Decision Theory

✰ Introduce framework or model for process of decision-making

✰ Simplest form - three elements

(1) choice/action/alternative

(2) chance/uncertainty

(3) consequence/payoff/outcome

Choice + Chance ⇨ Consequence

(1) Choice

✰ Central element - involves selection of alternative (or alternatives) that decision-maker's analysis has shown to be the 'best' approach to the solution of the problem

(2) Chance

✰ Refers to uncertain nature of outcomes (usually measured at some time in the future) resulting from choices (which are made at present)

(3) Consequence

✰ Consequence ('payoff', 'outcome') of an action (choice) and an event (chance) forms the objective function that we wish to optimise (maximise or minimise) in making the selection among available alternatives

✰ Nature of consequences may be positive ('profit', 'revenue', 'utility', 'sales') or negative ('cost', 'time', 'regret') which will determine the direction of our optimisation

☆ Three classes of decisions, based on the degree of knowledge (or lack of it) of the likelihoods of occurrence of various outcomes

(A) Certainty

(B) Uncertainty

(C) Risk

(A) Certainty

☆ Under conditions of certainty, we know at the time of decision what the eventual outcome of each alternative will be

☆ Decision-making process involves systematic evaluation of each available alternative and selection of alternative with most attractive result

(B) Uncertainty

- ☆ Decision making under conditions of uncertainty, requires that decisions be made with no knowledge of the likelihoods of the various outcomes

- ☆ Obviously chances of making a less than optimal decision under these conditions substantial

(C) Risk

- ☆ Relates to those decisions all of whose outcomes are known and each is able to be assigned a probability of occurrence

- ☆ Solution procedures involve applications of mathematical expectation

Decision Making Under Certainty

☆ Consider three projects that can be developed in each of three sites

☆ Assume cost of each project varies according to site:

\$00,000	A	B	C
Project 1 (P_1)	3	7	4
Project 2 (P_2)	4	6	6
Project 3 (P_3)	3	8	5

☆ There is only one state of nature (certainty)

☆ Complete enumeration of every payoff is:

Alternatives	Total cost
$P_1 \Rightarrow A$, $P_2 \Rightarrow B$, $P_3 \Rightarrow C$	3 + 6 + 5 = 14
$P_1 \Rightarrow A$, $P_2 \Rightarrow C$, $P_3 \Rightarrow B$	3 + 6 + 8 = 17
$P_1 \Rightarrow B$, $P_2 \Rightarrow A$, $P_3 \Rightarrow C$	7 + 4 + 5 = 16
$P_1 \Rightarrow B$, $P_2 \Rightarrow C$, $P_3 \Rightarrow A$	7 + 6 + 3 = 16
$P_1 \Rightarrow C$, $P_2 \Rightarrow B$, $P_3 \Rightarrow A$	4 + 6 + 3 = 13*
$P_1 \Rightarrow C$, $P_2 \Rightarrow A$, $P_3 \Rightarrow B$	4 + 4 + 8 = 16

* Minimum cost

Decision Making Under Uncertainty

☆ Decision making under uncertainty (as under risk) involves alternative actions whose payoff depend on the (random) states of nature (event, possible future)

☆ Specifically payoff matrix (decision matrix) of a decision problem with m alternative actions and n states of natures

	s_1	s_2		s_n
a_1	$v(a_1,s_1)$	$v(a_1,s_2)$	...	$v(a_1,s_n)$
a_2	$v(a_2,s_1)$	$v(a_2,s_2)$	...	$v(a_2,s_n)$
.	.	.		.
.	.	.		.
.	.	.		.
a_m	$v(a_m,s_1)$	$v(a_m,s_2)$	...	$v(a_m,s_n)$

a_i = action (alternative) i (i = 1,,m)

s_j = state of nature j (j = 1,,n)

$v(a_i,s_j)$ = outcome (payoff) of action i under state of nature j

☆ Under uncertainty, probability distribution associated with states s_j unknown or cannot be determined

☆ Following criteria used for analysing decision problem under uncertainty

(1) Laplace

(2) Minimax (Wald)

(3) Maximax

(4) Minimax regret (Savage)

(5) Hurwicz

☆ Criteria differ in degree of conservatism the decision maker exhibits in the face of uncertainty

(1) Laplace Criterion

☆ Laplace criterion based on 'principle of insufficient reason'

☆ Because the probability distributions of the states of nature unknown, there is no reason to believe they are different

☆ Thus alternatives are evaluated using optimistic assumption that the are equal, i.e.

$$P(s_1) = P(s_2) = \ldots = P(s_n) = 1/n$$

☆ Given that payoffs, $v(a_i,s_j)$ represent gain (i.e. are positive), best alternative is the one that yields

$$\max_{a_i}\left\{\frac{1}{n}\sum_{j=1}^{n} v(a_i,s_j)\right\}$$

☆ If payoffs, $v(a_i,s_j)$ represent loss (i.e. are negative), then best alternative is the one that yields

$$\min_{a_i}\left\{\frac{1}{n}\sum_{j=1}^{n} v(a_i,s_j)\right\}$$

Payoff Table

	State of Nature		
	P(f) = 0.5	P(u) = 0.5	
Decision Purchase	Favourable Economic Conditions (f)	Unfavourable Economic Conditions (u)	
Apartment Building	$50,000	$30,000	$40,000*
Office Building	$100,000	-$40,000	$30,000
Warehouse	$30,000	$10,000	$20,000

*Laplace solution

(Apart Bld

$50,000(0.5) + $30,000(0.5) = $40,000*

(Office Bld)

$100,000(0.5) - $40,000(0.5) = $30,000

(Warehouse)

$30,000(0.5) + $10,000(0.5) = $20,000

(2) Maximin (Minimax) Wald Criterion

✰ Maximin (minimax) criterion based on the conservative attitude of making the best out of the worst possible conditions

✰ Maximin (gain), minimax (loss) criterion of pessimism

✰ If payoffs, $v(a_i,s_j)$ represent gain (i.e. are positive), then best alternative is the one that corresponds to maximin criterion

$$\max_{a_i}\left\{\min_{s_j} v(a_i, s_j)\right\}$$

✰ If payoffs, $v(a_i,s_j)$ represent loss (i.e. are negative), then best alternative is the one that corresponds to minimax criterion

$$\min_{a_i}\left\{\max_{s_j} v(a_i, s_j)\right\}$$

Payoff Table

	State of Nature		
Decision Purchase	Favourable Economic Conditions (f)	Unfavourable Economic Conditions (u)	Row Minimum
Apartment Building	$50,000	$30,000	$30,000*
Office Building	$100,000	-$40,000	-$40,000
Warehouse	$30,000	$10,000	$10,000

* maximin solution

(3) Maximax

☆ Maximax criterion is optimistic

☆ Appeals to highly venturesome individual

☆ If payoffs, $v(a_i,s_j)$ represent gain (i.e. are positive), then best alternative is the one that corresponds to maximax criterion

$$\max_{a_i}\left\{\max_{s_j} v(a_i,s_j)\right\}$$

☆ If payoffs, $v(a_i,s_j)$ represent loss (i.e. are negative), then best alternative is the one that corresponds to minimin criterion

$$\min_{a_i}\left\{\min_{s_j} v(a_i,s_j)\right\}$$

Payoff Table

	State of Nature		
Decision Purchase	Favourable Economic Conditions (f)	Unfavourable Economic Conditions (u)	Row Maximum
Apartment Building	$50,000	$30,000	$50,000
Office Building	$100,000	-$40,000	$100,000*
Warehouse	$30,000	$10,000	$30,000

* maximax solution

(note possibility of significant loss of $40,000)

(4) Savage Minimax Regret Criterion

☆ Savage regret criterion aims at moderating conservatism in the minimax (maximin) criterion by replacing the (gain or loss) payoff matrix $v(a_i,s_j)$ with a loss (or 'regret') $r(a_i,s_j)$ matrix

$$r(a_i,s_j)=\begin{cases}\max_{a_k}\left\{v(a_k,s_j)\right\}-v(a_i,s_j), & \text{if v is a gain}\\ v(a_i,s_j)-\min_{a_k}\left\{v(a_k,s_j)\right\}, & \text{if v is a loss}\end{cases}$$

☆ Savage criterion 'moderates' the minimax (maximin) criterion

Payoff Table

	State of Nature	
Decision Purchase	Favourable Economic Conditions (f)	Unfavourable Economic Conditions (u)
Apartment Building	$50,000	$30,000
Office Building	$100,000	-$40,000
Warehouse	$30,000	$10,000
Column Maximum	$100,000	$30,000

Opportunity Loss Table

	State of Nature		
Decision Purchase	Favourable Economic Conditions (f)	Unfavourable Economic Conditions (u)	Maximum Regret
Apartment Building	$50,000	$0	$50,000*
Office Building	$0	$70,000	$70,000
Warehouse	$70,000	$20,000	$70,000

* minimax regret

☆ Consider the following *loss*, $v(a_i,s_j)$ matrix

	s_1	s_2	row max
a_1	\$11,000	\$90	\$11,000
a_2	\$10,000	\$10,000	\$10,000 <minimax

☆ Application of minimax criterion shows that a_2 with a definite loss of \$10,000, is preferable

☆ However, we may select a_1 because there is a chance of a loss of only \$90 if s_2 is realised

☆ If use regret $r(a_i,s_j)$ matrix instead

	s_1	s_2	row max
a_1	\$1,000	\$0	\$1000 <minimax
a_2	\$0	\$9910	\$9910

☆ Thus, minimax criterion, when applied to regret matrix, will select a_1, as desired

(5) Hurwicz Criterion

☆ Hurwicz criterion designed to reflect range of decision-making attitudes from most optimistic to most pessimistic (conservative)

☆ Define $0 \leq \alpha \leq 1$, and assume that $v(a_i,s_j)$ represents a gain

☆ Then selected action must be associated with

$$\max_{a_i}\left\{\alpha \max_{s_j} v(a_i,s_j) + (1-\alpha)\min_{s_j} v(a_i,s_j)\right\}$$

☆ Parameter known as 'index of optimism'

☆ If $\alpha = 0$, criterion is conservative because equivalent to applying regular maximin (best of the worst) criterion

☆ If $\alpha = 1$, criterion yield optimistic results (equivalent to applying a maximax (best of the best) criterion)

☆ Can adjust degree of optimism/pessimism in range (0,1) range - in absence of strong feeling regarding optimism/pessimism, set $\alpha = 05$

☆ If $v(a_i,s_j)$ represents a loss, then Hurwicz criterion is

$$\min_{a_i}\left\{\alpha \min_{s_j} v(a_i,s_j) + (1-\alpha)\max_{s_j} v(a_i,s_j)\right\}$$

Payoff Table

	State of Nature		
Decision Purchase	Favourable Economic Conditions (f)	Unfavourable Economic Conditions (u)	α = 0.4
Apartment Building	$50,000	$30,000	$38,000*
Office Building	$100,000	-$40,000	$16,000
Warehouse	$30,000	$10,000	$18,000

*Hurwicz (Coefficient of optimism, α = 0.4)

(Apart Bld)	$50,000(0.4)	+	$30,000(0.6)	=	$38,000*
(Office Bld)	$100,000(0.4)	-	$40,000(0.6)	=	$16,000
(Warehouse)	$30,000(0.4)	+	$10,000(0.6)	=	$18,000

Decision Making Under Risk

✰ Under conditions of risk, payoffs associated with each decision alternative are usually described by probability distributions

✰ Decision making under risk usually based on ‘expected value’ criterion in which alternatives are compared based on the maximisation of ‘expected profit’ or the minimisation of ‘expected cost’

✰ Decision problem may include n states of nature and m alternatives.

$p_j > 0$ = probability of occurrence for state of nature j

$a_{ij} = v(a_i,s_j)$ = payoff of alternative i given state of nature j

$(i = 1,...,m;\ j = 1,..,n)$

☆ Expected payoff for alternative i computed as

$$EV_i = \sum_{j=1,n} a_{ij} p_j \qquad (i = 1,\ldots,m)$$

☆ By definition, $\sum_{j=1,n} p_j = 1$.

☆ Best alternative is the one associated with

$$EV_i^* = \max_{i=1,m}\{EV_i\}$$

or

$$EV_i^* = \min_{i=1,m}\{EV_i\}$$

depending on whether the payoff of the problem represents profit (income) or loss (expense), respectively.

Payoff Table

	State of Nature		
	P(f) = 0.6	P(u) = 0.4	
Decision Purchase	Favourable Economic Conditions (f)	Unfavourable Economic Conditions (u)	
Apartment Building	$50,000	$30,000	$42,000
Office Building	$100,000	-$40,000	$44,000*
Warehouse	$30,000	$10,000	$22,000

*Expected value solution

(Apart Bld)

$50,000(0.6) + $30,000(0.4) = $42,000

(Office Bld)

$100,000(0.6) - $40,000(0.4) = $44,000*

(Warehouse)

$30,000(0.6) + $10,000(0.4) = $22,000

Expected Opportunity Loss (EOL)

- ☆ Decision criterion closely related to EV is the expected opportunity loss
- ☆ To use this, multiply probabilities by the regret (opportunity loss) for each decision outcome
- ☆ As with the minimax regret criterion, the best decision results from minimising the regret, or, in this case minimising the expected regret or opportunity loss
- ☆ Decisions recommended by EV and EOL are the same
- ☆ These two methods always result in the same decision

Opportunity Loss Table

	State of Nature		
	P(f) = 0.6	P(u) = 0.4	
Decision Purchase	Favourable Economic Conditions (f)	Unfavourable Economic Conditions (u)	Expected Opportunity Loss (EOL)
Apartment Building	$50,000	$0	$30,000
Office Building	$0	$70,000	$28,000*
Warehouse	$70,000	$20,000	$70,000
Column Maximum	$100,000	$30,000	

* minimum expected opportunity loss

(Apart Bld)

$50,000(0.6) + $0(0.4) = $30,000

(Office Bld)

$0(0.6) - $70,000(0.4) = $28,000*

(Warehouse)

$70,000(0.6) + $20,000(0.4) = $50,000

Payoff Table

	State of Nature		
	P(f) = 0.6	P(u) = 0.4	
Decision Purchase	Favourable Economic Conditions (f)	Unfavourable Economic Conditions (u)	
Apartment Building	$50,000	$30,000	$42,000
Office Building	$100,000	-$40,000	$44,000*
Warehouse	$30,000	$10,000	$22,000
Column Maximum	$100,00	$30,000	

*Expected value solution

(Apart Bld)

$50,000(0.6) + $30,000(0.4) = $42,000

(Office Bld)

$100,000(0.6) - $40,000(0.4) = $44,000*

(Warehouse)

$30,000(0.6) + $10,000(0.4) = $22,000

☆ Expected value of the decision **given** perfect information is given as

$100,000(0.6) + $30,000(0.4) = $72,000

☆ **Expected value of perfect information** (EVPI) is maximum amount that would be paid to gain information that would result in a decision better than the one made without perfect information (e.g. expected value solution)

EVPI = $72,000 - $44,000 = $28,000

☆ Note that EVPI is the same as expected opportunity loss (EOL)

☆ To see this, let $o_{ij} = \max_i\{a_{ij}\} - a_{ij}$

☆ Then,

$$\begin{aligned} EOL &= \min_i\{\textstyle\sum_{j=1,n} o_{ij}p_j\} \\ &= \min_i\{\textstyle\sum_{j=1,n} [\max_i\{a_{ij}\} - a_{ij}]p_j\} \\ &= \min_i\{\textstyle\sum_{j=1,n} \max_i\{a_{ij}\}p_j - \sum_{j=1,n}a_{ij}p_j\} \\ &= \textstyle\sum_{j=1,n} \max_i\{a_{ij}\}p_j - \max_i\{\sum_{j=1,n} a_{ij}p_j\} \\ &= EVWPI - EV \\ &= EVPI \end{aligned}$$

Most Probable State of Nature

- ✰ If payoffs, $v(a_i,s_j)$ represent gain (i.e. are positive), then best alternative under the most probable state of nature criterion involves selecting that alternative with the maximum payoff for the most probable state (i.e. the state for which p_j is a maximum)

- ✰ If payoffs, $v(a_i,s_j)$ represent loss (i.e. are negative), then best alternative under the most probable state of nature criterion involves selecting that alternative with the minimum payoff for the most probable state (i.e. the state for which p_j is a maximum)

Payoff Table

	State of Nature		
	P(f) = 0.6	P(u) = 0.4	
Decision Purchase	Favourable Economic Conditions (f)	Unfavourable Economic Conditions (u)	
Apartment Building	$50,000	$30,000	$50,000
Office Building	$100,000	-$40,000	$100,000*
Warehouse	$30,000	$10,000	$30,000

*Most probable state of nature

	A	B	C	D	E	F	G	H	I
1	**Payoff**							**alpha**	
2	**Matrix**							0.4	
3			**State of Nature**						
4			**Favourable**	**Unfavourable**	row	row			
5		**Probability**	0.5	0.5	maximum	minimum			
6									
7		**Apartment Building**	$50,000	$30,000	$50,000	$30,000	$40,000	$38,000	
8		**Office Building**	$100,000	-$40,000	$100,000	-$40,000	$30,000	$16,000	
9		**Warehouse**	$30,000	$10,000	$30,000	$10,000	$20,000	$18,000	
10									
11		column maximum	$100,000	$30,000	$100,000	$30,000	$40,000	$38,000	
12					**maximax**	**maximin**	**Laplace**	**Hurwicz**	
13									
14	**Regret**								
15	**Matrix**								
16			**State of Nature**						
17			**Favourable**	**Unfavourable**					
18		**Probability**	0.5	0.5					
19									
20		**Apartment Building**	$50,000	$0	$50,000				
21		**Office Building**	$0	$70,000	$70,000				
22		**Warehouse**	$70,000	$20,000	$70,000				
23									
24					$50,000				
25					**minimax**				
26					**regret**				
27									

	A	B	C	D	E	F	G	H
1	**Payoff**							**alpha**
2	**Matrix**							0.4
3			**State of Nature**					
4			**Favourable**	**Unfavourable**	row	row		
5		**Probability**	0.5	0.5	maximum	minimum		
6								
7		**Apartment Building**	50000	30000	=MAX(C7:D7)	=MIN(C7:D7)	=SUMPRODUCT($C$5:$D$5,C7:D7)	=$H$2*E7+(1-$H$2)*F7
8		**Office Building**	100000	-40000	=MAX(C8:D8)	=MIN(C8:D8)	=SUMPRODUCT($C$5:$D$5,C8:D8)	=$H$2*E8+(1-$H$2)*F8
9		**Warehouse**	30000	10000	=MAX(C9:D9)	=MIN(C9:D9)	=SUMPRODUCT($C$5:$D$5,C9:D9)	=$H$2*E9+(1-$H$2)*F9
10								
11		column maximum	=MAX(C7:C9)	=MAX(D7:D9)	=MAX(E7:E9)	=MAX(F7:F9)	=MAX(G7:G9)	=MAX(H7:H9)
12					**maximax**	**maximin**	**Laplace**	**Hurwicz**
13								
14	**Regret**							
15	**Matrix**							
16			**State of Nature**					
17			**Favourable**	**Unfavourable**				
18		**Probability**	0.5	0.5				
19								
20		**Apartment Building**	=$C$11-C7	=$D$11-D7	=MAX(C20:D20)			
21		**Office Building**	=$C$11-C8	=$D$11-D8	=MAX(C21:D21)			
22		**Warehouse**	=$C$11-C9	=$D$11-D9	=MAX(C22:D22)			
23								
24					=MIN(E20:E22)			
25					**minimax**			
26					**regret**			
27								

	A	B	C	D	E
1	**Payoff**				
2	**Matrix**				
3			**State of Nature**		
4			**Favourable**	**Unfavourable**	
5		**Probability**	0.6	0.4	
6					
7		**Apartment Building**	$50,000	$30,000	$42,000
8		**Office Building**	$100,000	-$40,000	$44,000
9		**Warehouse**	$30,000	$10,000	$22,000
10					
11					$44,000
12					**expected**
13					**value**
14					
15		column maximum	$100,000	$30,000	$72,000
16					
17					$28,000
18					**EVPI**

	A	B	C	D	E
1	**Payoff**				
2	**Matrix**				
3			**State of Nature**		
4			**Favourable**	**Unfavourable**	
5		**Probability**	0.6	0.4	
6					
7		**Apartment Building**	50000	30000	=SUMPRODUCT($C$5:$D$5,C7:D7)
8		**Office Building**	100000	-40000	=SUMPRODUCT($C$5:$D$5,C8:D8)
9		**Warehouse**	30000	10000	=SUMPRODUCT($C$5:$D$5,C9:D9)
10					
11					=MAX(E7:E9)
12					**expected**
13					**value**
14					
15		column maximum	=MAX(C7:C9)	=MAX(D7:D9)	=SUMPRODUCT($C$5:$D$5,C15:D15)
16					
17					=E15-E11
18					**EVPI**