Analytic Hierarchy Process

☆ *Analytic hierarchy process* (AHP) methodology introduced by Thomas Saaty (1977) has had numerous applications in a wide range of contexts

☆ AHP explicitly recognises the *hierarchical value structure* of evaluation problems.

☆ In addition to hierarchical structuring, AHP is based on two other compelling and widely accepted concepts,

- ✱ use of *pairwise*, *relative comparisons*

- ✱ use of *redundancy* in judgements to improve accuracy and deal with 'fuzziness'

☆ The limited *cognitive capacity* of individuals in terms of both *short term memory* and *discriminability* (channel capacity) is cited in support of the AHP.

☆ Hierarchical structures are a means of coping with complexity

☆ *Relative judgements* assumed to be more easily generated and more meaningful than *absolute judgements* (which, in a sense, are relative to information stored in long term memory (and perhaps, in new situations, relative to no information)).

☆ Redundancy reduces errors and provides a measure of consistency.

☆ AHP process represents an evaluation problem hierarchically and involves pairwise comparisons of elements (projects, criteria, sub-criteria, etc.) at each level with respect to elements at the adjacent higher level.

☆ In a three level hierarchy, each project is compared to each other project with respect to each criterion (criterion), and each criterion is compared relative to each other criterion with respect some overriding goal.

☆ Comparison is in terms of the extent to which one project 'dominates' another.

- ☆ Such subjectively determined pairwise comparisons (values) are commonly expressed on a 1-9 scale of 'dominance' (or preference).

- ☆ For example, if project A performs outstandingly relative to project B with respect to criterion C_1 then '9' might be used to represent this dominance.

- ☆ If A and B perform equally with respect to C_1, then a score of '1' would be used, and other scores used as appropriate to represent intermediate degrees of dominance.

- ☆ Pairwise comparisons are considered to be 'reciprocal' such that, for example, if the dominance of A relative to B for C_1 is say '5', then the dominance of B relative to A for C_1 must be '1/5'.

- ☆ Numbers 1,3,5,7,9 are associated with verbal expressions of dominance (respectively, 'equal', 'weak', 'strong', 'very strong', 'absolute') and the numbers 2,4,6,8 represent intermediate values between adjacent scale values.

☆ Criteria are then compared to each other in terms their importance in achieving some overall goal (e.g. select a 'best' project), again using a 1-9 scale.

☆ Numbers 1,3,5,7,9 are now associated with verbal expressions of relative importance (respectively, 'equal', 'weak', 'strong', 'very strong', 'absolute') and the numbers 2,4,6,8 represent intermediate values between adjacent scale values.

☆ For each reciprocal pairwise comparison matrix of Q elements, $\mathbf{A} = [a_{ij}]$, 'scores' representing the 'dominance' of elements may be derived by solving the matrix equation

$$\boxed{\mathbf{A}\,\mathbf{q} = \lambda_{max}\,\mathbf{q}}$$

for $\mathbf{q} = [q_1, q_2,...,q_Q]$, the eigenvector associated with the largest eigenvalue, λ_{max}, of $\mathbf{A}$.

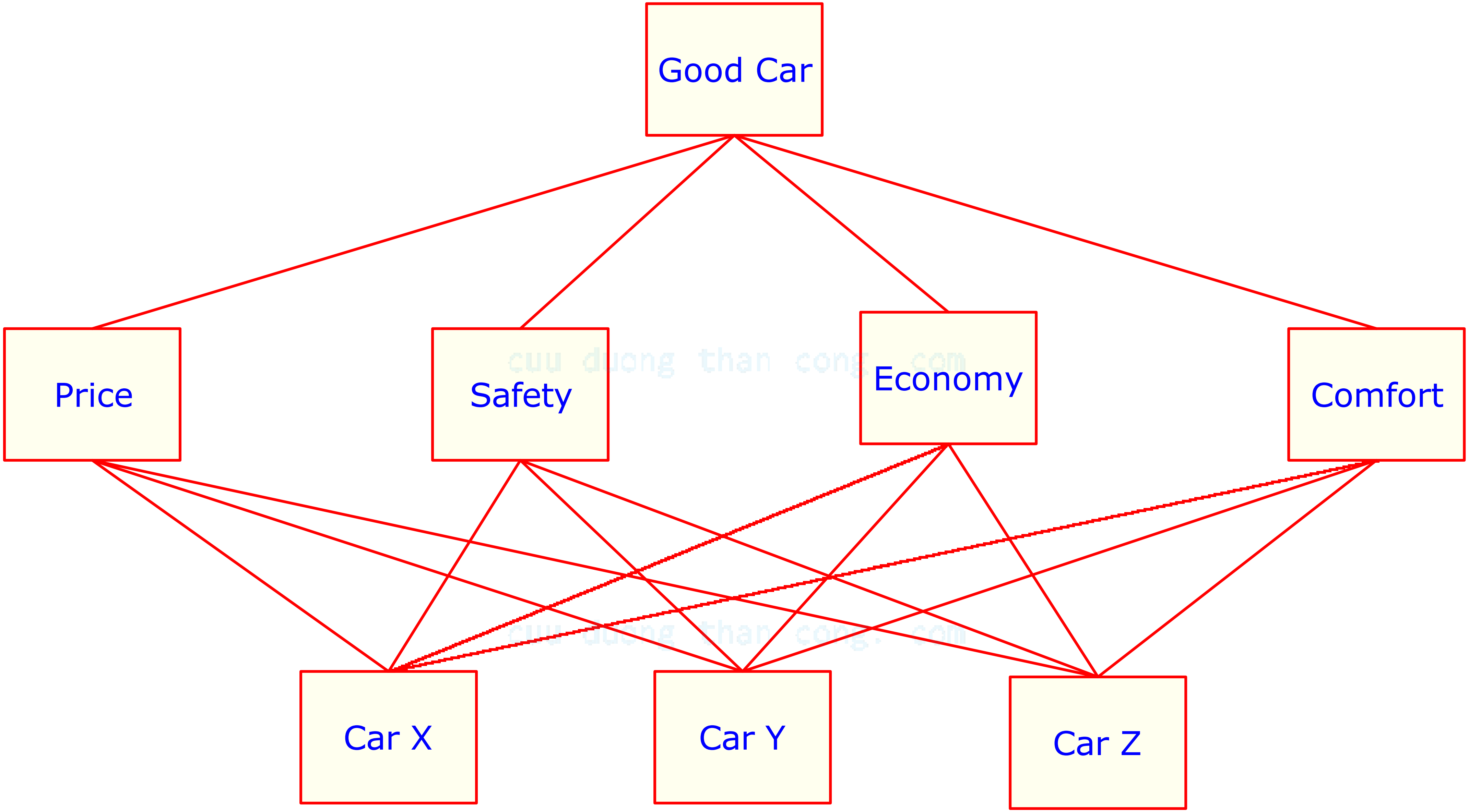
☆ These scores are normalised to $\mathbf{p} = [p_1, p_2,...,p_Q]$ where $p_i = q_i/\sum_k q_k$ $(i = 1,...,Q)$.

☆ Alternatively, 'scores' may be approximated by calculating the arithmetic means of the normalised rows of **A**, that is,

$$q_i = (1/Q)\ \{a_{i1}/\sum_{k=1,Q} a_{k1} + a_{i2}/\sum_{k=1,Q} a_{k2} \ldots + a_{iQ}/\sum_{k=1,Q} a_{kQ}\}$$

☆ Normalised scores associated with each pairwise comparison matrix are concatenated throughout the hierarchical structure to form scores for each lowest level project.

Car Selection Using AHP



Car Selection Using AHP

Criteria:

1. Price
2. Safety
3. Economy
4. Comfort

	Price	Safety	Economy	Comfort	Priority
Price	1	1/7	1/2	1/5	0.0655
Safety	7	1	4	2	0.5177
Economy	2	1/4	1	1/2	0.1335
Comfort	5	1/2	2	1	0.2833

λ_{max} = 4.012, CI = 0.004
CR = CI/RI = 0.004/0.9 = 0.0044

Price					
	Car X	**Car Y**	**Car Z**	**Normalised Priorities**	**Idealised Priorities**
Car X	1	1/4	3	0.2109	0.2992
Car Y	4	1	7	0.7049	1.0000
Car Z	1/3	1/7	1	0.0841	0.1194

λ_{max} = 3.032, CI = 0.016
CR = CI/RI = 0.016/0.58 = 0.0276

Safety					
	Car X	**Car Y**	**Car Z**	**Normalised Priorities**	**Idealised Priorities**
Car X	1	1/2	3	0.2854	0.4543
Car Y	2	1	8	0.6282	1.0000
Car Z	1/3	1/8	1	0.0864	0.1376

λ_{max} = 3.009, CI = 0.005
CR = CI/RI = 0.005/0.58 = 0.0086

Economy					
	Car X	**Car Y**	**Car Z**	**Normalised Priorities**	**Idealised Priorities**
Car X	1	1/3	1/6	0.0953	0.1456
Car Y	3	1	1/3	0.2499	0.3816
Car Z	6	3	1	0.6548	1.0000

λ_{max} = 3.018, CI = 0.009
CR = CI/RI = 0.009/0.58 = 0.0155

Comfort					
	Car X	**Car Y**	**Car Z**	**Normalised Priorities**	**Idealised Priorities**
Car X	1	1/4	1/8	0.0732	0.1092
Car Y	4	1	1/3	0.2560	0.3816
Car Z	8	3	1	0.6708	1.0000

λ_{max} = 3.018, CI = 0.009
CR = CI/RI = 0.009/0.58 = 0.0155

Distributive mode

	Price	Safety	Economy	Comfort	
	0.0655	0.5177	0.1335	0.2833	
Car X	0.2109	0.2854	0.0953	0.0732	**0.1950**
Car Y	0.7049	0.6282	0.2499	0.2560	**0.4773**
Car Z	0.0841	0.0864	0.6548	0.6708	**0.3277**

Ideal mode

	Price	Safety	Economy	Comfort	
	0.0655	0.5177	0.1335	0.2833	
Car X	0.2992	0.4543	0.1456	0.1092	**0.3051**
Car Y	1.0000	1.0000	0.3816	0.3816	**0.7422**
Car Z	0.1194	0.1376	1.0000	1.0000	**0.4959**

AHP: The Fundamental Scale

Intensity of importance	Definition	Explanation
1	**Equal Importance**	**Two activities contribute equally to the objective**
2	Weak	
3	**Moderate Importance**	**Experience and judgement slightly favour one activity over another**
4	Moderate Plus	
5	**Strong Importance**	**Experience and judgement strongly favour one activity over another**
6	Strong Plus	
7	**Very Strong or Demonstrated Importance**	**An activity is favoured very strongly over another; its dominance is demonstrated in practice**
8	Very, very strong	
9	**Extreme Importance**	**The evidence favouring one activity over another is of the highest possible order of affirmation**

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Reciprocals:

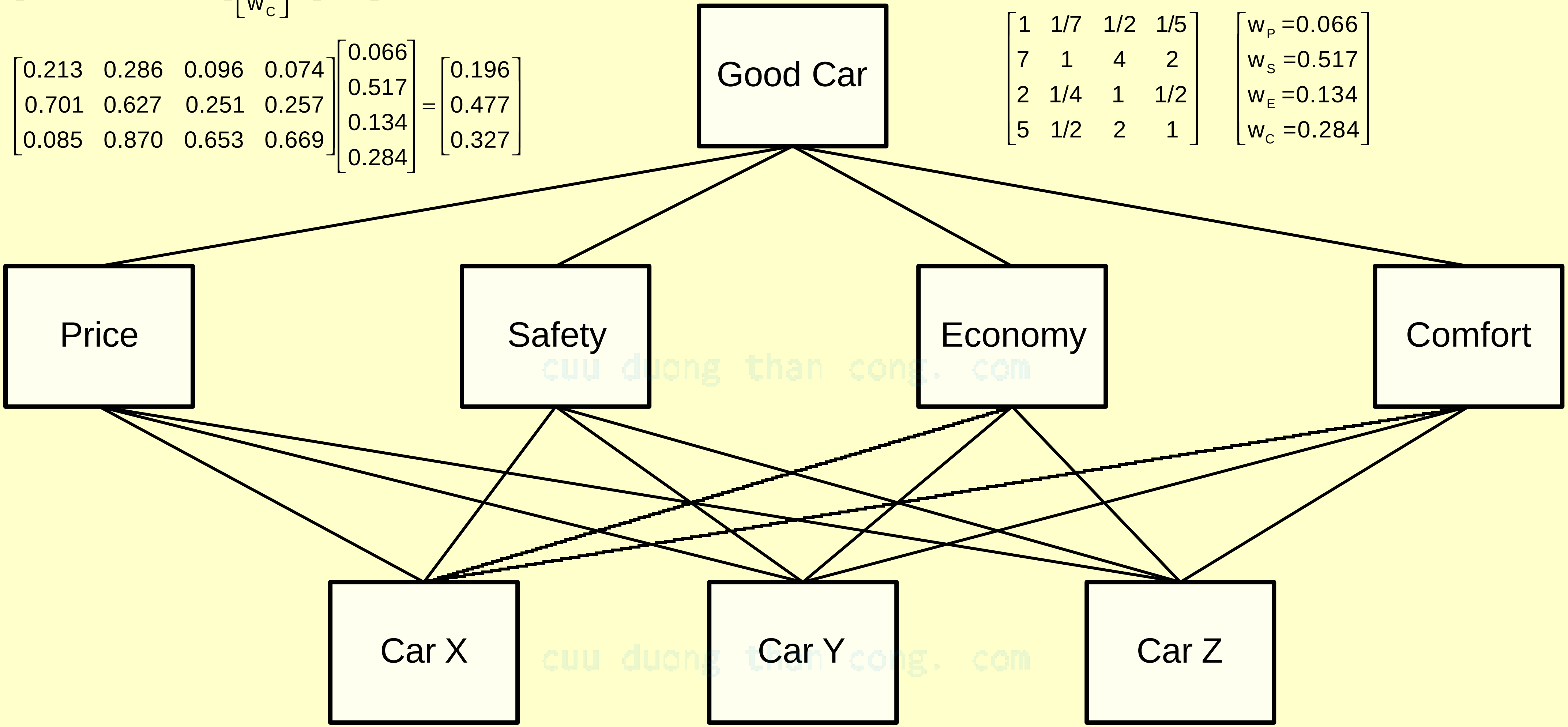
If activity i has one of the nonzero numbers {**1**, 2, **3**, 4, **5**, 6, **7**, 8, **9**} assigned to it when compared with activity j, the j has the reciprocal value when compared with i, that is {**1**, 1/2, **1/3**, 1/4, **1/5**, 1/6, **1/7**, 1/8, **1/9**}.

Car Selection Using AHP

$$\begin{bmatrix} s_{XP} & s_{XS} & s_{XE} & s_{XC} \\ s_{YP} & s_{YS} & s_{YE} & s_{YC} \\ s_{ZP} & s_{ZS} & s_{ZE} & s_{ZC} \end{bmatrix} \begin{bmatrix} w_P \\ w_S \\ w_E \\ w_C \end{bmatrix} = \begin{bmatrix} V(X) \\ V(Y) \\ V(Z) \end{bmatrix}$$

$$\begin{bmatrix} 0.213 & 0.286 & 0.096 & 0.074 \\ 0.701 & 0.627 & 0.251 & 0.257 \\ 0.085 & 0.870 & 0.653 & 0.669 \end{bmatrix} \begin{bmatrix} 0.066 \\ 0.517 \\ 0.134 \\ 0.284 \end{bmatrix} = \begin{bmatrix} 0.196 \\ 0.477 \\ 0.327 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1/7 & 1/2 & 1/5 \\ 7 & 1 & 4 & 2 \\ 2 & 1/4 & 1 & 1/2 \\ 5 & 1/2 & 2 & 1 \end{bmatrix} \quad \begin{bmatrix} w_P = 0.066 \\ w_S = 0.517 \\ w_E = 0.134 \\ w_C = 0.284 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1/4 & 3 \\ 4 & 1 & 7 \\ 1/3 & 1/8 & 1 \end{bmatrix} \begin{bmatrix} s_{XP} = 0.213 \\ s_{YP} = 0.701 \\ s_{ZP} = 0.085 \end{bmatrix} \quad \begin{bmatrix} 1 & 1/2 & 3 \\ 2 & 1 & 8 \\ 1/3 & 1/8 & 1 \end{bmatrix} \begin{bmatrix} s_{XS} = 0.286 \\ s_{YS} = 0.627 \\ s_{ZS} = 0.870 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1/3 & 1/6 \\ 3 & 1 & 1/3 \\ 6 & 3 & 1 \end{bmatrix} \begin{bmatrix} s_{XE} = 0.096 \\ s_{YE} = 0.251 \\ s_{ZE} = 0.653 \end{bmatrix} \quad \begin{bmatrix} 1 & 1/4 & 1/8 \\ 4 & 1 & 1/3 \\ 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} s_{XC} = 0.074 \\ s_{YC} = 0.257 \\ s_{ZC} = 0.669 \end{bmatrix}$$

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Scores on Price

Scores on Safety

Scores on Economy

Scores on Comfort

Weight for Price

Weight for Safety

Overall Score for Car X

Car X

Car Y

Car Z

$$\begin{bmatrix} s_{XP} = 0.213 & s_{XS} = 0.286 & s_{XE} = 0.096 & s_{XC} = 0.074 \\ s_{YP} = 0.701 & s_{YS} = 0.627 & s_{YE} = 0.251 & s_{YC} = 0.257 \\ s_{ZP} = 0.085 & s_{ZS} = 0.868 & s_{ZE} = 0.653 & s_{ZC} = 0.669 \end{bmatrix} \begin{bmatrix} w_P = 0.066 \\ w_S = 0.517 \\ w_E = 0.134 \\ w_C = 0.284 \end{bmatrix} = \begin{bmatrix} \sum_{j=P,S,E,C} w_j s_{Xj} = 0.196 \\ \sum_{j=P,S,E,C} w_j s_{Yj} = 0.477 \\ \sum_{j=P,S,E,C} w_j s_{Zj} = 0.327 \end{bmatrix}$$

Overall Score for Car Y

Overall Score for Car Z

Weight for Economy

Weight for Comfort

Car Selection Using Analytical Hierarchy Process

Pairwise comparisons among objectives

	Price	Safety	Economy	Comfort
Price	1	1/7	1/2	1/5
Safety	7	1	4	2
Economy	2	1/4	1	1/2
Comfort	5	1/2	2	1

Normalized matrix

0.0667	0.0755	0.0667	0.0541
0.4667	0.5283	0.5333	0.5405
0.1333	0.1321	0.1333	0.1351
0.3333	0.2642	0.2667	0.2703

Weights

0.0657
0.5172
0.1335
0.2836

Pairwise comparisons among cars on price

	Car X	Car Y	Car Z
Car X	1	1/4	3
Car Y	4	1	7
Car Z	1/3	1/7	1

Normalized matrix

0.1875	0.1795	0.2727
0.7500	0.7179	0.6364
0.0625	0.1026	0.0909

Scores

0.2132
0.7014
0.0853

Pairwise comparisons among cars on safety

	Car X	Car Y	Car Z
Car X	1	1/2	3
Car Y	2	1	8
Car Z	1/3	1/8	1

Normalized matrix

0.3000	0.3077	0.2500
0.6000	0.6154	0.6667
0.1000	0.0769	0.0833

Scores

0.2859
0.6274
0.0868

Pairwise comparisons among cars on economy

	Car X	Car Y	Car Z
Car X	1	1/3	1/6
Car Y	3	1	1/3
Car Z	6	3	1

Normalized matrix

0.1000	0.0769	0.1111
0.3000	0.2308	0.2222
0.6000	0.6923	0.6667

Scores

0.0960
0.2510
0.6530

Pairwise comparisons among cars on comfort

	Car X	Car Y	Car Z
Car X	1	1/4	1/8
Car Y	4	1	1/3
Car Z	8	3	1

Normalized matrix

0.0769	0.0588	0.0857
0.3077	0.2353	0.2286
0.6154	0.7059	0.6857

Scores

0.0738
0.2572
0.6690

Determining best car

Matrix of scores

	Price	Safety	Economy	Comfort
Car X	0.2132	0.286	0.096	0.074
Car Y	0.7014	0.627	0.251	0.257
Car Z	0.0853	0.087	0.653	0.669

Overall scores

0.1956
0.4770
0.3274

Car Y has the highest score

To use matrix multiplication: select G37:G39, *select* MMULT, *select* B37:E39 *as array 1*; *select* L5:L8 *as array 2*; *before closing* MMULT, *press* ***control-shift-enter***. *This will carry out all row (performance scores) and column (criterion weights) multiplications at the same time*.

	A	B	C	D	E	F	G
1	**Car Selection Using Analytical Hierarchy Process**						
2							
3	**Pairwise comparisons among objectives**						**Normalized matrix**
4		Price	Safety	Economy	Comfort		
5	Price	1	0.142857142857143	0.5	0.2		=B5/SUM(B$5:B$8)
6	Safety	7	1	4	2		=B6/SUM(B$5:B$8)
7	Economy	2	0.25	1	0.5		=B7/SUM(B$5:B$8)
8	Comfort	5	0.5	2	1		=B8/SUM(B$5:B$8)
9							
10	**Pairwise comparisons among cars on price**						**Normalized matrix**
11		Car X	Car Y	Car Z			
12	Car X	1	0.25	3			=B12/SUM(B$12:B$14)
13	Car Y	4	1	7			=B13/SUM(B$12:B$14)
14	Car Z	0.333333333333333	0.142857142857143	1			=B14/SUM(B$12:B$14)
15							
16	**Pairwise comparisons among cars on safety**						**Normalized matrix**
17		Car X	Car Y	Car Z			
18	Car X	1	0.5	3			=B18/SUM(B$18:B$20)
19	Car Y	2	1	8			=B19/SUM(B$18:B$20)
20	Car Z	0.333333333333333	0.125	1			=B20/SUM(B$18:B$20)
21							
22	**Pairwise comparisons among cars on economy**						**Normalized matrix**
23		Car X	Car Y	Car Z			
24	Car X	1	0.333333333333333	0.166666666666667			=B24/SUM(B$24:B$26)
25	Car Y	3	1	0.333333333333333			=B25/SUM(B$24:B$26)
26	Car Z	6	3	1			=B26/SUM(B$24:B$26)
27							
28	**Pairwise comparisons among cars on comfort**						**Normalized matrix**
29		Car X	Car Y	Car Z			
30	Car X	1	0.25	0.125			=B30/SUM(B$30:B$32)
31	Car Y	4	1	0.333333333333333			=B31/SUM(B$30:B$32)
32	Car Z	8	3	1			=B32/SUM(B$30:B$32)
33							
34	**Determining best car**						
35	Matrix of scores						Overall scores
36		Price	Safety	Economy	Comfort		
37	Car 1	=L12	=L18	=L24	=L30		=MMULT(B37:E39,L5:L8)
38	Car 2	=L13	=L19	=L25	=L31		=MMULT(B37:E39,L5:L8)
39	Car 3	=L14	=L20	=L26	=L32		=MMULT(B37:E39,L5:L8)
40							

	H	I	J	K	L
1					
2					
3					**Weights**
4					
5	=C5/SUM(C$5:C$8)	=D5/SUM(D$5:D$8)	=E5/SUM(E$5:E$8)		=AVERAGE(G5:J5)
6	=C6/SUM(C$5:C$8)	=D6/SUM(D$5:D$8)	=E6/SUM(E$5:E$8)		=AVERAGE(G6:J6)
7	=C7/SUM(C$5:C$8)	=D7/SUM(D$5:D$8)	=E7/SUM(E$5:E$8)		=AVERAGE(G7:J7)
8	=C8/SUM(C$5:C$8)	=D8/SUM(D$5:D$8)	=E8/SUM(E$5:E$8)		=AVERAGE(G8:J8)
9					
10					**Scores**
11					
12	=C12/SUM(C$12:C$14)	=D12/SUM(D$12:D$14)			=AVERAGE(G12:I12)
13	=C13/SUM(C$12:C$14)	=D13/SUM(D$12:D$14)			=AVERAGE(G13:I13)
14	=C14/SUM(C$12:C$14)	=D14/SUM(D$12:D$14)			=AVERAGE(G14:I14)
15					
16					**Scores**
17					
18	=C18/SUM(C$18:C$20)	=D18/SUM(D$18:D$20)			=AVERAGE(G18:I18)
19	=C19/SUM(C$18:C$20)	=D19/SUM(D$18:D$20)			=AVERAGE(G19:I19)
20	=C20/SUM(C$18:C$20)	=D20/SUM(D$18:D$20)			=AVERAGE(G20:I20)
21					
22					**Scores**
23					
24	=C24/SUM(C$24:C$26)	=D24/SUM(D$24:D$26)			=AVERAGE(G24:I24)
25	=C25/SUM(C$24:C$26)	=D25/SUM(D$24:D$26)			=AVERAGE(G25:I25)
26	=C26/SUM(C$24:C$26)	=D26/SUM(D$24:D$26)			=AVERAGE(G26:I26)
27					
28					**Scores**
29					
30	=C30/SUM(C$30:C$32)	=D30/SUM(D$30:D$32)			=AVERAGE(G30:I30)
31	=C31/SUM(C$30:C$32)	=D31/SUM(D$30:D$32)			=AVERAGE(G31:I31)
32	=C32/SUM(C$30:C$32)	=D32/SUM(D$30:D$32)			=AVERAGE(G32:I32)
33					
34					
35					
36					
37					
38					
39					
40					

	A	B	C	D	E	F	G	H	I	J	K	L
4		Price	Safety	Economy	Comfort							
5	Price	1	1/7	1/2	1/5		0.0667	0.0755	0.0667	0.0541		0.0657
6	Safety	7	1	4	2		0.4667	0.5283	0.5333	0.5405		0.5172
7	Economy	2	1/4	1	1/2		0.1333	0.1321	0.1333	0.1351		0.1335
8	Comfort	5	1/2	2	1		0.3333	0.2642	0.2667	0.2703		0.2836
9												
10	**Pairwise comparisons among cars on price**						**Normalized matrix**					**Scores**
11		Car X	Car Y	Car Z								
12	Car X	1	1/4	3			0.1875	0.1795	0.2727			0.2132
13	Car Y	4	1	7			0.7500	0.7179	0.6364			0.7014
14	Car Z	1/3	1/7	1			0.0625	0.1026	0.0909			0.0853
15												
16	**Pairwise comparisons among cars on safety**						**Normalized matrix**					**Scores**
17		Car X	Car Y	Car Z								
18	Car X	1	1/2	3			0.3000	0.3077	0.2500			0.2859
19	Car Y	2	1	8			0.6000	0.6154	0.6667			0.6274
20	Car Z	1/3	1/8	1			0.1000	0.0769	0.0833			0.0868
21												
22	**Pairwise comparisons among cars on economy**						**Normalized matrix**					**Scores**
23		Car X	Car Y	Car Z								
24	Car X	1	1/3	1/6			0.1000	0.0769	0.1111			0.0960
25	Car Y	3	1	1/3			0.3000	0.2308	0.2222			0.2510
26	Car Z	6	3	1			0.6000	0.6923	0.6667			0.6530
27												
28	**Pairwise comparisons among cars on comfort**						**Normalized matrix**					**Scores**
29		Car X	Car Y	Car Z								
30	Car X	1	1/4	1/8			0.0769	0.0588	0.0857			0.0738
31	Car Y	4	1	1/3			0.3077	0.2353	0.2286			0.2572
32	Car Z	8	3	1			0.6154	0.7059	0.6857			0.6690
33												
34	**Determining best car**											
35	Matrix of scores						Overall scores					
36		Price	Safety	Economy	Comfort							
37	Car 1	0.2132	0.286	0.096	0.074		0.1956					
38	Car 2	0.7014	0.627	0.251	0.257		0.4770					
39	Car 3	0.0853	0.087	0.653	0.669		0.3274					

	M	N	O	P	Q	R	S
4							
5	0.2631	4.0030					
6	2.0783	4.0183					
7	0.5360	4.0159					
8	1.1377	4.0116					
9	**CI**	0.0041	**CI/RI**	0.0045			
10							
11					**Random Indices**		
12	0.6446	3.022773856			**n**	**RI**	
13	2.1517	3.067502077			2	0	
14	0.2566	3.00745318			3	0.58	
15	**CI**	0.016288186	**CI/RI**	0.0281	4	0.9	
16					5	1.12	
17					6	1.24	
18	0.8598	3.0075			7	1.32	
19	1.8932	3.0177			8	1.41	
20	0.2605	3.0025			9	1.45	
21	**CI**	0.0046	**CI/RI**	0.0079	10	1.59	
22							
23							
24	0.2885	3.0049					
25	0.7567	3.0148					
26	1.9821	3.0353					
27	**CI**	0.0092	**CI/RI**	0.0158			
28							
29							
30	0.2217	3.0038					
31	0.7755	3.0152					
32	2.0311	3.0361					
33	**CI**	0.0092	**CI/RI**	0.0158			
34							
35							
36							
37							
38							
39							

if CI/RI < 0.1, then consistency is satifactory

	A	B	C	D	E	F	G	H
1	**Car Selection Using Analytical Hierarchy Process**							
2								
3	**Pairwise comparisons among objectives**						**Normalized matrix**	
4		Price	Safety	Economy	Comfort			
5	Price	1	0.14285714	0.5	0.2		=B5/SUM(B$5:B$8)	=C5/SUM(C$5:C$8)
6	Safety	7	1	4	2		=B6/SUM(B$5:B$8)	=C6/SUM(C$5:C$8)
7	Economy	2	0.25	1	0.5		=B7/SUM(B$5:B$8)	=C7/SUM(C$5:C$8)
8	Comfort	5	0.5	2	1		=B8/SUM(B$5:B$8)	=C8/SUM(C$5:C$8)
9								
10	**Pairwise comparisons among cars on price**						**Normalized matrix**	
11		Car X	Car Y	Car Z				
12	Car X	1	0.25	3			=B12/SUM(B$12:B$14	=C12/SUM(C$12:C$14
13	Car Y	4	1	7			=B13/SUM(B$12:B$14	=C13/SUM(C$12:C$14
14	Car Z	0.33333	0.14285714	1			=B14/SUM(B$12:B$14	=C14/SUM(C$12:C$14
15								
16	**Pairwise comparisons among cars on safety**						**Normalized matrix**	
17		Car X	Car Y	Car Z				
18	Car X	1	0.5	3			=B18/SUM(B$18:B$20	=C18/SUM(C$18:C$20
19	Car Y	2	1	8			=B19/SUM(B$18:B$20	=C19/SUM(C$18:C$20
20	Car Z	0.33333	0.125	1			=B20/SUM(B$18:B$20	=C20/SUM(C$18:C$20
21								
22	**Pairwise comparisons among cars on economy**						**Normalized matrix**	
23		Car X	Car Y	Car Z				
24	Car X	1	0.33333333	0.166666666			=B24/SUM(B$24:B$26	=C24/SUM(C$24:C$26
25	Car Y	3	1	0.333333333			=B25/SUM(B$24:B$26	=C25/SUM(C$24:C$26
26	Car Z	6	3	1			=B26/SUM(B$24:B$26	=C26/SUM(C$24:C$26
27								
28	**Pairwise comparisons among cars on comfort**						**Normalized matrix**	
29		Car X	Car Y	Car Z				
30	Car X	1	0.25	0.125			=B30/SUM(B$30:B$32	=C30/SUM(C$30:C$32
31	Car Y	4	1	0.333333333			=B31/SUM(B$30:B$32	=C31/SUM(C$30:C$32
32	Car Z	8	3	1			=B32/SUM(B$30:B$32	=C32/SUM(C$30:C$32
33								
34	**Determining best car**							
35	Matrix of scores						Overall scores	
36		Price	Safety	Economy	Comfort			
37	Car 1	=L12	=L18	=L24	=L30		=MMULT(B37:E39,L5:	
38	Car 2	=L13	=L19	=L25	=L31		=MMULT(B37:E39,L5:	
39	Car 3	=L14	=L20	=L26	=L32		=MMULT(B37:E39,L5:	

	I	J	K	L	M	N	O	P
1					**Consistency Calculations**			
2								
3				**Weights**	**Product**	**Ratios**		
4								
5	=D5/SUM(D$5:D$8)	=E5/SUM(E$5:E$8)		=AVERAGE(G5:J5)	=MMULT(B5:E8,L5:L8)	=M5/L5		
6	=D6/SUM(D$5:D$8)	=E6/SUM(E$5:E$8)		=AVERAGE(G6:J6)	=MMULT(B5:E8,L5:L8)	=M6/L6		
7	=D7/SUM(D$5:D$8)	=E7/SUM(E$5:E$8)		=AVERAGE(G7:J7)	=MMULT(B5:E8,L5:L8)	=M7/L7		
8	=D8/SUM(D$5:D$8)	=E8/SUM(E$5:E$8)		=AVERAGE(G8:J8)	=MMULT(B5:E8,L5:L8)	=M8/L8		
9					**CI**	=(AVERAGE(N5:N8)-4)/3	**CI/RI**	=N9/0.9
10				**Scores**				
11								
12	=D12/SUM(D$12:D$1			=AVERAGE(G12:I12)	=MMULT(B12:D14,L12:L14)	=M12/L12		
13	=D13/SUM(D$12:D$1			=AVERAGE(G13:I13)	=MMULT(B12:D14,L12:L14)	=M13/L13		
14	=D14/SUM(D$12:D$1			=AVERAGE(G14:I14)	=MMULT(B12:D14,L12:L14)	=M14/L14		
15					**CI**	=(AVERAGE(N12:N14)-3)/2	**CI/RI**	=N15/0.58
16				**Scores**				
17								
18	=D18/SUM(D$18:D$2			=AVERAGE(G18:I18)	=MMULT(B18:D20,L18:L20)	=M18/L18		
19	=D19/SUM(D$18:D$2			=AVERAGE(G19:I19)	=MMULT(B18:D20,L18:L20)	=M19/L19		
20	=D20/SUM(D$18:D$2			=AVERAGE(G20:I20)	=MMULT(B18:D20,L18:L20)	=M20/L20		
21					**CI**	=(AVERAGE(N18:N20)-3)/2	**CI/RI**	=N21/0.58
22				**Scores**				
23								
24	=D24/SUM(D$24:D$2			=AVERAGE(G24:I24)	=MMULT(B24:D26,L24:L26)	=M24/L24		
25	=D25/SUM(D$24:D$2			=AVERAGE(G25:I25)	=MMULT(B24:D26,L24:L26)	=M25/L25		
26	=D26/SUM(D$24:D$2			=AVERAGE(G26:I26)	=MMULT(B24:D26,L24:L26)	=M26/L26		
27					**CI**	=(AVERAGE(N24:N26)-3)/2	**CI/RI**	=N27/0.58
28				**Scores**				
29								
30	=D30/SUM(D$30:D$3			=AVERAGE(G30:I30)	=MMULT(B30:D32,L30:L32)	=M30/L30		
31	=D31/SUM(D$30:D$3			=AVERAGE(G31:I31)	=MMULT(B30:D32,L30:L32)	=M31/L31		
32	=D32/SUM(D$30:D$3			=AVERAGE(G32:I32)	=MMULT(B30:D32,L30:L32)	=M32/L32		
33					**CI**	=(AVERAGE(N30:N32)-3)/2	**CI/RI**	=N33/0.58
34								
35								
36								
37								
38								
39								

Checking for Consistency in AHP

☆ Pairwise comparison matrices can suffer from inconsistencies.

☆ Entries in pairwise comparison matrix have a built-in pairwise consistency since require that $a_{ij} = 1/a_{ji}$, i.e. the matrix is a 'reciprocal' matrix.

☆ However, if $a_{12} = 5$ (C_1 'strongly more important' than C_2), $a_{23} = 2$ (C_2 'very slightly more important' than C_3) and $a_{13} = 2$ (C_1 'very slightly more important' than C_3)
Are these judgements consistent?

$$a_{12} \times a_{23} = 5 \times 2 = 10 \neq a_{13} = 2$$

☆ Slight inconsistencies are common in pairwise comparisons.

☆ Major inconsistencies must be resolved.

☆ Checking for inconsistency: consider the criteria in the car selection problem:

price

safety

economy

comfort

☆ (1) Perform matrix multiplication of pairwise comparisons and normalised criteria weights

$$\begin{bmatrix} 1 & 1/7 & 1/2 & 1/5 \\ 7 & 1 & 4 & 2 \\ 2 & 1/4 & 1 & 1/2 \\ 5 & 1/2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0.0657 \\ 0.5172 \\ 0.1351 \\ 0.2703 \end{bmatrix} = \begin{bmatrix} 0.2631 \\ 2.0783 \\ 0.5360 \\ 1.1377 \end{bmatrix}$$

☆ (2) Calculate the average of ratios of the result of matrix multiplication [0.2631, 2.0783, 0.5360, 1.1377] to original normalised weights [0.0657, 0.5172, 0.1351, 0.2703], i.e. calculate

$$\frac{\frac{0.2631}{0.0657} + \frac{2.0783}{0.5172} + \frac{0.5360}{0.1335} + \frac{1.1377}{0.2836}}{4} = 4.0383$$

☆ (3) Calculate 'consistency index' (CI) defined as

$$CI = \frac{(4.0383-4)}{4-1} = 0.0041$$

☆ (4) Calculate ratio of CI to the given 'random index' (RI), i.e. calculate CI/RI

$$\frac{CI}{RI} = \frac{0.0041}{0.9} = 0.0045$$

☆ Random indices are give as

Random Indices	
n	RI
2	0
3	0.58
4	0.90
5	1.12
6	1.24
7	1.32
8	1.41
9	1.45
10	1.49
11	1.51
12	1.48
13	1.56
14	1.57
15	1.59

- ☆ Values of RI give the average value of CI if the entries in the pairwise comparison matrix, $\mathbf{A} = [a_{ij}]_{n\times n}$, were chosen at random (subject to the constraints that $a_{ii} = 1$, and $a_{ji} = 1/a_{ij}$).

- ☆ If the ratio of CI to RI is sufficiently small, then the pairwise comparisons are probably consistent enough to be useful.

- ☆ Saaty suggests that, if CI/RI < 0.10, then the degree of consistency is satisfactory.

- ☆ If CI/RI > 0.10, then serious inconsistencies exist and AHP may not yield meaningful results.