Optimisation: Linear programming

☆ Linear programming model expressed as

max

$$Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \qquad (1)$$

subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\le b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad &\vdots \qquad (2) \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\le b_m \end{aligned}$$

$$x_1 \ge 0,\ x_2 \ge 0,\ \ldots,\ x_n \ge 0 \qquad (3)$$

(1) objective function

(2) constraint set

(3) nonnegativity conditions

x_j = level of activity j (decision variables)

a_{ij}, b_j, c_j = known constants

☆ More concisely LP model expressed as

max

$$Z = \sum_{j=1,n} c_j x_j$$

subject to

$$\sum_{j=1,n} a_{ij} x_j \leq b_i \qquad (i = 1,2,...,m)$$

$$x_j \geq 0 \qquad (j = 1,2,...,n)$$

Minimisation Problem

min

$$Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \qquad (1)$$

subject to

$$\begin{aligned} a_{11}x_1 &+ a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1 \\ a_{21}x_1 &+ a_{22}x_2 + \ldots + a_{2n}x_n \geq b_2 \\ &\vdots \\ a_{m1}x_1 &+ a_{m2}x_2 + \ldots + a_{mn}x_n \geq b_m \end{aligned} \qquad (2)$$

$$x_1 \geq 0,\ x_2 \geq 0,\ \ldots,\ x_n \geq 0 \qquad (3)$$

(1) objective function

(2) constraint set

(3) nonnegativity conditions

x_j = level of activity j (decision variables)

a_{ij}, b_j, c_j = known constants

☆ More concisely LP model expressed as

min

$$Z = \sum_{j=1,n} c_j x_j$$

subject to

$$\sum_{j=1,n} a_{ij} x_j \geq b_i \qquad (i = 1,2,...,m)$$

$$x_j \geq 0 \qquad (j = 1,2,...,n)$$

Example LP - Crop Selection

- Farmer has 10 hectares (ha) in which to grow potatoes and/or tomatoes in the combination that will yield greatest profit
- A contract has been signed in which a purchaser guarantees to buy the produce from 2 ha of tomatoes
- After meeting other commitments, farmer has enough labour available to spend 12 hours a week on the area
- Each hectare of potatoes requires 2 hours of labour per week, while tomatoes require 0.5 hours per week/ha
- Potatoes yield gross returns (before deducting costs) of $400/ha, while tomatoes yield $300/ha
- Fertiliser costs $100 per tonne and must be applied as follows:

  1 tonne/ha for potatoes and 0.5 tonne/ha for tomatoes
- Farmer requires that no land remain unused
- To find objective function: for each activity, determine net profit per unit of the activity
- For potatoes, gross returns are $400 per hectare and cost of fertiliser is

  1 (tonne per ha) × $100 per tonne
- Thus objective function coefficients is

  $400 - (1 × $100) = $300
- For tomatoes, value is

  $300 - (0.5 × $100) = $250
- Thus, objective function

  $Z = 300P + 250T$
- Constraints as follows

Land	1 P	+	1 T	≤	10
Contract	0 P	+	1 T	≥	2
Labour	2 P	+	0.5 T	≤	12

- Non-negativity conditions

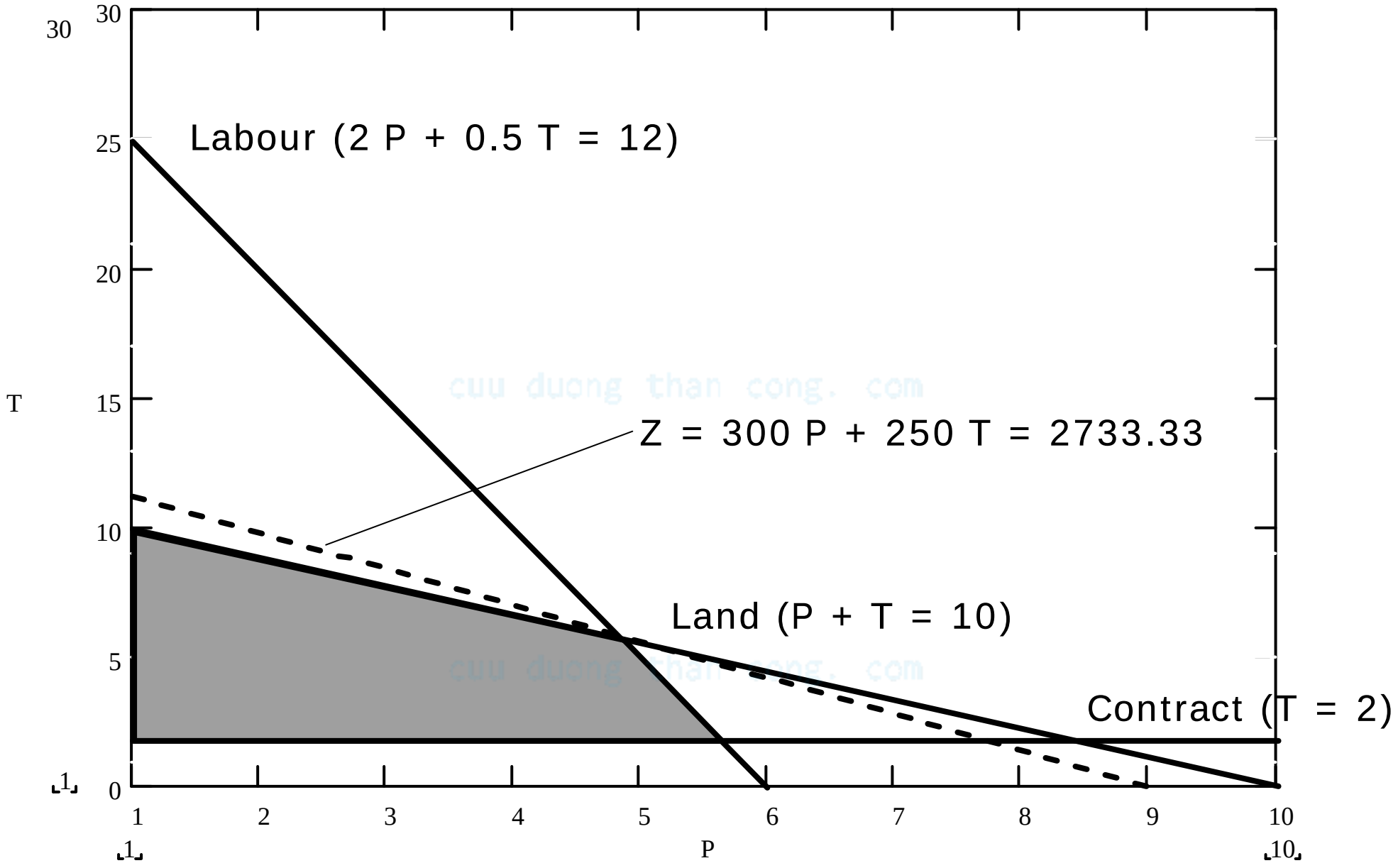
  $P, T \geq 0$
- More simply, constraints are

Land	P	+	T	≤	10
Contract			T	≥	2
Labour	2 P	+	0.5 T	≤	12

- Non-negativity conditions

  $P, T \geq 0$

CROP SELECTION



	A	B	C	D	E	F	G
1	**Crop selection**						
2	**Product:**	P	T				
3	**Production Qty.**	4.666666667	5.33333	**Profit**			
4	**Unit Contri. Mar.**	**$300**	**$250**	**$2,733**			
5	**Constraints**	**Resource Usage**		**Total LHS**		**RHS**	
6	**Land**	1	1	10	=	10	
7	**Labour**	2	0.5	12	≤	12	0.000
8	**Contract**	0	1	5.333333333	≥	2	3.333
9							
10							
11							

Target Cell

Slack/Surplus

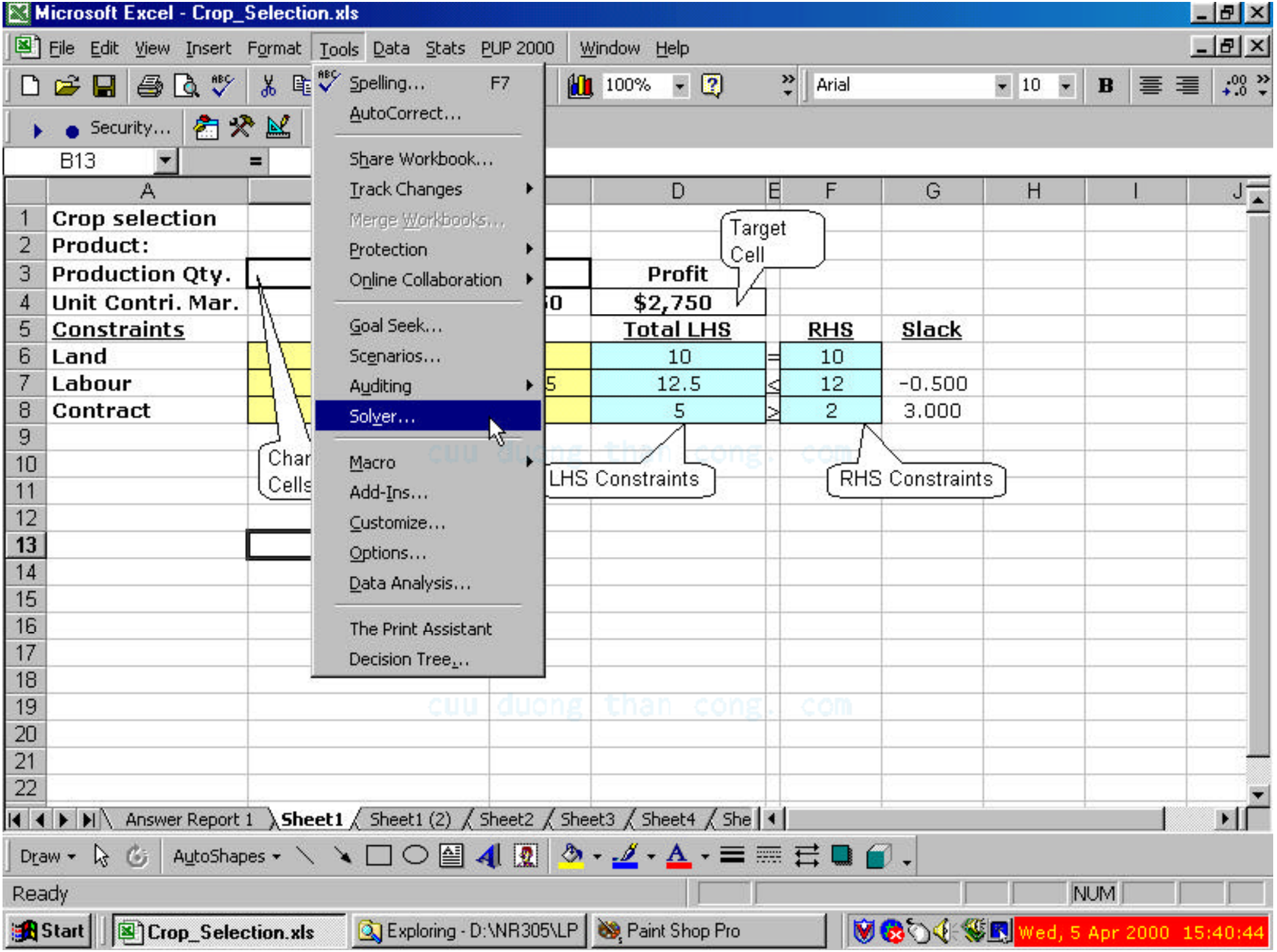
Changing Cells

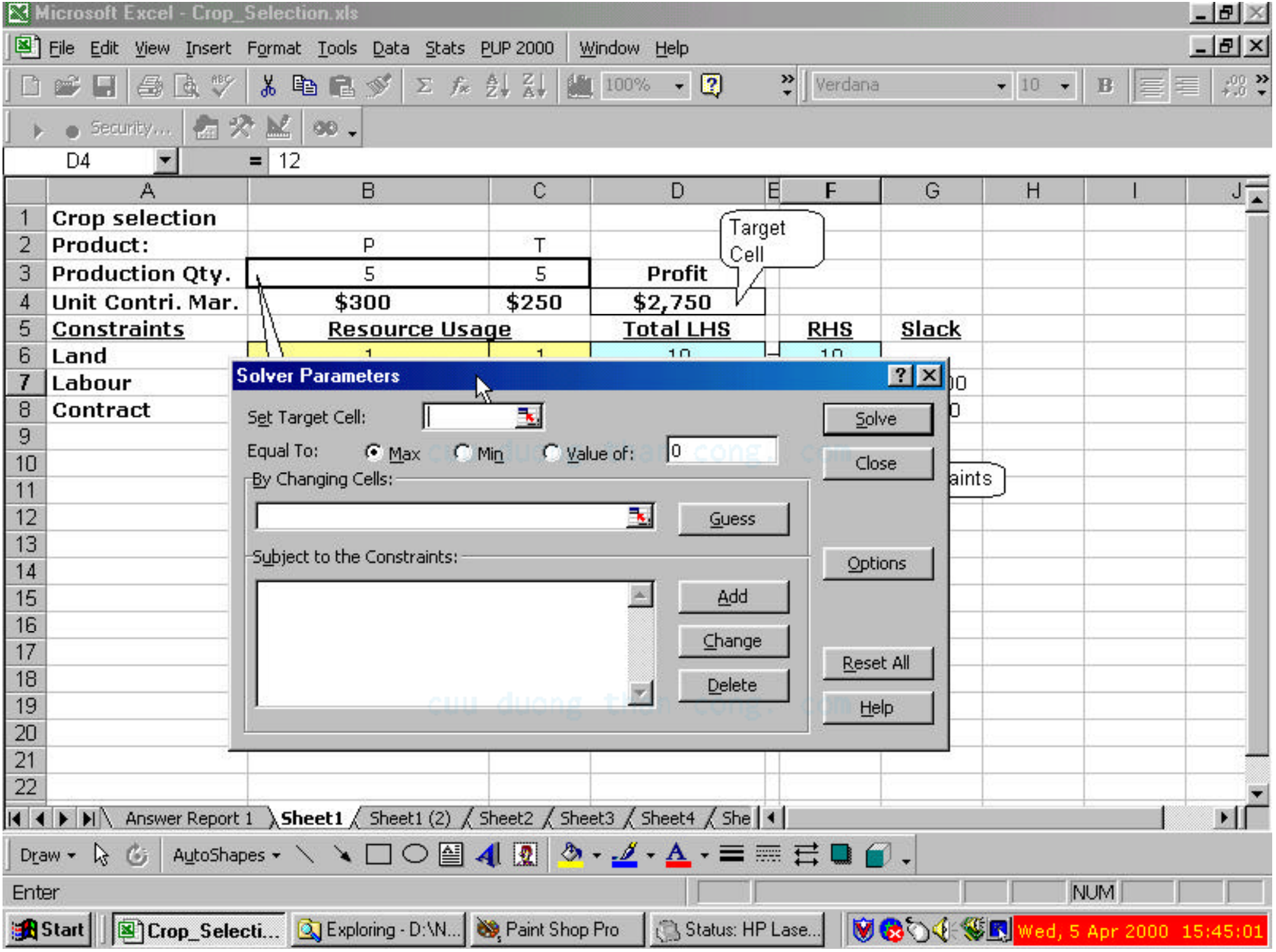
LHS Constraints

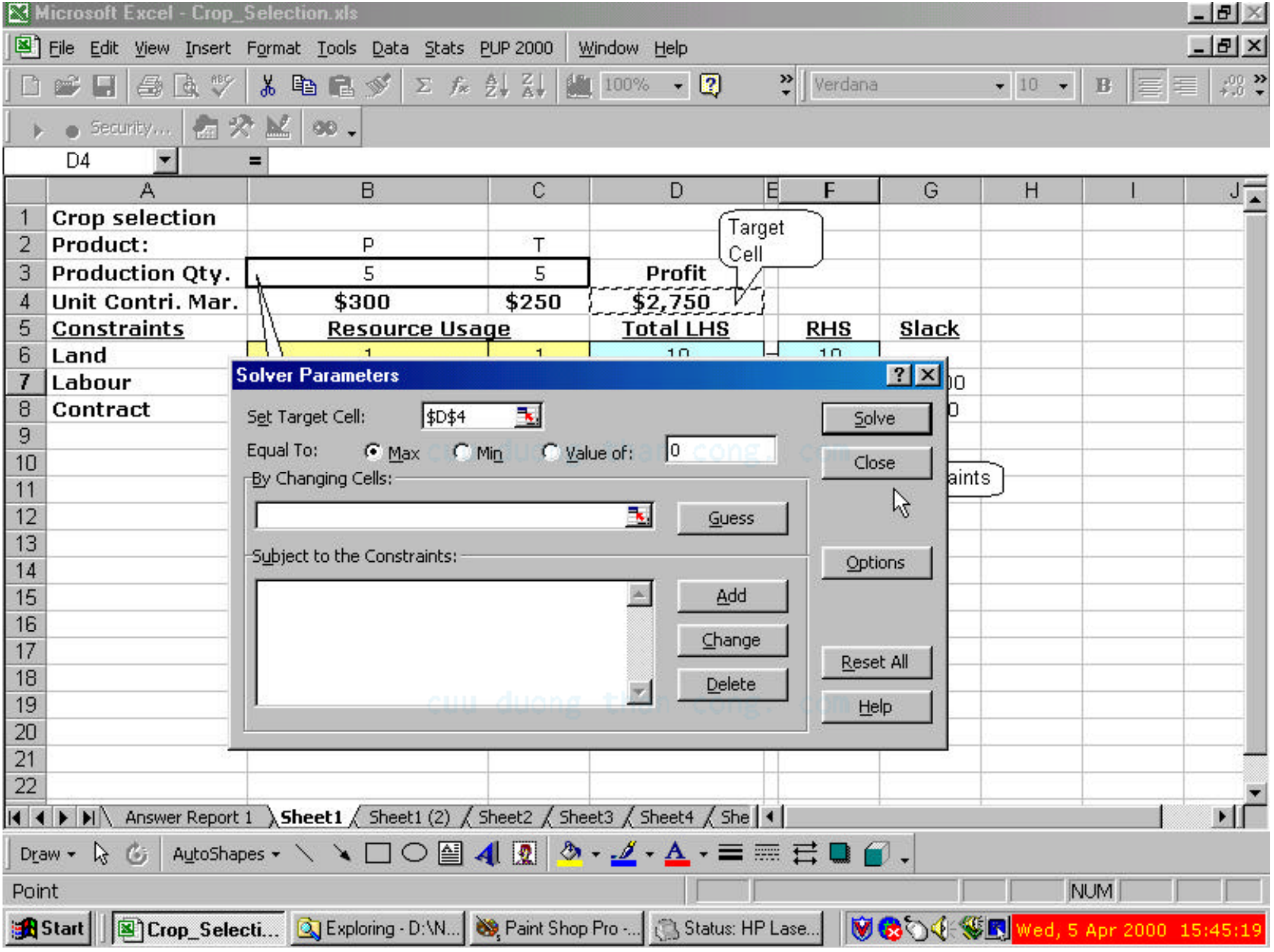
RHS Constraints

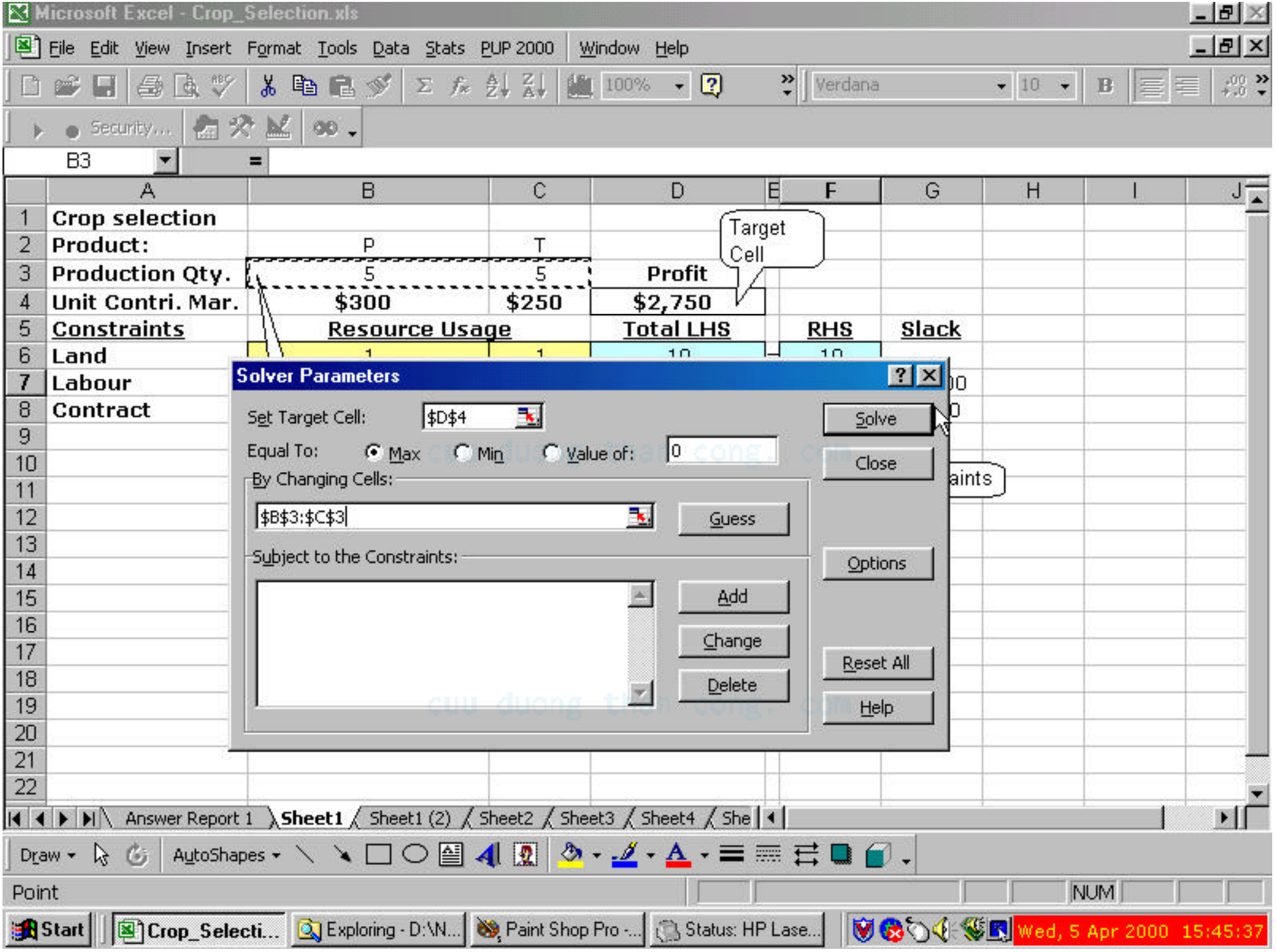
	A	B	C	D	E	F	G
1	**Crop selection**						
2	**Product:**	P	T				
3	**Production Qty.**	5	5	**Profit**			
4	**Unit Contri. Mar.**	**300**	**250**	**=SUMPRODUCT($B$3:$C$3,B4:C4)**			
5	**Constraints**	**Resource Usage**		**Total LHS**		**RHS**	
6	**Land**	1	1	=SUMPRODUCT($B$3:$C$3,B6:C6)	=	10	
7	**Labour**	2	0.5	=SUMPRODUCT($B$3:$C$3,B7:C7)	≤	12	=F7-D7
8	**Contract**	0	1	=SUMPRODUCT($B$3:$C$3,B8:C8)	≥	2	=D8-F8

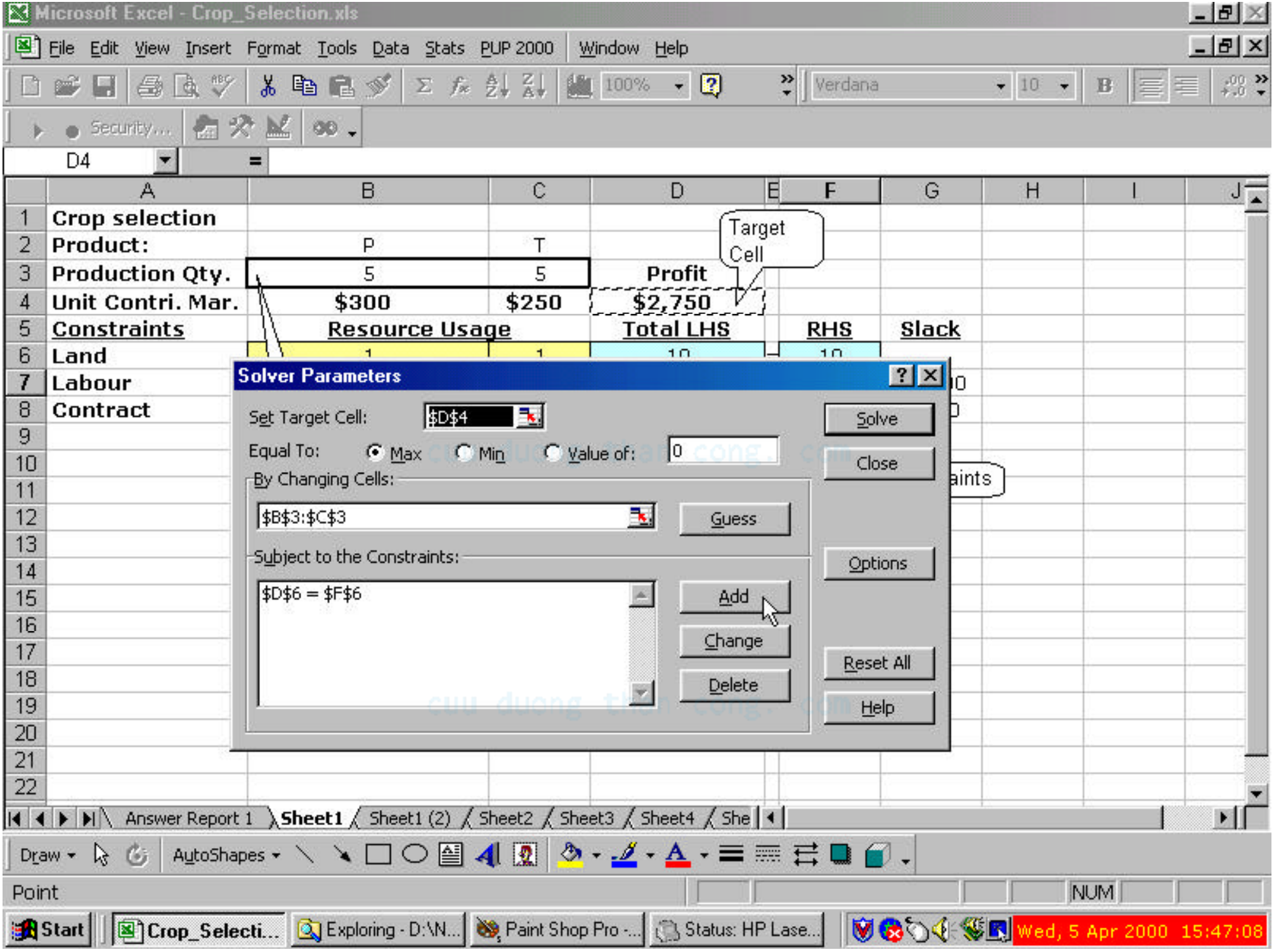
	A	B	C	D	E	F	G
1	**Microsoft Excel 8.0d Answer Report**						
2	**Worksheet: [Crop_Selection.xls]Sheet1**						
3	**Report Created: 27/02/99 10:40:15**						
4							
5							
6	Target Cell (Max)						
7		**Cell**	**Name**	**Original Value**	**Final Value**		
8		$D$4	Unit Contri. Mar. Profit	$2,750	$2,733		
9							
10							
11	Adjustable Cells						
12		**Cell**	**Name**	**Original Value**	**Final Value**		
13		$B$3	Production Qty. P	5	4.666666667		
14		$C$3	Production Qty. T	5	5.333333333		
15							
16							
17	Constraints						
18		**Cell**	**Name**	**Cell Value**	**Formula**	**Status**	**Slack**
19		$D$6	Land Total LHS	10	$D$6=$F$6	Binding	0
20		$D$7	Labour Total LHS	12	$D$7<=$F$7	Binding	0
21		$D$8	Contract Total LHS	5.333333333	$D$8>=$F$8	Not Binding	3.333333333

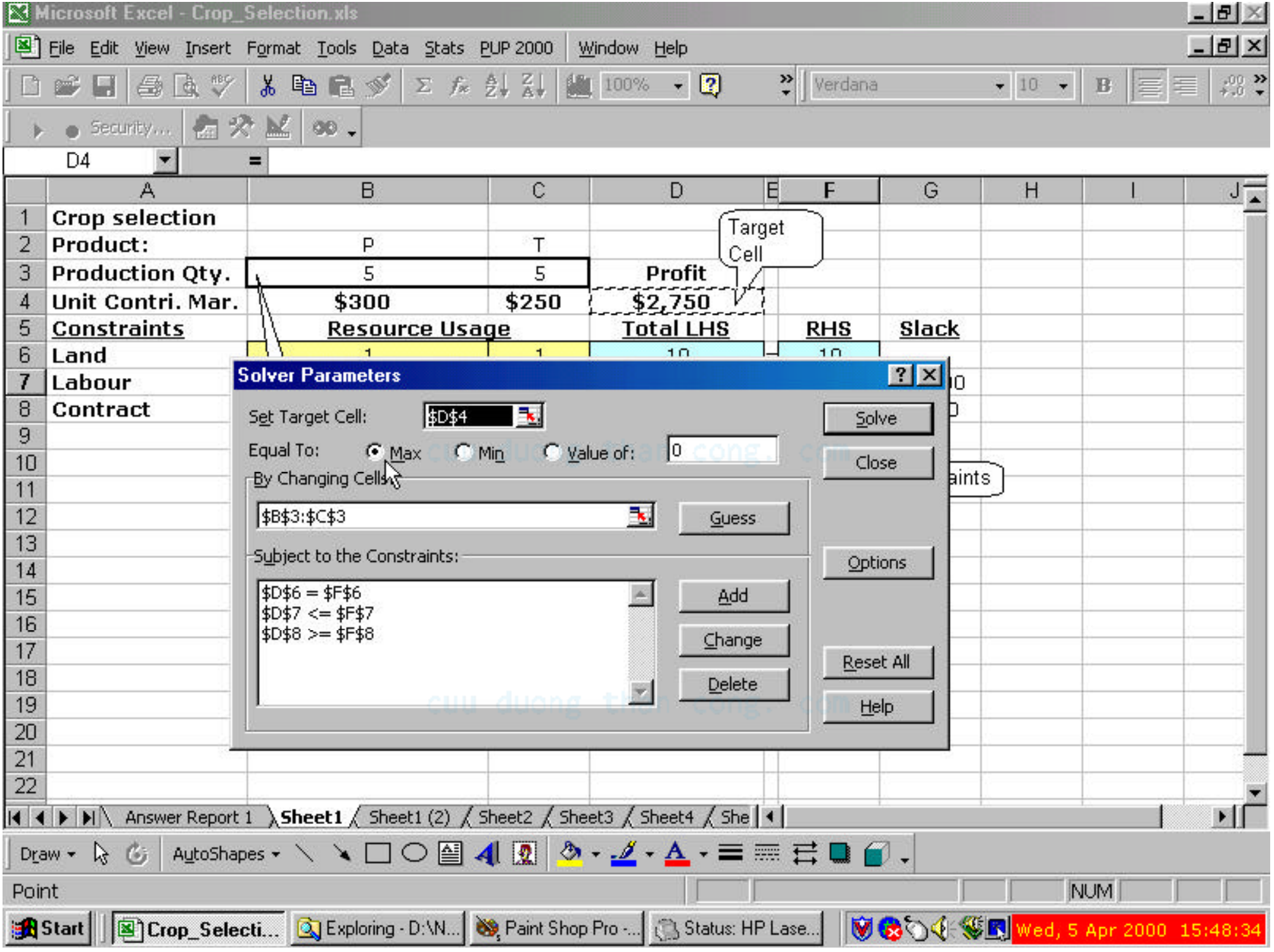


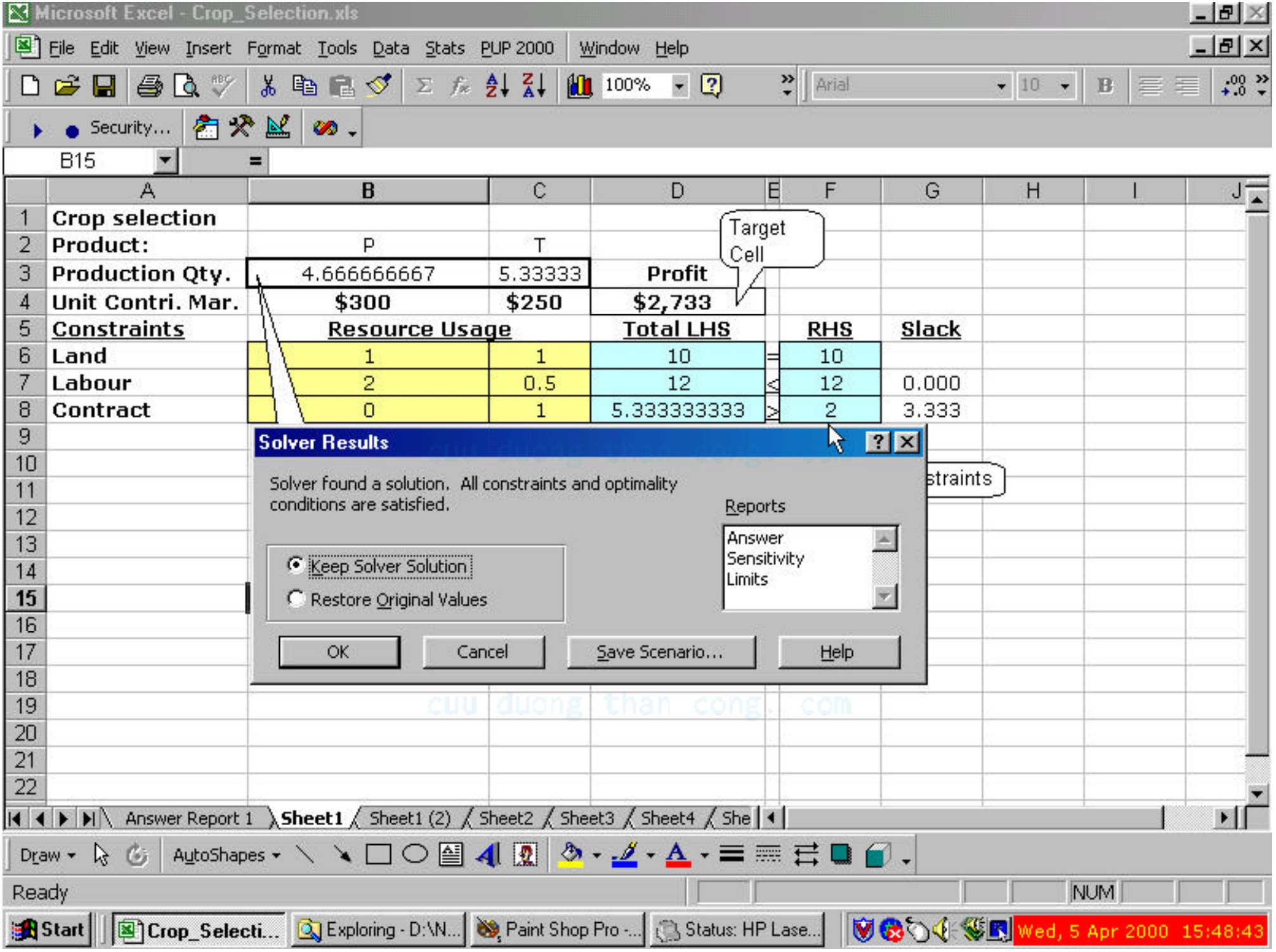












Linear Programming: Planning Application

☆ Suppose that a large piece of land can be put to two different built-up uses or left as open space.

☆ All these uses may exist side by side.

☆ 'Industrial land' brings a higher income from rates than does 'residential land', while 'open space' brings none.

☆ The local authority wants to maximise its rate income but there are certain constraints on what it can do.

☆ Let,

F = acreage of industrial land

H = acreage of residential land

G = acreage of open space

☆ Total area of land is 1000 acres.

☆ Thus,

$$F + G + H = 1000$$

☆ Rewrite **constraint on total area** as

$$F + H \leq 1000$$

☆ Other constraints:

☆ (A) **Minimum job provision**: to ensure at least a certain number of local jobs, for every acre devoted to residential use, there must be at least 1/10 of an acre devoted to industrial use

$$F \geq 1/10\ H$$

☆ (B) **Minimum labour supply**: to help local factories to get a supply of labour, for every acre devoted to industrial use, there must be at least one acre devoted to residential use

$$H \geq F$$

☆ (C) **Water supply**: there is a maximum quantity of water that can be piped to the land at reasonable cost.

☆ An acre of residential use is expected to consume 18,000 gallons of water per week, while an acre of industrial use will consume 25,000 gallons per week.

☆ Maximum weekly supply is 20,000,000 gallons.

☆ Thus,

$$18\,H + 25\,F \leq 20{,}000$$

☆ (D) **Electricity supply**: a similar constraint to the water supply constraint applies

$$7\,H + 30\,F \leq 14{,}000$$

☆ Thus LP problem

Maximise $R = 1000F + 600F$

subject to:

$$F + H \leq 1{,}000$$
$$10F \geq H$$
$$H \geq F$$
$$25F + 18H \leq 20{,}000$$
$$30F + 7H \leq 14{,}000$$

$$F \geq 0,\ H \geq 0$$

☆ Coefficients matrix

F	H	SIGN	RHS	
1000	600	=	Z	Objective function
1	1	≤	1000	Land area
10	-1	≥	0	Min job provision
-1	1	≥	0	Min labour supply
25	18	≤	20,000	Water supply
30	7	≤	14,000	Electricity supply
1	0	≥	0	Non-negativity, H
0	1	≥	0	Non-negativity, F

Graphic LP Optimizer - [lp-land8.glp]

File Edit View Option Window Help

New | RESET | UNDO

X H Y F PAYOFF 600.0 H + 1000.0 F = 717740.0

-1.0 H + 10.0 F >= 0.0

7.0 H + 30.0 F <= 14000.0

1.0 H - 1.0 F >= 0.0

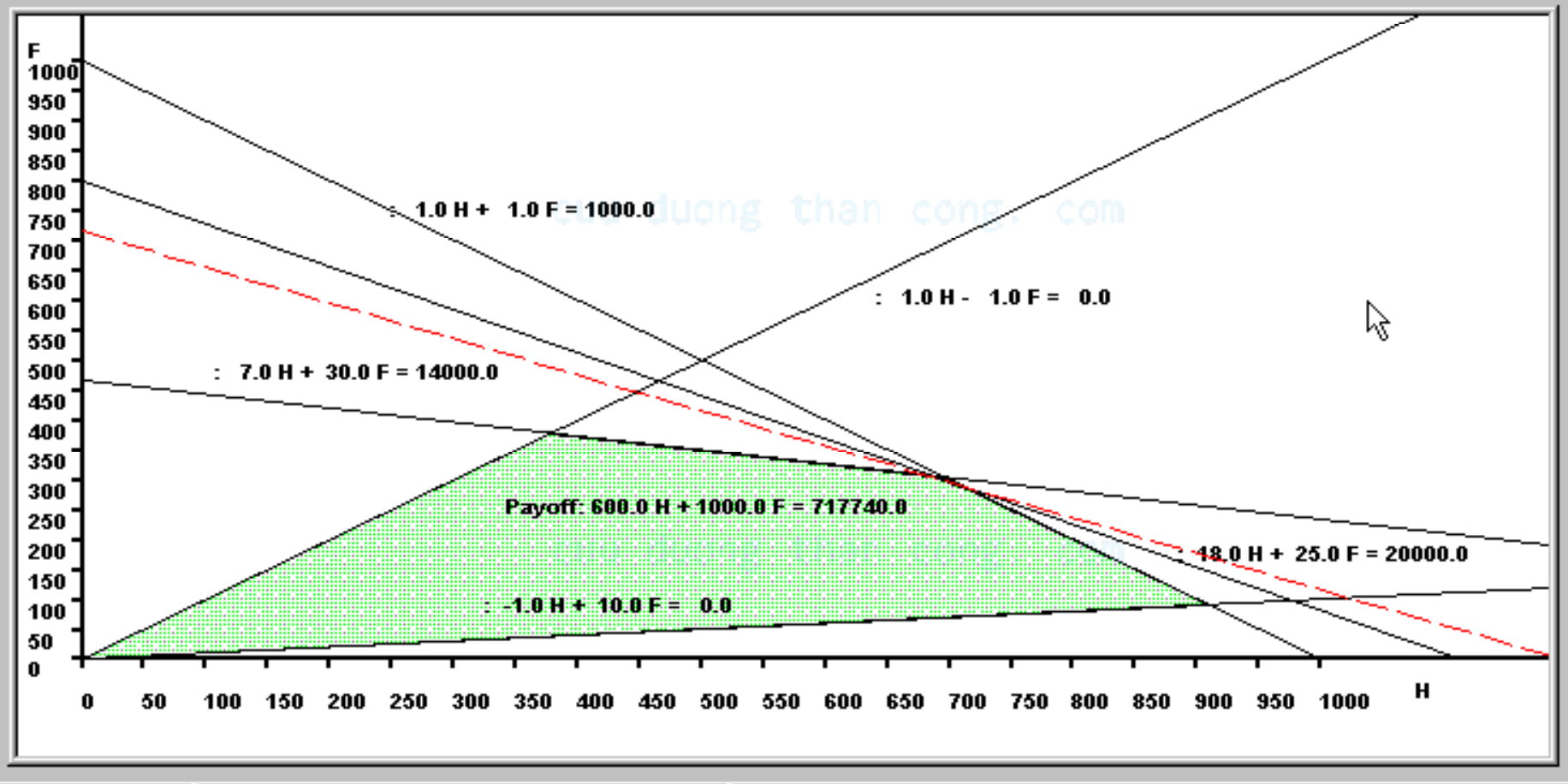
1.0 F + 1.0 F <= 1000.0

18.0 H + 25.0 F <= 20000.0

0.0 H + 0.0 F <= 0.0

X Min 0 X Max 1000 Y Min 0 Y Max 1000

X Zoom 0.578703 Y Zoom 0.279081 Decimal number 1



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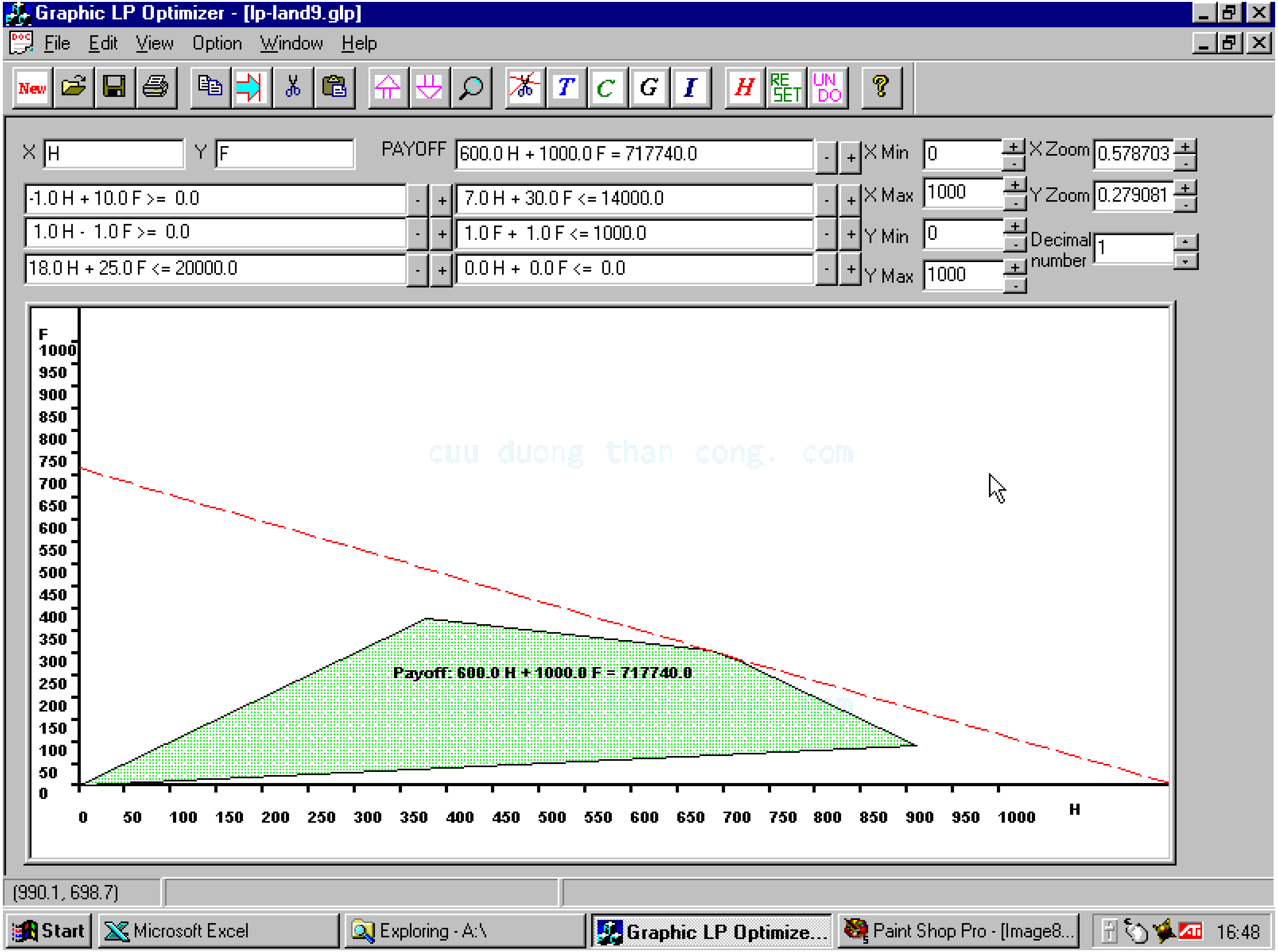
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File Edit View Option Window Help
New
X H
Y F
PAYOFF 600.0 H + 1000.0 F = 717740.0
-1.0 H + 10.0 F >= 0.0
1.0 H - 1.0 F >= 0.0
18.0 H + 25.0 F <= 20000.0
7.0 H + 30.0 F <= 14000.0
1.0 F + 1.0 F <= 1000.0
0.0 H + 0.0 F <= 0.0
X Min 0
X Max 1000
Y Min 0
Y Max 1000
X Zoom 0.578703
Y Zoom 0.279081
Decimal number 1
F
H
Payoff: 600.0 H + 1000.0 F = 717740.0
(990.1, 698.7)
Start
Microsoft Excel
Exploring - A:\
Graphic LP Optimize...
Paint Shop Pro - [Image8...
16:48

	A	B	C	D	E	F	G	H
1	**Land development**							
2	Land development	F	H					
3	Quantity	306.84932	684.93151	Profit				
4	Rate income/acre	$1,000	$600	$717,808				
5	Constraints	Resource Usage		Total LHS		RHS		
6	Job provision	10	-1	2383.561644	≥	0	2384	Surplus
7	Labour supply	-1	1	378.0821918	≥	0	378	Surplus
8	Water supply	25	18	20000	≤	20000	0.000000	Slack
9	Electricity supply	30	7	14000	≤	14000	0.000000	Slack
10	Land development	1	1	991.7808219	≤	1000	8.219178	Slack
11								
12								
13								

Target Cell

Changing Cells

LHS Constraints

RHS Constraints

	A	B	C	D	E	F	G	H
1	**Land development**							
2	Land development	H	F					
3	Quantity	684.931506846634	306.849315069743	Profit				
4	Rate income/acre	600	1000	=SUMPRODUCT($B$3:$C$3,B4:C4)				
5	Constraints	Resource Usage		Total LHS		RHS		
6	Job provision	-1	10	=SUMPRODUCT($B$3:$C$3,B6:C6)	≥	0	=F6-D6	Suplus
7	Labour supply	1	-1	=SUMPRODUCT($B$3:$C$3,B7:C7)	≥	0	=D7-F7	Surplus
8	Water supply	18	25	=SUMPRODUCT($B$3:$C$3,B8:C8)	≤	20000	=F8-D8	Slack
9	Electricity supply	7	30	=SUMPRODUCT($B$3:$C$3,B9:C9)	≤	14000	=F9-D9	Slack
10	Land development	1	1	=SUMPRODUCT($B$3:$C$3,B10:C10)	≤	1000	=F10-D10	Slack

	A	B	C	D	E	F	G
1	**Microsoft Excel 9.0 Answer Report**						
2	**Worksheet: [LP-Planning.xls]Sheet1**						
3	**Report Created: 5/04/00 4:02:45 PM**						
4							
5							
6	Target Cell (Max)						
7		**Cell**	**Name**	**Original Value**	**Final Value**		
8		$D$4	Rate income/acre Profit	$717,808	$717,808		
9							
10							
11	Adjustable Cells						
12		**Cell**	**Name**	**Original Value**	**Final Value**		
13		$B$3	Quantity F	306.8493151	306.8493151		
14		$C$3	Quantity H	684.9315068	684.9315068		
15							
16							
17	Constraints						
18		**Cell**	**Name**	**Cell Value**	**Formula**	**Status**	**Slack**
19		$D$9	Electricity supply Total LHS	14000	$D$9<=$F$9	Binding	0
20		$D$8	Water supply Total LHS	20000	$D$8<=$F$8	Binding	0
21		$D$10	Land development Total LHS	991.7808219	$D$10<=$F$10	Not Binding	8.219178082
22		$D$7	Labour supply Total LHS	378.0821918	$D$7>=$F$7	Not Binding	378.0821918
23		$D$6	Job provision Total LHS	2383.561644	$D$6>=$F$6	Not Binding	2383.561644