}_A M < \infty$.

To prove this, we first argue that if I is any injective A-module then

$$\mathrm{Ext}^i_A(M, \mathrm{Hom}_A(C, I)) = 0$$

for $i > 0$. For this, note that we have

$$\begin{aligned}
\mathrm{RHom}_A(M, \mathrm{Hom}_A(C, I)) &\cong \mathrm{RHom}_A(M, \mathrm{RHom}_A(C, I)) \\
&\overset{\text{(a)}}{\cong} \mathrm{RHom}_A(M \otimes^{\mathrm{L}}_A C, I) \\
&\overset{\text{(b)}}{\cong} \mathrm{RHom}_A(C, \mathrm{RHom}_A(M, I)) \\
&\cong \mathrm{RHom}_A(C, \mathrm{Hom}_A(M, I))
\end{aligned}$$

where (a) and (b) are by adjunction, and consequently,

$$\mathrm{Ext}^i_A(M, \mathrm{Hom}_A(C, I)) \cong \mathrm{Ext}^i_A(C, \mathrm{Hom}_A(M, I)) \tag{5}$$

for each i. The condition $\mathrm{pd}_A M < \infty$ implies $\mathrm{id}_A \mathrm{Hom}_A(M, I) < \infty$, and therefore $\mathrm{Hom}_A(M, I)$ belongs to ${}_C\mathcal{B}(A)$ by [3, prop. (4.4)]. Thus [3, obs. (4.10)] implies that the last module in (5) is zero for $i > 0$.

Now set $n = \operatorname{pd}_A M$. Repeated use of lemma 4.1 shows that there is an exact sequence of A-modules

(6)
$$0 \to H' \longrightarrow \operatorname{Hom}_A(C, I_{n-1}) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_A(C, I_0) \longrightarrow H \to 0,$$

where $I_0, \ldots, I_{n-1}$ are injective A-modules. Applying $\operatorname{Hom}_A(M, -)$ to (6) and using $\operatorname{Ext}^i_A(M, \operatorname{Hom}_A(C, I_j)) = 0$ for each $i > 0$ and each j, we obtain

$$\operatorname{Ext}^1_A(M, H) \cong \operatorname{Ext}^{n+1}_A(M, H') = 0$$

as desired. Here the last equality holds because $\operatorname{pd}_A M = n$. $\square$

Lemma 4.4. *Let C be a semi-dualizing module for A and let M be in ${}_C\mathcal{A}(A)$. Set $s = \sup\{\, i \mid \mathrm{H}_i M \neq 0 \,\}$ and suppose that M satisfies*

$$\mathrm{H}_{-(s+1)} \operatorname{RHom}_A(M, H) = 0$$

for each A-module H which is Gorenstein injective over $A \ltimes C$. Then

$$\operatorname{Gid}_{A\ltimes C} M = \operatorname{id}_A(C \otimes^{\mathrm{L}}_A M).$$

Proof. To prove the lemma's equation, let us first prove the inequality $\leqslant$. Let $t = \sup\{\, i \mid \mathrm{H}_i(C \otimes^{\mathrm{L}}_A M) \neq 0 \,\}$ and $n = \operatorname{id}_A(C \otimes^{\mathrm{L}}_A M)$. We may clearly suppose that n is finite. Let

$$J = \cdots \longrightarrow 0 \longrightarrow J_t \longrightarrow \cdots \longrightarrow \cdots \longrightarrow J_{-n} \longrightarrow 0 \longrightarrow \cdots$$

be an injective resolution of $C \otimes^{\mathrm{L}}_A M$. The complex M is in ${}_C\mathcal{A}(A)$, so we get the first $\cong$ in

$$M \cong \operatorname{RHom}_A(C, C \otimes^{\mathrm{L}}_A M) \cong \operatorname{RHom}_A(C, J) \cong \operatorname{Hom}_A(C, J).$$

Lemma 3.3(2) implies that $\operatorname{Hom}_A(C, J)$ is a complex of Gorenstein injective modules over $A \ltimes C$. Since $\operatorname{Hom}_A(C, J)_\ell = \operatorname{Hom}_A(C, J_\ell) = 0$ for $\ell < -n$, we see that

$$\operatorname{Gid}_{A\ltimes C} M \leqslant n = \operatorname{id}_A(C \otimes^{\mathrm{L}}_A M).$$

Let us next prove the inequality $\geqslant$. Recall that $s = \sup\{i \mid \mathrm{H}_i M \neq 0\,\}$. We may clearly suppose that $\operatorname{Gid}_{A\ltimes C} M$ is finite. By lemma 4.2 there is a distinguished triangle in $\mathsf{D}(A)$,

$$\Sigma^s H \longrightarrow Y \xrightarrow{f} M \longrightarrow, \tag{7}$$

where H is an A-module which is Gorenstein injective over $A \ltimes C$, and where

$$\operatorname{id}_A(C \otimes^{\mathrm{L}}_A Y) \leqslant \operatorname{Gid}_{A\ltimes C} M. \tag{8}$$

Applying $\operatorname{RHom}_A(M, -)$ to (7) gives another distinguished triangle whose long exact homology sequence contains

$$\mathrm{H}_0 \operatorname{RHom}_A(M, Y) \longrightarrow \mathrm{H}_0 \operatorname{RHom}_A(M, M) \longrightarrow \mathrm{H}_{-1} \operatorname{RHom}_A(M, \Sigma^s H)$$

which can also be written

$$\mathrm{Hom}_{\mathsf{D}(A)}(M,Y) \longrightarrow \mathrm{Hom}_{\mathsf{D}(A)}(M,M) \longrightarrow \mathrm{H}_{-(s+1)}\,\mathrm{RHom}_A(M,H) = 0,$$

where the last zero comes from the assumptions on M. Consequently, there exists a morphism $g : M \longrightarrow Y$ in $\mathsf{D}(A)$ with $fg = 1_M$. That is, the distinguished triangle (7) is split, so $Y \cong \Sigma^s H \oplus M$.

This implies

$$C \otimes^{\mathrm{L}}_A Y \cong (C \otimes^{\mathrm{L}}_A \Sigma^s H) \oplus (C \otimes^{\mathrm{L}}_A M)$$

from which clearly follows

$$\mathrm{id}_A(C \otimes^{\mathrm{L}}_A M) \leqslant \mathrm{id}_A(C \otimes^{\mathrm{L}}_A Y). \tag{9}$$

Combining the inequalities (8) and (9) now shows

$$\mathrm{id}_A(C \otimes^{\mathrm{L}}_A M) \leqslant \mathrm{Gid}_{A \ltimes C}\, M$$

as desired. □

Proposition 4.5. *Assume that the ring A is local. Let C be a semi-dualizing module for A, and let M be a complex of A-modules with homology bounded to the right and* $\mathrm{fd}_A\, M < \infty$. *Then*

$$\mathrm{id}_A\, C \leqslant \mathrm{Gid}_{A \ltimes C}\, M + \mathrm{width}_A\, M.$$

Proof. Denote by k the residue class field of A. Since $\mathrm{fd}_A\, M < \infty$, the isomorphism [2, (A.4.23)] gives

$$\mathrm{RHom}_A(k, C \otimes^{\mathrm{L}}_A M) \cong \mathrm{RHom}_A(k, C) \otimes^{\mathrm{L}}_A M.$$

This implies (a) in

$$\begin{aligned}
&\inf\{\, i \mid \mathrm{H}_i\,\mathrm{RHom}_A(k, C \otimes^{\mathrm{L}}_A M) \neq 0 \,\} \\
&\quad \overset{(a)}{=} \inf\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_A(k, C) \otimes^{\mathrm{L}}_A M) \neq 0 \,\} \\
&\quad \overset{(b)}{=} \inf\{\, i \mid \mathrm{H}_i\,\mathrm{RHom}_A(k, C) \neq 0 \,\} + \inf\{\, i \mid \mathrm{H}_i(M \otimes^{\mathrm{L}}_A k) \neq 0 \,\} \\
&\quad = -\,\mathrm{id}_A\, C + \mathrm{width}_A\, M,
\end{aligned}$$

where (b) is by the "Accounting Principle" [2, (A.7.9.2)]. Consequently,

$$\begin{aligned}
\mathrm{id}_A\, C &= -\inf\{\, i \mid \mathrm{H}_i\,\mathrm{RHom}_A(k, C \otimes^{\mathrm{L}}_A M) \neq 0 \,\} + \mathrm{width}_A\, M \\
&\leqslant \mathrm{id}_A(C \otimes^{\mathrm{L}}_A M) + \mathrm{width}_A\, M \\
&= \mathrm{Gid}_{A \ltimes C}\, M + \mathrm{width}_A\, M.
\end{aligned}$$

The last $=$ follows from lemmas 4.3 and 4.4. Lemma 4.3 applies to M because $\mathrm{fd}_A\, M < \infty$ implies $\mathrm{pd}_A\, M < \infty$ by [12, Seconde partie, cor. (3.2.7)], and lemma 4.4 applies because $\mathrm{fd}_A\, M < \infty$ implies $M \in {}_C\mathcal{A}(A)$ by [3, prop. (4.4)]. □

Lemma 4.6. *Let C be a semi-dualizing module for A, let I be a faithfully injective A-module, and let M be a complex of A-modules with right-bounded homology. Then*

$$\operatorname{Gid}_{A\ltimes C}\operatorname{Hom}_A(M,I)=\operatorname{Gfd}_{A\ltimes C}M.$$

Proof. From lemma 3.1(1) it follows that $E=\operatorname{Hom}_A(A\ltimes C,I)$ is a faithfully injective $(A\ltimes C)$-module. Hence

$$\operatorname{Gid}_{A\ltimes C}\operatorname{Hom}_{A\ltimes C}(M,E)=\operatorname{Gfd}_{A\ltimes C}M$$

as follows from [2, thm. (6.4.2)].

But equation (1) in the proof of lemma 3.1 shows $\operatorname{Hom}_{A\ltimes C}(M,E)\cong\operatorname{Hom}_A(M,I)$, so accordingly,

$$\operatorname{Gid}_{A\ltimes C}\operatorname{Hom}_A(M,I)=\operatorname{Gfd}_{A\ltimes C}M.$$

□

Proposition 4.7. *Assume that the ring A is local. Let C be a semi-dualizing A-module, and let N be a complex of A-modules with homology bounded to the right and $\operatorname{id}_A N<\infty$. Then*

$$\operatorname{id}_A C\leqslant\operatorname{Gfd}_{A\ltimes C}N+\operatorname{depth}_A N.$$

Proof. Apply lemma 4.6 and Matlis duality to proposition 4.5. □

5. Properties of the Cohen-Macaulay dimensions

Theorem 5.1. *Let A be a local ring with residue class field k. Then the following conditions are equivalent.*

1. *A is a Cohen-Macaulay ring with a dualizing module.*
2. *$\operatorname{CMid}_A M<\infty$ holds when M is any complex of A-modules with bounded homology.*
3. *There is a complex M of A-modules with bounded homology, $\operatorname{CMid}_A M<\infty$, $\operatorname{fd}_A M<\infty$, and $\operatorname{width}_A M<\infty$.*
4. *$\operatorname{CMid}_A k<\infty$.*
5. *$\operatorname{CMpd}_A M<\infty$ holds when M is any complex of A-modules with bounded homology.*
6. *There is a complex M of A-modules with bounded homology, $\operatorname{CMpd}_A M<\infty$, $\operatorname{id}_A M<\infty$, and $\operatorname{depth}_A M<\infty$.*
7. *$\operatorname{CMpd}_A k<\infty$.*
8. *$\operatorname{CMfd}_A M<\infty$ holds when M is any complex of A-modules with bounded homology.*
9. *There is a complex M of A-modules with bounded homology, $\operatorname{CMfd}_A M<\infty$, $\operatorname{id}_A M<\infty$, and $\operatorname{depth}_A M<\infty$.*

(10) $\mathrm{CMfd}_A\, k < \infty$.

Proof. Let us prove that conditions (1), (2), (3), and (4) are equivalent. Similar proofs give that so are (1), (5), (6), and (7) as well as (1), (8), (9), and (10).

(1) $\Rightarrow$ (2). Let A be Cohen-Macaulay with dualizing module C. Then $A \ltimes C$ is Gorenstein by lemma 3.5. If M is a complex of A-modules with bounded homology, then M is also a complex of $(A \ltimes C)$-modules with bounded homology, so

$$\mathrm{Gid}_{A \ltimes C}\, M < \infty$$

by [2, thm. (6.2.7)]. But as C is in particular a semi-dualizing module, this clearly implies

$$\mathrm{CMid}_A\, M < \infty.$$

(2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4). Trivial.

(3) $\Rightarrow$ (1). For $\mathrm{CMid}_A\, M < \infty$, the definition of CMid implies that A has a semi-dualizing module C with $\mathrm{Gid}_{A \ltimes C}\, M < \infty$. But when $\mathrm{fd}_A\, M < \infty$ and $\mathrm{width}_A\, M < \infty$ also hold, then proposition 4.5 implies $\mathrm{id}_A\, C < \infty$. So A is Cohen-Macaulay with dualizing module C.

(4) $\Rightarrow$ (1). When $\mathrm{CMid}_A\, k < \infty$ then A has a semi-dualizing module C with

$$\mathrm{Gid}_{A \ltimes C}\, k < \infty.$$

Then the homology of $\mathrm{RHom}_{A \ltimes C}(E_{A \ltimes C}(k), k)$ is bounded by [7, thm. 2.22]. However,

$$\begin{aligned}
\mathrm{RHom}_{A \ltimes C}(E_{A \ltimes C}(k), k) &\overset{\text{(a)}}{\cong} \mathrm{RHom}_{A \ltimes C}(k^\vee, E_{A \ltimes C}(k)^\vee) \\
&\cong \mathrm{RHom}_{A \ltimes C}(k, \widehat{A \ltimes C}) \\
&\overset{\text{(b)}}{\cong} \mathrm{RHom}_{A \ltimes C}(k, A \ltimes C),
\end{aligned}$$

where (a) is by Matlis duality and (b) is by [11, exer. 7.7]. So the homology of $\mathrm{RHom}_{A \ltimes C}(k, A \ltimes C)$ is bounded, whence $A \ltimes C$ is a Gorenstein ring. But then A is a Cohen-Macaulay ring with dualizing module C by lemma 3.5. $\square$

Remark 5.2. In condition (3) of the above theorem, one could consider for M either the ring A itself, or the Koszul complex $K(x_1, \ldots, x_r)$ on any sequence of elements $x_1, \ldots, x_r$ in the maximal ideal. These complexes satisfy $\operatorname{fd}_A M < \infty$ and $\operatorname{width}_A M < \infty$, and so either of the conditions

$$\operatorname{CMid}_A A < \infty \quad \text{and} \quad \operatorname{CMid}_A K(x_1, \ldots, x_r) < \infty$$

is equivalent to A being a Cohen-Macaulay ring with a dualizing module.

Similarly, in conditions (6) and (9), one could consider for M either the injective hull of the residue class field $E_A(k)$, or a dualizing complex D (if one is known to exist). These complexes satisfy $\operatorname{id}_A M < \infty$ and $\operatorname{depth}_A M < \infty$, and so either of the conditions

$$\operatorname{CMpd}_A E_A(k) < \infty, \quad \text{and} \quad \operatorname{CMpd}_A D < \infty,$$

and

$$\operatorname{CMfd}_A E_A(k) < \infty, \quad \text{and} \quad \operatorname{CMfd}_A D < \infty$$

is equivalent to A being a Cohen-Macaulay ring with a dualizing module.

The following results use CMdim, the Cohen-Macaulay dimension introduced by Gerko in [6], and Gdim, the G-dimension originally introduced by Auslander and Bridger in [1].

Lemma 5.3. *Let A be a local ring with a semi-dualizing module C and a finitely generated module M. If*

$$\operatorname{Gpd}_{A \ltimes C} M < \infty$$

then

$$\operatorname{CMdim}_A M = \operatorname{Gpd}_{A \ltimes C} M.$$

Proof. Combining [6, proof of thm. 3.7] with [6, def. 3.2'] shows

$$\operatorname{CMdim}_A M \leqslant \operatorname{Gpd}_{A \ltimes C} M.$$

So $\operatorname{CMdim}_A M$ is finite and hence

$$\operatorname{CMdim}_A M = \operatorname{depth}_A A - \operatorname{depth}_A M$$

by [6, thm. 3.8].

On the other hand,

$$\operatorname{Gpd}_{A \ltimes C} M = C\text{-}\operatorname{Gdim}_A M$$

by [8, prop. 3.1], where C-Gdim_A is the dimension introduced in [3, def. (3.11)] under the name G-$\dim_C$. So C-$\operatorname{Gdim}_A M$ is finite and hence

$$C\text{-}\operatorname{Gdim}_A M = \operatorname{depth}_A A - \operatorname{depth}_A M$$

by [3, thm. (3.14)].

Combining the last three equations shows

$$\operatorname{CMdim}_A M = \operatorname{Gpd}_{A \ltimes C} M$$

as desired. □

Theorem 5.4. *Let A be a local ring with a finitely generated module M. Then*

$$\operatorname{CMdim}_A M \leqslant \operatorname{CMpd}_A M \leqslant \operatorname{Gdim}_A M,$$

and if one of these numbers is finite then the inequalities to its left are equalities.

Proof. The first inequality is clear from lemma 5.3, since $\operatorname{CMpd}_A M$ is defined as the infimum of all $\operatorname{Gpd}_{A \ltimes C} M$.

For the second inequality, note that the ring A is itself a semi-dualizing module, so the definition of CMpd gives $\leqslant$ in

$$\operatorname{CMpd}_A M \leqslant \operatorname{Gpd}_{A \ltimes A} M = \operatorname{Gpd}_A M = \operatorname{Gdim}_A M,$$

where the first $=$ is by [8, cor. 2.17] and the second $=$ follows as M is finitely generated.

Equalities: If $\operatorname{Gdim}_A M < \infty$ then $\operatorname{CMdim}_A M < \infty$ by [6, thm. 3.7]. But $\operatorname{Gdim}_A M < \infty$ implies

$$\operatorname{Gdim}_A M = \operatorname{depth}_A A - \operatorname{depth}_A M$$

by [2, thm. (2.3.13)], and similarly, $\operatorname{CMdim}_A M < \infty$ implies

$$\operatorname{CMdim}_A M = \operatorname{depth}_A A - \operatorname{depth}_A M$$

by [6, thm. 3.8]. So it follows that $\operatorname{CMdim}_A M = \operatorname{Gdim}_A M$, and hence both inequalities in the theorem must be equalities.

If $\operatorname{CMpd}_A M < \infty$ then by the definition of CMpd there exists a semi-dualizing module C over A with $\operatorname{Gpd}_{A \ltimes C} M < \infty$. But by lemma 5.3, any such C has

$$\operatorname{CMdim}_A M = \operatorname{Gpd}_{A \ltimes C} M,$$

and so it follows that the first inequality in the theorem is an equality. □

Since much is known about CMdim_A, this theorem has several immediate consequences for CMpd_A. The following is even clear from the proof of the theorem.

Theorem 5.5 (Auslander-Buchsbaum formula)**.** *Let A be a local ring with a finitely generated module M. If $\operatorname{CMpd}_A M$ is finite, then*

$$\operatorname{CMpd}_A M = \operatorname{depth}_A A - \operatorname{depth}_A M.$$

Acknowledgment. The diagrams were typeset with Paul Taylor's `diagrams.tex`.

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A PROOF OF A LEMMA

A NOTE FOR THE READER

The purpose of this note is, for the convenience of the reader, to provide a proof of the claim [5, lem. 4.2], stating that:

Lemma 0.1. *Let C be a semi-dualizing module for A, let M in ${}_C\mathcal{A}(A)$ satisfy $\mathrm{Gid}_{A \ltimes C} M < \infty$, and write $s = \sup\{\, i \mid \mathrm{H}_i M \neq 0 \,\}$. Then there is a distinguished triangle in $\mathsf{D}(A)$,*

$$\Sigma^s H \longrightarrow Y \longrightarrow M \longrightarrow,$$

where H is an A-module which is Gorenstein injective over $A \ltimes C$, and where

$$\mathrm{id}_A(C \otimes^{\mathrm{L}}_A Y) \leqslant \mathrm{Gid}_{A \ltimes C} M.$$

Lemma 0.1 is a generalization of a result which is to appear in Frankild–Holm [4]. The paper [4] is delayed due to unfortunate circumstances concerning A. Frankild's health.

It is in accordance with the wishes of A. Frankild and P. Jørgensen that the generalized lemma 0.1 is to appear Frankild–Holm [4] and not in Holm–Jørgensen [5].

For the sake of the self-containedness of this thesis, we have chosen to include this note.

Before proving Lemma 0.1, we need to recall [5, lem. 3.3 and 4.1]:

Lemma 0.2. *The A-modules A and C are Gorenstein projective over $A \ltimes C$. If I is an injective A-module, then $\mathrm{Hom}_A(A, I) \cong I$ and $\mathrm{Hom}_A(C, I)$ are Gorenstein injective over $A \ltimes C$.* □

Lemma 0.3. *Let M be an A-module which is Gorenstein injective over $A \ltimes C$. Then there exists a short exact sequence of A-modules,*

$$0 \to M' \longrightarrow \mathrm{Hom}_A(C, J) \longrightarrow M \to 0,$$

where J is injective over A and M' is Gorenstein injective over $A \ltimes C$. Furthermore, the sequence stays exact if one applies to it the functor $\mathrm{Hom}_A(\mathrm{Hom}_A(C, I), -)$ for any injective A-module I. □

Proof of Lemma 0.1. Consider the integers $n = \mathrm{Gid}_{A \ltimes C} M$ and

$$(\dagger) \qquad s \;=\; \sup\{\, i \mid \mathrm{H}_i M \neq 0 \,\} \;=\; \sup\{\, i \mid \mathrm{H}_i(C \otimes^{\mathrm{L}}_A M) \neq 0 \,\},$$

where the last equality is by [1, prop. (4.8)(a)]. Let

$$C\otimes^{\mathrm{L}}_{A}M \;\simeq\; I = 0 \to I_s \to I_{s-1} \to \cdots$$

be an injective resolution of $C\otimes^{\mathrm{L}}_{A}M$. Since $M \in {}_{C}\mathcal{A}(A)$, we get the first isomorphism in:

$$M \;\simeq\; \mathrm{RHom}_A(C, C\otimes^{\mathrm{L}}_{A}M) \;\simeq\; \mathrm{RHom}_A(C, I) \;\simeq\; \mathrm{Hom}_A(C, I).$$

Define $I' = \mathrm{Hom}_A(C, I)$, which by lemma 0.2 is a complex of Gorenstein injective $(A \ltimes C)$-modules. Since $\mathrm{Gid}_{A\ltimes C}M = n$, [2, thm. (2.5)] implies that the A-module $Z^{I'}_{-n}$ is Gorenstein injective over $A \ltimes C$. Consequently, repeated use of lemma 0.3 gives an exact complex of A-modules:

$$J' \;=\; 0 \to J'_{s+1} \to J'_s \to \cdots \to J'_{-n+1} \to Z^{I'}_{-n} \to 0$$

satisfying the following properties:

(i) For indices $-n+1 \leqslant i \leqslant s$ we have $J'_i = \mathrm{Hom}_A(C, J_i)$ for some injective A-module J_i.

(ii) J'_{s+1} is Gorenstein injective over $A \ltimes C$.

(iii) The complex $\mathrm{Hom}_A(\mathrm{Hom}_A(C, I), J')$ is exact for every injective A-module I.

By the property (iii) we get a chain map of complexes:

$$\begin{tikzcd}
I'_{-n}\supset \arrow[d, dotted, "\alpha"] & = & 0 \arrow[r] & 0 \arrow[r] \arrow[d, dotted, "0"] & I'_s \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & I'_{-n+1} \arrow[r] \arrow[d, dotted] & Z^{I'}_{-n} \arrow[r] \arrow[d, equal] & 0 \\
J' & = & 0 \arrow[r] & J'_{s+1} \arrow[r] & J'_s \arrow[r] & \cdots \arrow[r] & J'_{-n+1} \arrow[r] & Z^{I'}_{-n} \arrow[r] & 0
\end{tikzcd}$$

The short exact sequence $0 \to J' \to \mathsf{Cone}\,\alpha \to \Sigma^1(I'_{-n}\supset) \to 0$, together with the fact that J' is exact, gives the first isomorphism in:

$$\mathsf{Cone}\,\alpha \;\simeq\; \Sigma^1(I'_{-n}\supset) \;\simeq\; \Sigma^1 I' \;=\; \Sigma^1\,\mathrm{Hom}_A(C, I) \;\simeq\; \Sigma^1 M.$$

The second isomorphism is because of the inequality:

$$(\ddagger) \qquad -n \;=\; -\mathrm{Gid}_{A\ltimes C}M \;\leqslant\; \inf M \;=\; \inf I'.$$

It is easy to see that the exact complex $0 \to Z^{I'}_{-n} \xrightarrow{=} Z^{I'}_{-n} \to 0$ is a subcomplex of $\mathsf{Cone}\,\alpha$, and thus the quotient complex Q satisfies:

$$Q \;\simeq\; \mathsf{Cone}\,\alpha \;\simeq\; \Sigma^1 M.$$

If we introduce the A-module $H = J'_{s+1} \oplus I'_s$, which is Gorenstein injective over $A \ltimes C$, together with the injective A-modules:

$$K_{-n+1} = J_{-n+1} \quad \text{and} \quad K_i = J_i \oplus I_{i-1} \;\text{ for }\; -n+2 \leqslant i \leqslant s,$$

then we may write Q as:

$$Q \;=\; 0 \to H \xrightarrow{\varepsilon} \operatorname{Hom}_A(C, K_s) \to \cdots \to \operatorname{Hom}_A(C, K_{-n+1}) \to 0,$$

where H sits in degree $s+1$. Note that if U_1 and U_2 are injective A-modules, then every homomorphism $\varphi\colon \operatorname{Hom}_A(C, U_1) \to \operatorname{Hom}_A(C, U_2)$ is induced from a homomorphism $\psi\colon U_1 \to U_2$; namely $\psi = C \otimes_A \varphi$ because of the Hom-evaluation isomorphism [1, p. 11]:

$$C \otimes_A \operatorname{Hom}_A(C, U_i) \xrightarrow{\cong} U_i \quad , \quad i = 1, 2.$$

Consequently, Q induces a complex of injective A-modules:

$$K \;=\; 0 \to K_s \to \cdots \to K_{-n+1} \to 0,$$

and this complex has the property that the mapping cone of the chain map $\varepsilon\colon \Sigma^s H \to \operatorname{Hom}_A(C, K)$ has Q as its mapping cone. Thus there is a distinguished triangle in $\mathsf{D}(A)$:

$$\Sigma^s H \to \operatorname{Hom}_A(C, K) \to Q \rightsquigarrow .$$

Rotating this triangle and using that $\Sigma^{-1} Q \simeq M$ we get another distinguished triangle in $\mathsf{D}(A)$:

$$(\Delta 1) \qquad M \to \Sigma^s H \to \operatorname{Hom}_A(C, K) \rightsquigarrow .$$

Since H is a Gorenstein injective over $A \ltimes C$, lemma 0.3 gives an exact sequence of A-modules, $0 \to H' \to \operatorname{Hom}_A(C, L) \to H \to 0$, where L is injective, and H' is Gorenstein injective over $A \ltimes C$. This sequence induces a distinguished triangle in $\mathsf{D}(A)$:

$$(\Delta 2) \qquad \Sigma^s H' \to \Sigma^s \operatorname{Hom}_A(C, L) \to \Sigma^s H \rightsquigarrow .$$

The triangles $(\Delta 1)$ and $(\Delta 2)$ appear as the left and right triangle, respectively, in the "lower cap" diagram:

$$(*) \qquad \begin{tikzcd}
\operatorname{Hom}_A(C,K) \arrow[dd, squiggly] & & \Sigma^s \operatorname{Hom}_A(C,L) \arrow[ll] \arrow[dl] \\
& \Sigma^s H \arrow[ul] \arrow[dr, squiggly] & \\
M \arrow[ur] \arrow[rr, squiggly] & & \Sigma^s H' \arrow[uu]
\end{tikzcd}$$

where the horizontal maps are given by composition. Applying the octahedron axiom [3, (TR4') p. 123] in the triangulated category $\mathsf{D}(A)$, we now get a complex Y together with an “upper cap” diagram:

$$(**)\qquad \begin{tikzcd}
\operatorname{Hom}_A(C,K) \arrow[dr, rightsquigarrow] \arrow[dd, rightsquigarrow] & & \Sigma^s \operatorname{Hom}_A(C,L) \arrow[ll] \\
& Y \arrow[ur] \arrow[dl] & \\
M \arrow[rr, rightsquigarrow] & & \Sigma^s H' \arrow[ul] \arrow[uu]
\end{tikzcd}$$

where the top and bottom parts of $(**)$ are distinguished triangles in $\mathsf{D}(A)$. Applying the triangulated functor $C\otimes^{\mathrm{L}}_A -$ to the upper triangle in $(**)$ we get another distinguished triangle:

$$(\natural)\qquad C\otimes^{\mathrm{L}}_A Y \to \Sigma^s L \to K \rightsquigarrow .$$

Here we have use that $\operatorname{Hom}_A(C,L) \simeq \operatorname{RHom}_A(C,L)$ and that:

$$C\otimes^{\mathrm{L}}_A\big(\Sigma^s \operatorname{RHom}_A(C,L)\big) \;\simeq\; \Sigma^s\big(C\otimes^{\mathrm{L}}_A \operatorname{RHom}_A(C,L)\big) \;\simeq\; \Sigma^s L,$$

where the last isomorphism comes from [1, (A.4.24)]. A similar argument gives that $C\otimes^{\mathrm{L}}_A \operatorname{Hom}_A(C,K) \simeq K$.

If $i \in \mathbb{Z}$ with $i < -n$, then we have $i+1 < -(n-1)$ and $i < s$, where the last inequality comes from (†) and (‡). Hence, for any A-module T and any integer $i < -n$, the long exact homology sequence induced by $(\natural)$ looks like:

$$\begin{aligned} 0 \overset{\text{(a)}}{=} \mathrm{H}_{i+1} \operatorname{RHom}_A(T,K) &\longrightarrow \mathrm{H}_i \operatorname{RHom}_A(T, C\otimes^{\mathrm{L}}_A Y) - \\ &\to \mathrm{H}_i \operatorname{RHom}_A(T,\Sigma^s L) = \mathrm{H}_{i-s} \operatorname{RHom}_A(T,L) \overset{\text{(b)}}{=} 0, \end{aligned}$$

which implies that $\mathrm{H}_i \operatorname{RHom}_A(T, C\otimes^{\mathrm{L}}_A Y) = 0$. The first zero (a) is because $\operatorname{id}_A K \leqslant n-1$, whereas the second zero (b) comes from the fact that $\operatorname{RHom}_A(T,L) \simeq \operatorname{Hom}_A(C,L)$ is a module. This proves that:

$$-\inf\{\, i \mid \mathrm{H}_i \operatorname{RHom}_A(T, C\otimes^{\mathrm{L}}_A Y) \neq 0 \,\} \;\leqslant\; n$$

for every A-module T. By [1, (A.5.2.1)], we get an estimate for the injective dimension:

$$\operatorname{id}_A(C\otimes^{\mathrm{L}}_A Y) \;\leqslant\; n \;=\; \operatorname{Gid}_{A\ltimes C} M,$$

and the lower triangle in $(**)$ is the desired one. $\square$

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Part VI

Semi-dualizing modules and related Gorenstein homological dimensions

SEMI-DUALIZING MODULES AND RELATED GORENSTEIN HOMOLOGICAL DIMENSIONS

HENRIK HOLM AND PETER JØRGENSEN

Abstract. A semi-dualizing module over a commutative noetherian ring A is a finitely generated module C with $\mathrm{RHom}_A(C,C) \simeq A$ in the derived category $\mathsf{D}(A)$.

We show how each such module gives rise to three new homological dimensions which we call C–Gorenstein projective, C–Gorenstein injective, and C–Gorenstein flat dimension, and investigate the properties of these dimensions.

Introduction

It is by now a well-established fact that over any associative ring A, there exists a Gorenstein injective, Gorenstein projective and Gorenstein flat dimension defined for complexes of A–modules. These are usually denoted $\mathrm{Gid}_A(-)$, $\mathrm{Gpd}_A(-)$ and $\mathrm{Gfd}_A(-)$, respectively. Some references are [2], [4], [11], and [14].

In this paper, we need to consider *semi-dualizing* A–modules C (see Definition 1.1), and in order to make things less technical, we only consider commutative and noetherian rings.

For any semi-dualizing module (in fact, complex) C over A, and any complex Z with bounded and finitely generated homology, Christensen [3] introduced the dimension $G\text{–}\mathrm{dim}_C Z$, and developed a satisfactory theory for this new invariant.

If C is a semi-dualizing A–module and M is any A–complex, then we suggested in [10] the viewpoint that one should change rings from A to $A \ltimes C$ (the *trivial extension* of A by C; see Definition 1.2), and then consider the three "ring changed" Gorenstein dimensions:

$$\mathrm{Gid}_{A\ltimes C}M \quad , \quad \mathrm{Gpd}_{A\ltimes C}M \quad \text{and} \quad \mathrm{Gfd}_{A\ltimes C}M.$$

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The usefulness of this viewpoint was demonstrated as it enabled us to introduce three new *Cohen-Macaulay dimensions*, which characterize Cohen-Macaulay rings in a way one could hope for.

In this paper, we define for every semi-dualizing A–module C, three new Gorenstein dimensions:

$$C\text{-}\mathrm{Gid}_A(-) \quad , \quad C\text{-}\mathrm{Gpd}_A(-) \quad \text{and} \quad C\text{-}\mathrm{Gfd}_A(-),$$

which are called the *C–Gorenstein injective*, *C–Gorenstein projective* and *C–Gorenstein flat dimension*, respectively (see Definition 2.9).

It is worth pointing out that the, say, C–Gorenstein injective dimension is defined in terms of resolutions consisting of so-called *C–Gorenstein injective A–modules* (see Definition 2.7); and it does not involve a change of rings. The dimensions (†) have at least five nice properties:

(1) For complexes with bounded and finitely generated homology, our $C\text{-}\mathrm{Gpd}_A(-)$ agrees with Christensen's $G\text{–}\mathrm{dim}_C(-)$ (Proposition 3.1).

(2) The three C–Gorenstein dimensions always agree with the "ring changed" dimensions $\mathrm{Gid}_{A\ltimes C}(-)$, $\mathrm{Gpd}_{A\ltimes C}(-)$ and $\mathrm{Gfd}_{A\ltimes C}(-)$, which were so important in [10] (Theorem 2.16).

(3) If $C = A$, the C–Gorenstein dimensions agree with the classical Gorenstein dimensions $\mathrm{Gid}_A(-)$, $\mathrm{Gpd}_A(-)$ and $\mathrm{Gfd}_A(-)$.

If A admits a dualizing complex D; cf. [4, Definition (1.1)], then finiteness of the C–Gorenstein dimensions can be interpreted in terms of *Auslander and Bass categories* (see Remark 4.1):

(4) If we define $C^\dagger = \mathrm{RHom}_A(C, D)$, then for all (appropriately homologically bounded) A–complexes M and N, we have the following implications (Theorem 4.6):

$$M \in \mathsf{A}_{C^\dagger}(A) \Leftrightarrow C\text{-}\mathrm{Gpd}_A M < \infty \Leftrightarrow C\text{-}\mathrm{Gfd}_A M < \infty;$$
$$N \in \mathsf{B}_{C^\dagger}(A) \Leftrightarrow C\text{-}\mathrm{Gid}_A N < \infty.$$

This generalizes the main results in [4, Theorems (4.3) and (4.5)].

Finally, each of the three C–Gorenstein dimensions has a related *proper* variant, giving us three additional dimensions (Definitions 5.2 and 5.3):

$$C\text{-}\mathsf{Gid}_A(-) \quad , \quad C\text{-}\mathsf{Gpd}_A(-) \quad \text{and} \quad C\text{-}\mathsf{Gfd}_A(-).$$

It turns out that the best one could hope for really happens, as we in Theorems 5.6, 5.8 and 5.11 prove:

(5) The proper C–Gorenstein dimensions (whenever these are defined) agree with the ordinary C–Gorenstein dimensions.

The paper is organized as follows:

In Section 1 we have collected some fundamental facts about the trivial extension $A \ltimes C$, which will be important later on. Section 2 defines the three new C–Gorenstein dimensions and proves how they are related to the "ring changed" Gorenstein dimensions over $A \ltimes C$. Section 3 compares our $C\text{-Gpd}_A(-)$ with Christensen's $G\text{–dim}_C(-)$. In Section 4 we interpret the C–Gorenstein dimensions in terms of Auslander and Bass categories. Finally, Section 5 investigates the proper C–Gorenstein dimensions.

Setup and notation. Throughout this paper, A is a fixed commutative and noetherian ring with unit, and C is a fixed semi-dualizing A–module; cf. Definition 1.1 below.

We work within the derived category $\mathsf{D}(A)$ of the category of A–modules; cf. e.g. [9, Chapter I] and [15, Chapter 10]; and complexes $M \in \mathsf{D}(A)$ have differentials going to the right:

$$M = \cdots \longrightarrow M_{i+1} \xrightarrow{\partial^M_{i+1}} M_i \xrightarrow{\partial^M_i} M_{i-1} \longrightarrow \cdots$$

We consistently use the hyper-homological notation from [2, Appendix], in particular we use $\mathrm{RHom}_A(-,-)$ for the right derived Hom functor, and $-\otimes^{\mathrm{L}}_A-$ for the left derived tensor product functor.

1. A few results about the trivial extension

In this section we collect some fundamental results about the trivial extension, which will be important later on.

Definition 1.1. A finitely generated A–module C with $\mathrm{RHom}_A(C,C) \simeq A$ in $\mathsf{D}(A)$ is called *semi-dualizing* ($C = A$ is such an example).

Definition 1.2. If C is any A–module, then the direct sum $A \oplus C$ can be equipped with the product:

$$(a,c)\cdot(a',c') \;=\; (aa', ac' + a'c).$$

This turns $A \oplus C$ into a ring which is called the *trivial extension* of A by C and denoted $A \ltimes C$.

We import from [10, Lemma 3.2] the following facts about the interplay between the rings A and $A \ltimes C$:

Lemma 1.3. *Let A be a ring with a semi-dualizing module C.*

(1) *There is an isomorphism in* $\mathsf{D}(A \ltimes C)$*:*
$$\mathrm{RHom}_A(A \ltimes C, C) \cong A \ltimes C.$$
(2) *There is a natural equivalence of functors on* $\mathsf{D}(A)$*:*
$$\mathrm{RHom}_{A\ltimes C}(-, A \ltimes C) \simeq \mathrm{RHom}_A(-, C).$$
(3) *If* M *is in* $\mathsf{D}(A)$ *then the two biduality morphisms:*
$$M \longrightarrow \mathrm{RHom}_A(\mathrm{RHom}_A(M, C), C) \quad \textit{and}$$
$$M \longrightarrow \mathrm{RHom}_{A\ltimes C}(\mathrm{RHom}_{A\ltimes C}(M, A \ltimes C), A \ltimes C)$$
are equal.
(4) *There is an isomorphism in* $\mathsf{D}(A \ltimes C)$*:*
$$\mathrm{RHom}_{A\ltimes C}(A, A \ltimes C) \;\cong\; C. \qquad \square$$

Furthermore, we have the next result [10, Lemma 3.1] about injective modules over A and $A \ltimes C$:

Lemma 1.4. *The following two conclusions hold:*
(1) *If* I *is a (faithfully) injective* A*–module then* $\mathrm{Hom}_A(A \ltimes C, I)$ *is a (faithfully) injective* $(A \ltimes C)$*–module.*
(2) *Each injective* $(A\ltimes C)$*–module is a direct summand in a module* $\mathrm{Hom}_A(A \ltimes C, I)$ *where* I *is some injective* A*–module.* $\square$

Using the same methods, we obtain:

Lemma 1.5. *The following two conclusions hold:*
(1) *If* P *is a projective* A*–module then* $(A\ltimes C)\otimes_A P$ *is a projective* $(A \ltimes C)$*–module.*
(2) *Each projective* $(A\ltimes C)$*–module is a direct summand in a module* $(A \ltimes C) \otimes_A P$ *where* P *is some projective* A*–module.* $\square$

2. C–Gorenstein homological dimensions

Let M be an (appropriately homologically bounded) A–complex. In [10] we demonstrated the usefulness of changing rings from A to $A\ltimes C$, and then considering the "ring changed" Gorenstein dimensions:
$$\mathrm{Gid}_{A\ltimes C}M \quad , \quad \mathrm{Gpd}_{A\ltimes C}M \quad \text{and} \quad \mathrm{Gfd}_{A\ltimes C}M.$$
This point of view enabled us to introduce three *Cohen-Macaulay dimensions* which characterize Cohen-Macaulay local rings in a way one could hope for. The next result is taken from [10, Lemma 4.6].

Proposition 2.1. *If* E *is a faithfully injective* A*–module, and* M *is any homologically right-bounded* A*–complex, then:*
$$\mathrm{Gid}_{A\ltimes C}\,\mathrm{Hom}_A(M, E) \;=\; \mathrm{Gfd}_{A\ltimes C}M. \qquad \square$$

Lemma 2.2. *Let J be an injective A–module and Q a projective A–module. Then we have a natural equivalence of functors on $\mathsf{D}(A \ltimes C)$:*

(1) $\mathrm{RHom}_{A\ltimes C}(\mathrm{Hom}_A(A \ltimes C, J), -) \simeq \mathrm{RHom}_A(\mathrm{Hom}_A(C, J), -)$.

(2) $\mathrm{RHom}_{A\ltimes C}(-, (A \ltimes C) \otimes_A Q) \simeq \mathrm{RHom}_A(-, C \otimes_A Q)$.

Proof. (1) is [10, Lemma 3.4], and (2) is proved similarly. ☐

Corollary 2.3. *For any A–module M, and integer n we have:*

(1) $\mathrm{Ext}^n_A(\mathrm{Hom}_A(C, J), M) = 0$ *for all injective A–modules J if and only if* $\mathrm{Ext}^n_{A\ltimes C}(U, M) = 0$ *for all injective $(A \ltimes C)$–modules U.*

(2) $\mathrm{Ext}^n_A(M, C \otimes_A P) = 0$ *for all projective A–modules P if and only if* $\mathrm{Ext}^n_{A\ltimes C}(M, S) = 0$ *for all projective $(A \ltimes C)$–modules S.*

Proof. (1) follows from Lemmas 2.2(1) and 1.4, while (2) is a consequence of Lemmas 2.2(2) and 1.5. ☐

We need to recall the next result from [10, Lemma 4.1]. Its proof uses, in fact, Lemmas 2.2(1) and 1.4.

Lemma 2.4. *Let M be an A–module which is Gorenstein injective over $A \ltimes C$. Then there exists a short exact sequence of A–modules,*

$$0 \to M' \longrightarrow \mathrm{Hom}_A(C, I) \longrightarrow M \to 0,$$

where I is injective over A and M' is Gorenstein injective over $A \ltimes C$. Furthermore, the sequence stays exact if one applies to it the functor $\mathrm{Hom}_A(\mathrm{Hom}_A(C, J), -)$ *for any injective A–module J.* ☐

"Dualizing" the proof of Lemma 2.4; this time using Lemmas 2.2(2) and 1.5, we establish the next:

Lemma 2.5. *Let M be an A–module which is Gorenstein projective over $A \ltimes C$. Then there exists a short exact sequence of A–modules,*

$$0 \to M \longrightarrow C \otimes_A P \longrightarrow M' \to 0,$$

where P is projective over A and M' is Gorenstein projective over $A \ltimes C$. Furthermore, the sequence stays exact if one applies to it the functor $\mathrm{Hom}_A(-, C \otimes_A Q)$ *for any projective A–module Q.* ☐

The last result we will need to get started is [10, Lemma 3.3]:

Lemma 2.6. *The A–modules A and C are Gorenstein projective over $A \ltimes C$. If I is an injective A–module, then* $\mathrm{Hom}_A(A, I) \cong I$ *and* $\mathrm{Hom}_A(C, I)$ *are Gorenstein injective over $A \ltimes C$.* ☐

Next, we introduce three new classes of modules:

Definition 2.7. An A–module M is called *C–Gorenstein injective* if:

(I1) $\mathrm{Ext}_A^{\geqslant 1}(\mathrm{Hom}_A(C,I),M)=0$ for all injective A–modules I.
(I2) There exist injective A–modules $I_0, I_1, \ldots$ together with an exact sequence:

$$\cdots \to \mathrm{Hom}_A(C,I_1) \to \mathrm{Hom}_A(C,I_0) \to M \to 0,$$

and also, this sequence stays exact when we apply to it the functor $\mathrm{Hom}_A(\mathrm{Hom}_A(C,J),-)$ for any injective A–module J.

M is called *C–Gorenstein projective* if:

(P1) $\mathrm{Ext}_A^{\geqslant 1}(M, C\otimes_A P)=0$ for all projective A–modules P.
(P2) There exist projective A–modules $P^0, P^1, \ldots$ together with an exact sequence:

$$0 \to M \to C\otimes_A P^0 \to C\otimes_A P^1 \to \cdots,$$

and furthermore, this sequence stays exact when we apply to it the functor $\mathrm{Hom}_A(-, C\otimes_A Q)$ for any projective A–module Q.

Finally, M is called *C–Gorenstein flat* if:

(F1) $\mathrm{Tor}^A_{\geqslant 1}(\mathrm{Hom}_A(C,I),M)=0$ for all injective A–modules I.
(F2) There exist flat A–modules $F^0, F^1, \ldots$ together with an exact sequence:

$$0 \to M \to C\otimes_A F^0 \to C\otimes_A F^1 \to \cdots,$$

and furthermore, this sequence stays exact when we apply to it the functor $\mathrm{Hom}_A(C,I)\otimes_A -$ for any injective A–module I.

Example 2.8. (a) If I is an injective A–module, then $\mathrm{Hom}_A(C,I)$ and I are C–Gorenstein injective because:

It is easy to see that $\mathrm{Hom}_A(C,I)$ is C–Gorenstein injective. Concerning I itself it is clear that condition (I1) of Definition 2.7 is satisfied. From Lemma 2.6 it follows that I is Gorenstein injective over $A \ltimes C$, so iterating Lemma 2.4 we also get condition (I2).

(b) Similarly, if P is a projective A–module, then $C\otimes_A P$ and P are C–Gorenstein projective. The last claim uses Lemmas 2.6 and 2.5.

(c) If F is a flat A–module, then $C\otimes_A F$ and F are C–Gorenstein flat. The last claim uses (a) together with Propositions 2.1, 2.13(1), 2.15 (the last two can be found below).

Definition 2.9. By Example 2.8(a), there exists for every homologically left-bounded complex N a left-bounded complex Y of C–Gorenstein injective modules with $Y \simeq N$ in $\mathsf{D}(A)$. Every such Y is called a *C–Gorenstein injective resolution* of N.

C–Gorenstein projective and *C–Gorenstein flat resolutions* of homologically right-bounded complexes are defined in a similar way, and they always exist by Examples 2.8(b) and (c). Thus, we may define:

For any homologically left-bounded A–complex N we introduce:

$$C\text{-}\mathrm{Gid}_A N \;=\; \inf_Y\Big(\sup\big\{n\in\mathbb{Z}\,|\,Y_{-n}\neq 0\big\}\Big),$$

where the infimum is taken over all C–Gorenstein injective resolutions Y of N. For a homologically right-bounded A–complex M we define:

$$C\text{-}\mathrm{Gpd}_A M \;=\; \inf_X\Big(\sup\big\{n\in\mathbb{Z}\,|\,X_{n}\neq 0\big\}\Big),$$

where the infimum is taken over all C–Gorenstein projective resolutions X of M. Finally, we define $C\text{-}\mathrm{Gfd}_A M$ analogously to $C\text{-}\mathrm{Gpd}_A M$.

Observation 2.10. Note that when $C = A$ in Definition 2.7, we recover the categories of ordinary Gorenstein injective, Gorenstein projective, and Gorenstein flat A–modules.

Thus, $A\text{-}\mathrm{Gid}_A(-)$, $A\text{-}\mathrm{Gpd}_A(-)$, and $A\text{-}\mathrm{Gfd}_A(-)$ are the usual Gorenstein injective, Gorenstein projective and Gorenstein flat dimensions over A, which one usually denotes $\mathrm{Gid}_A(-)$, $\mathrm{Gpd}_A(-)$ and $\mathrm{Gfd}_A(-)$, respectively.

Lemma 2.11. *Let M be an A–module which is C–Gorenstein injective. Then there exists a short exact sequence of $(A \ltimes C)$–modules,*

$$0 \to M' \longrightarrow U \longrightarrow M \to 0,$$

where U is injective over $A \ltimes C$ and M' is C–Gorenstein injective over A. Furthermore, the sequence stays exact if one applies to it the functor $\mathrm{Hom}_{A\ltimes C}(V,-)$ for any injective $(A \ltimes C)$–module V.

Proof. Since M is C–Gorenstein injective, we in particular get a short exact sequence of A–modules:

$$0 \to N \longrightarrow \mathrm{Hom}_A(C, I) \longrightarrow M \to 0,$$

where I is injective and N is C–Gorenstein injective, which stays exact under $\mathrm{Hom}_A(\mathrm{Hom}_A(C, J), -)$ when J is injective. Applying the functor $\mathrm{Hom}_A(-, I)$ to the exact sequence:

$$(*) \qquad 0 \to C \longrightarrow A \ltimes C \longrightarrow A \to 0$$

gives an exact sequence of $(A \ltimes C)$–modules:

$$(**) \qquad 0 \to I \longrightarrow \mathrm{Hom}_A(A \ltimes C, I) \longrightarrow \mathrm{Hom}_A(C, I) \to 0.$$

If viewed as a sequence of A–modules then this is split, because the same holds for $(*)$. Combining these data gives a commutative diagram of $(A \ltimes C)$–modules with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d] & \operatorname{Hom}_A(A \ltimes C, I) \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & \operatorname{Hom}_A(C, I) \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

We will prove that the upper row here has the properties claimed in the lemma:

First, $\operatorname{Hom}_A(A \ltimes C, I)$ is an injective $(A \ltimes C)$–module by Lemma 1.4(1). Secondly, using the Snake Lemma on the diagram embeds the vertical arrows into exact sequences. The leftmost of these is:

$$0 \to I \longrightarrow M' \longrightarrow N \to 0,$$

proving that as A–modules, $M' \cong I \oplus N$. Here N is C–Gorenstein injective by construction, and I is by Example 2.8(a). So M' is clearly also C–Gorenstein injective.

Finally, by construction, the lower row in the diagram stays exact under $\operatorname{Hom}_A(\operatorname{Hom}_A(C, J), -)$ when J is injective. If viewed as a sequence of A–modules then the sequence $(**)$ is split, so the surjection $\operatorname{Hom}_A(A \ltimes C, I) \longrightarrow \operatorname{Hom}_A(C, I)$ is split, and therefore the upper row in the diagram also stays exact under $\operatorname{Hom}_A(\operatorname{Hom}_A(C, J), -)$.

By applying $\mathrm{H}_0(-)$ to Lemma 2.2(1), we see that the upper row in the diagram stays exact under $\operatorname{Hom}_{A \ltimes C}(\operatorname{Hom}_A(A \ltimes C, J), -)$ when J is an injective A–module. Thus, it also stays exact under $\operatorname{Hom}_{A \ltimes C}(V, -)$ for any injective $(A \ltimes C)$–module V, because of Lemma 1.4(2). $\square$

By a similar argument we get:

Lemma 2.12. *Let M be an A–module which is C–Gorenstein projective. Then there exists a short exact sequence of $(A \ltimes C)$–modules,*

$$0 \to M \longrightarrow R \longrightarrow M' \to 0,$$

where R is projective over $A \ltimes C$ and M' is C–Gorenstein projective over A. Furthermore, the sequence stays exact if one applies to it the functor $\operatorname{Hom}_{A \ltimes C}(-, S)$ for any projective $(A \ltimes C)$–module S.

Proposition 2.13. *For any A–module M the two conclusions hold:*

1. *M is C–Gorenstein injective if and only if M is Gorenstein injective over $A \ltimes C$.*
2. *M is C–Gorenstein projective if and only if M is Gorenstein projective over $A \ltimes C$.*

Proof. (1) If M is C–Gorenstein injective, then Lemma 2.11 gives the "left half" of a complete injective resolution of M over $A \ltimes C$.

Conversely, if M is Gorenstein injective over $A \ltimes C$, then Lemma 2.4 gives the existence of a sequence like the one in Definition 2.7 (I2). Now, to finish the proof we only need to refer to Corollary 2.3(1).

(2) Similar, but using Lemmas 2.12, 2.5 and Corollary 2.3(2). □

Before turning to C–Gorenstein flat modules, we need to recall the notion of *Kaplansky classes* from [8, Definition 2.1], which is reformulated in Definition 5.4, Section 5. The following lemma will be central:

Lemma 2.14. *The class* $\mathsf{F} = \{C \otimes_A F \mid F \text{ flat } A\text{–module}\}$ *is Kaplansky, and furthermore it is closed under direct limits.*

Proof. Every homomorphism $\varphi\colon C \otimes_A F_1 \to C \otimes_A F_2$, where F_i is flat, has the form $\varphi = C \otimes_A \psi$ for some homomorphism $\psi\colon F_1 \to F_2$; namely $\psi = \mathrm{Hom}_A(C, \varphi)$, because $\mathrm{Hom}_A(C, C \otimes_A F_i) \cong F_i$.

With this observation in mind it is clear that F is closed under direct limits, since the class of flat modules has this property.

To see that F is Kaplansky, we first note that a finitely generated A–module has cardinality at most $\kappa = \max\{|A|, \aleph_0\}$.

Now, assume that x is an element of $G = C \otimes_A F$, where F is a flat A–module. Write $x = \sum_{i=1}^n c_i \otimes x_i$ for some $c_1, \ldots, c_n \in C$ and $x_1, \ldots, x_n \in F$. Let S be the A–submodule of F generated by $x_1, \ldots, x_n$, and then use [16, Lemma 2.5.2] (or [6, Lemma 5.3.12]) to enlarge S to a pure submodule F' in F with cardinality:

$$|F'| \leqslant \max\{|S| \cdot |A|, \aleph_0\} \leqslant \kappa.$$

Since F is flat and $F' \subseteq F$ is a pure submodule, then F' and F/F' are flat as well. Furthermore, exactness of:

$$0 \to C \otimes_A F' \to C \otimes_A F \to C \otimes_A (F/F') \to 0.$$

shows that $G' = C \otimes_A F'$ is a submodule of $G = C \otimes_A F$ which contains x. Clearly, G' and $G/G' \cong C \otimes_A (F/F')$ belong to F, and:

$$|G'| = |C \otimes_A F'| \leqslant |\mathbb{Z}^{(C \times F')}| \leqslant 2^{\kappa}.$$

Note that the cardinal number 2^{κ} only depends on the ring A. □

The next proof is modeled on that of [2, Theorem (6.4.2)].

Proposition 2.15. *Let M be an A–module. Then M is C–Gorenstein flat if and only if M is Gorenstein flat over $A \ltimes C$. In the affirmative case, M has the next property, which implies Definition 2.7* (F2)*:*

(F2') *There exist flat A–modules $F^0, F^1, \ldots$ together with an exact sequence:*

$$0 \to M \to C \otimes_A F^0 \to C \otimes_A F^1 \to \cdots,$$

and furthermore, this sequence stays exact when we apply to it the functor $\mathrm{Hom}_A(-, C \otimes_A G)$ *for any flat A–module G.*

Proof. For the first statement, it suffices by Propositions 2.1 and 2.13(1) to show that if E is a faithfully injective A–module, then:

$$M \text{ is } C\text{–Gorenstein flat} \;\Leftrightarrow\; \mathrm{Hom}_A(M, E) \text{ is } C\text{–Gorenstein injective.}$$

For any injective A–module I we have (adjointness) isomorphisms:

$$\begin{aligned}&\mathrm{Ext}^i_A(\mathrm{Hom}_A(C, I), \mathrm{Hom}_A(M, E)) \cong \\ &\quad \mathrm{Hom}_A(\mathrm{Tor}^A_i(\mathrm{Hom}_A(C, I), M), E).\end{aligned}$$

Thus, Definition 2.7 (F1) for M is equivalent to (I1) for $\mathrm{Hom}_A(M, E)$.

If $\mathbb{S} = 0 \to M \to C \otimes_A F^0 \to C \otimes_A F^1 \to \cdots$ is a sequence for M like the one in Definition 2.7 (F2), then, using adjointness, it is easy to see that $\mathrm{Hom}_A(\mathbb{S}, E)$ is a sequence for $\mathrm{Hom}_A(M, E)$ like the one in (I2). Therefore, we have proved the implication "$\Rightarrow$"

To show "$\Leftarrow$", we assume that $\mathrm{Hom}_A(M, E)$ is C–Gorenstein injective. First note that (F2') really implies Definition 2.7 (F2), since:

$$\begin{aligned}\mathrm{Hom}_A(\mathrm{Hom}_A(C, I) \otimes_A -, E) &\simeq \mathrm{Hom}_A(-, \mathrm{Hom}_A(\mathrm{Hom}_A(C, I), E)) \\ &\simeq \mathrm{Hom}_A(-, C \otimes_A \mathrm{Hom}_A(I, E)),\end{aligned}$$

and when I is injective, then $G = \mathrm{Hom}_A(I, E)$ is flat. In order prove (F2'), it suffices to show the existence of a short exact sequence:

$$(\dagger) \qquad 0 \to M \to C \otimes_A F \to M' \to 0,$$

satisfying the following three conditions:

(1) F is flat,
(2) $\mathrm{Hom}_A(M', E)$ is C–Gorenstein injective,
(3) $\mathrm{Hom}_A((\dagger), C \otimes_A G)$ is exact for any flat A–module G.

Because then one obtains the sequence in (F2') by iterating ($\dagger$). By Lemma 2.14, the class of A–modules:

$$\mathsf{F} = \{C \otimes_A F \mid F \text{ flat } A\text{–module}\}.$$

is Kaplansky. Furthermore, it is closed under arbitrary direct products; since C is finitely generated and A is noetherian, and hence [8, Theorem 2.5] implies that every A–module has an F–preenvelope.

Note that since $\mathrm{Hom}_A(M,E)$ is C–Gorenstein injective, there in particular exists an epimorphism $\mathrm{Hom}_A(C,I) \twoheadrightarrow \mathrm{Hom}_A(M,E)$, where I is injective. Applying $\mathrm{Hom}_A(-,E)$, we get a monomorphism:

$$\begin{aligned} M &\hookrightarrow \mathrm{Hom}_A(\mathrm{Hom}_A(M,E),E) \\ &\hookrightarrow \mathrm{Hom}_A(\mathrm{Hom}_A(C,I),E) \cong C \otimes_A \mathrm{Hom}_A(I,E) \in \mathsf{F}. \end{aligned}$$

Thus, M can be embedded into a module from F. Therefore, taking an F–preenvelope $\varphi\colon M \to C \otimes_A F$ of M, it is automaticly injective; and defining $M' = \mathrm{Coker}\,\varphi$, we certainly get an exact sequence (†) satisfying (1) and (3).

Finally, we argue that (2) is true. Keeping Proposition 2.13(1) in mind we must prove that $\mathrm{Hom}_A(M',E)$ is Gorenstein injective over $A \ltimes C$. Applying $\mathrm{Hom}_A(-,E)$ to (†) we get:

$$(\ddagger) \qquad 0 \to \mathrm{Hom}_A(M',E) \to \mathrm{Hom}_A(C,J) \to \mathrm{Hom}_A(M,E) \to 0,$$

where $J \cong \mathrm{Hom}_A(F,E)$ is injective. $\mathrm{Hom}_A(C,J)$ and $\mathrm{Hom}_A(M,E)$ are both Gorenstein injective over $A \ltimes C$ — the last module by assumption. Hence, if we can prove that $\mathrm{Ext}^1_{A\ltimes C}(U, \mathrm{Hom}_A(M',E)) = 0$ for every injective $(A \ltimes C)$–module U, then [5, Theorem 2.13] gives the desired conclusion. Using Corollary 2.3(1), we must prove that:

$$(\natural) \qquad \mathrm{Ext}^1_A(\mathrm{Hom}_A(C,I), \mathrm{Hom}_A(M',E)) = 0$$

for all injective A–modules I. Consider the commutative diagram with exact columns:

$$\begin{array}{ccc} 0 & & \\ \uparrow & & \\ \mathrm{Ext}^1_A(\mathrm{Hom}_A(C,I), \mathrm{Hom}_A(M',E)) & & 0 \\ \uparrow & & \uparrow \\ \mathrm{Hom}_A(\mathrm{Hom}_A(C,I), \mathrm{Hom}_A(M,E)) & \xleftarrow{\cong} & \mathrm{Hom}_A(\mathrm{Hom}_A(C,I) \otimes_A M, E) \\ \uparrow & & \uparrow \\ \mathrm{Hom}_A(\mathrm{Hom}_A(C,I), \mathrm{Hom}_A(C,J)) & \xleftarrow{\cong} & \mathrm{Hom}_A(\mathrm{Hom}_A(C,I) \otimes_A (C \otimes_A F), E) \end{array}$$

The first column is the induced long exact sequence which comes from applying $\mathrm{Hom}_A(\mathrm{Hom}_A(C,I),-)$ to (‡). We get another monomorphism when we apply $\mathrm{Hom}_A(C,I) \otimes_A -$ to the one $0 \to M \to C \otimes_A F$ from (†); this follows from the property (3) which (†) satisfies together with the calculation preceding (†). Turning this into an epimorphism with $\mathrm{Hom}_A(-,E)$ we get the second column. The vertical isomorphisms are by adjointness. The diagram implies that the module in ($\natural$) is zero. $\square$

Theorem 2.16. *For any (appropriately homologically bounded) A–complex M, we have the following equalities:*

$$\begin{aligned} C\text{-}\mathrm{Gid}_A M &= \mathrm{Gid}_{A\ltimes C} M,\\ C\text{-}\mathrm{Gpd}_A M &= \mathrm{Gpd}_{A\ltimes C} M,\\ C\text{-}\mathrm{Gfd}_A M &= \mathrm{Gfd}_{A\ltimes C} M. \end{aligned}$$

Proof. The proof uses Propositions 2.13(1),(2) and 2.15 in combination with [4, Theorems (2.5), (2.2) and (2.8)]. We only prove that $C\text{-}\mathrm{Gid}_A M = \mathrm{Gid}_{A\ltimes C} M$, since the proofs of the other two equalities are similar:

From Proposition 2.13(1) we get that every C–Gorenstein injective A–module is also Gorenstein injective over $A \ltimes C$, and this give us the inequality "$\geqslant$".

For the opposite inequality "$\leqslant$", we may assume that $n = \mathrm{Gid}_{A\ltimes C} M$ is an integer. Pick a left-bounded complex I of injective A–modules such that $I \simeq M$ in $\mathsf{D}(A)$. By Lemma 2.6 the modules I_i are Gorenstein injective over $A \ltimes C$, and therefore [4, Theorem (2.5)] implies that the A–module Z^I_{-n} is Gorenstein injective over $A \ltimes C$.

Now, Proposition 2.13(1) shows that Z^I_{-n} is C–Gorenstein injective. By Example 2.8(a), the complex $I_{-n}\supset \; = \; \cdots \to I_{-n+1} \to \mathrm{Z}^I_{-n} \to 0$ consists of C–Gorenstein injective A–modules, and since $I_{-n}\supset \; \simeq I \simeq M$ we see that $C\text{-}\mathrm{Gid}_A M \leqslant n$. $\square$

Corollary 2.17. *For any (appropriately homologically bounded) A–complex M, we have the following equalities:*

$$\begin{aligned} \mathrm{Gid}_{A\ltimes A} M &= \mathrm{Gid}_{A[x]/(x^2)}\, M &= \mathrm{Gid}_A\, M,\\ \mathrm{Gpd}_{A\ltimes A} M &= \mathrm{Gpd}_{A[x]/(x^2)}\, M &= \mathrm{Gpd}_A\, M,\\ \mathrm{Gfd}_{A\ltimes A} M &= \mathrm{Gfd}_{A[x]/(x^2)}\, M &= \mathrm{Gfd}_A\, M. \end{aligned}$$

Proof. This follows immediately from Theorem 2.16; we only have to note that $A \ltimes A \cong A[x]/(x^2)$ (sometimes referred to as the *dual numbers* over A). $\square$

Having realized that, on the level of A–complexes, the three (classical) Gorenstein dimensions can not distinguish between A and $A \ltimes A$, we can reap a nice result from the work of [10]:

Theorem 2.18. *If $(A, \mathfrak{m}, k)$ is local, then the following conditions are equivalent:*

(1) *A is Gorenstein.*
(2) *There exists an A–complex M such that all three numbers $\mathrm{fd}_A M$, $\mathrm{Gid}_A M$ and $\mathrm{width}_A M$ are finite.*
(3) *There exists an A–complex N such that all three numbers $\mathrm{id}_A N$, $\mathrm{Gpd}_A N$ and $\mathrm{depth}_A N$ are finite.*
(4) *There exists an A–complex N such that all three numbers $\mathrm{id}_A N$, $\mathrm{Gfd}_A N$ and $\mathrm{depth}_A N$ are finite.*

Proof. It is well-known that over a Gorenstein ring, every homologically bounded complex has finite Gorenstein injective, Gorenstein projective and Gorenstein flat dimension, and thus (1) $\Rightarrow$ (2), (3), (4).

Of course, (3) $\Rightarrow$ (4); and using Corollary 2.17, the remaining implications (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1) follow immediately from [10, Propositions 4.5 and 4.7]. □

Remark 2.19. There already exist special cases of this result in the literature: If A admits a dualizing complex, then [2, (3.3.5)] compared with [4, Theorems (4.3) and (4.5)] gives Theorem 2.18. If one drops the assumption that a dualizing complex should exists, then Theorem 2.18 is proved in [12, Corollary (3.3)], but only for modules.

3. Comparison with Christensen's G–$\mathrm{dim}_C(-)$

In [3, Definition (3.11)], Christensen introduced the number G–$\mathrm{dim}_C Z$ for any semi-dualizing complex C, and any complex Z with bounded and finitely generated homology. When $C = A$ (and Z is a module), we recover Auslander–Bridger's G–dimension by [2, Theorem (2.2.3)].

Proposition 3.1. *If C is a semi-dualizing A–module, and M an A–complex with bounded and finitely generated homology, then:*

$$C\text{-}\mathrm{Gpd}_A M \;=\; G\text{–}\mathrm{dim}_C M.$$

Proof. By Theorem 2.16, the proposition amounts to:

$$(*) \qquad \mathrm{Gpd}_{A\ltimes C}\, M \;=\; G\text{–}\mathrm{dim}_C M.$$

The homology of M is bounded and finitely generated over A, and hence it is also bounded and finitely generated over $A \ltimes C$. So by e.g. [4, Theorem (2.12)(b)] or [2, Theorem (4.2.6)], the left hand side in $(*)$ equals G–$\mathrm{dim}_{A\ltimes C} M$ (Auslander–Bridger's G–dimension over the ring $A \ltimes C$). We must therefore prove that:

$$(**) \qquad G\text{–}\mathrm{dim}_{A\ltimes C} M \;=\; G\text{–}\mathrm{dim}_C M.$$

The left hand side is finite precisely if the biduality morphism:

$$M \longrightarrow \mathrm{RHom}_{A\ltimes C}(\mathrm{RHom}_{A\ltimes C}(M, A \ltimes C), A \ltimes C)$$

is an isomorphism, and the right hand side is finite precisely when

$$M \longrightarrow \mathrm{RHom}_A(\mathrm{RHom}_A(M, C), C)$$

is an isomorphism. But these two morphisms are equal by Lemma 1.3(3), so the left hand side and right hand side of $(**)$ are simultaneously finite. When the left hand side of $(**)$ is finite, it equals:

$$-\inf \mathrm{RHom}_{A\ltimes C}(M, A \ltimes C),$$

and when the right hand side is finite, it is equal to:

$$-\inf \mathrm{RHom}_A(M, C)$$

But these two numbers are equal by Lemma 1.3(2). □

Observation 3.2. Christensen's G–$\dim_C(-)$ only works when the argument has bounded and finitely generated homology, but it has the advantage that C is allowed to be a semi-dualizing *complex.*

By Theorem 2.16, we get that for A–complexes M, the C–Gorenstein projective dimension $C\text{-}\mathrm{Gpd}_A M$ agrees with the "ring changed" Gorenstein projective dimension $\mathrm{Gpd}_{A\ltimes C} M$.

It is not immediately clear how one should make either of these dimensions work in the world of rings and modules/complexes when C is a semi-dualizing *complex.* Because in this case, $A \ltimes C$ becomes a differential graded algebra, and the C–Gorenstein projective objects in Definition 2.7 (from which we build our resolutions) become complexes.

In [1, Page 28] we find an interesting comment, which makes it even more clear why we run into trouble when C is a complex:

"*On the other hand, let C be a semi-dualizing complex with* $\mathrm{amp}\, C = s > 0$*. We are free to assume that* $\inf C = 0$*, and it is then immediate from the definition that G–$\dim_C C = 0$; but a resolution of C must have length at least s, so the G–dimension with respect to C can not be interpreted in terms of resolutions.*"

It is notable that the number $\mathrm{Gpd}_A \mathrm{RHom}_A(C, N)$, $N \in \mathsf{B}_C(A)$, occuring in Theorem 4.3 below makes perfect sense even if C is a complex.

4. Interpretations via Auslander and Bass categories

In this section, we interpret the C–Gorenstein homological dimensions from Section 2 in terms of Auslander and Bass categories.

Remark 4.1. Let C be a semi-dualizing A–complex. In [3, Section 4] is considered the adjoint pair of functors:

$$\mathsf{D}(A) \underset{\mathrm{RHom}_A(C,-)}{\overset{C\otimes^{\mathrm{L}}_A-}{\rightleftarrows}} \mathsf{D}(A)$$

and the full subcategories (where $\mathsf{D}_{\mathrm{b}}(A)$ is the full subcategory of $\mathsf{D}(A)$ consisting of homologically bounded complexes):

$$\mathsf{A}_C(A) = \left\{ M \in \mathsf{D}(A) \;\middle|\; \begin{array}{l} M \text{ and } C\otimes^{\mathrm{L}}_A M \text{ are in } \mathsf{D}_{\mathrm{b}}(A) \text{ and} \\ M \to \mathrm{RHom}_A(C, C\otimes^{\mathrm{L}}_A M) \text{ is an isomorphism} \end{array} \right\}$$

and

$$\mathsf{B}_C(A) = \left\{ N \in \mathsf{D}(A) \;\middle|\; \begin{array}{l} N \text{ and } \mathrm{RHom}_A(C,N) \text{ are in } \mathsf{D}_{\mathrm{b}}(A) \text{ and} \\ C\otimes^{\mathrm{L}}_A \mathrm{RHom}_A(C,N) \to N \text{ is an isomorphism} \end{array} \right\}.$$

It is an exercise in adjoint functors that the adjoint pair above restricts to a pair of quasi-inverse equivalences of categories:

$$\mathsf{A}_C(A) \underset{\mathrm{RHom}_A(C,-)}{\overset{C\otimes^{\mathrm{L}}_A-}{\rightleftarrows}} \mathsf{B}_C(A).$$

Theorem 4.2. *For any complex $M \in \mathsf{A}_C(A)$ we have an equality:*

$$C\text{-}\mathrm{Gid}_A M \;=\; \mathrm{Gid}_A(C\otimes^{\mathrm{L}}_A M).$$

Proof. Throughout the proof we make use of the nice descriptions of the *modules* in $\mathsf{A}_C(A)$ and $\mathsf{B}_C(A)$ from [3, Observation (4.10)].

STEP 1: In order to prove the equality $C\text{-}\mathrm{Gid}_A M = \mathrm{Gid}_A(C\otimes^{\mathrm{L}}_A M)$, we first justify the (necessary) bi-implication:

$$(\natural) \qquad \begin{array}{l} M \text{ is } C\text{–Gorenstein injective} \quad \Longleftrightarrow \\ C \otimes_A M \text{ is Gorenstein injective} \end{array}$$

for any *module* $M \in \mathsf{A}_C(A)$.

"$\Rightarrow$": By Definition 2.7(I2) there is an exact sequence:

$$(*) \qquad \cdots \to \mathrm{Hom}_A(C, I_1) \to \mathrm{Hom}_A(C, I_0) \to M \to 0,$$

where $I_0, I_1, \ldots$ are injective A–modules. Furthermore, we have exactness of $\mathrm{Hom}_A(\mathrm{Hom}_A(C,J),(*))$ for all injective A–modules J.

M belongs to $\mathsf{A}_C(A)$, and so does $\mathrm{Hom}_A(C,I)$ for any injective A–module I, since $I \in \mathsf{B}_C(A)$ by [3, Proposition (4.4)]. In particular, C is Tor-independent with both of the modules M and $\mathrm{Hom}_A(C,I)$. Hence the sequence $(*)$ stays exact if we apply to it the functor $C\otimes_A-$, and doing so we obtain:

$$(**) \qquad \cdots \to I_1 \to I_0 \to C \otimes_A M \to 0.$$

By similar arguments we see that if we apply $\mathrm{Hom}_A(C, -)$ to the sequence $(**)$, then we get $(*)$ back. If J is any injective A–module, then we have exactness of $\mathrm{Hom}_A(J, (**))$ because:

$$\begin{aligned}\mathrm{Hom}_A(J,(**)) &\cong \mathrm{Hom}_A(C \otimes_A \mathrm{Hom}_A(C,J),(**)) \\ &\cong \mathrm{Hom}_A(\mathrm{Hom}_A(C,J), \mathrm{Hom}_A(C,(**))) \\ &\cong \mathrm{Hom}_A(\mathrm{Hom}_A(C,J),(*)).\end{aligned}$$

Thus, $(**)$ is a "left half" of a complete injective resolution of the A–module $C \otimes_A M$. We also claim that $\mathrm{Ext}^i_A(J, C \otimes_A M) = 0$ for all $i > 0$ and all injective A–modules J. First note that:

$(\diamond)$

$$\begin{aligned}\mathrm{Ext}^i_A(J, C \otimes_A M) &\overset{\text{(a)}}{\cong} \mathrm{H}^i\,\mathrm{RHom}_A(C \otimes^{\mathrm{L}}_A \mathrm{RHom}_A(C,J), C \otimes^{\mathrm{L}}_A M) \\ &\overset{\text{(b)}}{\cong} \mathrm{H}^i\,\mathrm{RHom}_A(\mathrm{RHom}_A(C,J), \mathrm{RHom}_A(C, C \otimes^{\mathrm{L}}_A M)) \\ &\overset{\text{(c)}}{\cong} \mathrm{H}^i\,\mathrm{RHom}_A(\mathrm{RHom}_A(C,J), M) \\ &\cong \mathrm{Ext}^i_A(\mathrm{Hom}_A(C,J), M).\end{aligned}$$

Here (a) is follows as $J \in \mathsf{B}_C(A)$ by [3, Proposition (4.4)]; (b) is by adjointness; and (c) is because $M \in \mathsf{A}_C(A)$. This last module is zero because M is C–Gorenstein injective. These considerations prove that $C \otimes_A M$ is Gorenstein injective over A.

"$\Leftarrow$": If $C \otimes_A M$ is Gorenstein injective over A, we have by definition an exact sequence:

$$(\dagger) \qquad \cdots \to I_1 \to I_0 \to C \otimes_A M \to 0,$$

where $I_0, I_1, \ldots$ are injective A–modules. Furthermore, we have exactness of $\mathrm{Hom}_A(J, (\dagger))$ for all injective A–modules J.

Since $I_0, I_1, \ldots$ and $C \otimes_A M$ are modules from $\mathsf{B}_C(A)$, then so are all the kernels in $(\dagger)$, as $\mathsf{B}_C(A)$ is a full triangulated subcategory of $\mathsf{D}(A)$. If $N \in \mathsf{B}_C(A)$, then $\mathrm{Ext}^{\geqslant 1}_A(C, N) = 0$, and consequently, the sequence $(\dagger)$ stays exact if we apply to it the functor $\mathrm{Hom}_A(C, -)$. Doing so we obtain:

$$(\ddagger) \qquad \cdots \to \mathrm{Hom}_A(C, I_1) \to \mathrm{Hom}_A(C, I_0) \to M \to 0.$$

If J is any injective A–module, then we have exactness of the complex $\mathrm{Hom}_A(\mathrm{Hom}_A(C,J),(\ddagger))$ because:

$$\begin{aligned}\mathrm{Hom}_A(\mathrm{Hom}_A(C,J),(\ddagger)) &\cong \mathrm{Hom}_A(\mathrm{Hom}_A(C,J),\mathrm{Hom}_A(C,(\dagger)))\\ &\cong \mathrm{Hom}_A(C\otimes_A \mathrm{Hom}_A(C,J),(\dagger))\\ &\cong \mathrm{Hom}_A(J,(\dagger)).\end{aligned}$$

Furthermore, $(\diamond)$ above gives that:

$$\mathrm{Ext}_A^{\geqslant 1}(\mathrm{Hom}_A(C,J),M) \cong \mathrm{Ext}_A^{\geqslant 1}(J,C\otimes_A M) = 0,$$

for all injective A–modules J. The last zero is because $C\otimes_A M$ is Gorenstein injective over A. Hence M is C–Gorenstein injective.

Step 2: To prove the inequality $C\text{-}\mathrm{Gid}_A M \geqslant \mathrm{Gid}_A(C\otimes^{\mathrm{L}}_A M)$ for any complex $M \in \mathsf{A}_C(A)$, we may assume that $m = C\text{-}\mathrm{Gid}_A M = \mathrm{Gid}_{A\ltimes C} M$; cf. Theorem 2.16, is an integer. Since $C\otimes^{\mathrm{L}}_A M$ is homologically bounded, there exists a left-bounded injective resolution I of $C\otimes^{\mathrm{L}}_A M$, that is, $I \simeq C\otimes^{\mathrm{L}}_A M$ in $\mathsf{D}(A)$.

We wish to prove that the A–module Z^I_{-m} is Gorenstein injective. Since M belongs to $\mathsf{A}_C(A)$, we get isomorphisms:

$$M \simeq \mathrm{RHom}_A(C,C\otimes^{\mathrm{L}}_A M) \simeq \mathrm{RHom}_A(C,I) \simeq \mathrm{Hom}_A(C,I).$$

Now, $\mathrm{Hom}_A(C,I)$ is a complex of Gorenstein injective $A\ltimes C$–modules, and thus the A–module $L := \mathrm{Z}^{\mathrm{Hom}_A(C,I)}_{-m}$ is Gorenstein injective over $A\ltimes C$ by [4, Theorem (2.5)]. By Proposition 2.13(1), L is also C–Gorenstein injective. Note that:

$$-m = -\mathrm{Gid}_{A\ltimes C} M \leqslant \inf M \overset{\text{(a)}}{=} \inf(C\otimes^{\mathrm{L}}_A M) = \inf I,$$

where the equality (a) comes from [3, Lemma(4.11)(b)]. Therefore, $0\to \mathrm{Z}^I_{-m}\to I_{-m}\to I_{-m-1}$ is exact, and applying the left exact functor $\mathrm{Hom}_A(C,-)$ to this sequence we get an isomorphism of A–modules:

$$(\flat)\qquad L = \mathrm{Z}^{\mathrm{Hom}_A(C,I)}_{-m} \cong \mathrm{Hom}_A(C,\mathrm{Z}^I_{-m}).$$

We have a degreewise split exact sequence of complexes:

$$0\to \Sigma^{-m}\mathrm{Z}^I_{-m} \longrightarrow I_{-m}\supset \longrightarrow I_{-m+1}\sqsupset \longrightarrow 0,$$

where we have used the notation from [2, Appendix (A.1.14)] to denote soft and hard truncations. Since $I_{-m+1}\sqsupset$ has finite injective dimension it belongs to $\mathsf{B}_C(A)$ by [3, Proposition (4.4)], and furthermore,

$$I_{-m}\supset \simeq I \simeq C\otimes^{\mathrm{L}}_A M \in \mathsf{B}_C(A).$$

Thus, the module Z^I_{-m} is also in $\mathsf{B}_C(A)$, as $\mathsf{B}_C(A)$ is a full triangulated and shift-invariant subcategory of $\mathsf{D}(A)$. Consequently, the module L

from (♭) belongs to $\mathsf{A}_C(A)$ and has the property that $C \otimes_A L \cong \mathrm{Z}^I_{-m}$. Therefore, the implication "$\Rightarrow$" in (♮) gives that Z^I_{-m} is Gorenstein injective over A, as desired.

STEP 3: To prove the opposite inequality $C\text{-}\mathrm{Gid}_A M \leqslant \mathrm{Gid}_A(C \otimes^{\mathrm{L}}_A M)$ for any complex $M \in \mathsf{A}_C(A)$, we assume that $n = \mathrm{Gid}_A(C \otimes^{\mathrm{L}}_A M)$ is an integer. Pick any left-bounded injective resolution I of $C \otimes^{\mathrm{L}}_A M$. Then the A–module Z^I_{-n} is Gorenstein injective by [4, Theorem (2.5)].

As in STEP 2 we get $M \simeq \mathrm{Hom}_A(C, I)$, and thus it suffices to show that the module:

$$N \;:=\; \mathrm{Z}^{\mathrm{Hom}_A(C,I)}_{-n} \;\cong\; \mathrm{Hom}_A(C, \mathrm{Z}^I_{-n}).$$

is C–Gorenstein injective, because then $M \simeq \mathrm{Hom}_A(C, I)_{-n}\supset$ shows that $C\text{-}\mathrm{Gid}_A M \leqslant n$. As before we get that N is a module in $\mathsf{A}_C(A)$ with $C \otimes_A N \cong \mathrm{Z}^I_{-n}$, which this time is Gorenstein injective over A. Therefore, the implication "$\Leftarrow$" in (♮) gives that N is C–Gorenstein injective. □

Using Proposition 2.13(2), a similar argument gives:

Theorem 4.3. *For any complex $N \in \mathsf{B}_C(A)$ we have an equality:*

$$C\text{-}\mathrm{Gpd}_A N \;=\; \mathrm{Gpd}_A \mathrm{RHom}_A(C, N). \qquad \square$$

From Theorems 4.2 and 2.16, and Proposition 2.1 we can easily derive:

Theorem 4.4. *For any complex $N \in \mathsf{B}_C(A)$ we have an equality:*

$$C\text{-}\mathrm{Gfd}_A N \;=\; \mathrm{Gfd}_A \mathrm{RHom}_A(C, N).$$

Proof. Let E be a faithfully injective A–module. Since $N \in \mathsf{B}_C(A)$ it is easy to see that $\mathrm{RHom}_A(N, E) \simeq \mathrm{Hom}_A(N, E)$ is in $\mathsf{A}_C(A)$. Hence:

$$\begin{aligned} C\text{-}\mathrm{Gfd}_A N &= C\text{-}\mathrm{Gid}_A \mathrm{RHom}_A(N, E) \\ &= \mathrm{Gid}_A \big(C \otimes^{\mathrm{L}}_A \mathrm{RHom}_A(N, E)\big) \\ &= \mathrm{Gid}_A \mathrm{RHom}_A(\mathrm{RHom}_A(C, N), E) \\ &= \mathrm{Gfd}_A \mathrm{RHom}_A(C, N). \end{aligned}$$

The second last isomorphism comes from [2, (A.4.24)]. □

In the rest of this section, we assume that A admits a *dualizing complex* D^A; cf. [4, Definition (1.1)]. The canonical homomorphism of rings, $A \to A \ltimes C$, turns $A \ltimes C$ into a finitely generated A–module, and thus

$$D^{A \ltimes C} \;=\; \mathrm{RHom}_A(A \ltimes C, D^A)$$

is a dualizing complex for $A \ltimes C$.

Lemma 4.5. *There is an isomorphism over A,*

$$D^{A\ltimes C}\otimes^{\mathrm{L}}_{A\ltimes C}A \cong \mathrm{RHom}_A(C, D^A).$$

Proof. This is a computation:

$$\begin{aligned} D^{A\ltimes C}\otimes^{\mathrm{L}}_{A\ltimes C}A &= \mathrm{RHom}_A(A\ltimes C, D^A)\otimes^{\mathrm{L}}_{A\ltimes C}A \\ &\overset{(a)}{\cong} \mathrm{RHom}_A(\mathrm{RHom}_{A\ltimes C}(A, A\ltimes C), D^A) \\ &\overset{(b)}{\cong} \mathrm{RHom}_A(C, D^A), \end{aligned}$$

where (a) holds because D^A has finite injective dimension over A and (b) is by Lemma 1.3(4). ∎

By [3, Corollary (2.12)], the complex $C^\dagger = \mathrm{RHom}_A(C, D^A)$ is semi-dualizing for A. We now have the following generalization of the main results in [4, Theorems (4.3) and (4.5)]:

Theorem 4.6. *Let M and N be A–complexes such that the homology of M is right-bounded and the homology of N is left-bounded. Then:*

(1) $M \in \mathsf{A}_{C^\dagger}(A) \iff C\text{-}\mathrm{Gpd}_A M < \infty \iff C\text{-}\mathrm{Gfd}_A M < \infty$.
(2) $N \in \mathsf{B}_{C^\dagger}(A) \iff C\text{-}\mathrm{Gid}_A N < \infty$.

Proof. Recall that $D^{A\ltimes C} = \mathrm{RHom}_A(A\ltimes C, D^A)$ is a dualizing complex for $A\ltimes C$. If M is a complex of A–modules then

$$\begin{aligned} C^\dagger\otimes^{\mathrm{L}}_A M &= \mathrm{RHom}_A(C, D^A)\otimes^{\mathrm{L}}_A M \\ &\overset{(a)}{\cong} (D^{A\ltimes C}\otimes^{\mathrm{L}}_{A\ltimes C}A)\otimes^{\mathrm{L}}_A M \\ &\cong D^{A\ltimes C}\otimes^{\mathrm{L}}_{A\ltimes C}M \end{aligned}$$

and

$$\begin{aligned} \mathrm{RHom}_A(C^\dagger, M) &= \mathrm{RHom}_A(\mathrm{RHom}_A(C, D^A), M) \\ &\overset{(b)}{\cong} \mathrm{RHom}_A(D^{A\ltimes C}\otimes^{\mathrm{L}}_{A\ltimes C}A, M) \\ &\overset{(c)}{\cong} \mathrm{RHom}_{A\ltimes C}(D^{A\ltimes C}, \mathrm{RHom}_A(A, M)) \\ &\cong \mathrm{RHom}_{A\ltimes C}(D^{A\ltimes C}, M), \end{aligned}$$

where (a) and (b) are by Lemma 4.5 and (c) is by adjunction. So using the adjoint pair:

$$\mathsf{D}(A) \underset{\mathrm{RHom}_A(C^\dagger,-)}{\overset{C^\dagger\otimes^{\mathrm{L}}_A -}{\rightleftarrows}} \mathsf{D}(A)$$

on complexes of A–modules is the same as viewing these complexes as complexes of $(A \ltimes C)$–modules and using the adjoint pair:

$$\mathsf{D}(A \ltimes C) \underset{\mathrm{RHom}_{A\ltimes C}(D^{A\ltimes C},-)}{\overset{D^{A\ltimes C}\otimes^{\mathrm{L}}_{A\ltimes C}-}{\rightleftarrows}} \mathsf{D}(A \ltimes C)$$

Hence a complex M of A–modules is in $\mathsf{A}_{C^\dagger}(A)$ if and only if it is in $\mathsf{A}_{D^{A\ltimes C}}(A\ltimes C)$ when viewed as a complex of $(A\ltimes C)$–modules. If M has right-bounded homology, this is equivalent both to $\mathrm{Gpd}_{A\ltimes C} M < \infty$ and $\mathrm{Gfd}_{A\ltimes C} M < \infty$ by [4, Theorem (4.3)], and by Theorem 2.16 this is the same as $C\text{-}\mathrm{Gpd}_A M < \infty$ and $C\text{-}\mathrm{Gfd}_A M < \infty$.

So part (1) of the theorem follows, and a similar method using [4, Theorem (4.5)] deals with part (2). $\square$

5. Proper dimensions

In this section, we define and study the *proper* variants of the dimensions from Theorem 2.16. The results to follow depend highly on the work in [8].

In Definition 2.9 we introduced the dimensions $C\text{-}\mathrm{Gid}_A(-)$, $C\text{-}\mathrm{Gpd}_A(-)$ and $C\text{-}\mathrm{Gfd}_A(-)$ for A–complexes. When M is an A–module it is not hard to see that these dimensions specialize to:

$$C\text{-}\mathrm{Gid}_A M \;=\; \inf\left\{ n \in \mathbb{N}_0 \;\middle|\; \begin{array}{l} 0 \to M \to I^0 \to \cdots \to I^n \to 0 \text{ is exact} \\ \text{and } I^0, \ldots, I^n \text{ are } C\text{–Gorenstein injective} \end{array} \right\},$$

and similarly for $C\text{-}\mathrm{Gpd}_A M$ and $C\text{-}\mathrm{Gfd}_A M$.

Definition 5.1. Let Q be a class of A–modules (which contains the zero-module), and let M be any A–module. A *proper left* Q*–resolution* of M is a complex (not necessarily exact):

$$(\dagger) \qquad \cdots \to Q_1 \to Q_0 \to M \to 0,$$

where $Q_0, Q_1, \ldots \in \mathsf{Q}$ and such that $(\dagger)$ becomes exact when we apply to it the functor $\mathrm{Hom}_A(Q,-)$ for every $Q \in \mathsf{Q}$. A *proper right* Q*–resolution* of M is a complex (not necessarily exact):

$$(\ddagger) \qquad 0 \to M \to Q^0 \to Q^1 \to \cdots,$$

where $Q^0, Q^1, \ldots \in \mathsf{Q}$ and such that $(\ddagger)$ becomes exact when we apply to it the functor $\mathrm{Hom}_A(-,Q)$ for every $Q \in \mathsf{Q}$.

Definition 5.2. Let Q be a class of A–modules, and let M be any A–module. If M has a proper left Q–resolution, then we define the *proper left* Q*–dimension* of M by:

$$\mathcal{L}\text{-}\dim_{\mathsf{Q}} M \;=\; \inf\left\{ n \in \mathbb{N}_0 \;\middle|\; \begin{array}{l} 0 \to Q_n \to \cdots \to Q_0 \to M \to 0 \text{ is} \\ \text{a proper left } \mathsf{Q}\text{–resolution of } M \end{array} \right\}.$$

Similarly, if M has a proper right Q–resolution, then we define the *proper right* Q*–dimension* of M by:

$$\mathcal{R}\text{-}\dim_{\mathsf{Q}} M \;=\; \inf\left\{ n \in \mathbb{N}_0 \;\middle|\; \begin{array}{l} 0 \to M \to Q^0 \to \cdots \to Q^n \to 0 \text{ is} \\ \text{a proper right } \mathsf{Q}\text{–resolution of } M \end{array} \right\}.$$

Definition 5.3. We use $\mathsf{GI}_C(A)$, $\mathsf{GP}_C(A)$ and $\mathsf{GF}_C(A)$ to denote the classes of C–Gorenstein injective, C–Gorenstein projective and C–Gorenstein flat A–modules, respectively.

A proper right $\mathsf{GI}_C(A)$–resolution is called a *proper right C–Gorenstein injective resolution*, and with a similar meaning we use the terms *proper left C–Gorenstein projective/flat resolution*.

Finally, we introduce the (more natural) notation:

- C-$\mathsf{Gid}_A(-)$ for the proper right $\mathsf{GI}_C(A)$–dimension,
- C-$\mathsf{Gpd}_A(-)$ for the proper left $\mathsf{GP}_C(A)$–dimension,
- C-$\mathsf{Gfd}_A(-)$ for the proper left $\mathsf{GF}_C(A)$–dimension,

whenever these dimensions are defined.

The next definition is taken directly from [8, Definition 2.1]:

Definition 5.4. Let F be a class of A–modules. Then F is called *Kaplansky* if there exists a cardinal number κ such that for every module $M \in \mathsf{F}$ and every element $x \in M$ there is a submodule $N \subseteq M$ satisfying $x \in N$ and $N, M/N \in \mathsf{F}$ with $|N| \leqslant \kappa$.

Lemma 5.5. *The class of C–Gorenstein injective A–modules is Kaplansky.*

Proof. The class of Gorenstein injective $(A \ltimes C)$–modules is Kaplansky by [8, Proposition 2.6]. Let κ be a cardinal number which implements the Kaplansky property for this class.

Now assume that M is a C–Gorenstein injective A–module, and that $x \in M$ is an element. By Proposition 2.13(1), M is Gorenstein injective over $A \ltimes C$, and thus there exists a Gorenstein injective $(A \ltimes C)$–submodule $N \subseteq M$ with $x \in N$ and $|N| \leqslant \kappa$, and such that the $(A \ltimes C)$–module M/N is Gorenstein injective.

Since M is an A–module, when we consider it as a module over $A \ltimes C$, it is annihilated by the ideal $C \subseteq A \ltimes C$. Consequently, the two $(A \ltimes C)$–modules N and M/N are also annihilated by C. This means that N and M/N really are A–modules which are viewed as $(A \ltimes C)$–modules. Hence Proposition 2.13(1) implies that N and M/N are C–Gorenstein injective A–modules; and we are done. $\square$

Theorem 5.6. *Every A–module M has a proper right C–Gorenstein injective resolution, and we have an equality:*

$$C\text{-}\mathsf{Gid}_A M \;=\; C\text{-}\mathrm{Gid}_A M.$$

Proof. By Lemma 5.5 above, the class of C–Gorenstein injective A–modules is Kaplansky, and it is obviously also closed under arbitrary direct products. Therefore, [8, Theorem 2.5 and Remark 3] implies that every A–module admits a proper right C–Gorenstein injective resolution.

Every injective A–module is also Gorenstein injective by Example 2.8(a), and hence a proper right C–Gorenstein injective resolution is exact. Consequently, we immediately get the inequality:

$$C\text{-}\mathsf{Gid}_A M \;\geqslant\; C\text{-}\mathrm{Gid}_A M.$$

To show the opposite inequality, we may assume that $n = C\text{-}\mathrm{Gid}_A M$ is finite. Let $0 \to M \to E^0 \to E^1 \to \cdots$ be a proper right C–Gorenstein injective resolution of M. Defining $D^n = \mathrm{Coker}(E^{n-2} \to E^{n-1})$ we get an exact sequence:

$$0 \to M \to E^0 \to \cdots \to E^{n-1} \to D^n \to 0,$$

which also stays exact when we apply to it the (left exact) functor $\mathrm{Hom}_A(-, E)$ for every C–Gorenstein injective A–module E. Since $C\text{-}\mathrm{Gid}_A M = \mathrm{Gid}_{A\ltimes C} M = n$, we get by [11, Theorem 2.22] and Proposition 2.13(1) that D^n is C–Gorenstein injective, so $C\text{-}\mathsf{Gid}_A M \leqslant n$. $\square$

Sometimes, nice proper right C–Gorenstein injective resolutions exist:

Proposition 5.7. *If M is module in $\mathsf{A}_C(A)$ such that $n = C\text{-}\mathrm{Gid}_A M$ is finite, then there exists a proper right C–Gorenstein injective resolution of the form:*

$$(*) \qquad 0 \to M \to H^0 \to \mathrm{Hom}_A(C, I^1) \to \cdots \to \mathrm{Hom}_A(C, I^n) \to 0,$$

where H^0 is C–Gorenstein injective and $I^1, \ldots, I^n$ are injective.

Proof. As in the proof of Theorem 4.2, the assumption $M \in \mathsf{A}_C(A)$ gives the existence of an exact sequence of A–modules:

$$0 \to M \to \mathrm{Hom}_A(C, J^0) \to \ldots \to \mathrm{Hom}_A(C, J^{n-1}) \to D^n \to 0,$$

where $J^0, \dots, J^{n-1}$ are injective, and D^n is Gorenstein injective over $A \ltimes C$. Applying Lemma 2.4 to D^n we get a commutative diagram of A–modules with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, dotted] & \mathrm{Hom}_A(C, J^0) \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & \mathrm{Hom}_A(C, J^{n-1}) \arrow[r] \arrow[d, dotted] & D^n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & D^0 \arrow[r] & \mathrm{Hom}_A(C, U^0) \arrow[r] & \cdots \arrow[r] & \mathrm{Hom}_A(C, U^{n-1}) \arrow[r] & D^n \arrow[r] & 0
\end{tikzcd}$$

where $U^0, \dots, U^{n-1}$ are injective and D^0 is C–Gorenstein injective. The mapping cone of this chain map is of course exact, and furthermore, it has $0 \to D^n \xrightarrow{=} D^n \to 0$ as a subcomplex.

Consequently, we get the exact sequence $(*)$, where $I^i = U^{i-1} \oplus J^i$ for $i = 1, \dots, n-1$ together with $I^n = U^{n-1}$ are injective; and $H^0 = D^0 \oplus \mathrm{Hom}_A(C, J^0)$ is C–Gorenstein injective.

We claim that the sequence $(*)$ remains exact when we apply to it the functor $\mathrm{Hom}_A(-, N)$ for any C–Gorenstein injective A–module N (and this will finish the proof):

Splitting $(*)$ into short exact sequences, we get sequences of the form $0 \to X \to Y \to Z \to 0$, where Z has the property that it fits into an exact sequence:

$$0 \to Z \to \mathrm{Hom}_A(C, E^0) \to \mathrm{Hom}_A(C, E^m) \to 0,$$

where $E^0, \dots, E^m$ are injective. Therefore, it suffices to prove that every such module Z satisfies $\mathrm{Ext}^1_A(Z, N) = 0$ for all C–Gorenstein injective modules N. But as $\mathrm{Ext}^{\geqslant 1}_A(\mathrm{Hom}_A(C, E^i), N) = 0$ for $i = 0, \dots, m$, this follows easily. $\square$

We do not know if every module has a proper left C–Gorenstein projective resolution. However, in the case where A admits a dualizing complex and where $C = A$, then the answer is positive by [13, Theorem 3.2].

"Dualizing" the proof of Theorem 5.6 (except the first part about existence of proper resolutions) and Proposition 5.7, we get:

Theorem 5.8. *Assume that M is an A–module which has a proper left C–Gorenstein projective resolution. Then we have an equality:*

$$C\text{-}\mathsf{Gpd}_A M \;=\; C\text{-}\mathrm{Gpd}_A M. \qquad \square$$

Proposition 5.9. *If M is module in $\mathsf{B}_C(A)$ such that $n = C\text{-}\mathrm{Gpd}_A M$ is finite, then there exists a proper right C–Gorenstein projective resolution of the form:*

$$0 \to C \otimes_A P_n \to \cdots \to C \otimes_A P_1 \to G_0 \to M \to 0$$

where G_0 is C–Gorenstein projective and $P_1, \ldots, P_n$ are projective. Furthermore, if M is finitely generated, then $G_0, P_1, \ldots, P_n$ may be taken to be finitely generated as well. □

The C–Gorenstein flat case is more subtle. We begin with the next:

Lemma 5.10. *The class of C–Gorenstein flat A–modules is Kaplansky, and closed under direct limits.*

Proof. As in the proof of Lemma 5.10; this time using [8, Proposition 2.10], we see that the class of C–Gorenstein flat A–modules is Kaplansky.

By Proposition 2.15, a module M is C–Gorenstein flat if and only if M satisfies conditions (F1) in Definition 2.7 and (F2') in Proposition 2.15. Clearly, the condition (F1) is closed under direct limits.

Concerning condition (F2'), we recall from Lemma 2.14 that the class of A–modules $\mathsf{F} = \{C \otimes_A F \mid F \text{ flat } A\text{–module}\}$ is closed under direct limits. Condition (F2') states that M admits an infinite proper right F–resolution, or in the language of [7, 8], that $\mu_{\mathsf{F}}(M) = \infty$. Hence [8, Theorem 2.4] implies that also (F2') is closed under direct limits. □

Theorem 5.11. *Every A–module M has a proper left C–Gorenstein flat resolution, and there is an equality:*

$$C\text{-}\mathsf{Gfd}_A M \;=\; C\text{-}\mathrm{Gfd}_A M.$$

Proof. The class $\mathsf{GF}_C(A)$ of C–Gorenstein flat modules contains the projective (in fact, flat) modules by Example 2.8(c), and furthermore, it is closed under extensions by [11, Theorem 3.7] and Proposition 2.15.

Thus, by Lemma 5.10 above and [8, Theorem 2.9] we conclude that the pair $(\mathsf{GF}_C(A), \mathsf{GF}_C(A)^{\perp})$ is a *perfect cotorsion theory* according to [8, Definition 2.2]. In particular, every module admits a C–Gorenstein flat (pre)cover, and hence proper left C–Gorenstein flat resolutions always exist.

The equality $C\text{-}\mathsf{Gfd}_A M = C\text{-}\mathrm{Gfd}_A M$ follows as in Theorem 5.6; this time using [11, Theorem 3.14] instead of [11, Theorem 2.22]. □

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