Lesson 5: Producer Behavior

1. **Technology**
2. **Costs**
3. **Costs minimization**

Technologies

- A technology is a process by which inputs are converted to an output.
- *E.g.* labor, a computer, a projector, electricity, and software are being combined to produce this lecture.

Technologies

- Usually several technologies will produce the same product -- a blackboard and chalk can be used instead of a computer and a projector.
- Which technology is “best”?
- How do we compare technologies?

Input Bundles

- x_i denotes the amount used of input i; *i.e.* the level of input i.
- An **input bundle** is a vector of the input levels; $(x_1, x_2, \ldots, x_n)$.
- *E.g.* $(x_1, x_2, x_3) = (6, 0, 9{\cdot}3)$.

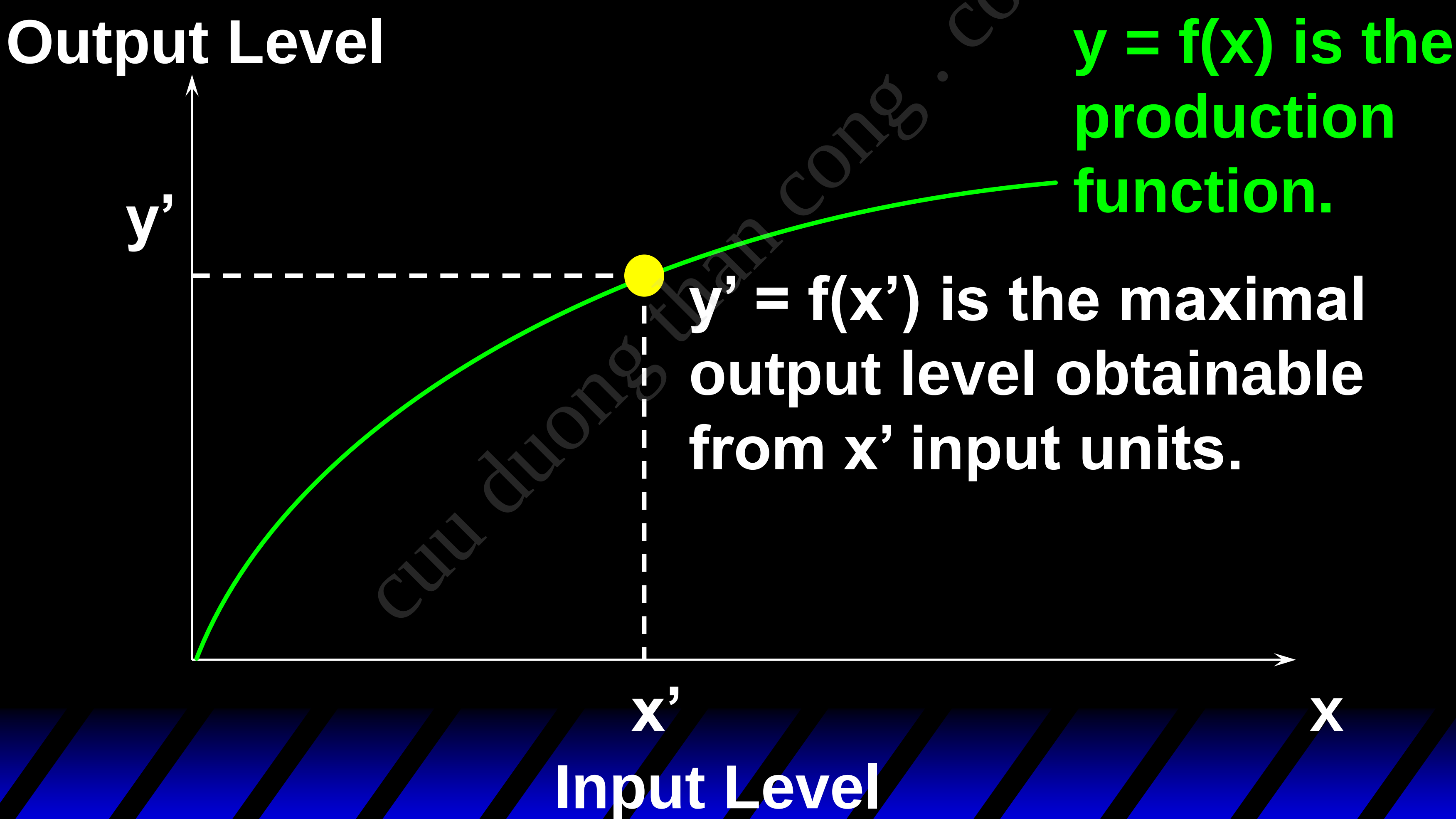
Production Functions

- y denotes the output level.
- The technology's **production function** states the **maximum** amount of output possible from an input bundle.

$$y = f(x_1, \ldots, x_n)$$

Production Functions

One input, one output



Technology Sets

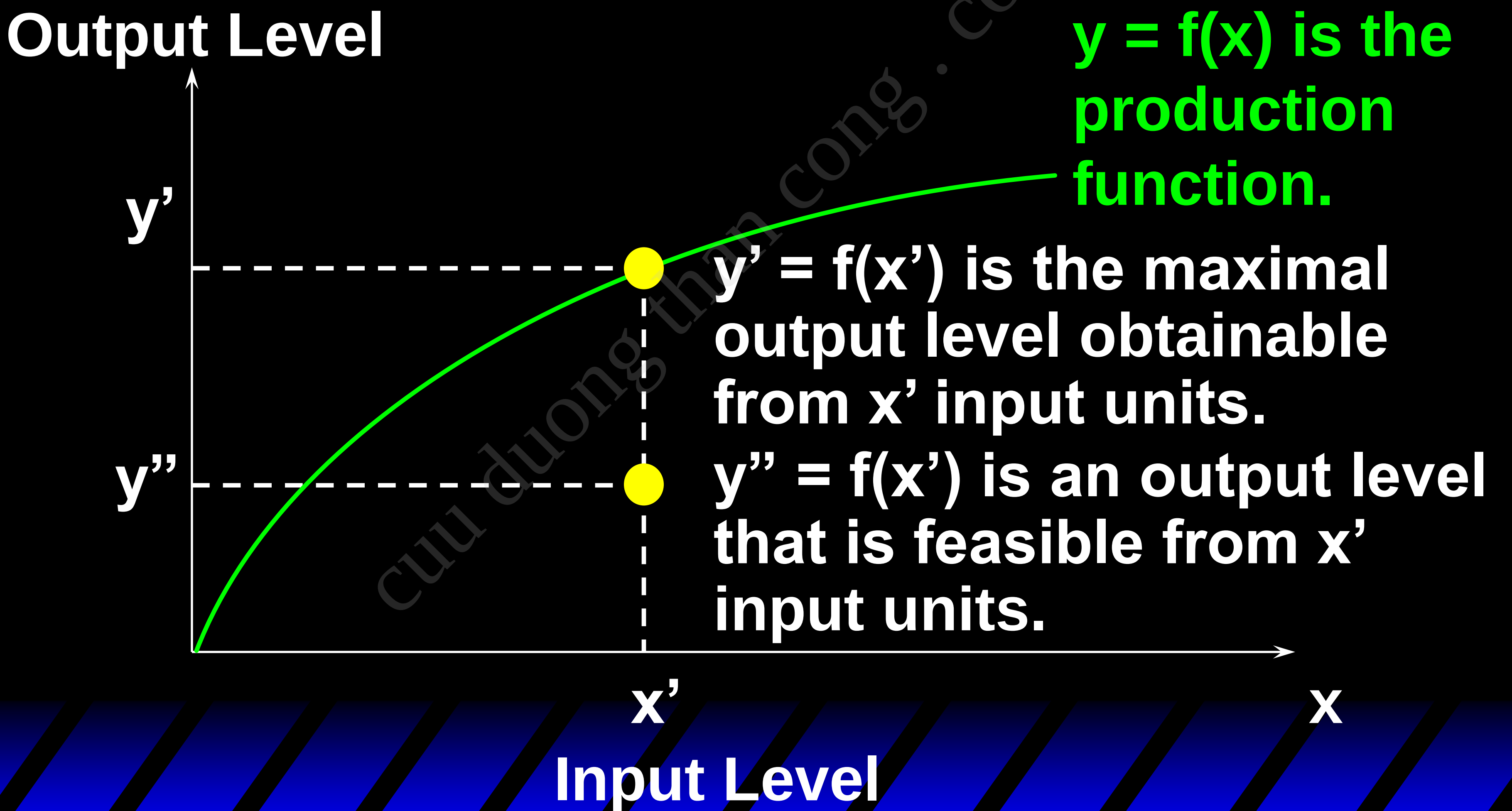
- A **production plan** is an input bundle and an output level; $(x_1, \dots, x_n, y)$.
- A production plan is **feasible** if

$$y \le f(x_1, \Lambda, x_n)$$

- The collection of all feasible production plans is the **technology set**.

Technology Sets

One input, one output



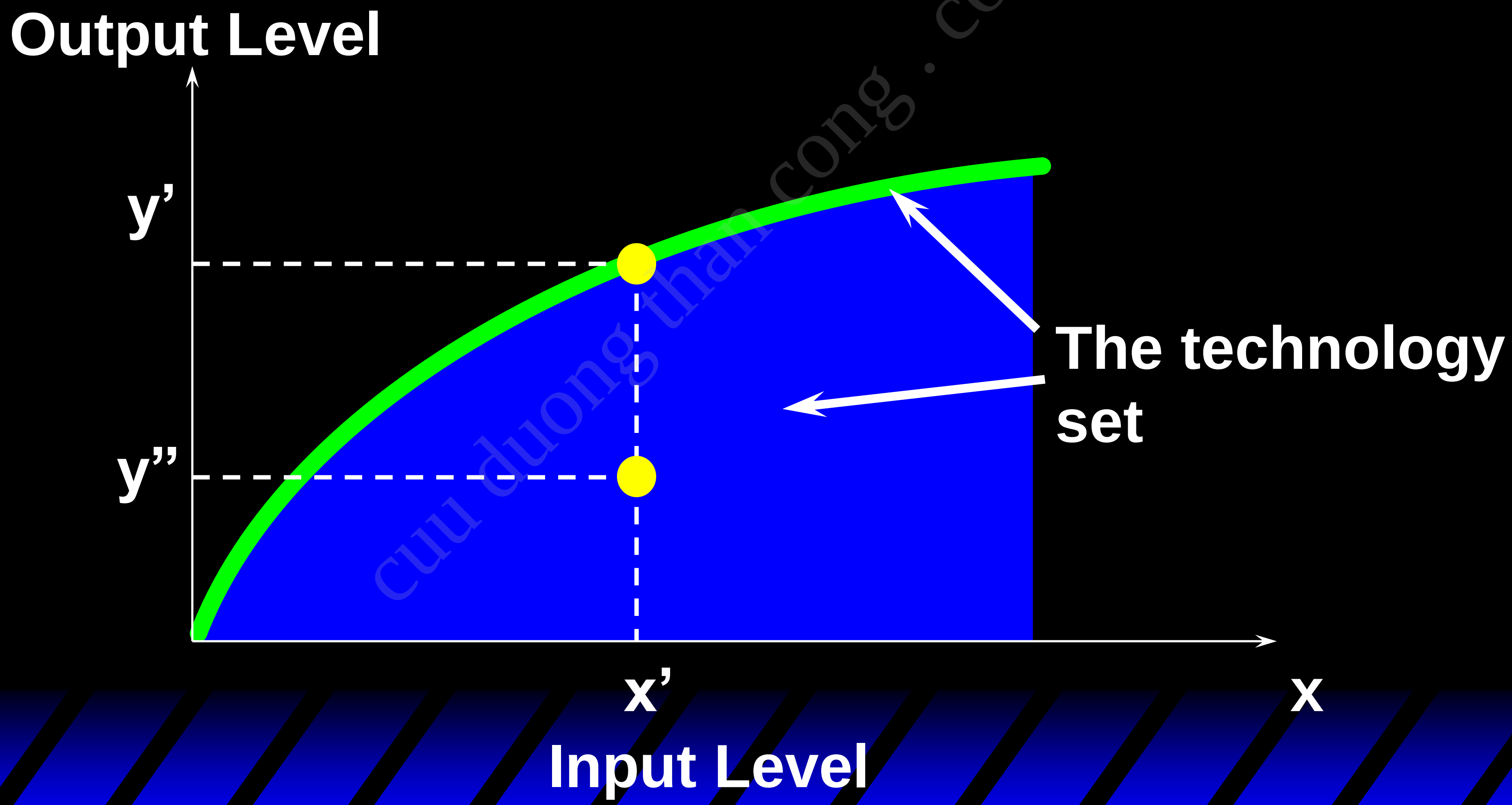
Technology Sets

The **technology set** is

$$T = \{(x_1,\ldots,x_n,y) \mid y \le f(x_1,\ldots,x_n) \text{ and } x_1 \ge 0,\ldots,x_n \ge 0\}.$$

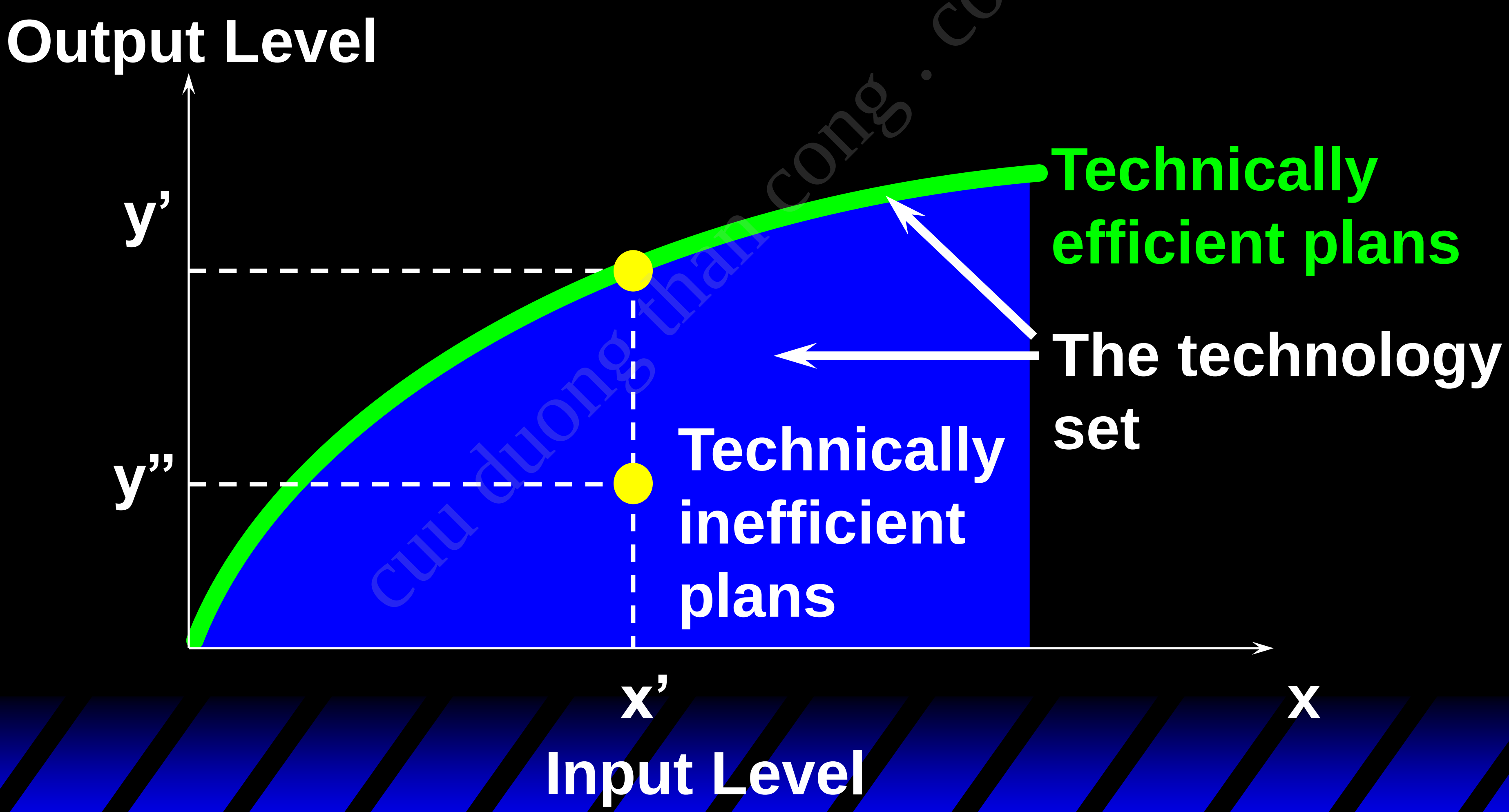
Technology Sets

One input, one output



Technology Sets

One input, one output



Technologies with Multiple Inputs

- What does a technology look like when there is more than one input?
- The two input case: Input levels are x_1 and x_2. Output level is y.
- Suppose the production function is

$$y = f(x_1, x_2) = 2x_1^{1/3}x_2^{1/3}.$$

Technologies with Multiple Inputs

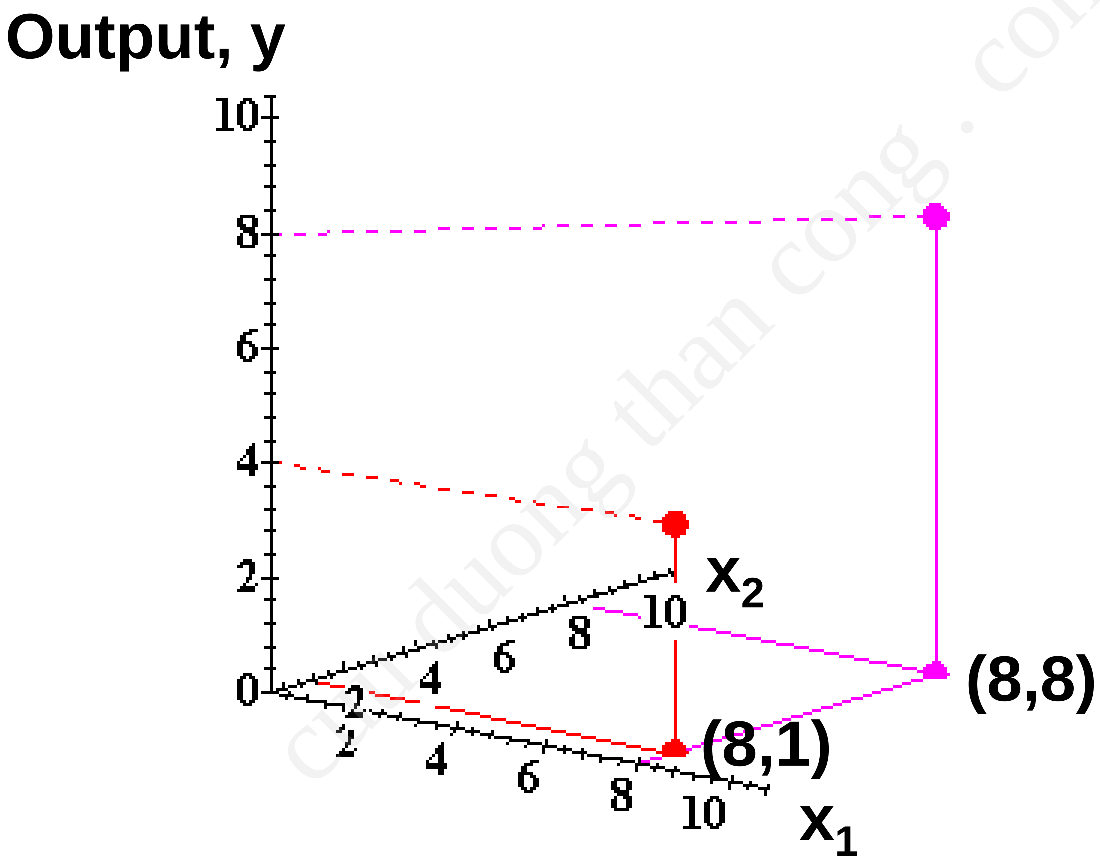
- *E.g.* the maximal output level possible from the input bundle $(x_1, x_2) = (1, 8)$ is

$$y = 2x_1^{1/3}x_2^{1/3} = 2 \times 1^{1/3} \times 8^{1/3} = 2 \times 1 \times 2 = 4.$$

- And the maximal output level possible from $(x_1,x_2) = (8,8)$ is

$$y = 2x_1^{1/3}x_2^{1/3} = 2 \times 8^{1/3} \times 8^{1/3} = 2 \times 2 \times 2 = 8.$$

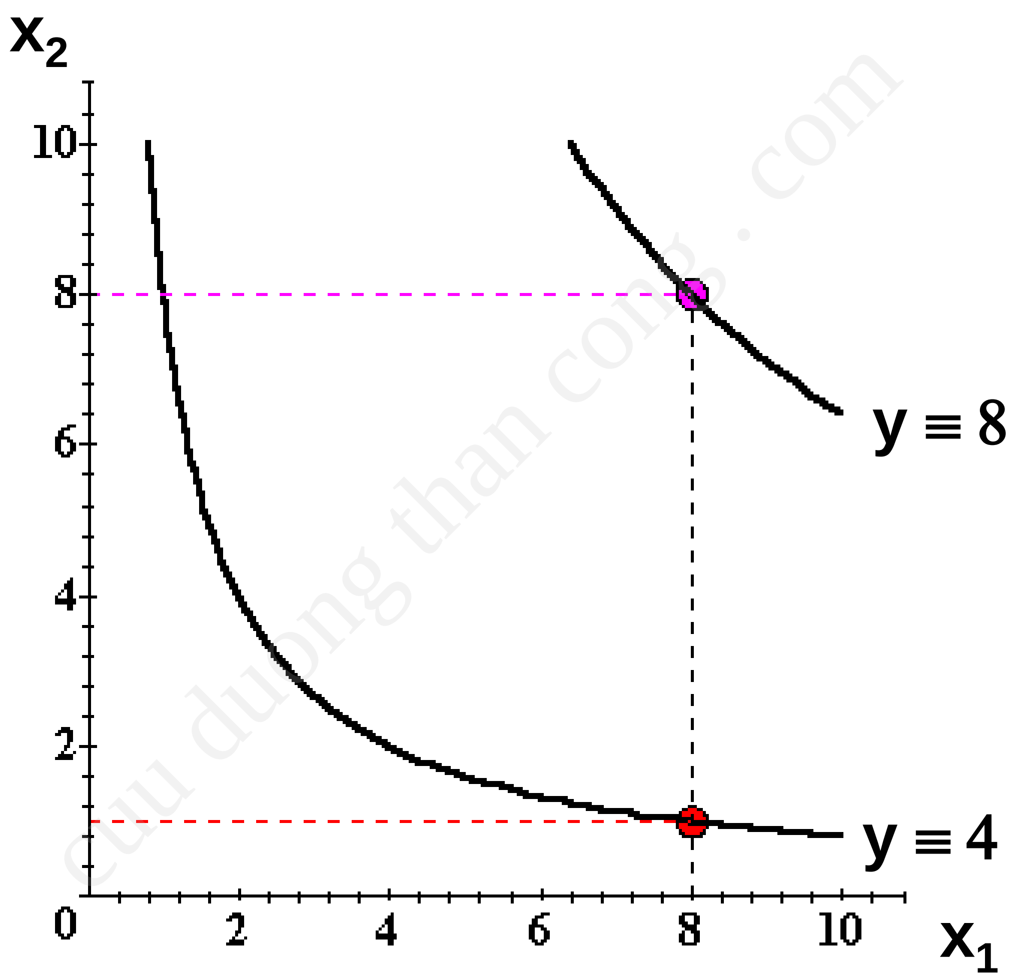
Technologies with Multiple Inputs



Technologies with Multiple Inputs

- The y output unit **isoquant** is the set of all input bundles that yield at most the same output level y.

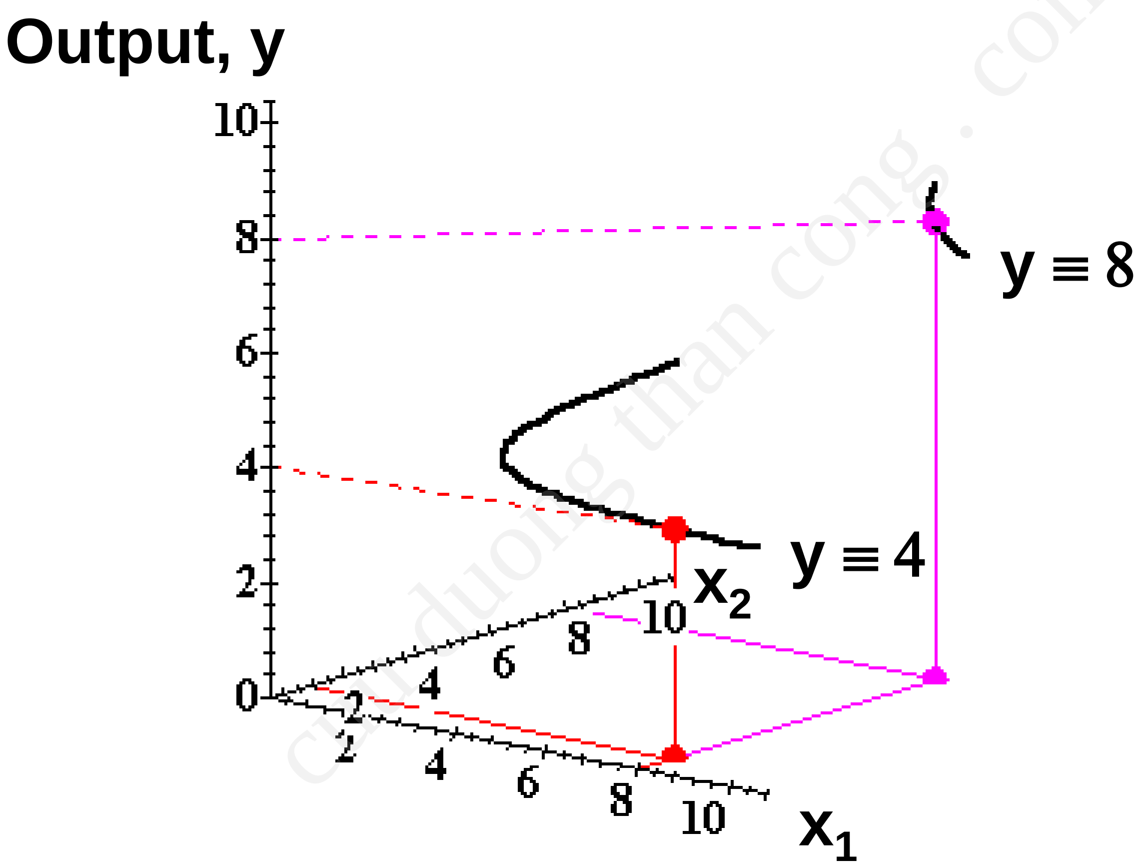
Isoquants with Two Variable Inputs



Isoquants with Two Variable Inputs

- **Isoquants can be graphed by adding an output level axis and displaying each isoquant at the height of the isoquant's output level.**

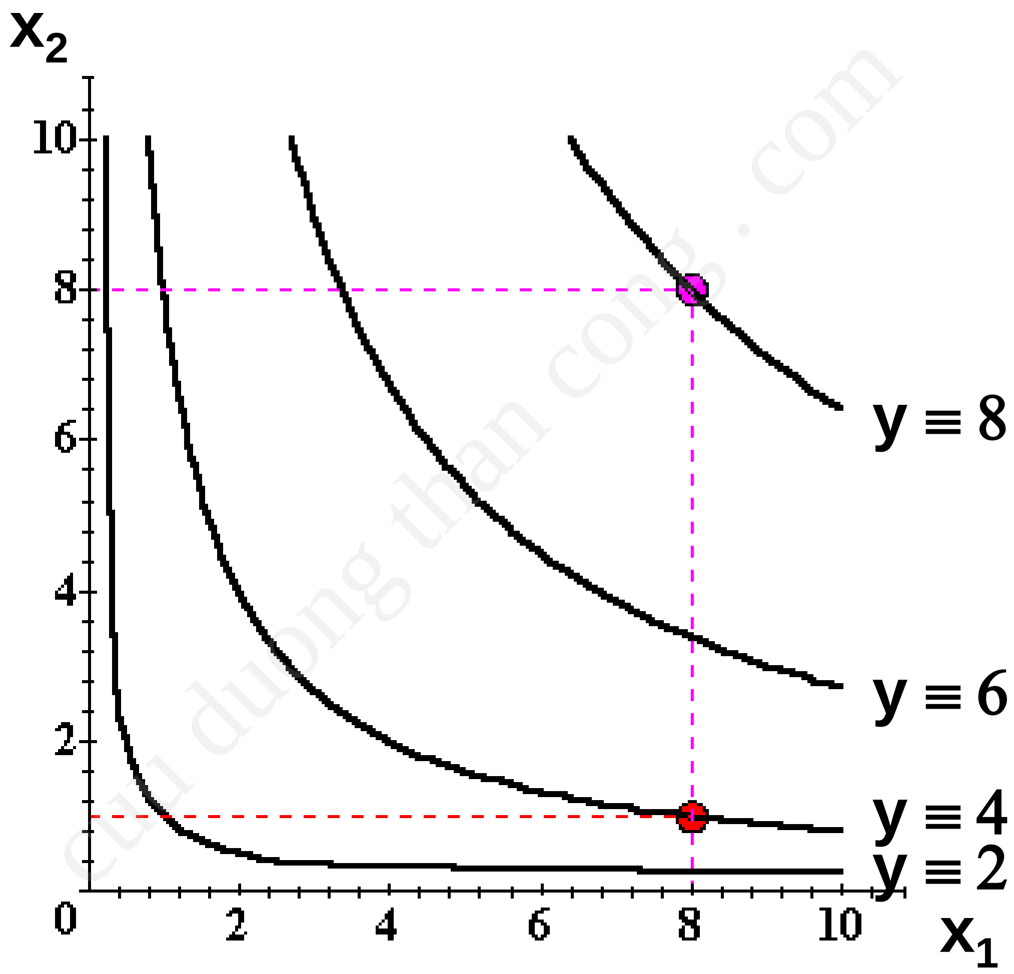
Isoquants with Two Variable Inputs



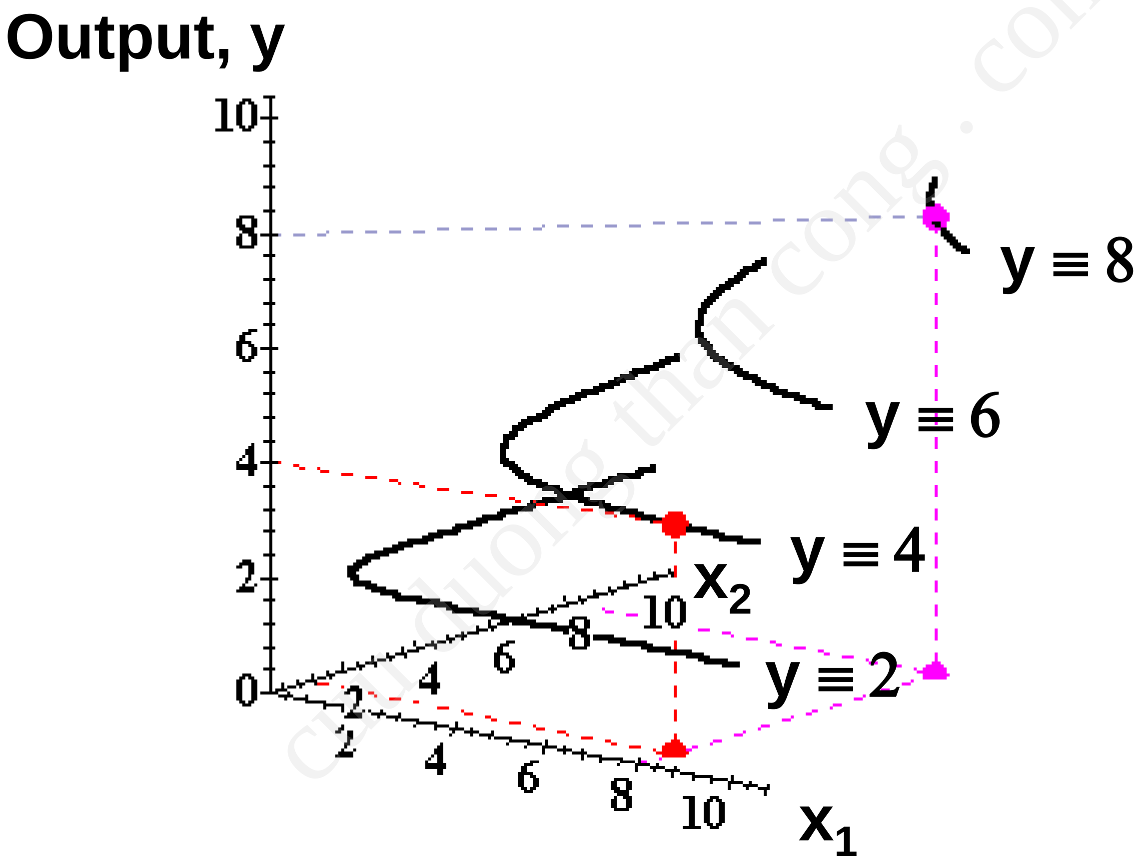
Isoquants with Two Variable Inputs

- More isoquants tell us more about the technology.

Isoquants with Two Variable Inputs



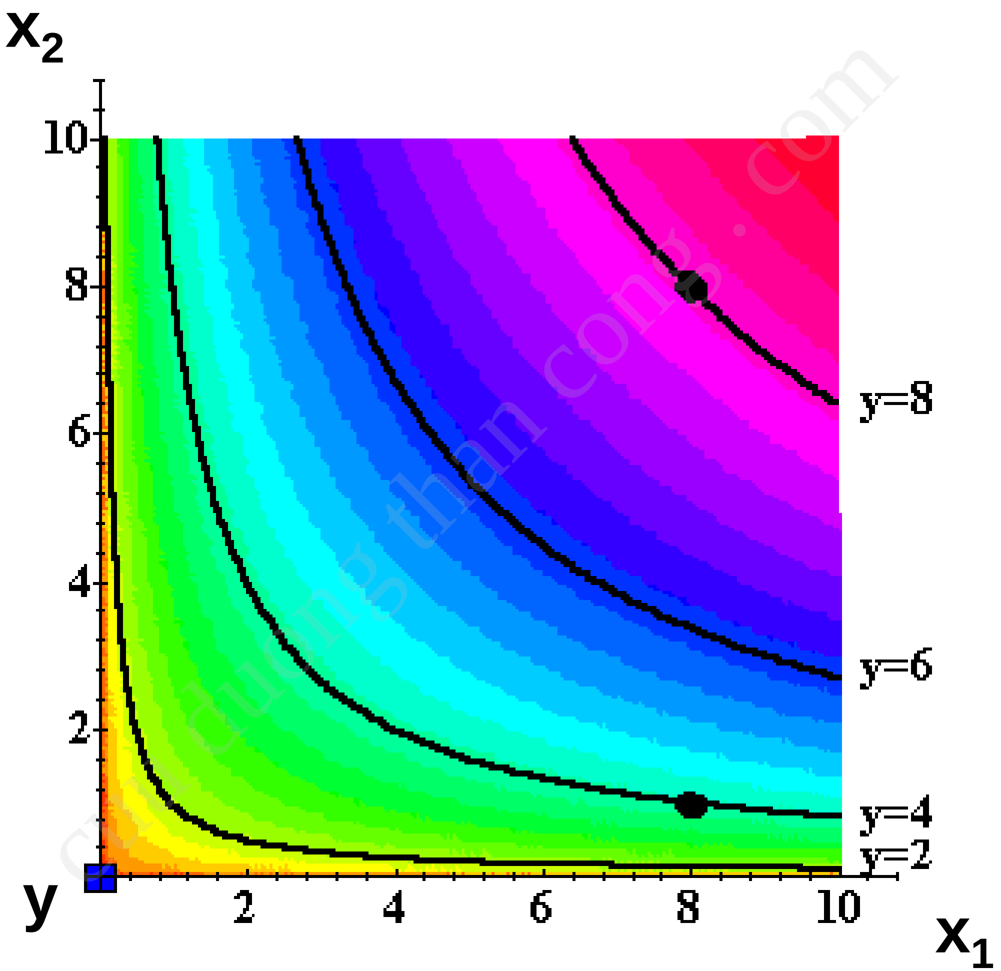
Isoquants with Two Variable Inputs



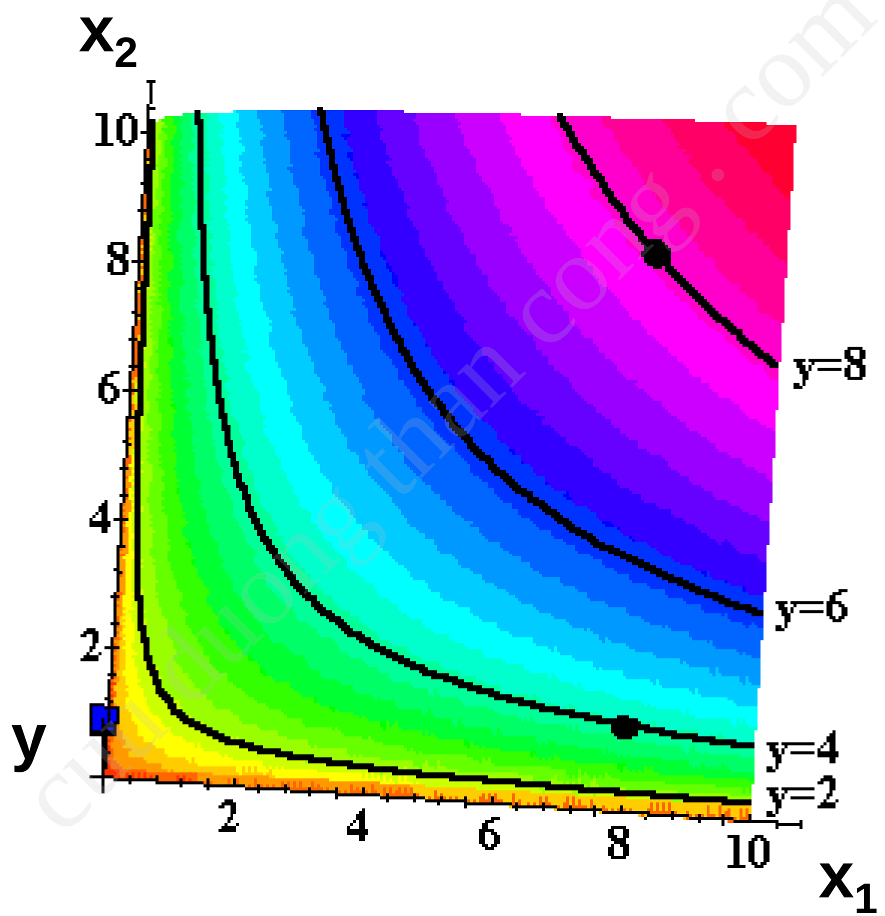
Technologies with Multiple Inputs

- The complete collection of isoquants is the **isoquant map.**
- **The isoquant map is equivalent to the production function** -- each is the other.
- E.g. $y = f(x_1, x_2) = 2x_1^{1/3}x_2^{1/3}$

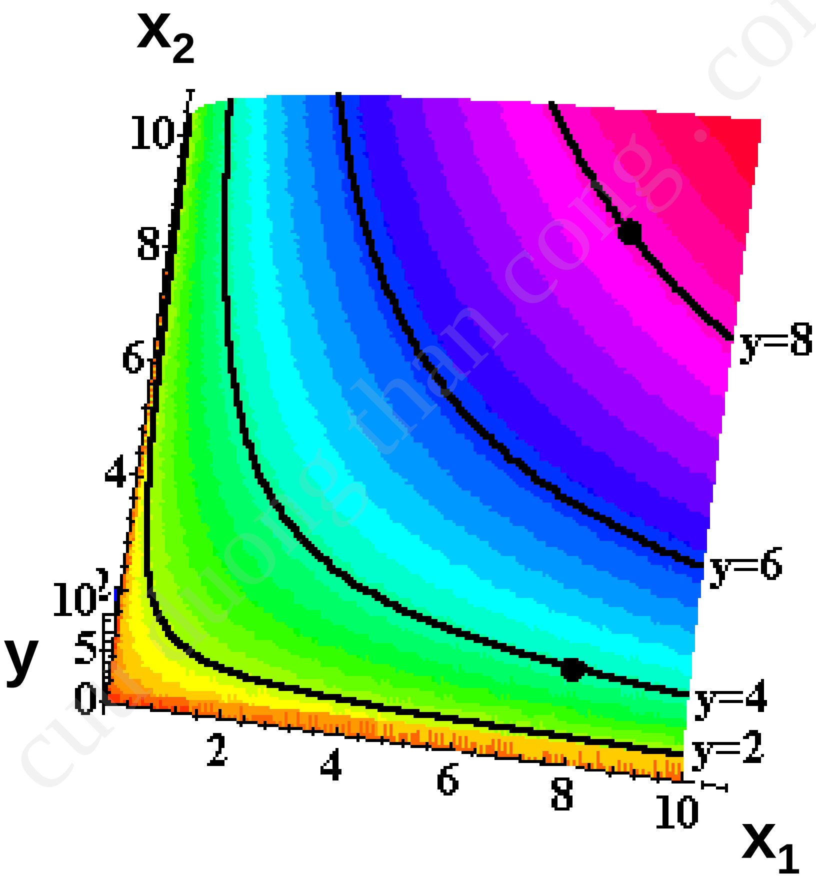
Technologies with Multiple Inputs



Technologies with Multiple Inputs



Technologies with Multiple Inputs



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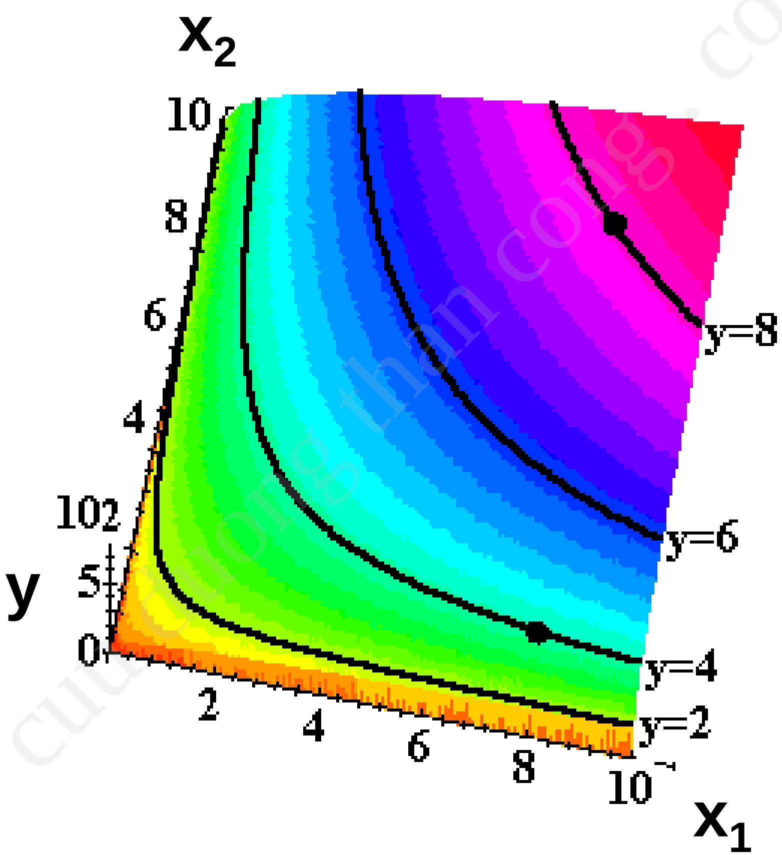

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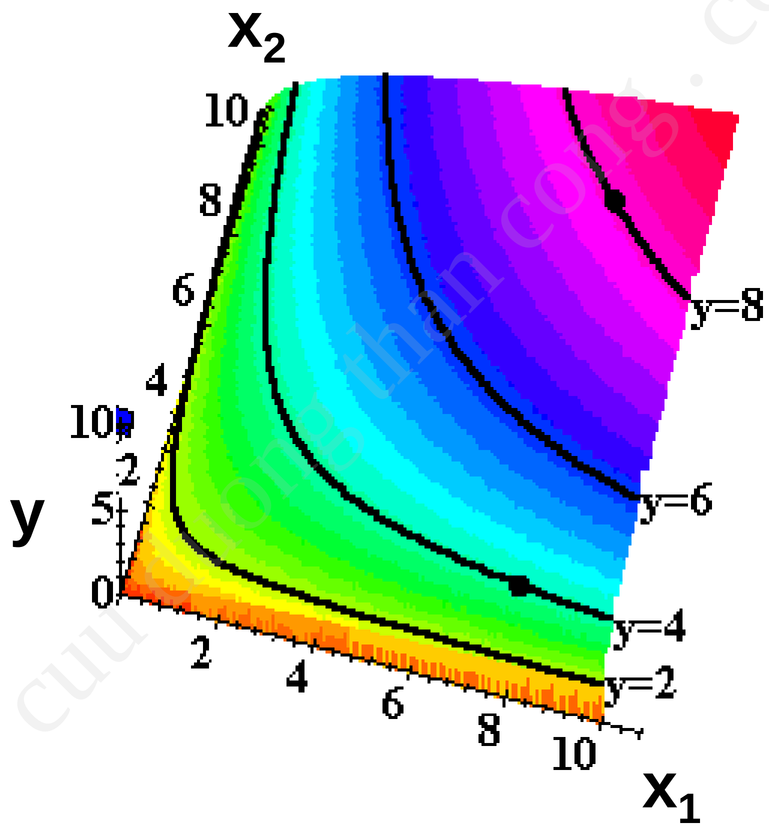
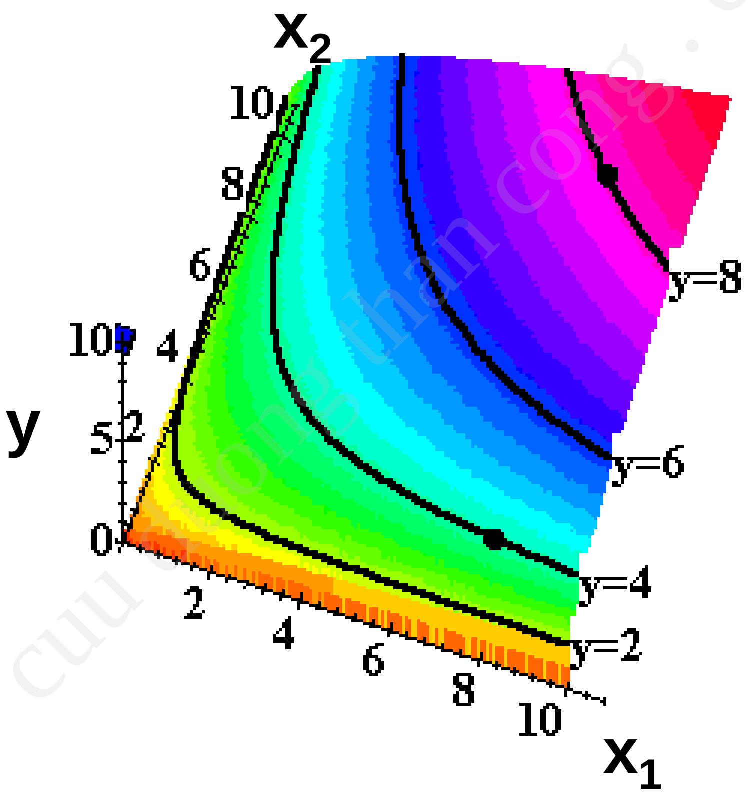
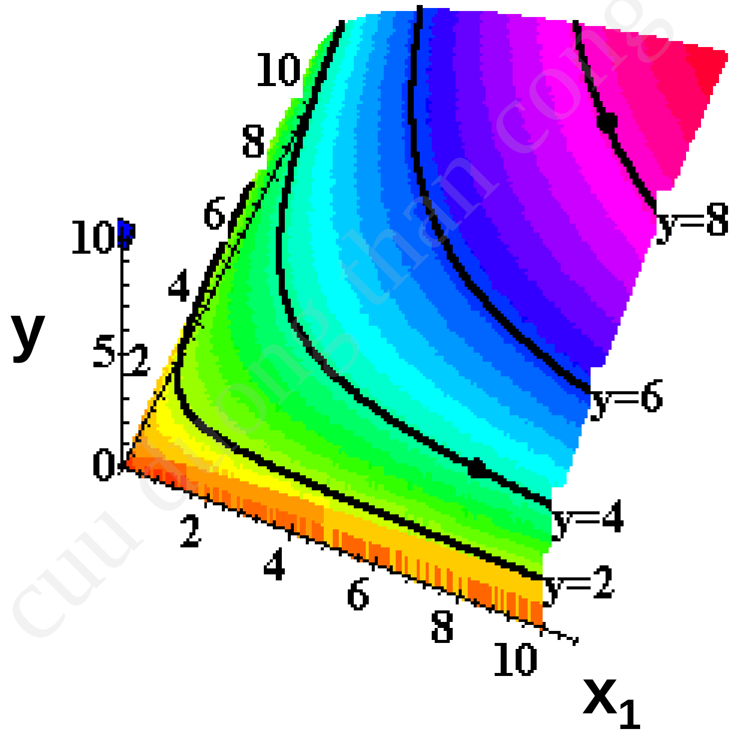


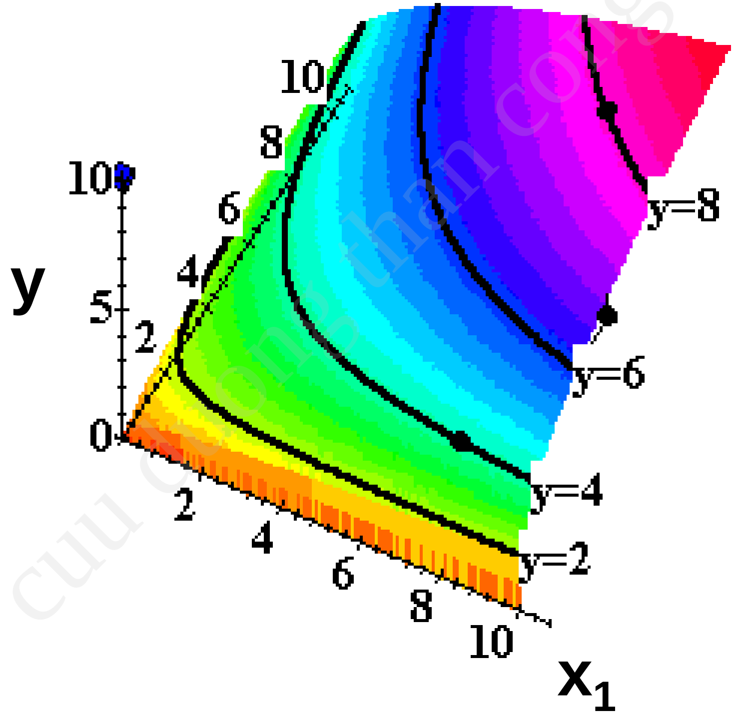
Technologies with Multiple Inputs



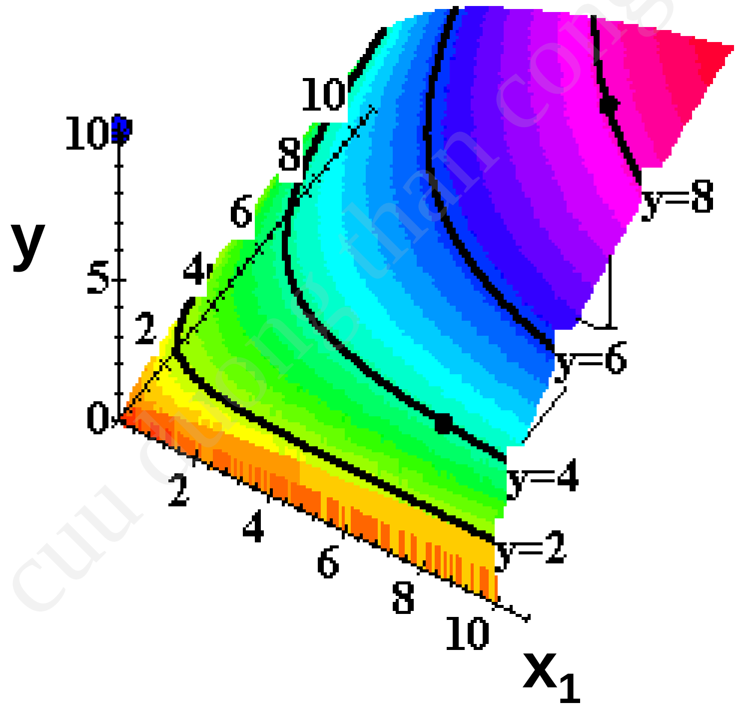
Technologies with Multiple Inputs



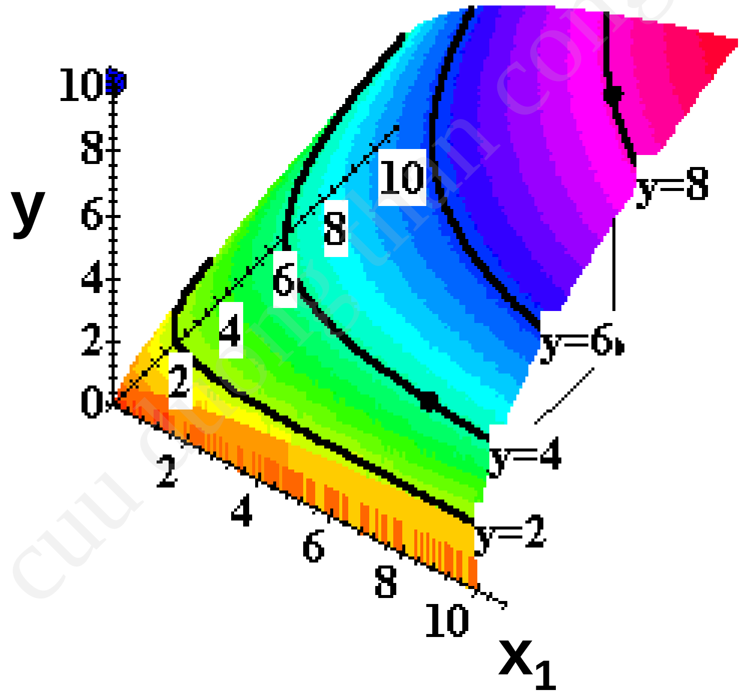
Technologies with Multiple Inputs



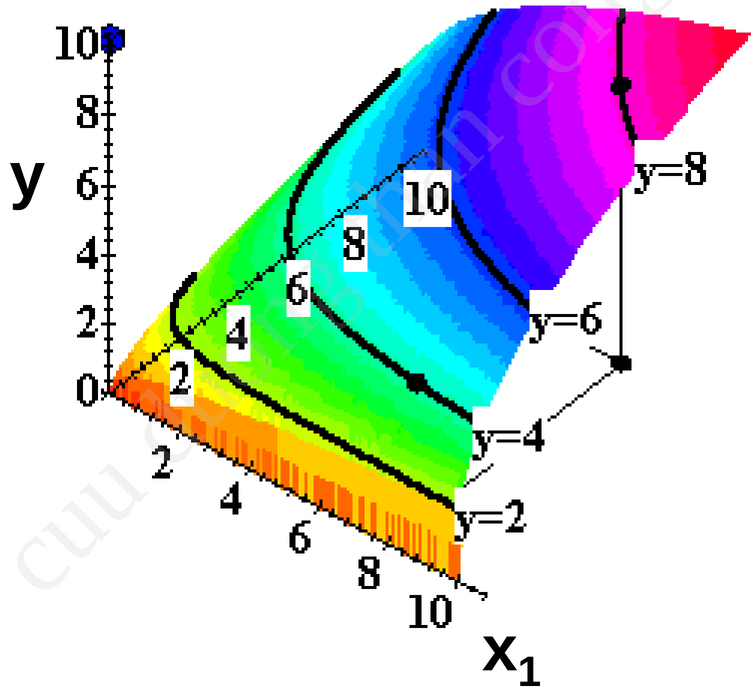
Technologies with Multiple Inputs



Technologies with Multiple Inputs



Technologies with Multiple Inputs



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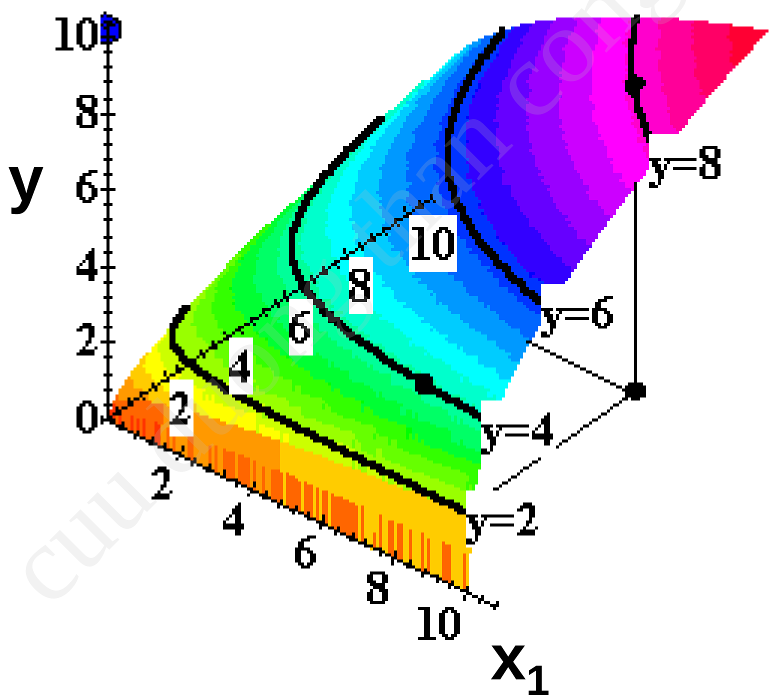

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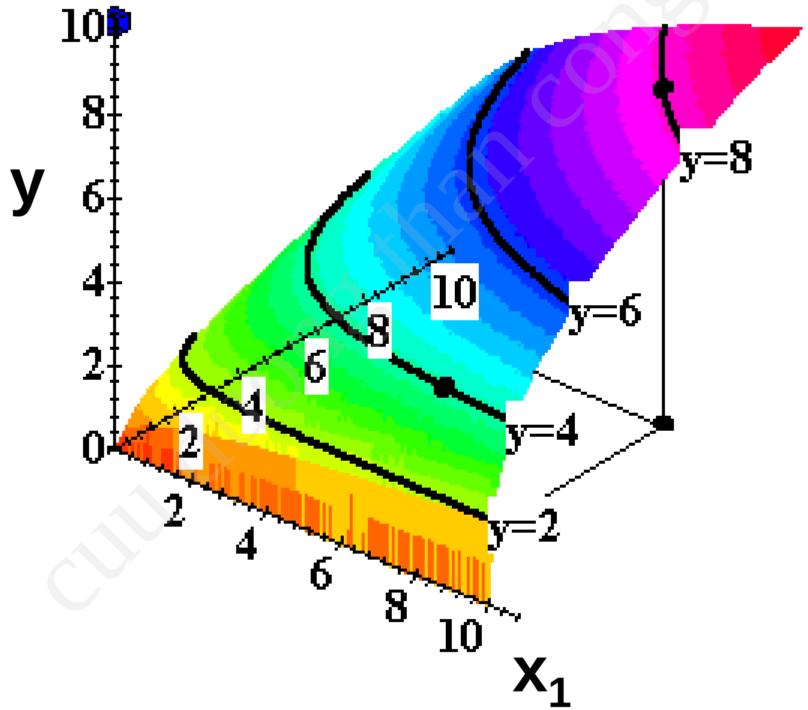

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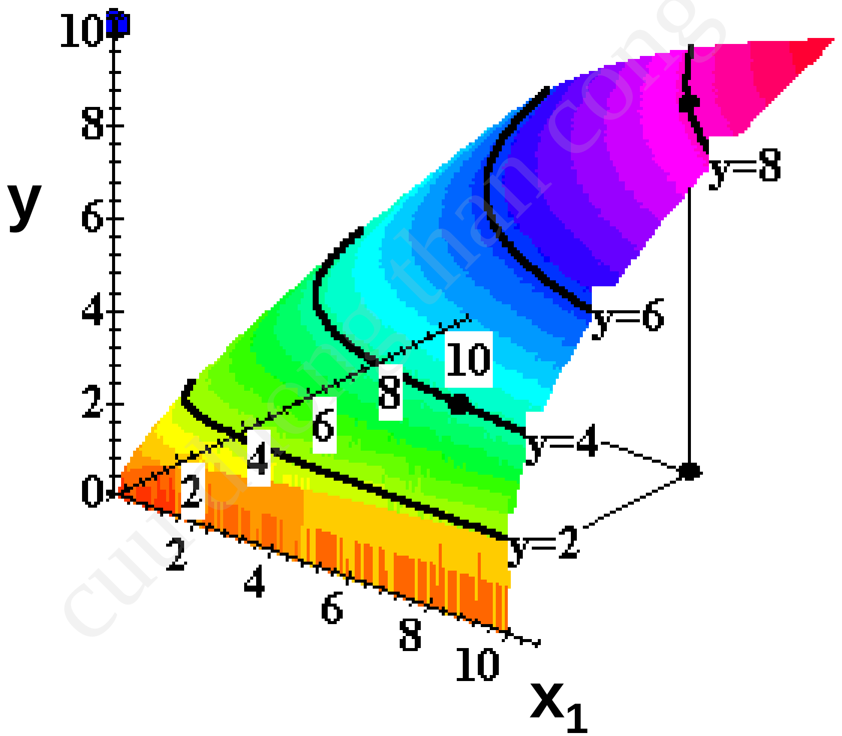

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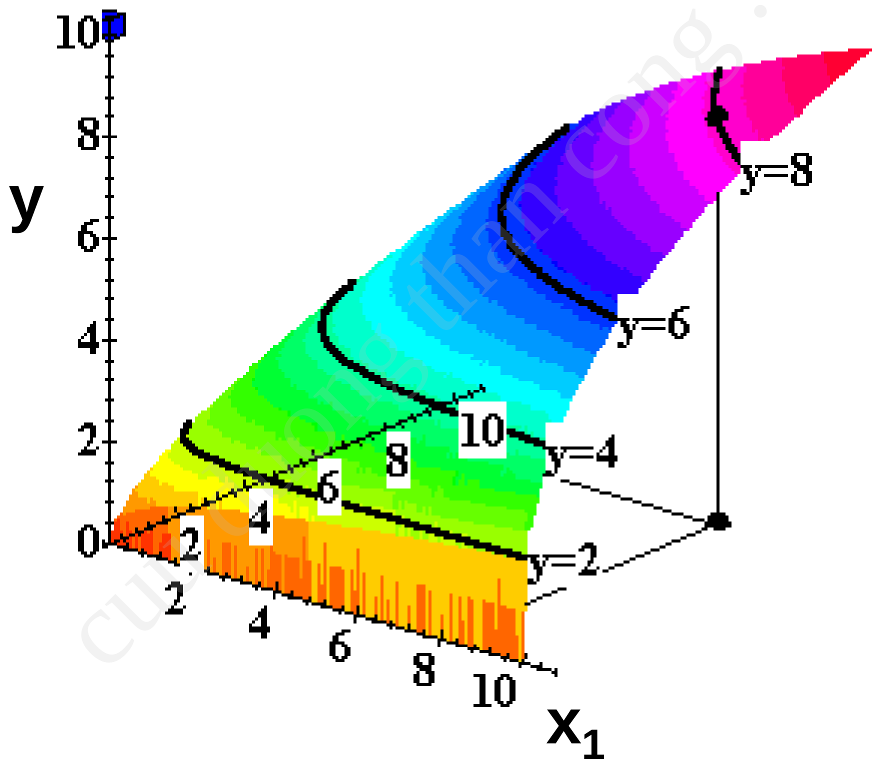

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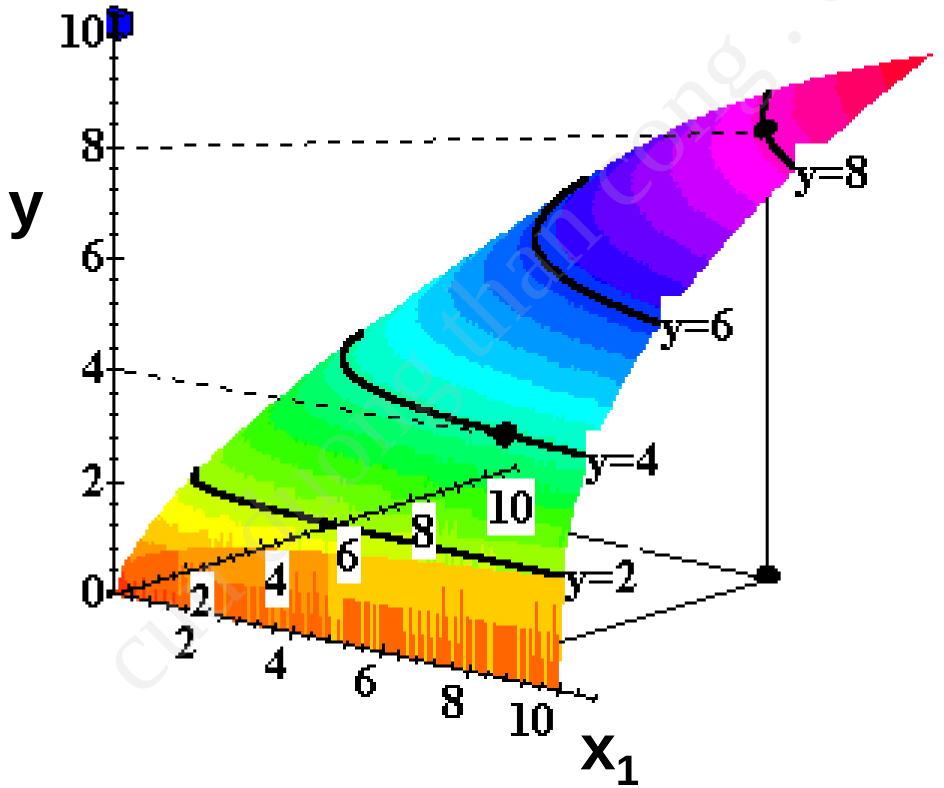
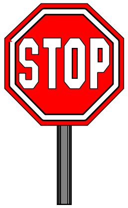




Cobb-Douglas Technologies

- A Cobb-Douglas production function is of the form

$$y = A x_1^{a_1} x_2^{a_2} \times \Lambda \times x_n^{a_n}.$$

- E.g.

$$y = x_1^{1/3} x_2^{1/3}$$

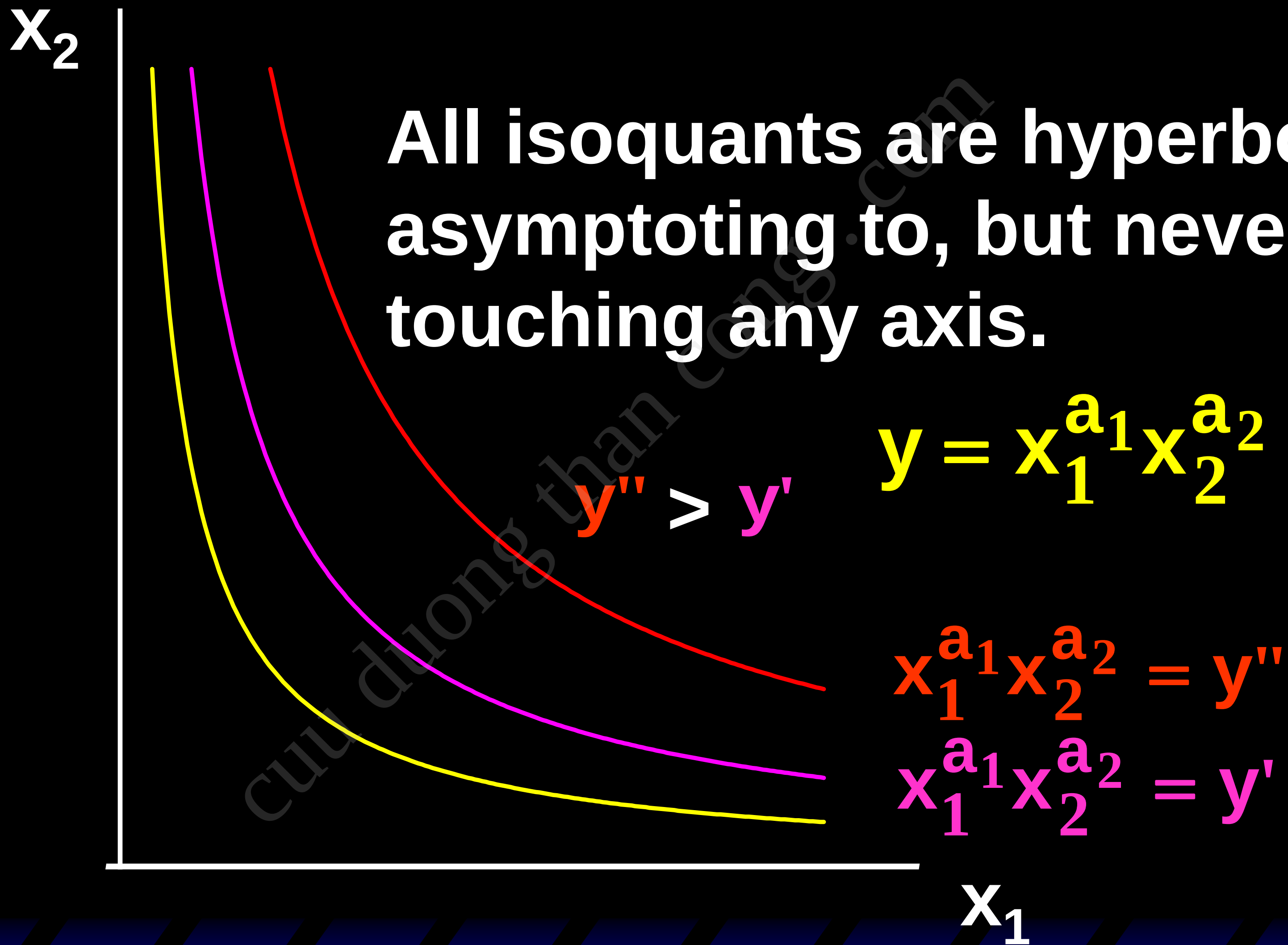
with

$$n = 2, A = 1, a_1 = \frac{1}{3} \text{ and } a_2 = \frac{1}{3}.$$

Cobb-Douglas Technologies

All isoquants are hyperbolic, asymptoting to, but never touching any axis.

$$y = x_1^{a_1} x_2^{a_2}$$



Fixed-Proportions Technologies

- A fixed-proportions production function is of the form

$$y = \min\{a_1 x_1, a_2 x_2, \Lambda, a_n x_n\}.$$

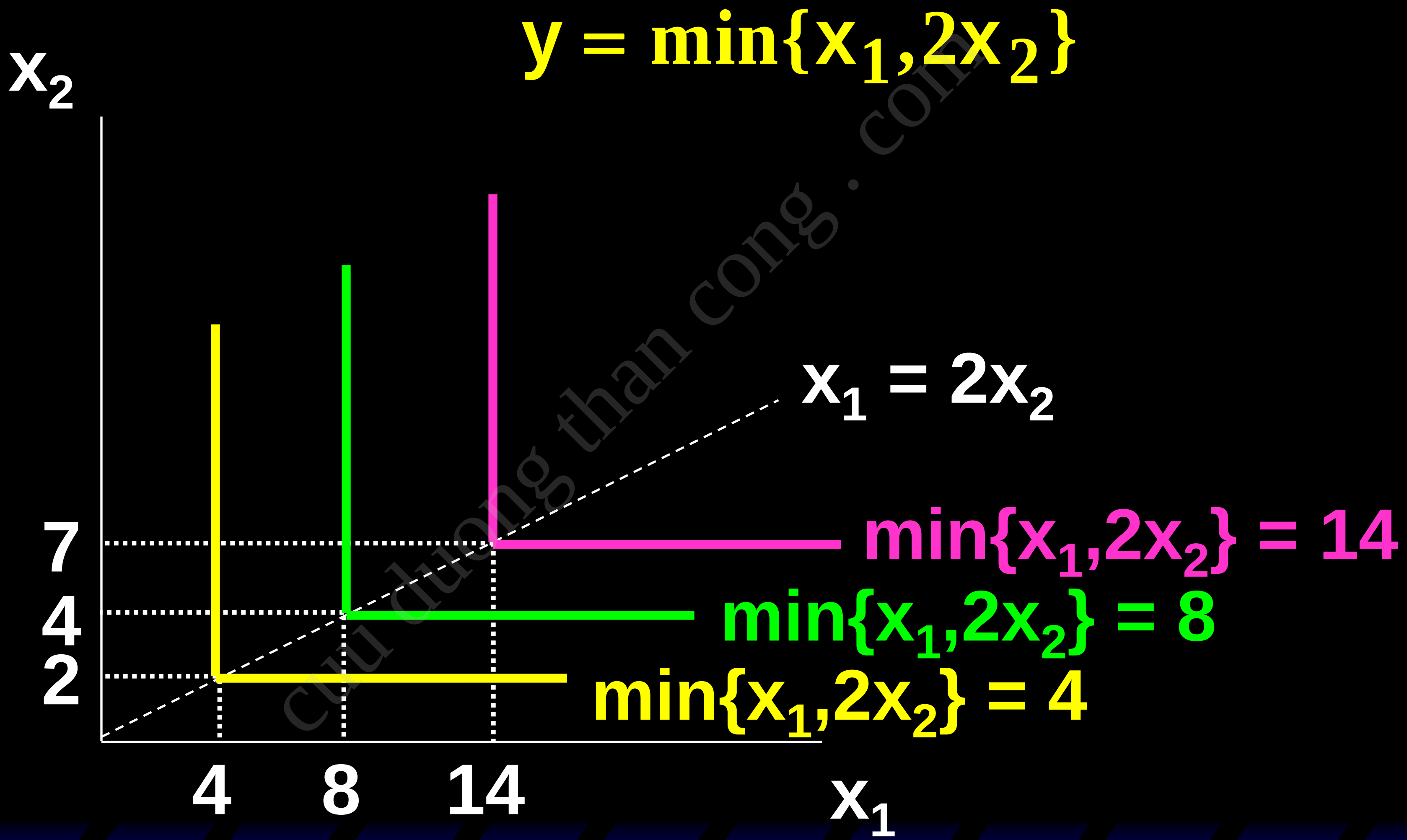
- E.g.

$$y = \min\{x_1, 2x_2\}$$

with

$$n = 2, a_1 = 1 \text{ and } a_2 = 2.$$

Fixed-Proportions Technologies



Perfect-Substitutes Technologies

- **A perfect-substitutes production function is of the form**

$$y = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n .$$

- **E.g.**

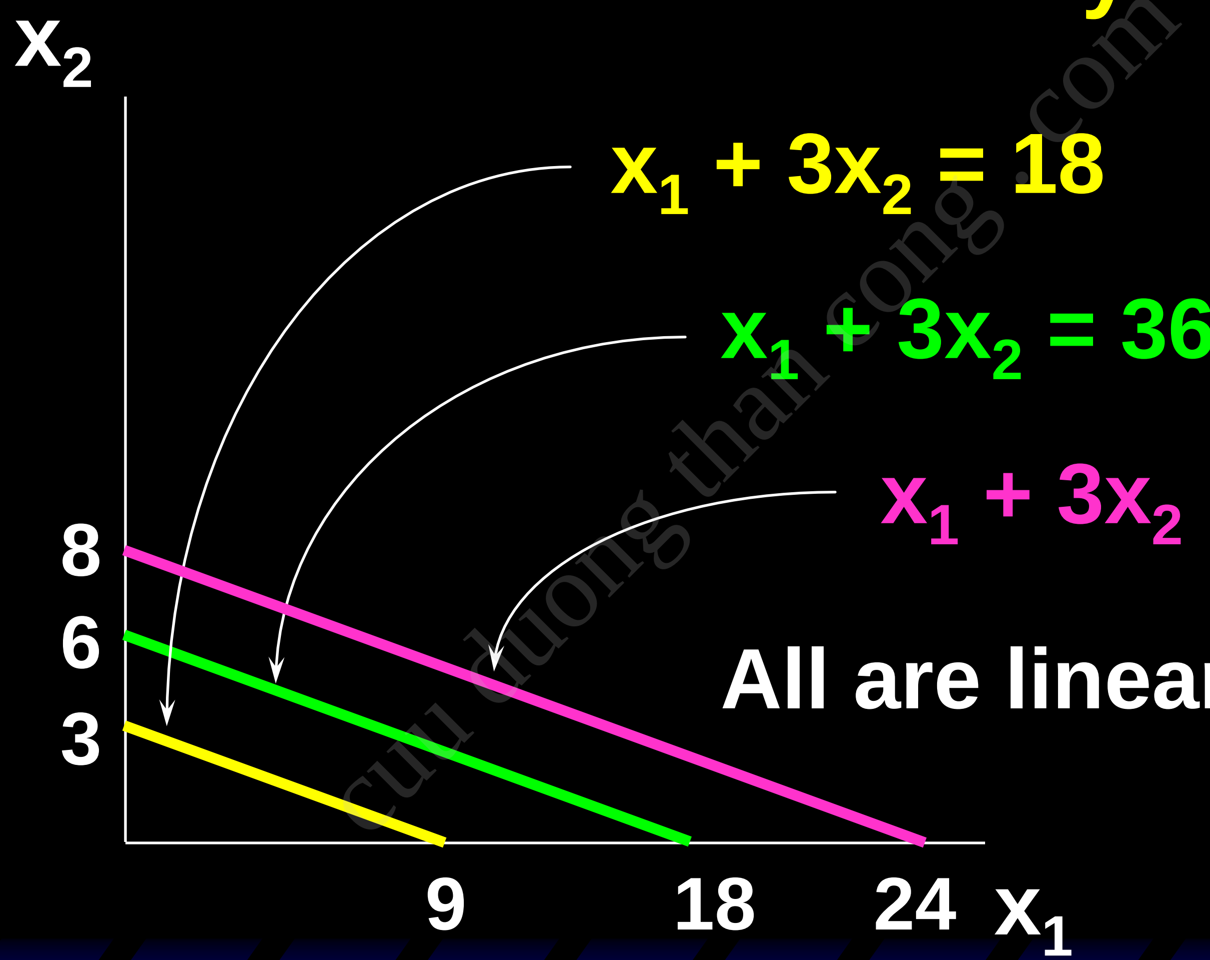
$$y = x_1 + 3x_2$$

with

$$n = 2, a_1 = 1 \text{ and } a_2 = 3.$$

Perfect-Substitution Technologies

$y = x_1 + 3x_2$



All are linear and parallel

Marginal (Physical) Products

$$y = f(x_1, \ldots, x_n)$$

- The **marginal product of input i is the rate-of-change of the output level as the level of input i changes, holding all other input levels fixed.**
- That is,

$$MP_i = \frac{\partial y}{\partial x_i}$$

Marginal (Physical) Products

E.g. if

$$y = f(x_1, x_2) = x_1^{1/3} x_2^{2/3}$$

then the marginal product of input 1 is

$$MP_1 = \frac{\partial y}{\partial x_1} = \frac{1}{3} x_1^{-2/3} x_2^{2/3}$$

and the marginal product of input 2 is

$$MP_2 = \frac{\partial y}{\partial x_2} = \frac{2}{3} x_1^{1/3} x_2^{-1/3}.$$

Marginal (Physical) Products

Typically the marginal product of one input depends upon the amount used of other inputs. E.g. if

$MP_1 = \frac{1}{3} x_1^{-2/3} x_2^{2/3}$ then,

if $x_2 = 8$, $MP_1 = \frac{1}{3} x_1^{-2/3} 8^{2/3} = \frac{4}{3} x_1^{-2/3}$

and if $x_2 = 27$ then

$$MP_1 = \frac{1}{3} x_1^{-2/3} 27^{2/3} = 3 x_1^{-2/3}.$$

Marginal (Physical) Products

- The marginal product of input i is **diminishing** if it becomes smaller as the level of input i increases. That is, if

$$\frac{\partial MP_i}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\partial y}{\partial x_i}\right) = \frac{\partial^2 y}{\partial x_i^2} < 0.$$

Marginal (Physical) Products

E.g. if $y = x_1^{1/3} x_2^{2/3}$ then

$MP_1 = \frac{1}{3} x_1^{-2/3} x_2^{2/3}$ and $MP_2 = \frac{2}{3} x_1^{1/3} x_2^{-1/3}$

so

$$\frac{\partial MP_1}{\partial x_1} = -\frac{2}{9} x_1^{-5/3} x_2^{2/3} < 0$$

and

$$\frac{\partial MP_2}{\partial x_2} = -\frac{2}{9} x_1^{1/3} x_2^{-4/3} < 0.$$

Both marginal products are diminishing.

Returns-to-Scale

- Marginal products describe the change in output level as a **single** input level changes.
- **Returns-to-scale** describes how the output level changes as **all** input levels change in **direct proportion** (*e.g.* all input levels doubled, or halved).

Returns-to-Scale

If, for any input bundle ($x_1,\ldots,x_n$),

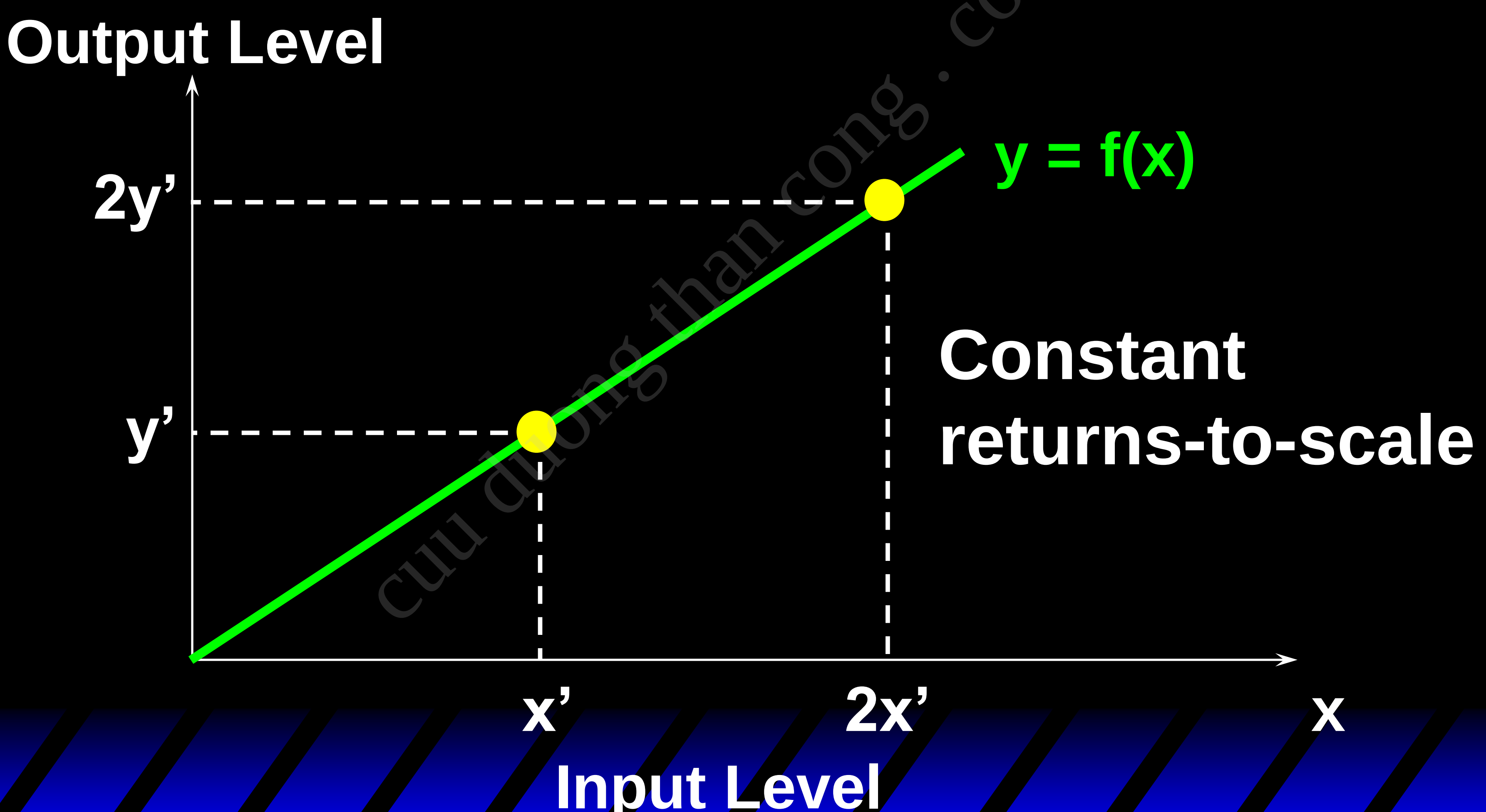
$$f(kx_1, kx_2, \Lambda, kx_n) = kf(x_1, x_2, \Lambda, x_n)$$

then the technology described by the production function f exhibits constant returns-to-scale.

***E.g.* (k = 2) doubling all input levels doubles the output level.**

Returns-to-Scale

One input, one output



Returns-to-Scale

If, for any input bundle $(x_1,\ldots,x_n)$,

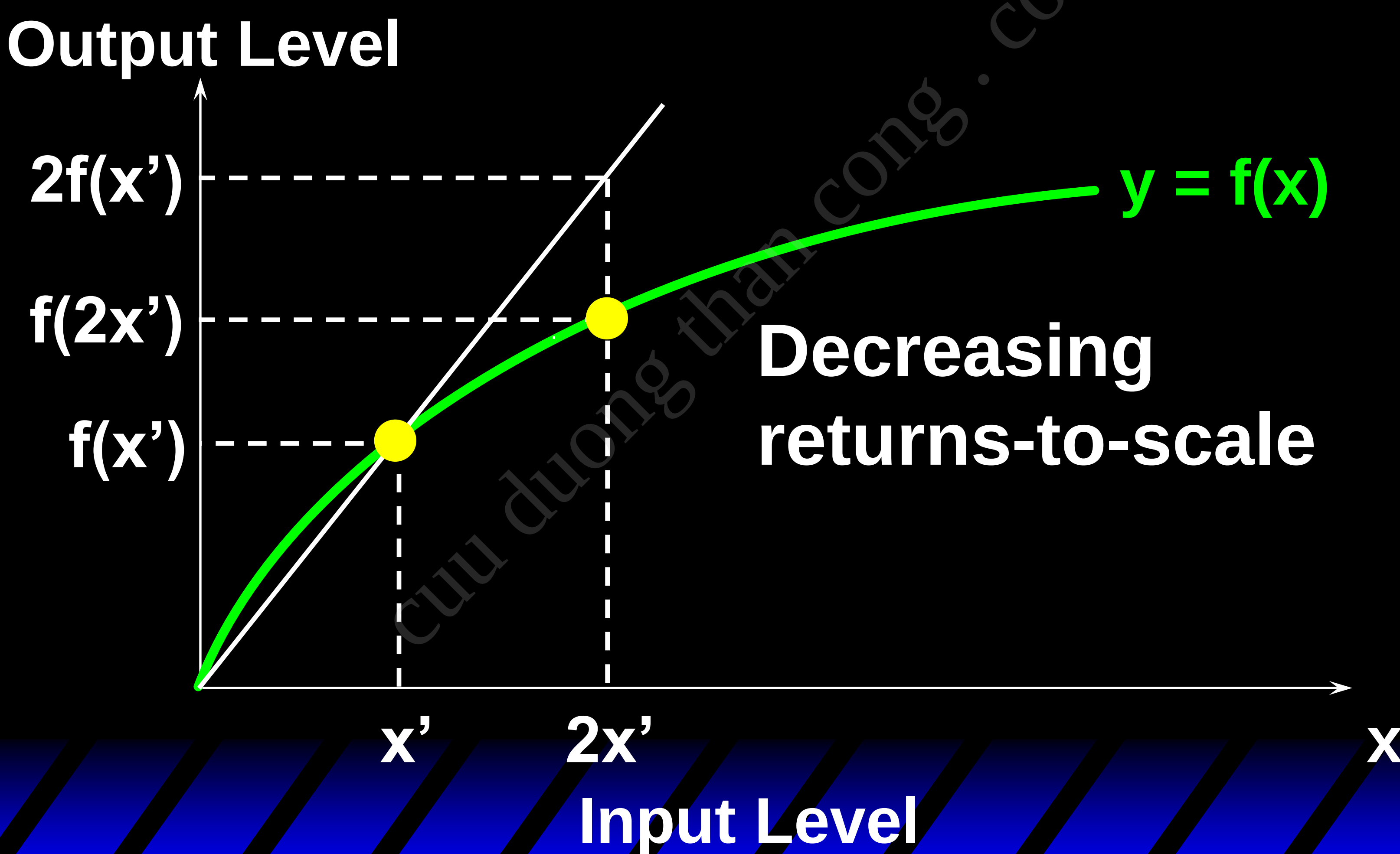
$$f(kx_1, kx_2, \Lambda, kx_n) < kf(x_1, x_2, \Lambda, x_n)$$

then the technology exhibits **diminishing returns-to-scale**.

E.g. (k = 2) doubling all input levels less than doubles the output level.

Returns-to-Scale

One input, one output



Returns-to-Scale

If, for any input bundle $(x_1,\ldots,x_n)$,

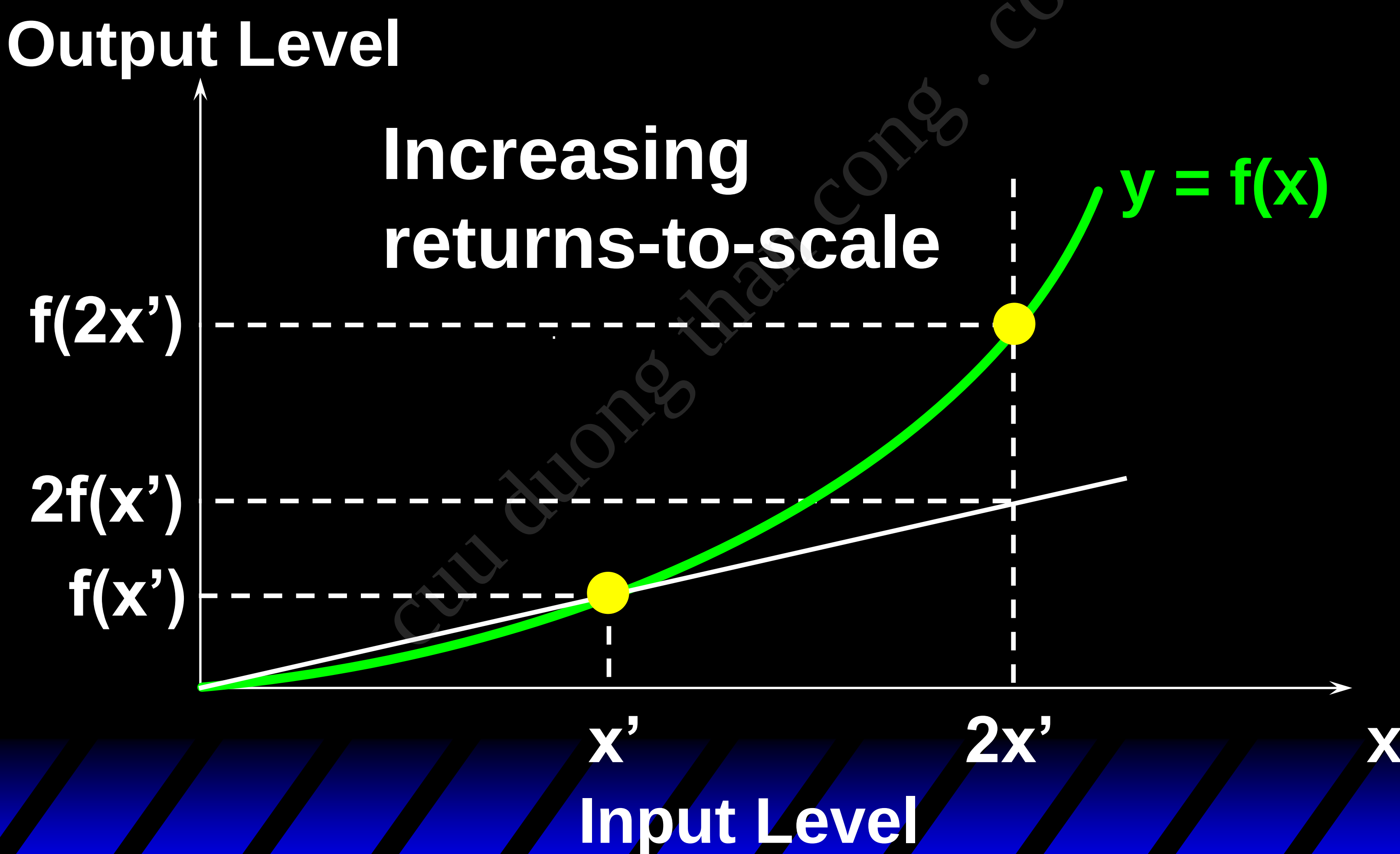
$$f(kx_1,kx_2,\Lambda\ ,kx_n) > kf(x_1,x_2,\Lambda\ ,x_n)$$

then the technology exhibits increasing returns-to-scale.

***E.g.*. (k = 2) doubling all input levels more than doubles the output level.**

Returns-to-Scale

One input, one output

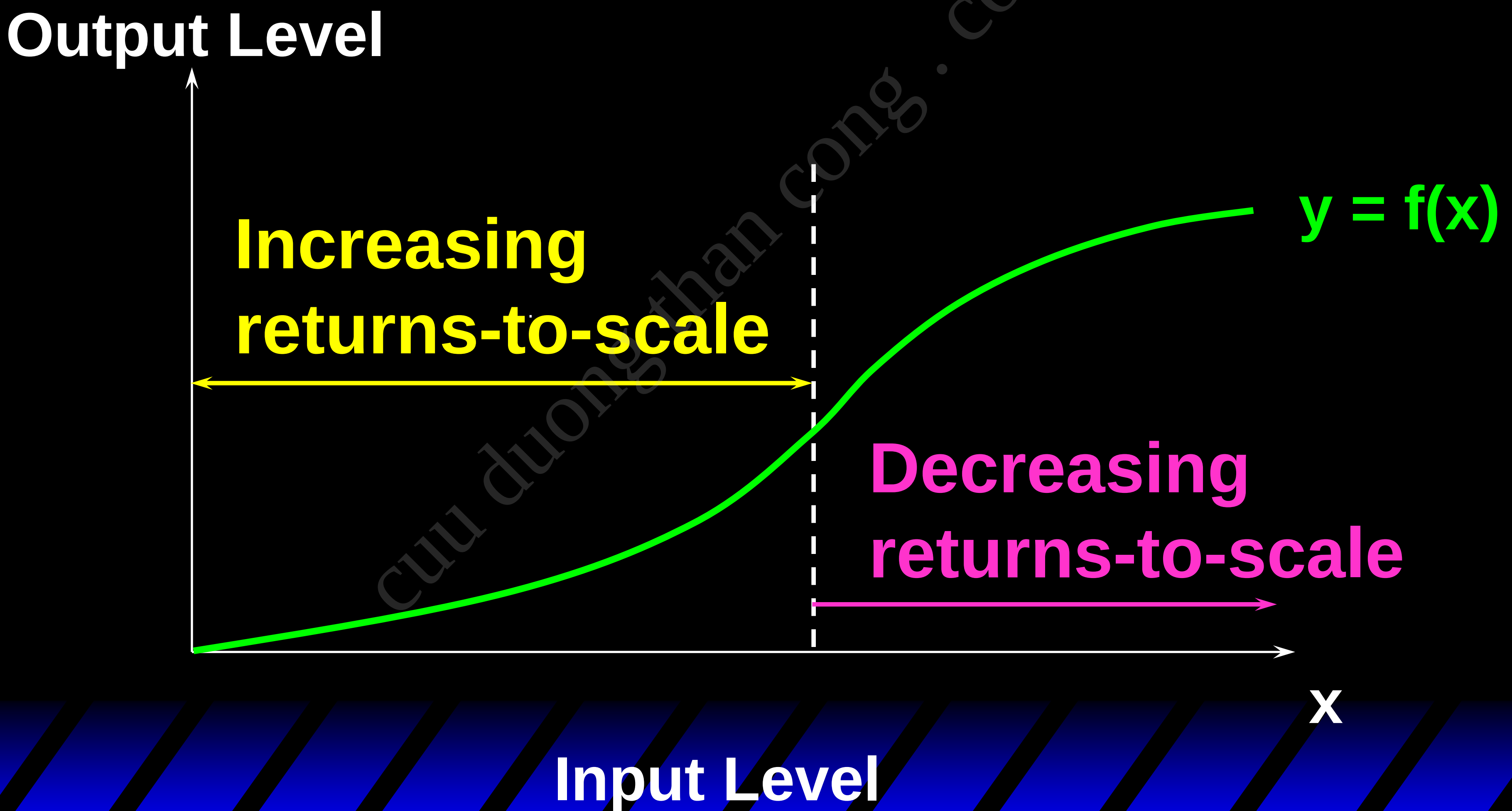


Returns-to-Scale

- A single technology can ‘locally’ exhibit different returns-to-scale.

Returns-to-Scale

One input, one output



Examples of Returns-to-Scale

The perfect-substitutes production function is

$$y = a_1 x_1 + a_2 x_2 + \Lambda + a_n x_n.$$

Expand all input levels proportionately by k. The output level becomes

$$a_1(kx_1) + a_2(kx_2) + \Lambda + a_n(kx_n)$$
$$= k(a_1 x_1 + a_2 x_2 + \Lambda + a_n x_n)$$
$$= ky.$$

The perfect-substitutes production function exhibits constant returns-to-scale.

Examples of Returns-to-Scale

The perfect-complements production function is

$$y = \min\{a_1 x_1, a_2 x_2, \Lambda, a_n x_n\}.$$

Expand all input levels proportionately by k. The output level becomes

$$\min\{a_1(kx_1), a_2(kx_2), \Lambda, a_n(kx_n)\}$$

$$= k(\min\{a_1x_1, a_2x_2, \Lambda, a_nx_n\})$$

$$= ky.$$

The perfect-complements production function exhibits constant returns-to-scale.

Examples of Returns-to-Scale

The Cobb-Douglas production function is

$$y = x_1^{a_1} x_2^{a_2} \Lambda\ x_n^{a_n}.$$

Expand all input levels proportionately by k. The output level becomes

$$(kx_1)^{a_1}(kx_2)^{a_2} \Lambda\ (kx_n)^{a_n}$$

Examples of Returns-to-Scale

The Cobb-Douglas production function is

$$y = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

Expand all input levels proportionately by k. The output level becomes

$$(kx_1)^{a_1}(kx_2)^{a_2} \cdots (kx_n)^{a_n}$$

$$= k^{a_1} k^{a_2} \cdots k^{a_n} x^{a_1} x^{a_2} \cdots x^{a_n}$$

$$= k^{a_1 + a_2 + \cdots + a_n} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

$$= k^{a_1 + \cdots + a_n} y.$$

Examples of Returns-to-Scale

The Cobb-Douglas production function is

$$y = x_1^{a_1} x_2^{a_2} \Lambda\ x_n^{a_n}.$$

$$(kx_1)^{a_1}(kx_2)^{a_2}\Lambda\ (kx_n)^{a_n} = k^{a_1 + \Lambda + a_n} y.$$

The Cobb-Douglas technology's returns-to-scale is

constant if $a_1 + \ldots + a_n = 1$

increasing if $a_1 + \ldots + a_n > 1$

decreasing if $a_1 + \ldots + a_n < 1$.

Returns-to-Scale

- **Q: Can a technology exhibit increasing returns-to-scale even though all of its marginal products are diminishing?**

Returns-to-Scale

- Q: Can a technology exhibit increasing returns-to-scale even if all of its marginal products are diminishing?
- A: Yes.
- E.g. $y = x_1^{2/3} x_2^{2/3}.$

Returns-to-Scale

$$y = x_1^{2/3} x_2^{2/3} = x_1^{a_1} x_2^{a_2}$$

$a_1 + a_2 = \frac{4}{3} > 1$ so this technology exhibits increasing returns-to-scale.

But $MP_1 = \frac{2}{3} x_1^{-1/3} x_2^{2/3}$ diminishes as x_1 increases and

$MP_2 = \frac{2}{3} x_1^{2/3} x_2^{-1/3}$ diminishes as x_2 increases.

Returns-to-Scale

- **So a technology can exhibit increasing returns-to-scale even if all of its marginal products are diminishing. Why?**

Returns-to-Scale

- A marginal product is the rate-of-change of output as **one** input level increases, holding all other input levels fixed.
- Marginal product diminishes because the other input levels are fixed, so the increasing input's units have each less and less of other inputs with which to work.

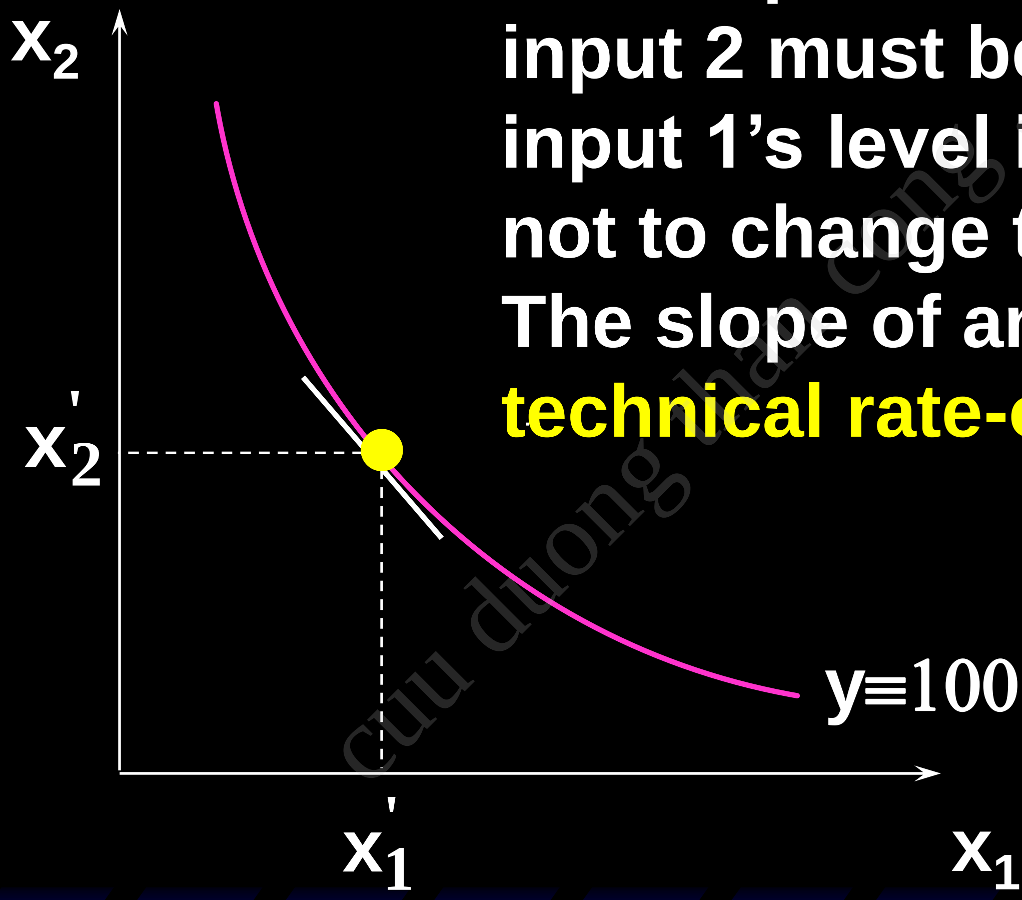
Returns-to-Scale

- When all input levels are increased proportionately, there need be no diminution of marginal products since each input will always have the same amount of other inputs with which to work. Input productivities need not fall and so returns-to-scale can be constant or increasing.

Technical Rate-of-Substitution

- **At what rate can a firm substitute one input for another without changing its output level?**

Technical Rate-of-Substitution



The slope is the rate at which input 2 must be given up as input 1's level is increased so as not to change the output level. The slope of an isoquant is its technical rate-of-substitution.

Technical Rate-of-Substitution

- **How is a technical rate-of-substitution computed?**

Technical Rate-of-Substitution

- How is a technical rate-of-substitution computed?
- The production function is $y = f(x_1, x_2)$.
- A small change (dx_1, dx_2) in the input bundle causes a change to the output level of

$$dy = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2.$$

Technical Rate-of-Substitution

$$dy = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2.$$

But dy = 0 since there is to be no change to the output level, so the changes dx_1 and dx_2 to the input levels must satisfy

$$0 = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2.$$

Technical Rate-of-Substitution

$$0 = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2$$

rearranges to

$$\frac{\partial y}{\partial x_2} dx_2 = -\frac{\partial y}{\partial x_1} dx_1$$

so

$$\frac{dx_2}{dx_1} = -\frac{\partial y / \partial x_1}{\partial y / \partial x_2}.$$

Technical Rate-of-Substitution

$$\frac{dx_2}{dx_1} = -\frac{\partial y / \partial x_1}{\partial y / \partial x_2}$$

is the rate at which input 2 must be given up as input 1 increases so as to keep the output level constant. It is the slope of the isoquant.

Technical Rate-of-Substitution; A Cobb-Douglas Example

$$y = f(x_1, x_2) = x_1^a x_2^b$$

so $\dfrac{\partial y}{\partial x_1} = a x_1^{a-1} x_2^b$ and $\dfrac{\partial y}{\partial x_2} = b x_1^a x_2^{b-1}$.

The technical rate-of-substitution is

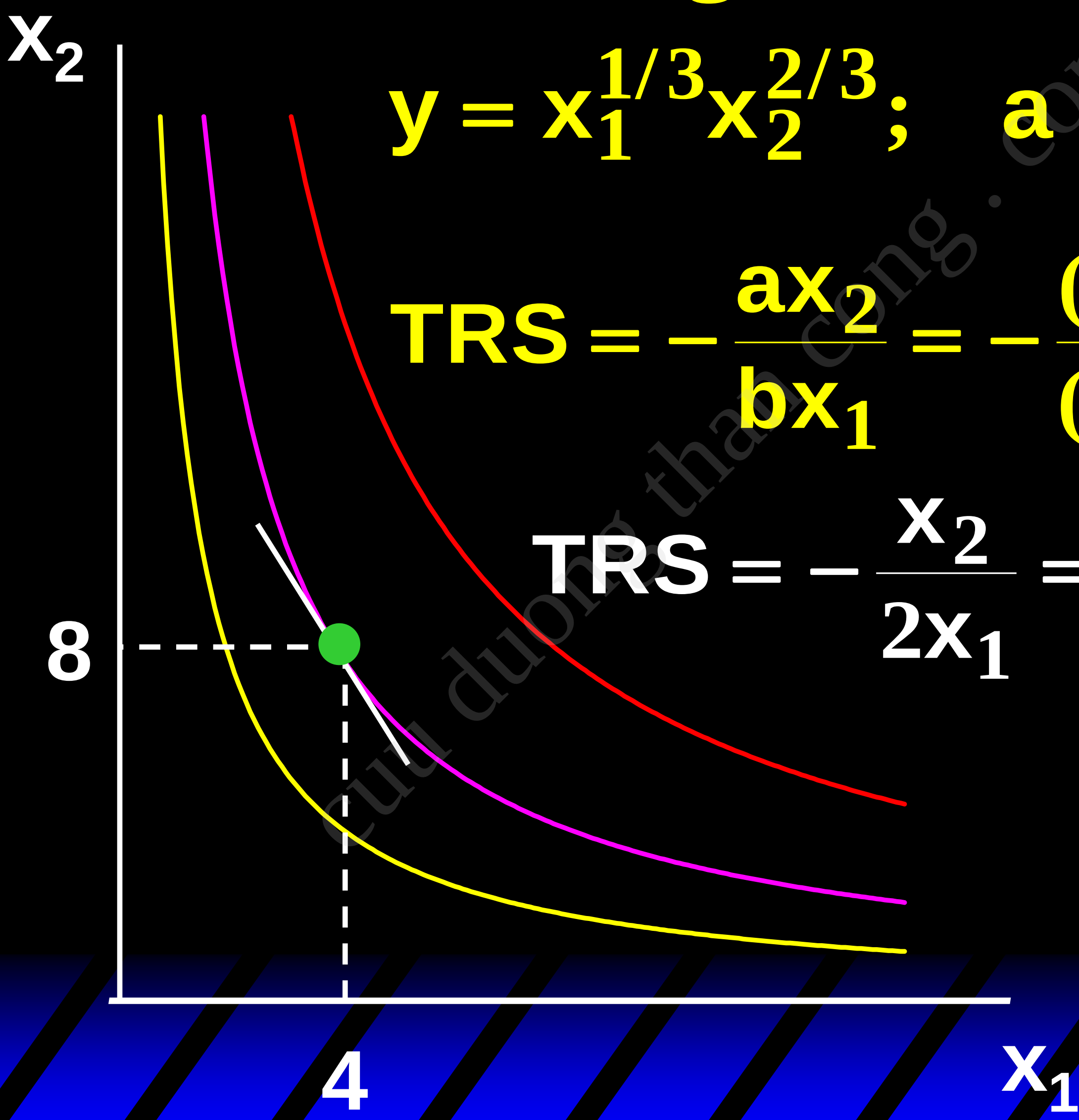
$$\frac{dx_2}{dx_1} = -\frac{\partial y / \partial x_1}{\partial y / \partial x_2} = -\frac{a x_1^{a-1} x_2^b}{b x_1^a x_2^{b-1}} = -\frac{a x_2}{b x_1}.$$

Technical Rate-of-Substitution; A Cobb-Douglas Example

$$y = x_1^{1/3} x_2^{2/3}; \quad a = \frac{1}{3} \text{ and } b = \frac{2}{3}$$

$$TRS = -\frac{ax_2}{bx_1} = -\frac{(1/3)x_2}{(2/3)x_1} = -\frac{x_2}{2x_1}$$

$$TRS = -\frac{x_2}{2x_1} = -\frac{8}{2 \times 4} = -1$$

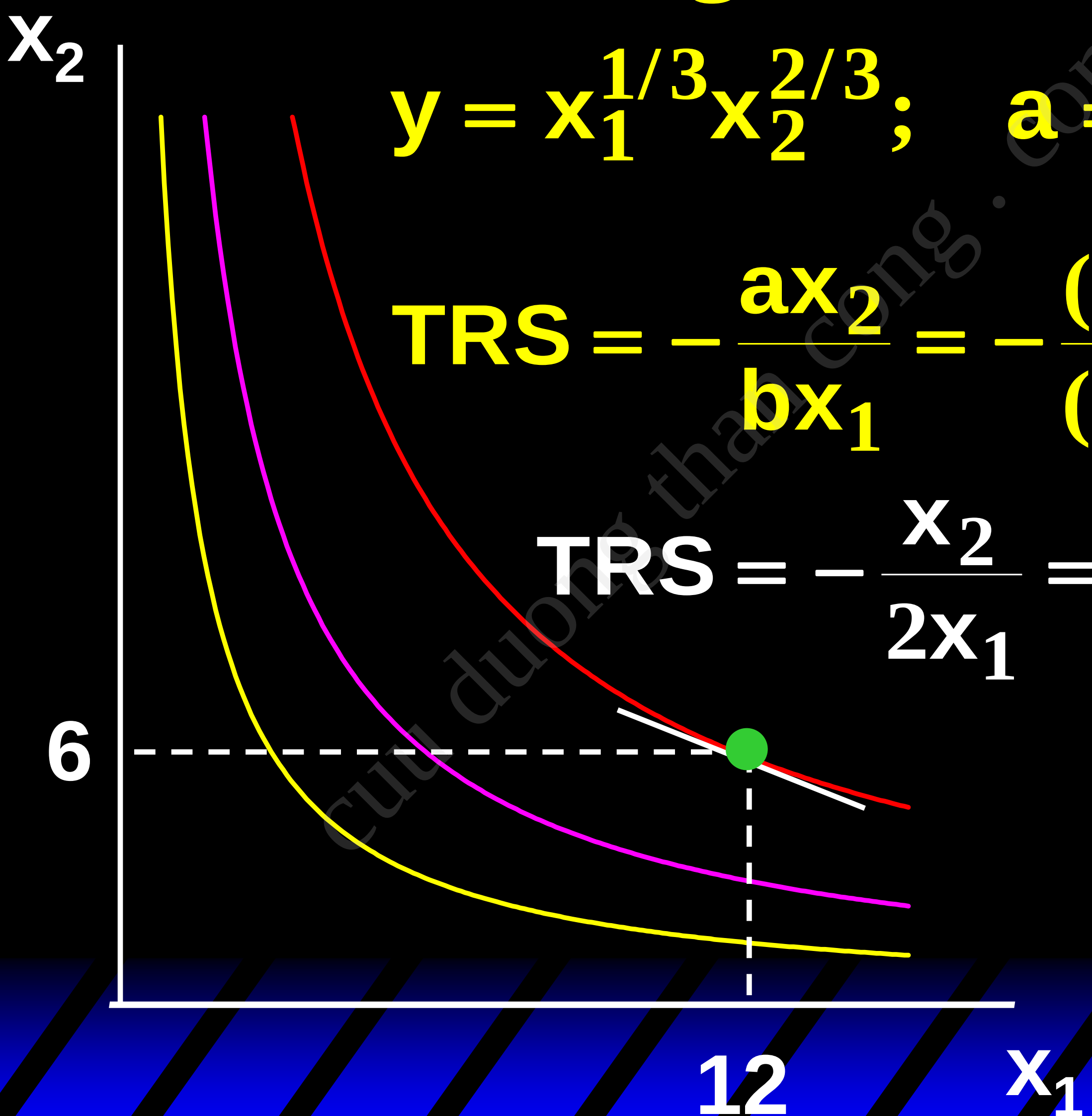


Technical Rate-of-Substitution; A Cobb-Douglas Example

$$y = x_1^{1/3} x_2^{2/3}; \quad a = \frac{1}{3} \text{ and } b = \frac{2}{3}$$

$$TRS = -\frac{a x_2}{b x_1} = -\frac{(1/3) x_2}{(2/3) x_1} = -\frac{x_2}{2 x_1}$$

$$TRS = -\frac{x_2}{2 x_1} = -\frac{6}{2 \times 12} = -\frac{1}{4}$$

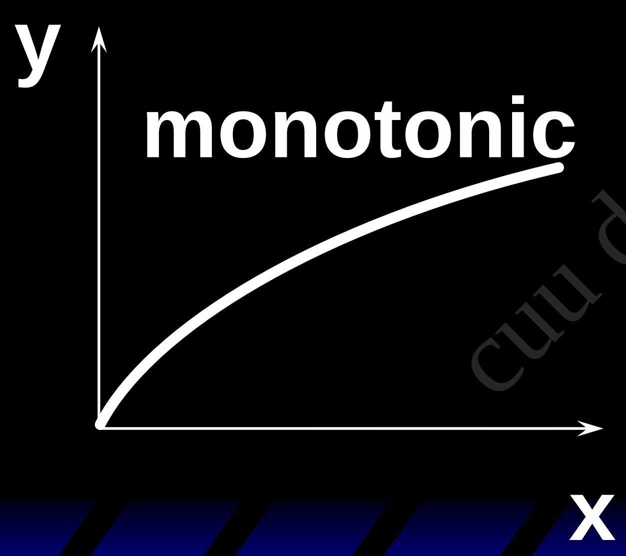


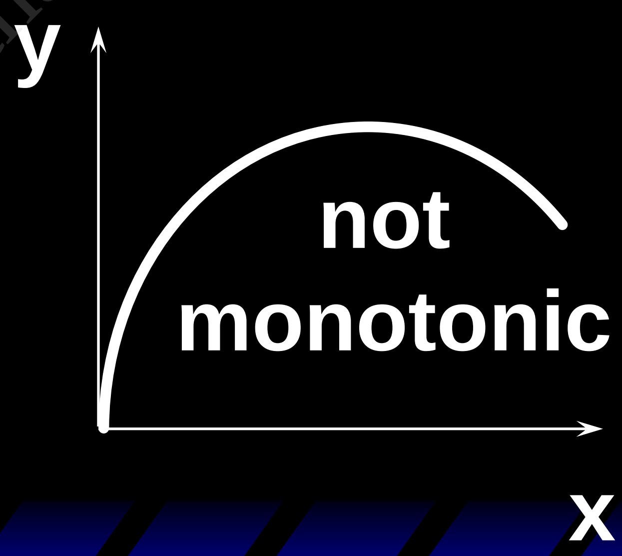
Well-Behaved Technologies

- A **well-behaved** technology is
  - **monotonic**, and
  - **convex**.

Well-Behaved Technologies - Monotonicity

- Monotonicity: More of **any** input generates more output.

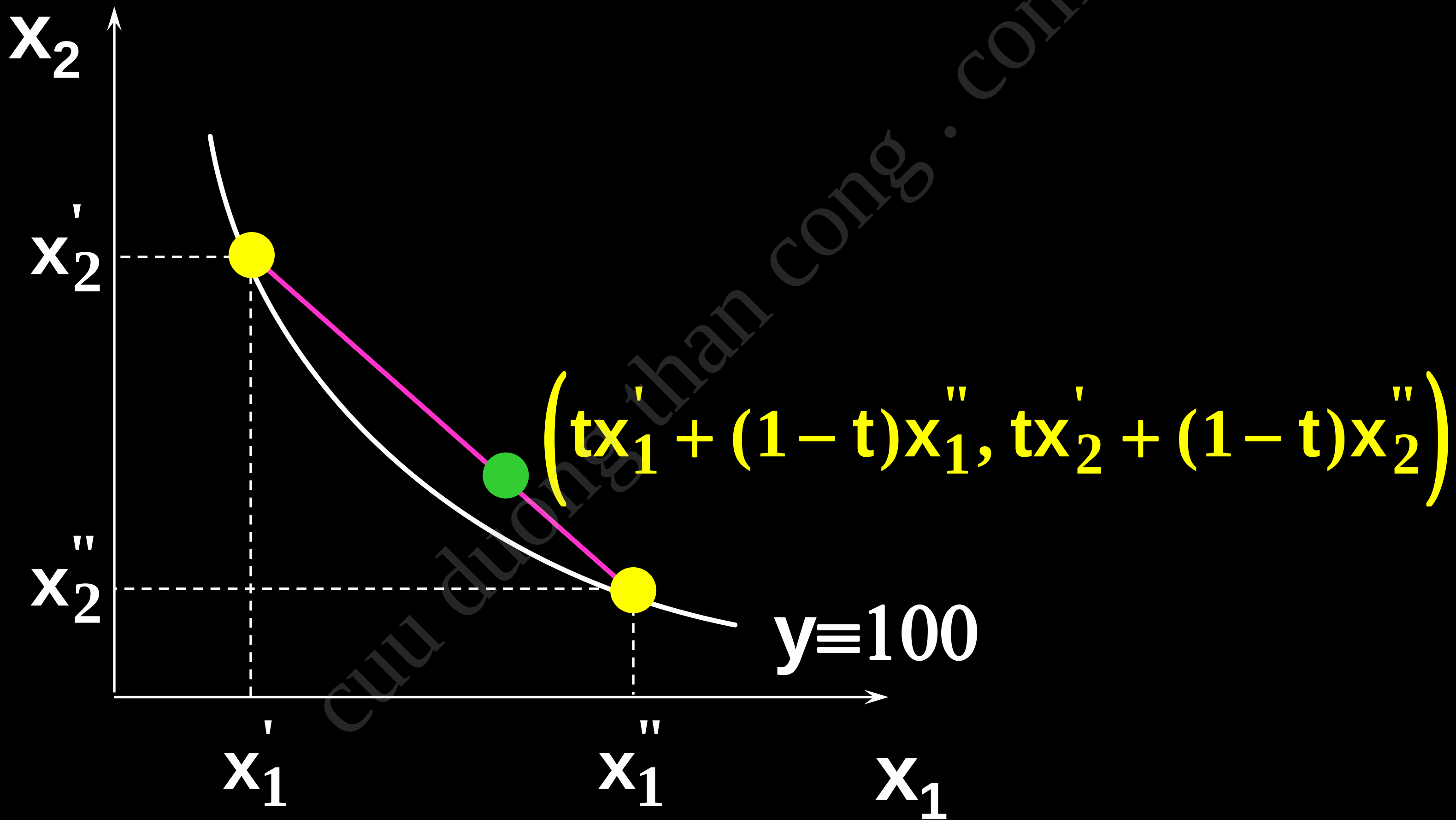




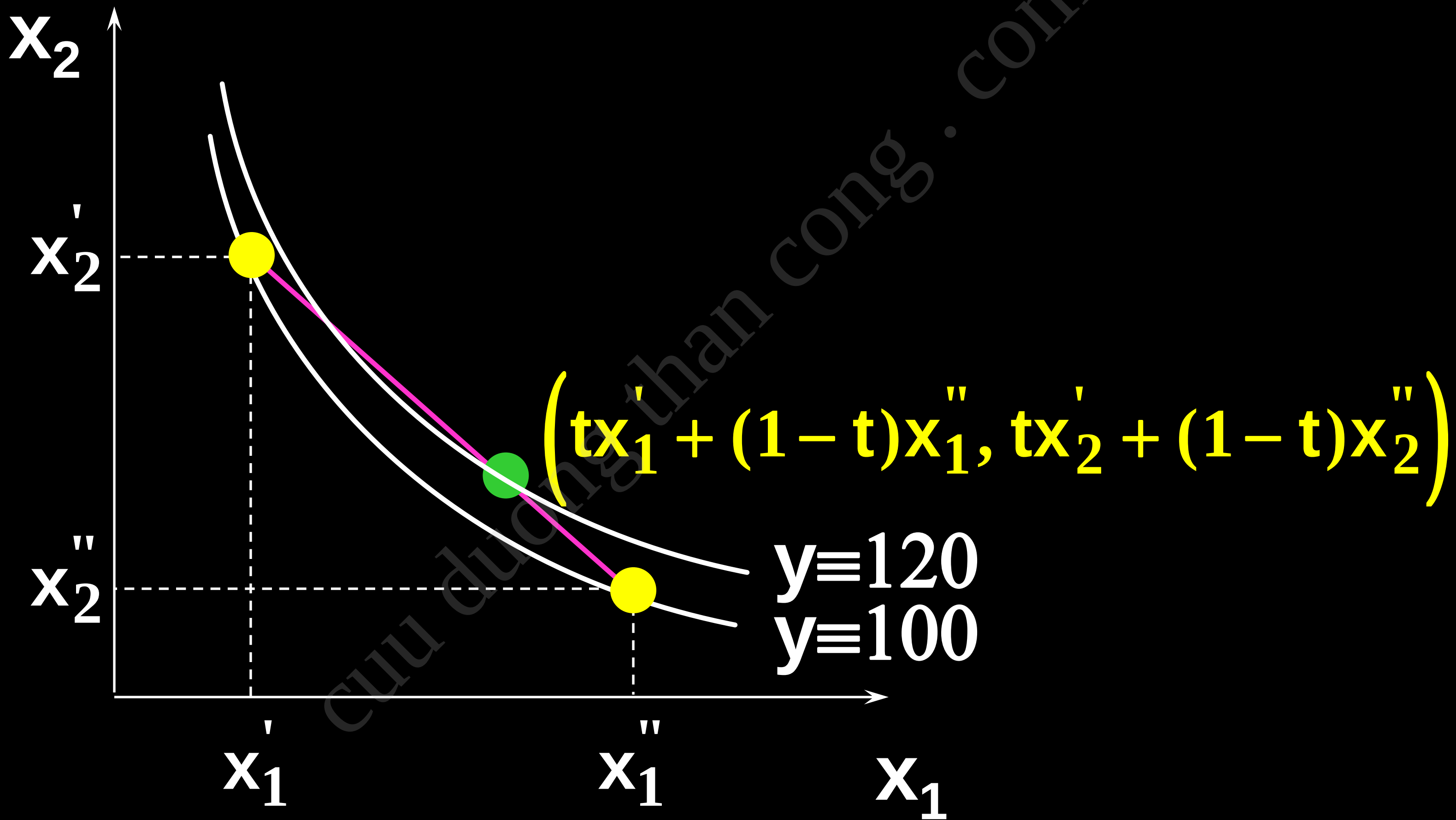
Well-Behaved Technologies - Convexity

- **Convexity**: **If the input bundles x' and x" both provide y units of output then the mixture tx' + (1-t)x" provides at least y units of output, for any 0 < t < 1.**

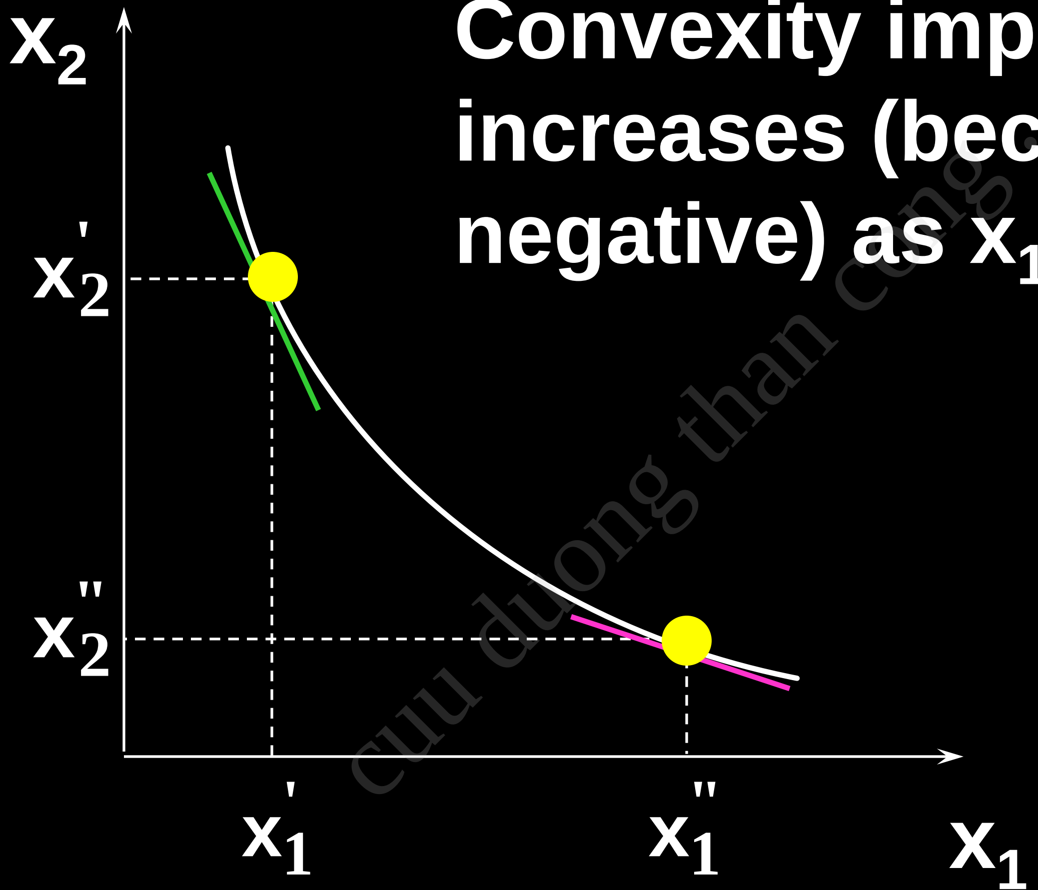
Well-Behaved Technologies - Convexity



Well-Behaved Technologies - Convexity

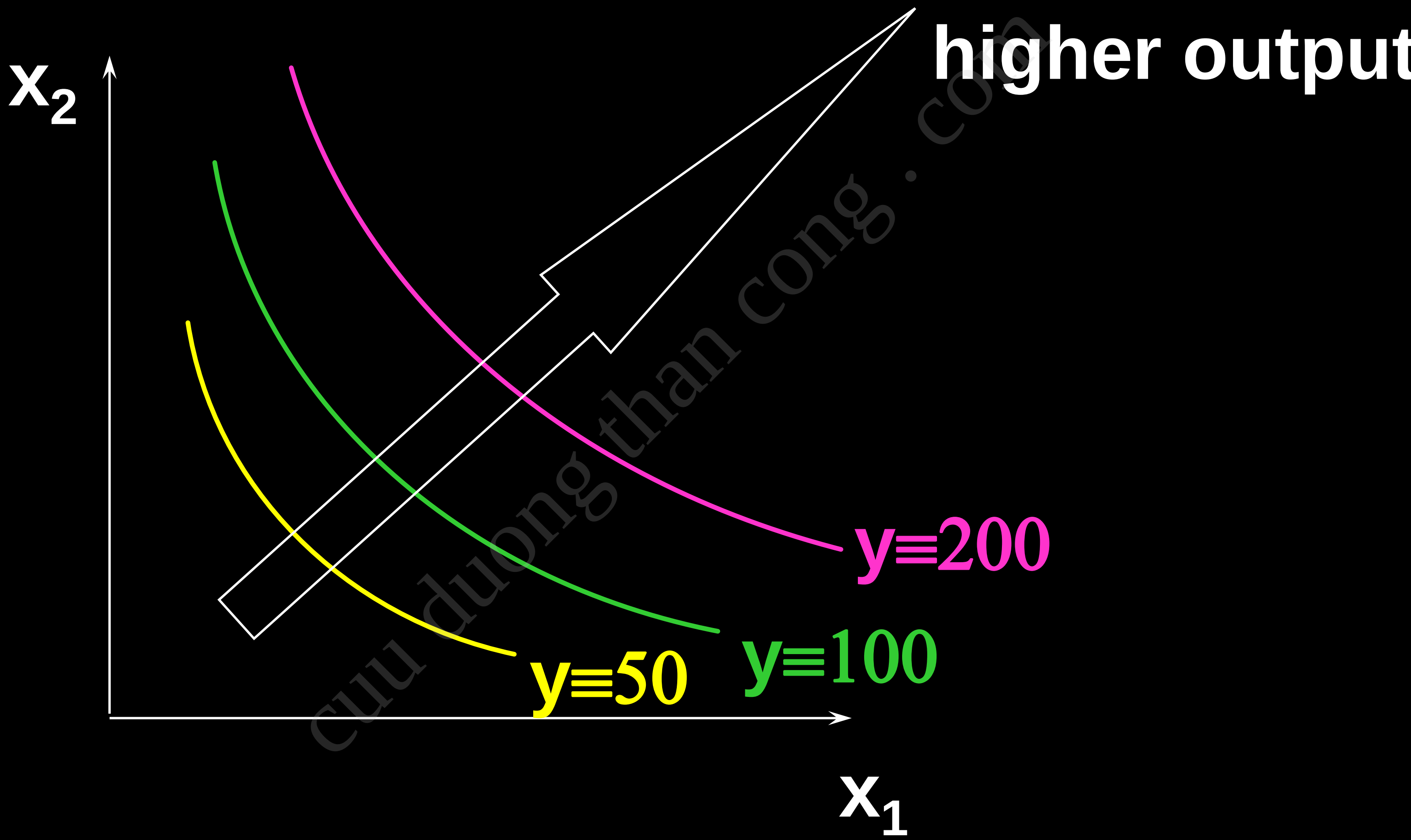


Well-Behaved Technologies - Convexity



Convexity implies that the TRS increases (becomes less negative) as x_1 increases.

Well-Behaved Technologies



The Long-Run and the Short-Runs

- **The long-run is the circumstance in which a firm is unrestricted in its choice of all input levels.**
- **There are many possible short-runs.**
- **A short-run is a circumstance in which a firm is restricted in some way in its choice of at least one input level.**

The Long-Run and the Short-Runs

- Examples of restrictions that place a firm into a short-run:
  - temporarily being unable to install, or remove, machinery
  - being required by law to meet affirmative action quotas
  - having to meet domestic content regulations.

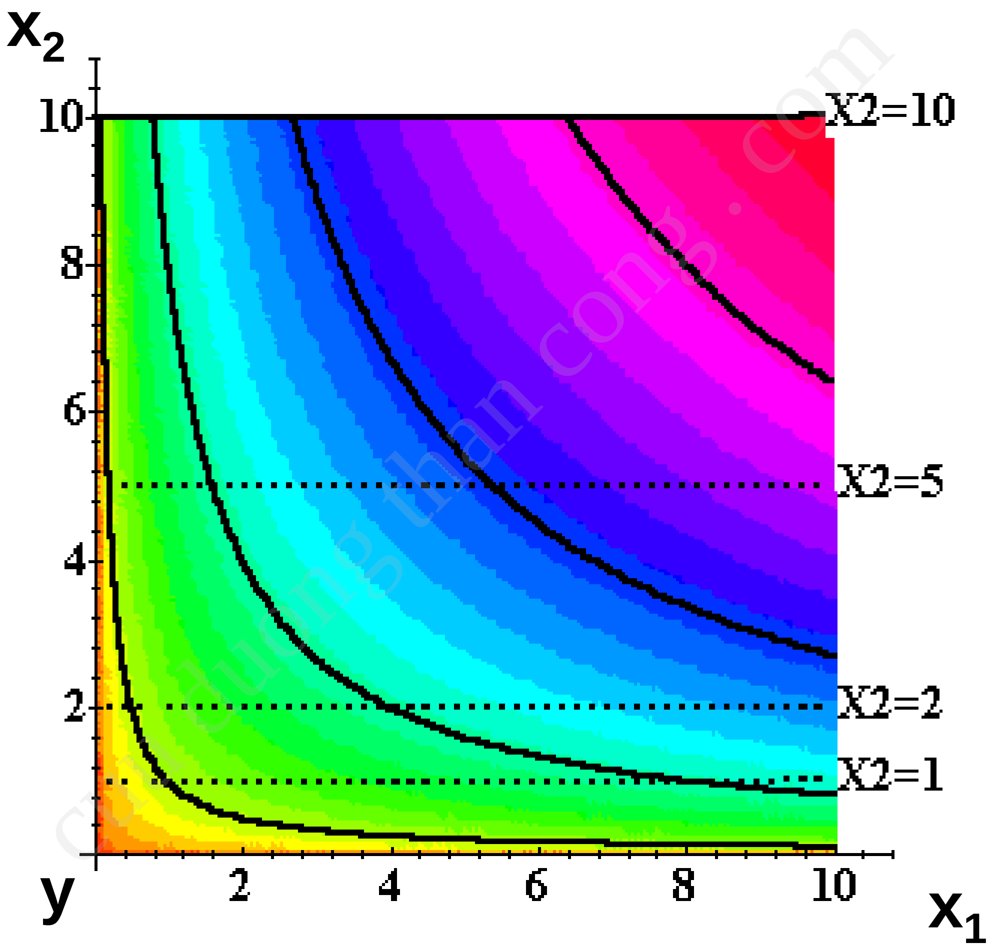
The Long-Run and the Short-Runs

- A useful way to think of the long-run is that the firm can choose as it pleases in which short-run circumstance to be.

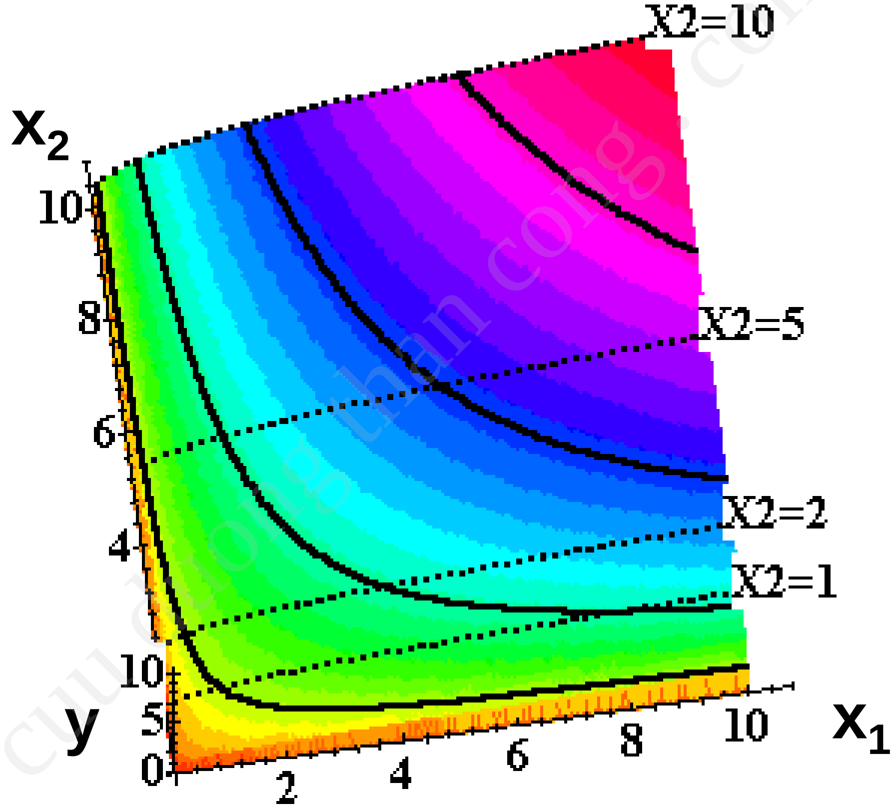
The Long-Run and the Short-Runs

- What do short-run restrictions imply for a firm's technology?
- Suppose the short-run restriction is fixing the level of input 2.
- Input 2 is thus a **fixed input** in the short-run. Input 1 remains **variable**.

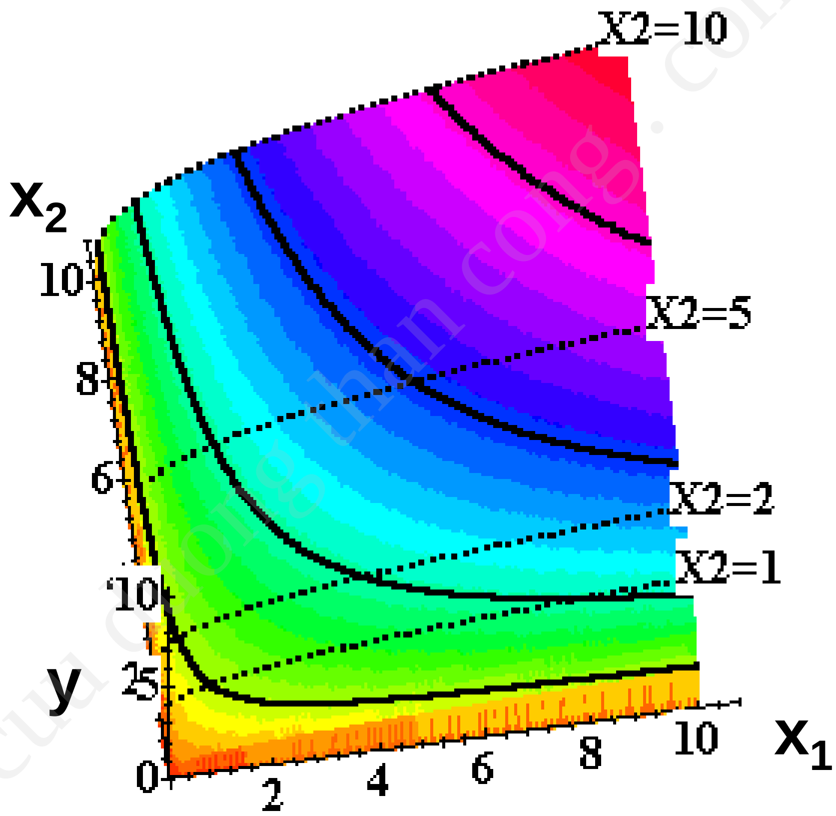
The Long-Run and the Short-Runs



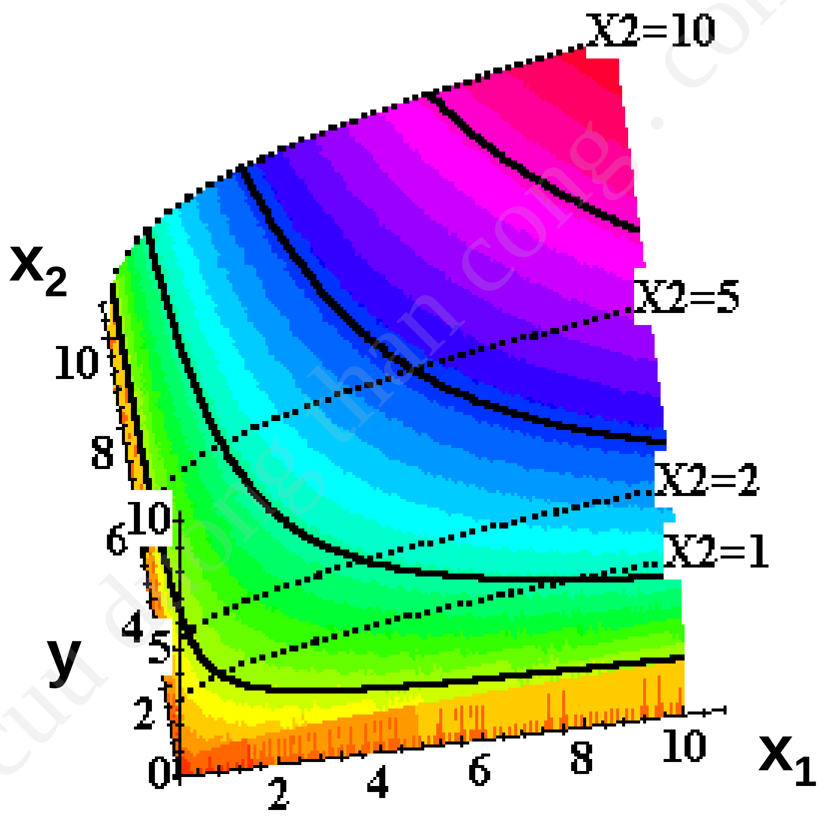
The Long-Run and the Short-Runs



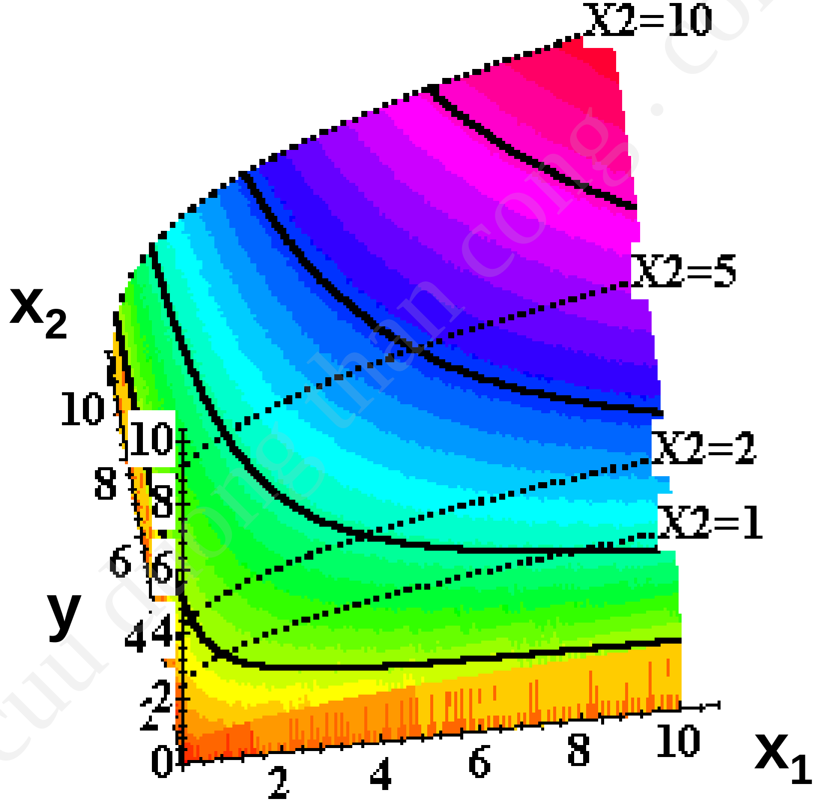
The Long-Run and the Short-Runs



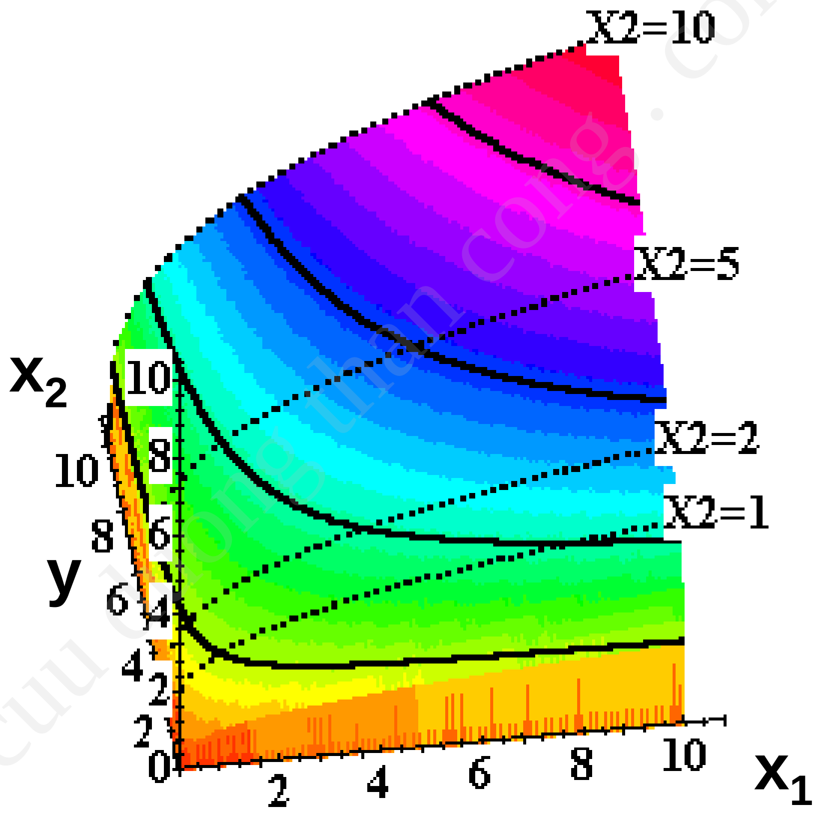
The Long-Run and the Short-Runs



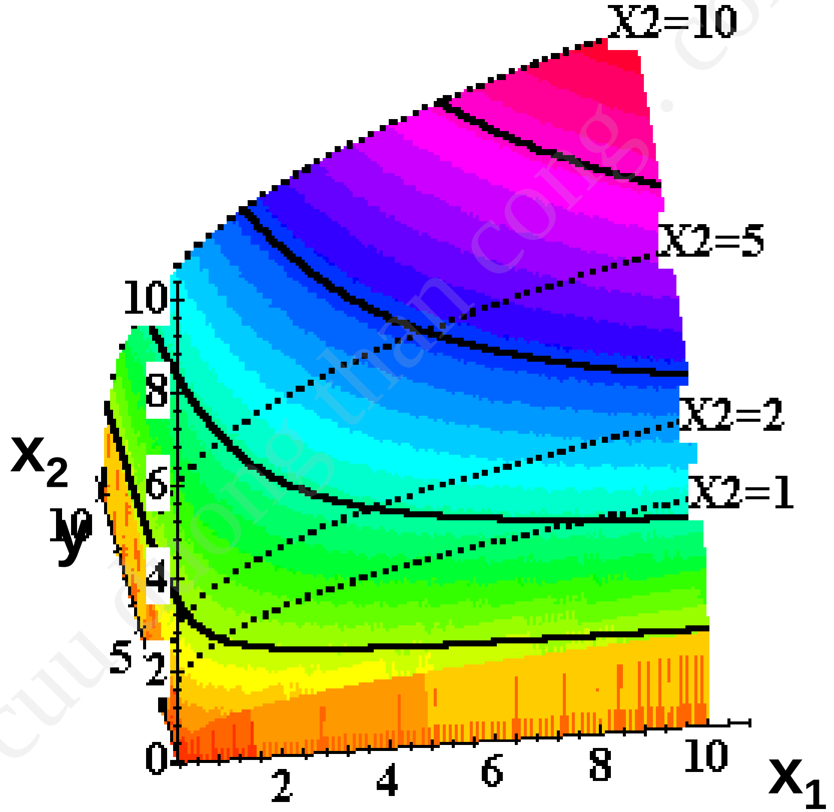
The Long-Run and the Short-Runs



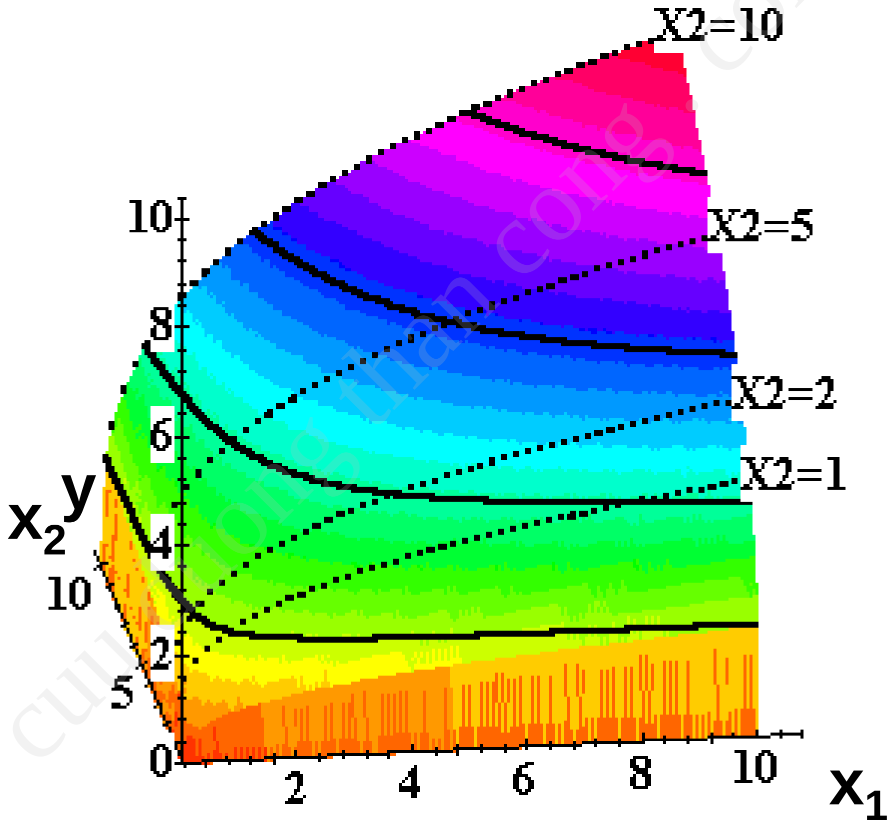
The Long-Run and the Short-Runs



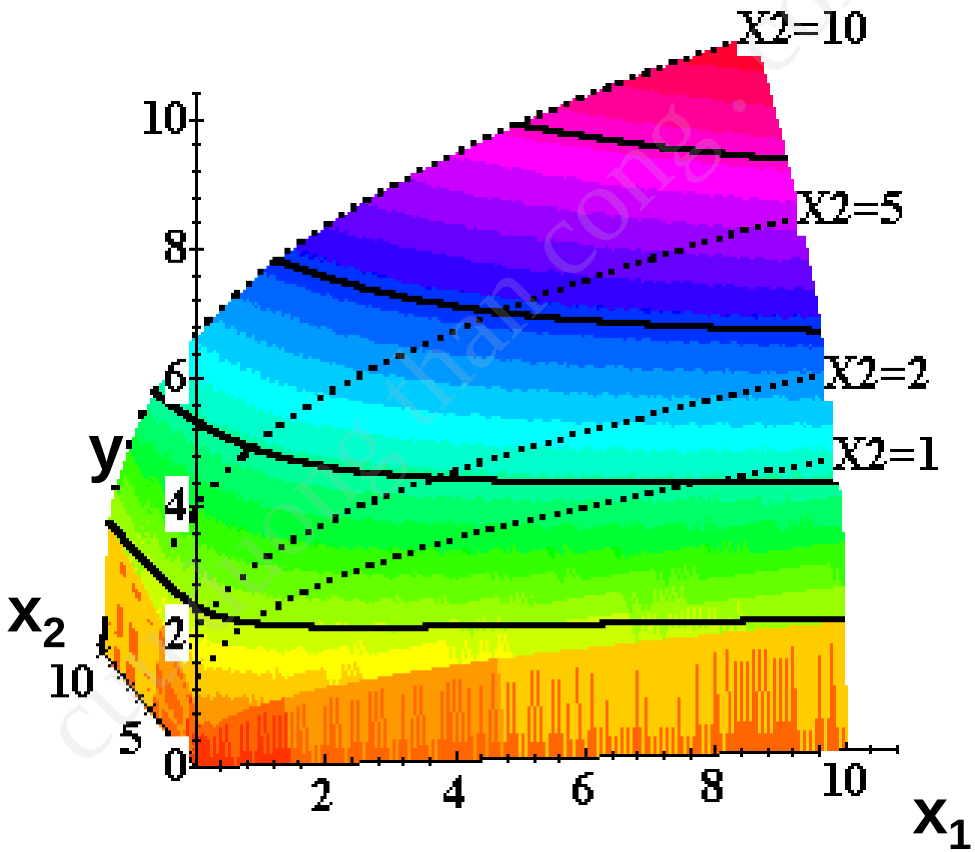
The Long-Run and the Short-Runs



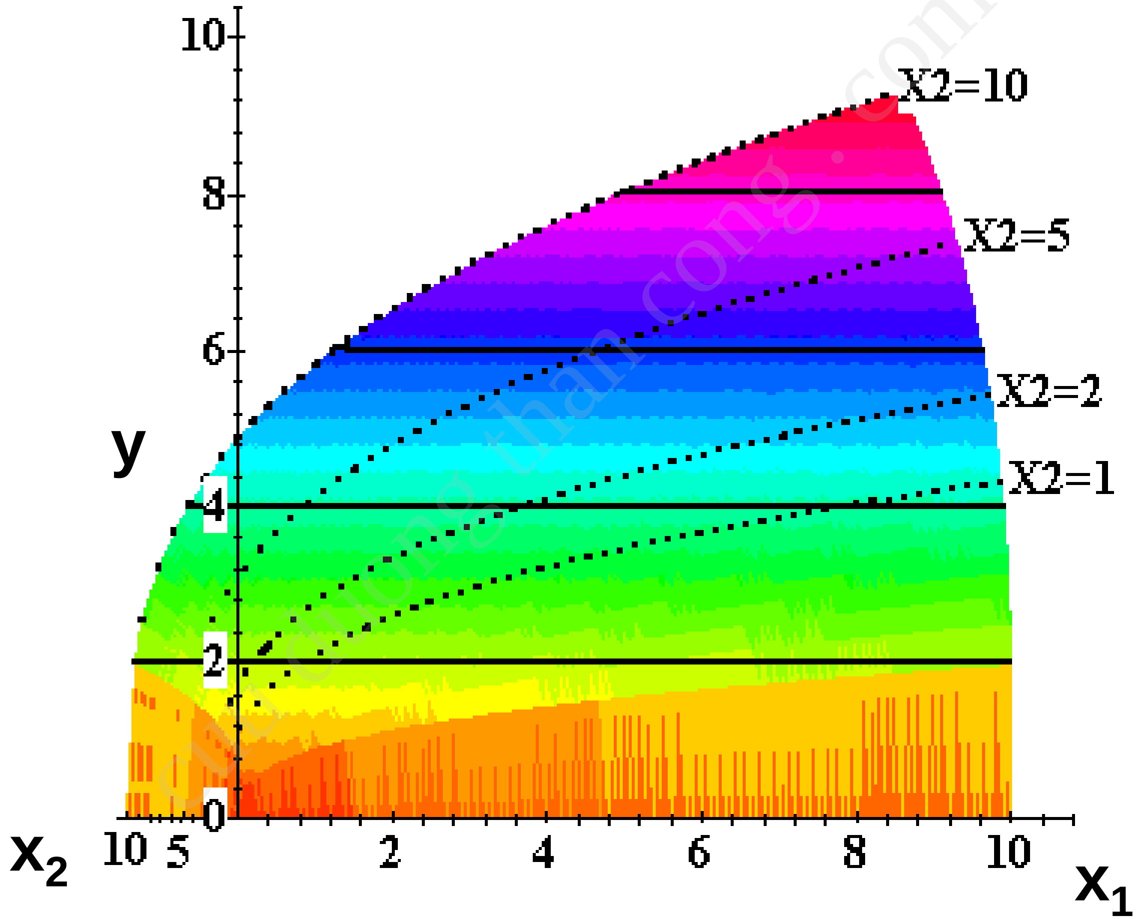
The Long-Run and the Short-Runs



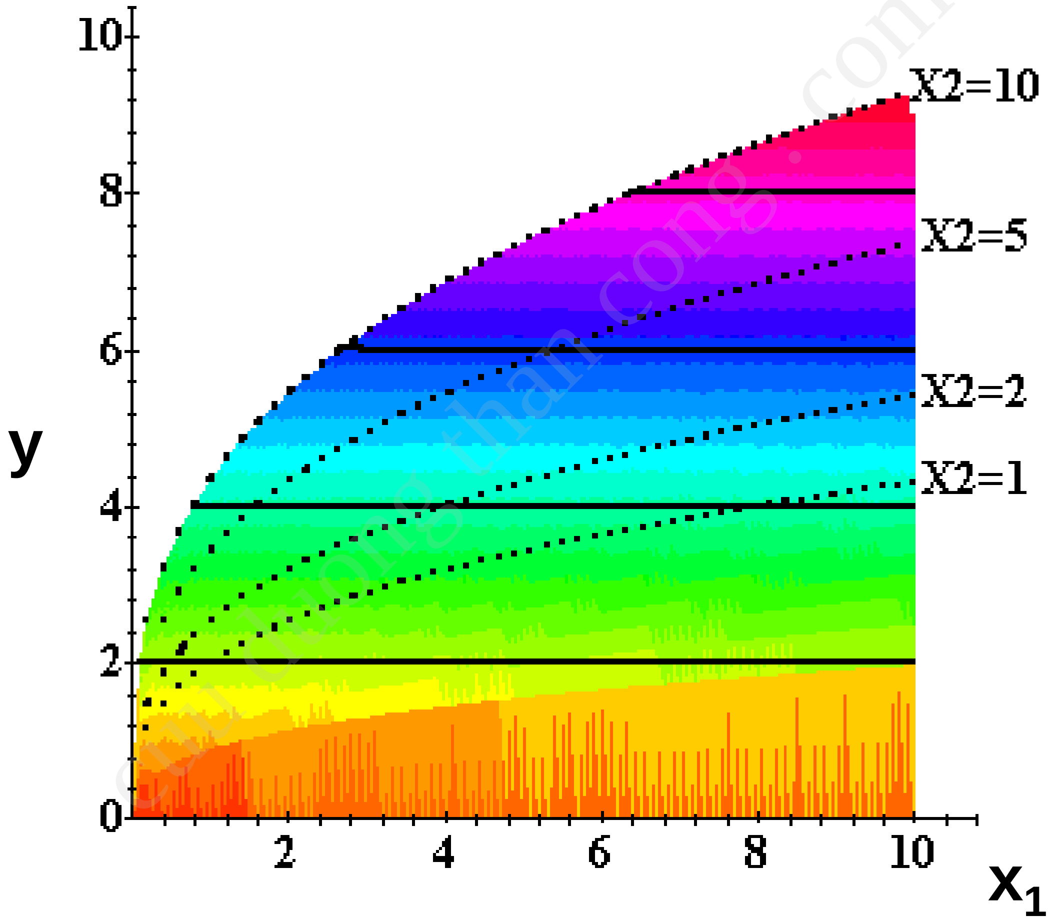
The Long-Run and the Short-Runs



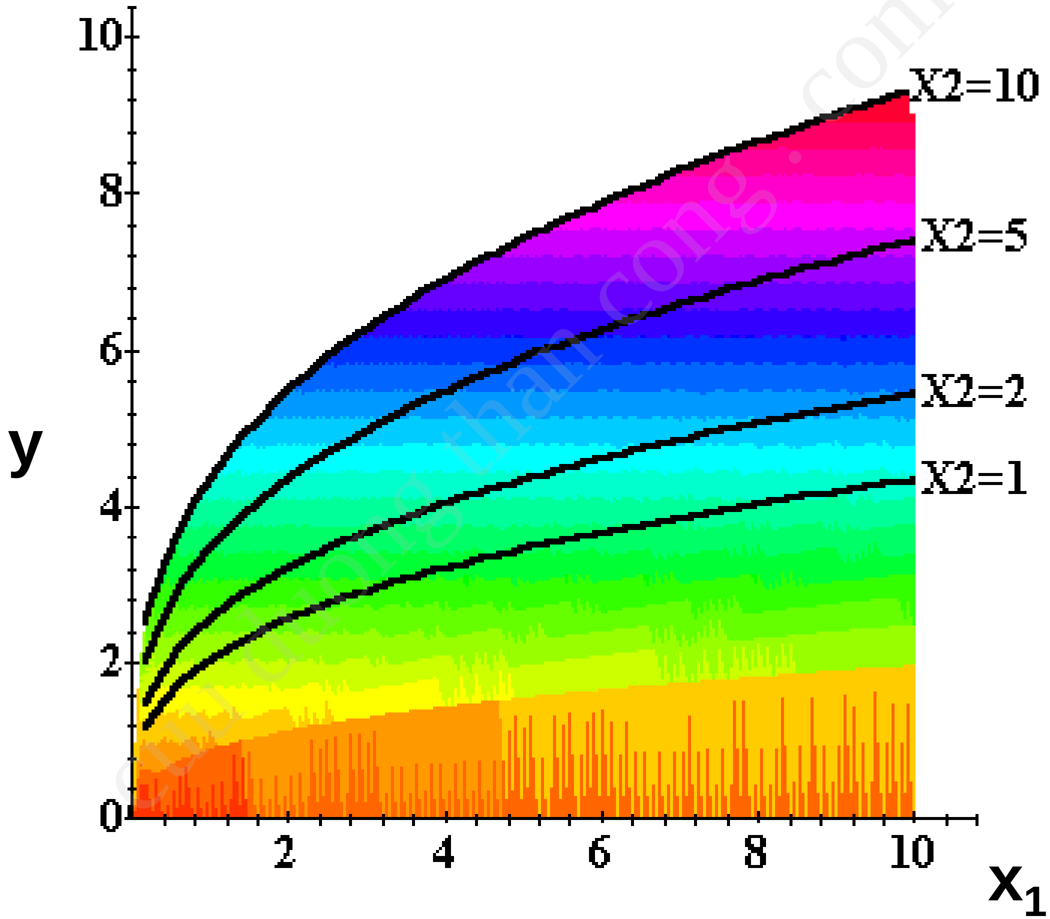
The Long-Run and the Short-Runs



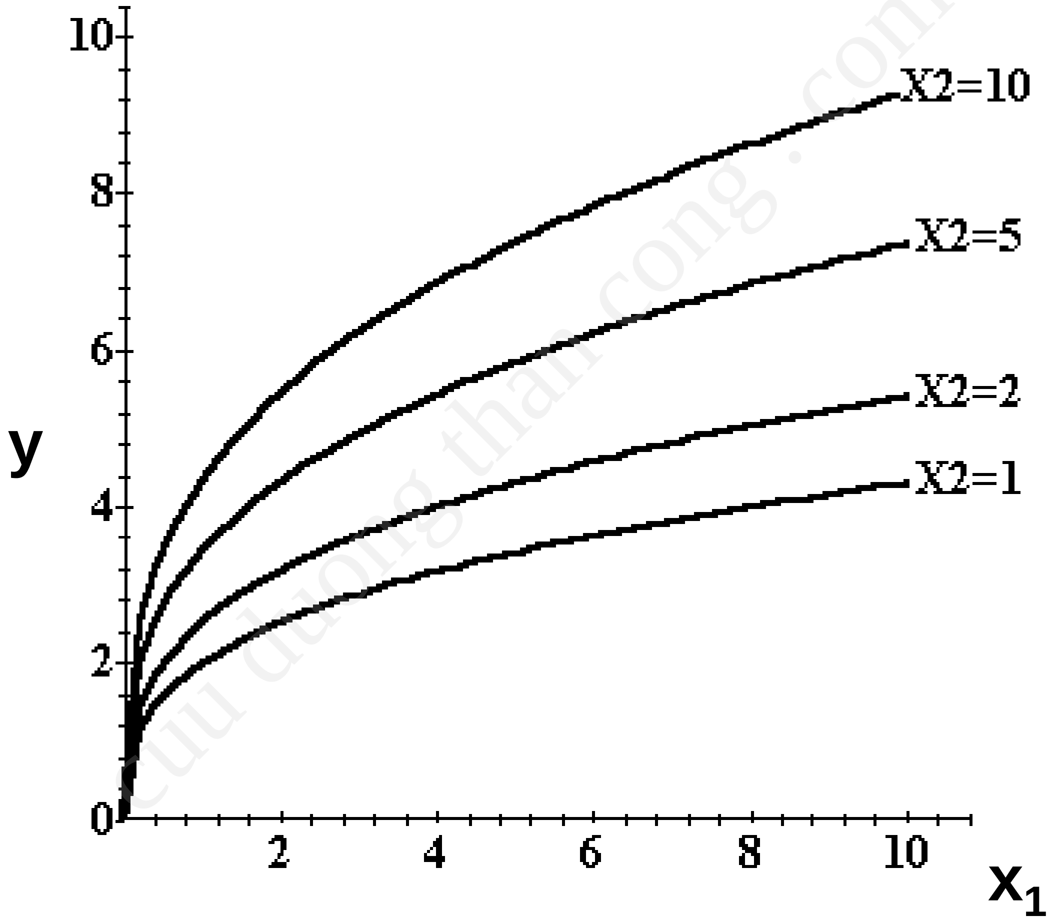
The Long-Run and the Short-Runs



The Long-Run and the Short-Runs



The Long-Run and the Short-Runs



Four short-run production functions.

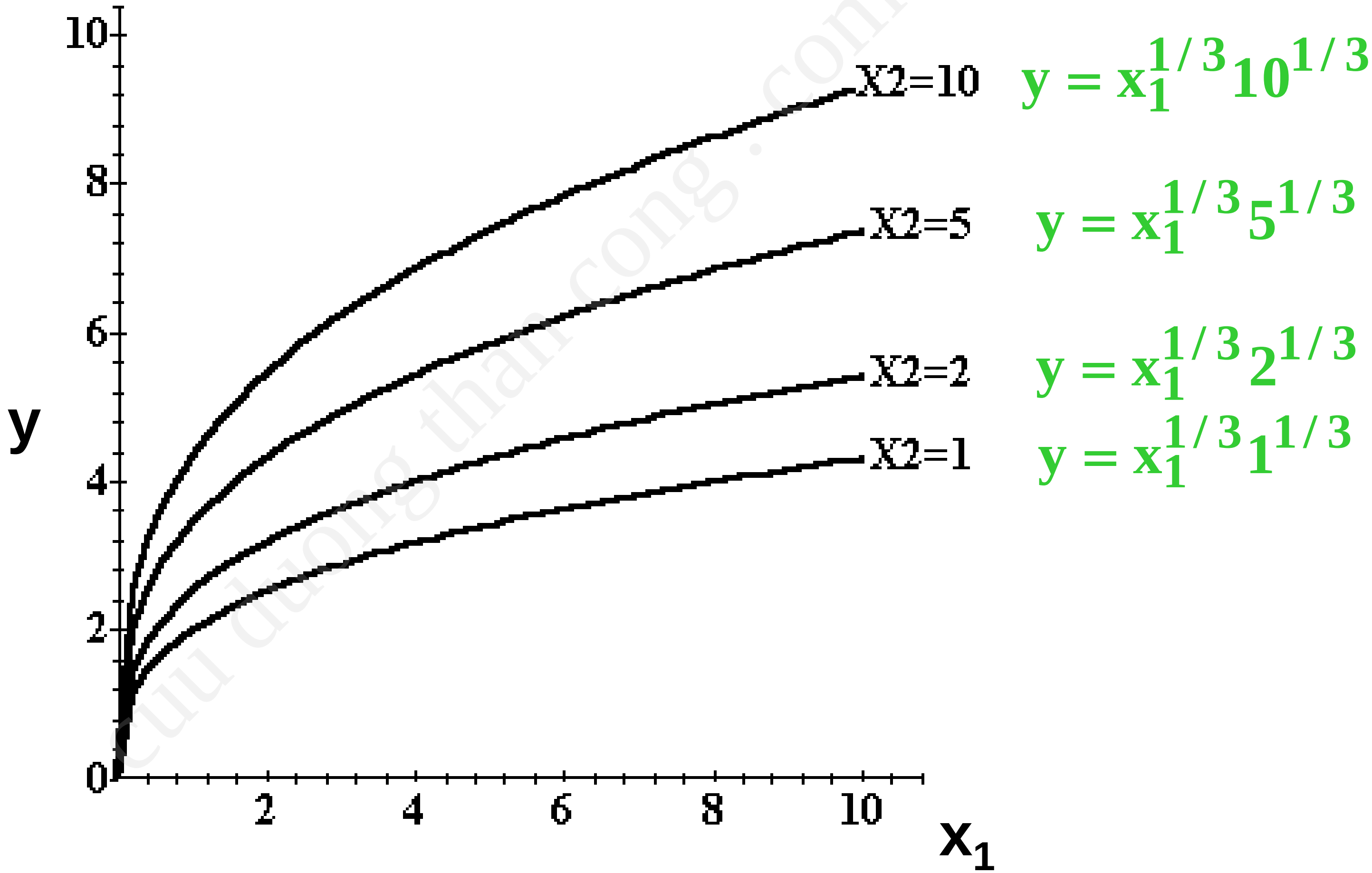
The Long-Run and the Short-Runs

$y = x_1^{1/3} x_2^{1/3}$ is the long-run production function (both x_1 and x_2 are variable).

The short-run production function when $x_2 \equiv 1$ is $y = x_1^{1/3} 1^{1/3} = x_1^{1/3}.$

The short-run production function when $x_2 \equiv 10$ is $y = x_1^{1/3} 10^{1/3} = 2 \cdot 15 x_1^{1/3}.$

The Long-Run and the Short-Runs



Four short-run production functions.

2. Cost Curves

Types of Cost Curves

- A **total cost curve** is the graph of a firm's total cost function.
- A **variable cost curve** is the graph of a firm's variable cost function.
- An **average total cost curve** is the graph of a firm's average total cost function.

Types of Cost Curves

- An **average variable cost curve** is the graph of a firm's average variable cost function.
- An **average fixed cost curve** is the graph of a firm's average fixed cost function.
- A **marginal cost curve** is the graph of a firm's marginal cost function.

Types of Cost Curves

- How are these cost curves related to each other?
- How are a firm’s long-run and short-run cost curves related?

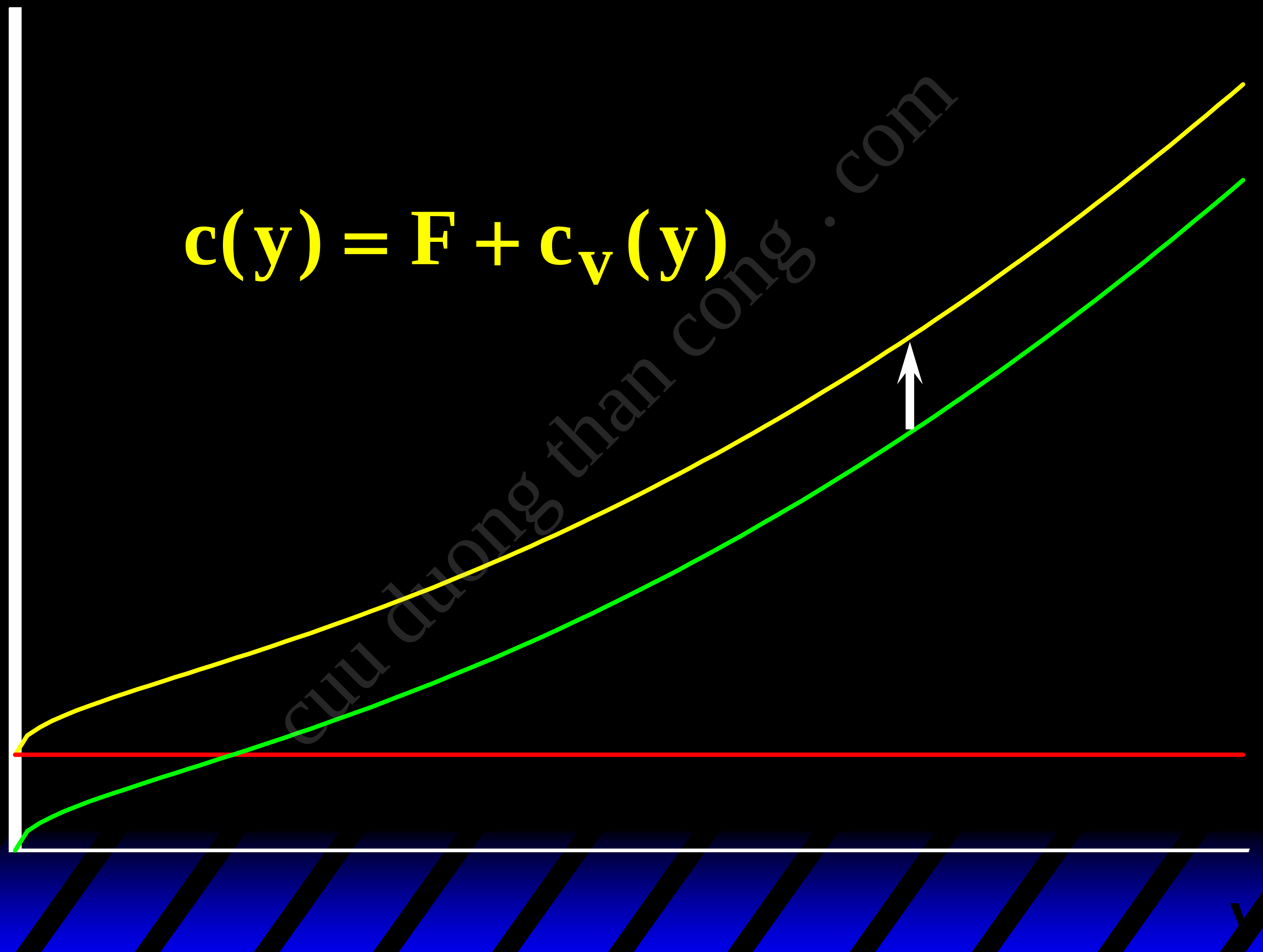
Fixed, Variable & Total Cost Functions

- F is the total cost to a firm of its **short-run fixed inputs**. F, the firm's **fixed cost**, does not vary with the firm's output level.
- $c_v(y)$ is the total cost to a firm of its **variable inputs** when producing y output units. $c_v(y)$ is the firm's **variable cost** function.
- $c_v(y)$ depends upon the levels of the fixed inputs.

Fixed, Variable & Total Cost Functions

- c(y) is the total cost of all inputs, **fixed and variable**, when producing y output units. c(y) is the firm's **total cost** function;

$$c(y) = F + c_v(y).$$



$c(y) = F + c_v(y)$
y

Av. Fixed, Av. Variable & Av. Total Cost Curves

- The firm's total cost function is

$$c(y) = F + c_v(y).$$

For y > 0, the firm's average total cost function is

$$AC(y) = \frac{F}{y} + \frac{c_v(y)}{y}$$

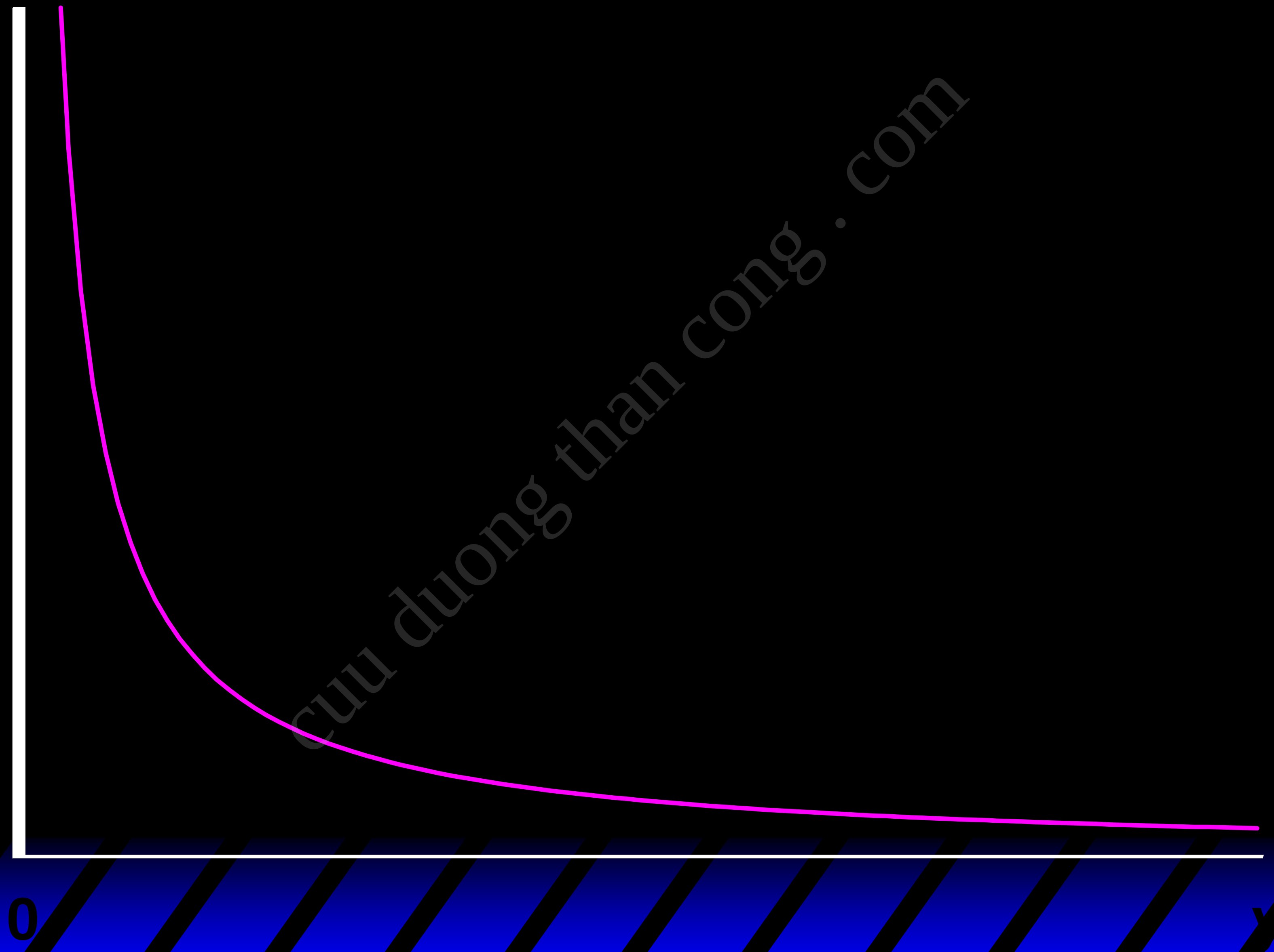
$$= AFC(y) + AVC(y).$$

Av. Fixed, Av. Variable & Av. Total Cost Curves

- What does an average fixed cost curve look like?

$$AFC(y) = \frac{F}{y}$$

- AFC(y) is a rectangular hyperbola so its graph looks like ...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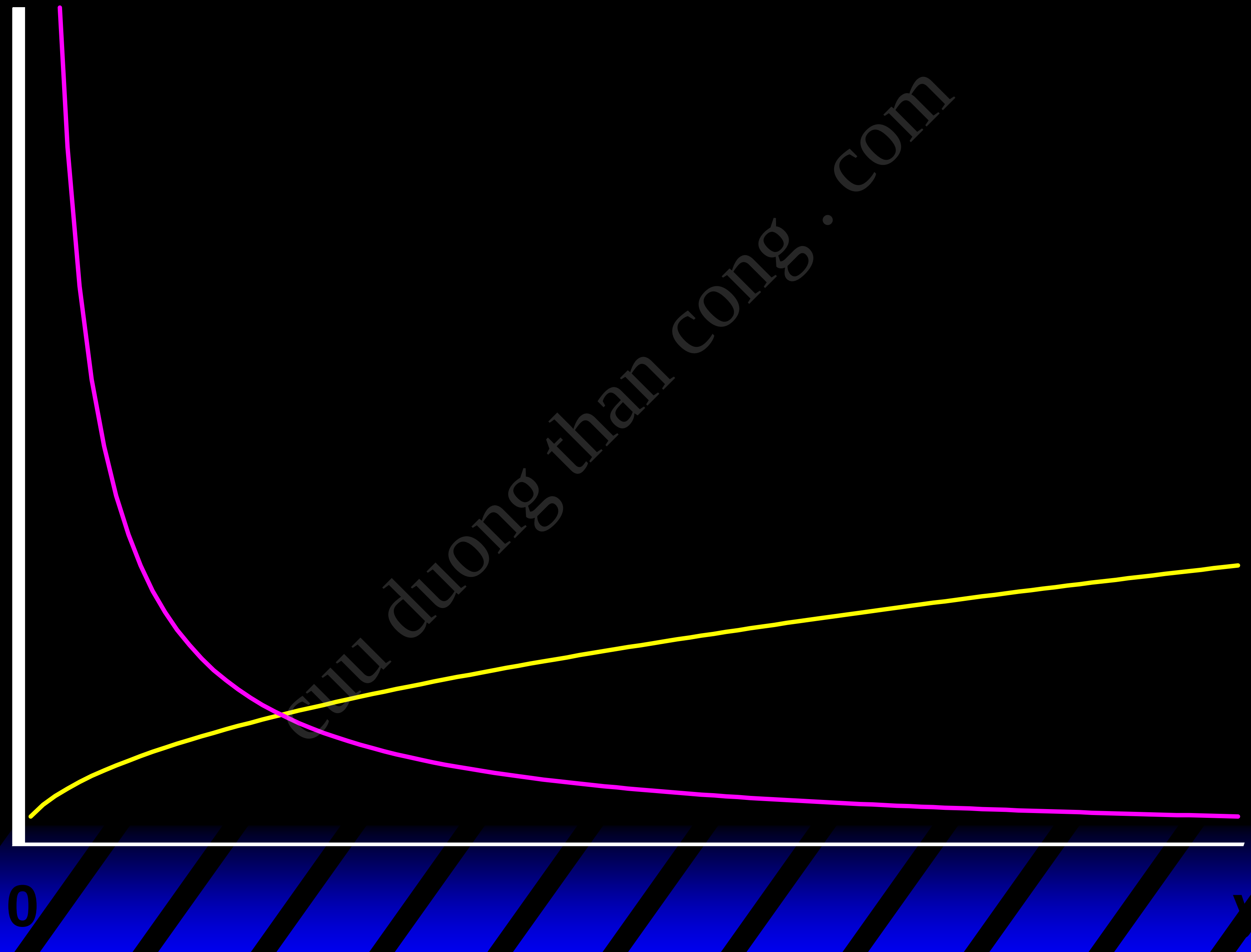
Av. Fixed, Av. Variable & Av. Total Cost Curves

- **In a short-run with a fixed amount of at least one input, the Law of Diminishing (Marginal) Returns must apply, causing the firm's average variable cost of production to increase eventually.**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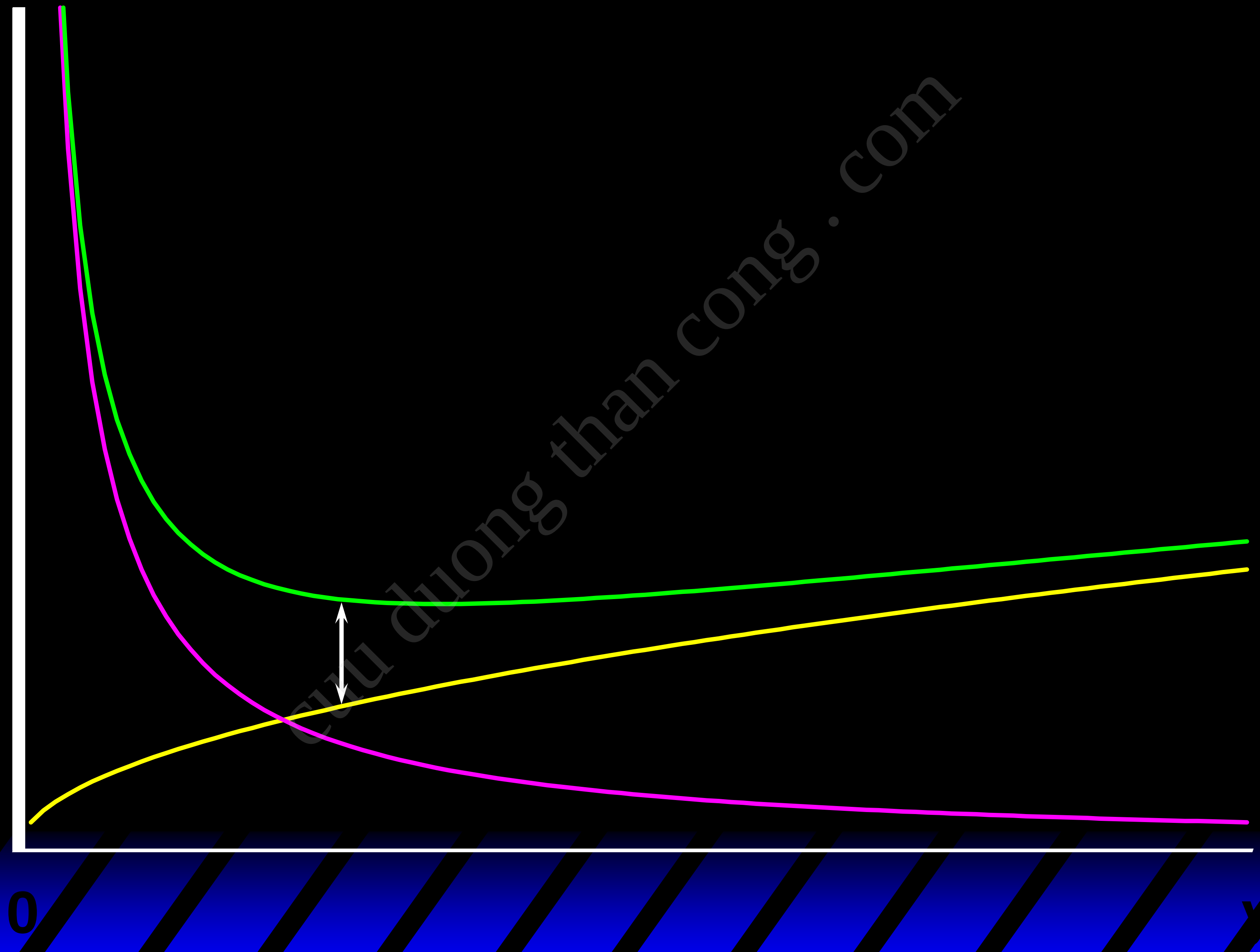
Av. Fixed, Av. Variable & Av. Total Cost Curves

- And   ATC(y) = AFC(y) + AVC(y)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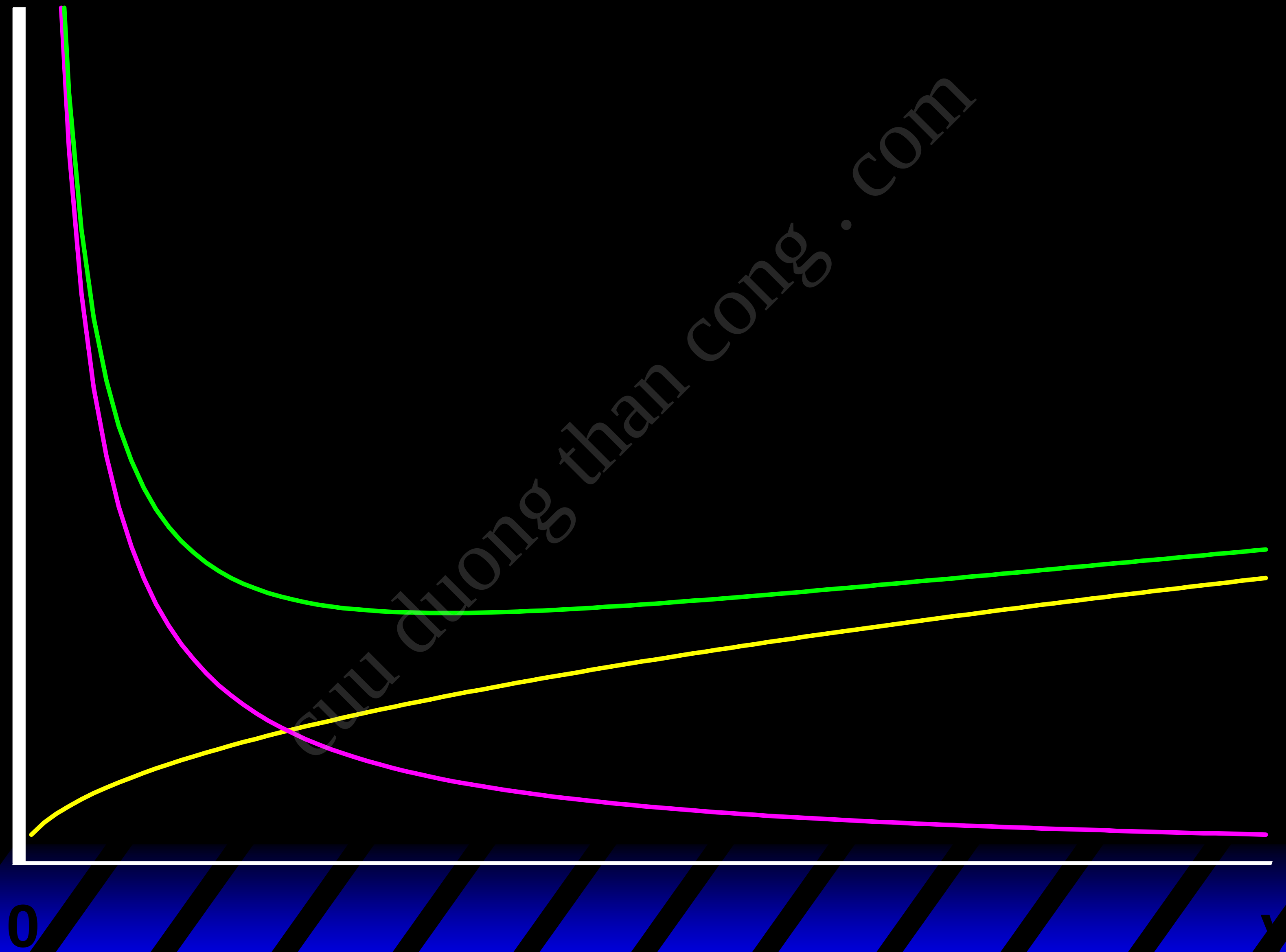
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y

Marginal Cost Function

- Marginal cost is the rate-of-change of variable production cost as the output level changes. That is,

$$MC(y) = \frac{\partial c_v(y)}{\partial y}.$$

Marginal Cost Function

- The firm's total cost function is

$$c(y) = F + c_v(y)$$

and the fixed cost F does not change with the output level y, so

$$MC(y) = \frac{\partial c_v(y)}{\partial y} = \frac{\partial c(y)}{\partial y}.$$

- MC is the slope of both the variable cost and the total cost functions.

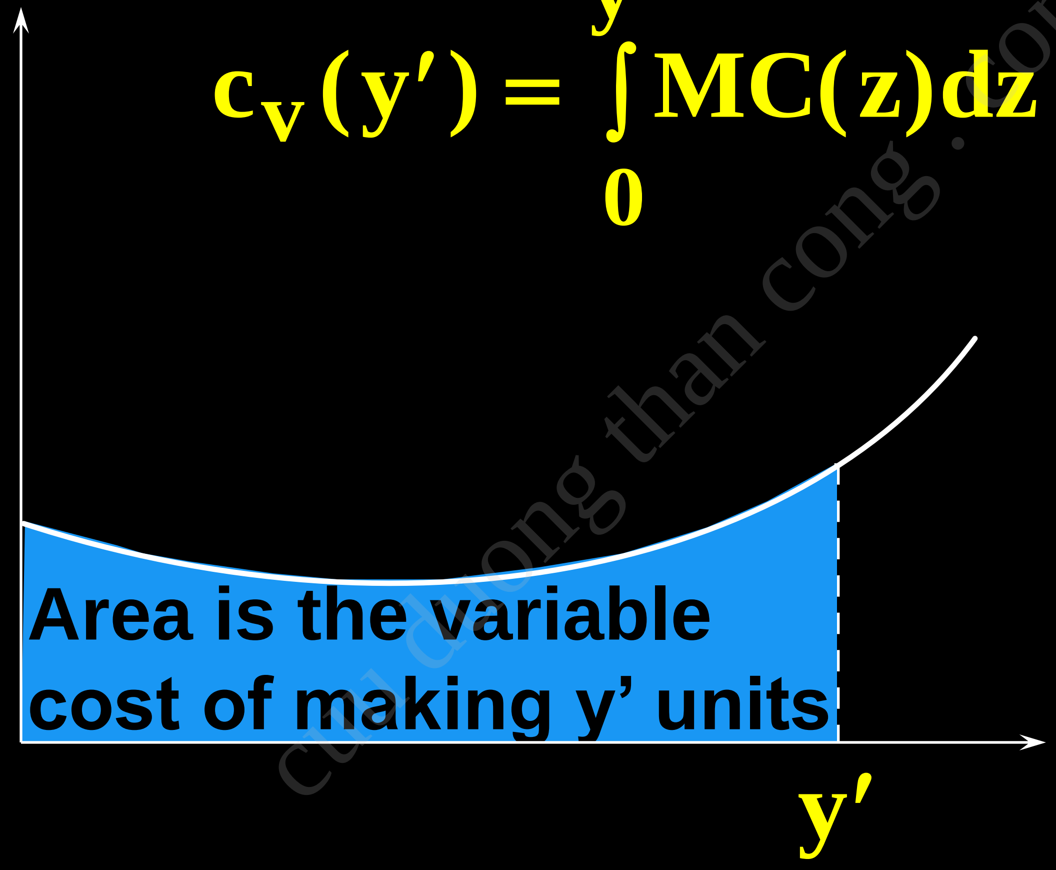
Marginal and Variable Cost Functions

- Since MC(y) is the derivative of $c_v(y)$, $c_v(y)$ must be the integral of MC(y). That is, $$MC(y) = \frac{\partial c_v(y)}{\partial y}$$

$$\Rightarrow \quad c_v(y) = \int_0^y MC(z)dz.$$

Marginal and Variable Cost Functions

$$c_v(y') = \int_0^{y'} MC(z)\,dz$$



Marginal & Average Cost Functions

- **How is marginal cost related to average variable cost?**

Marginal & Average Cost Functions

$$AVC(y) = \frac{c_v(y)}{y},$$

$$\frac{\partial\, AVC(y)}{\partial\, y} = \frac{y \times MC(y) - 1 \times c_v(y)}{y^2}.$$

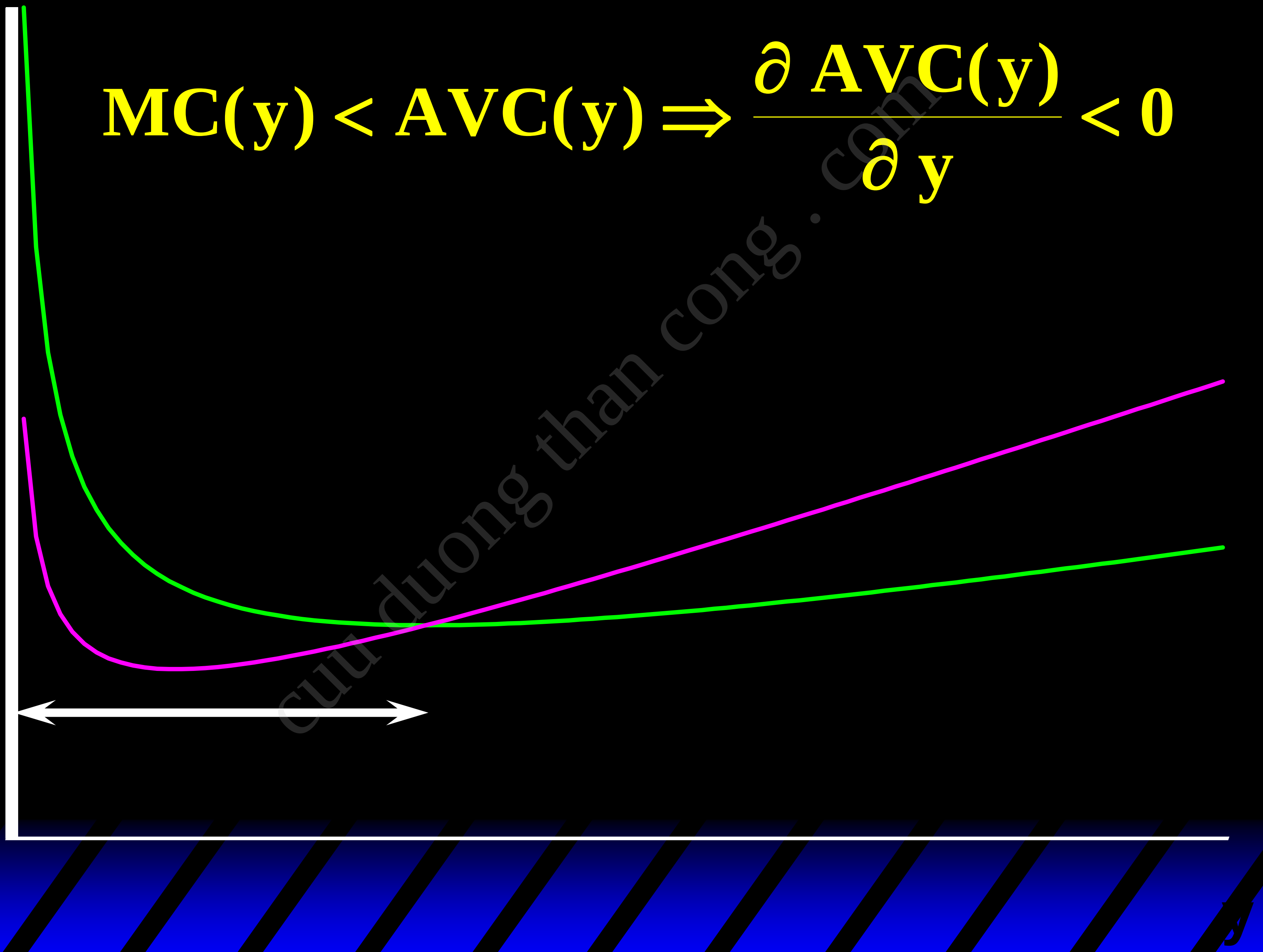
$$\frac{\partial\, AVC(y)}{\partial\, y} \begin{matrix} > \\ = \\ < \end{matrix} 0 \qquad y \times MC(y) \begin{matrix} > \\ = \\ < \end{matrix} c_v(y).$$

$$\frac{\partial\, AVC(y)}{\partial\, y} \begin{matrix} > \\ = \\ < \end{matrix} 0 \quad \text{as} \quad MC(y) \begin{matrix} > \\ = \\ < \end{matrix} \frac{c_v(y)}{y} = AVC(y).$$

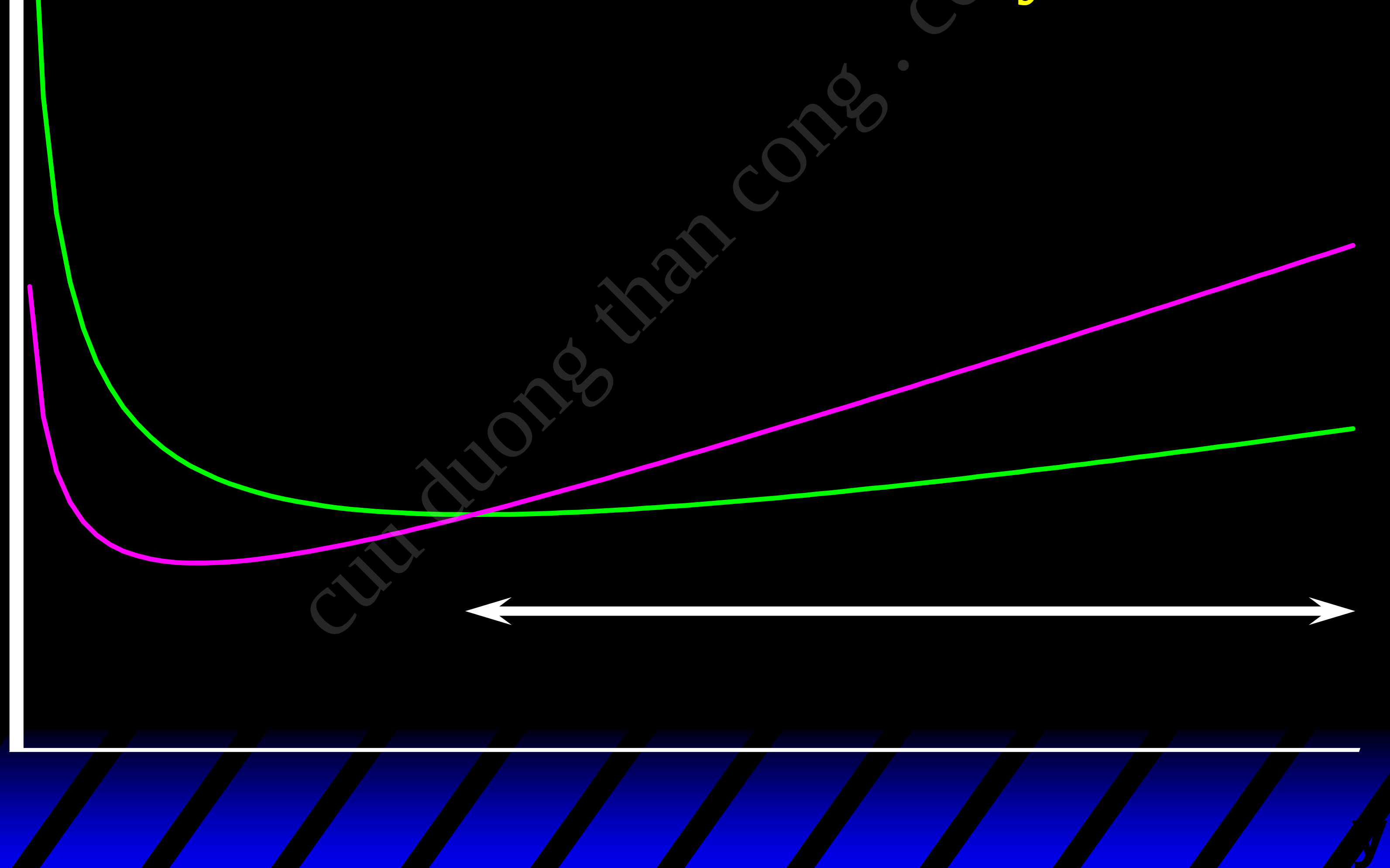
Marginal & Average Cost Functions

$$\frac{\partial \, AVC(y)}{\partial \, y} \gtreqless 0 \qquad MC(y) \gtreqless AVC(y).$$

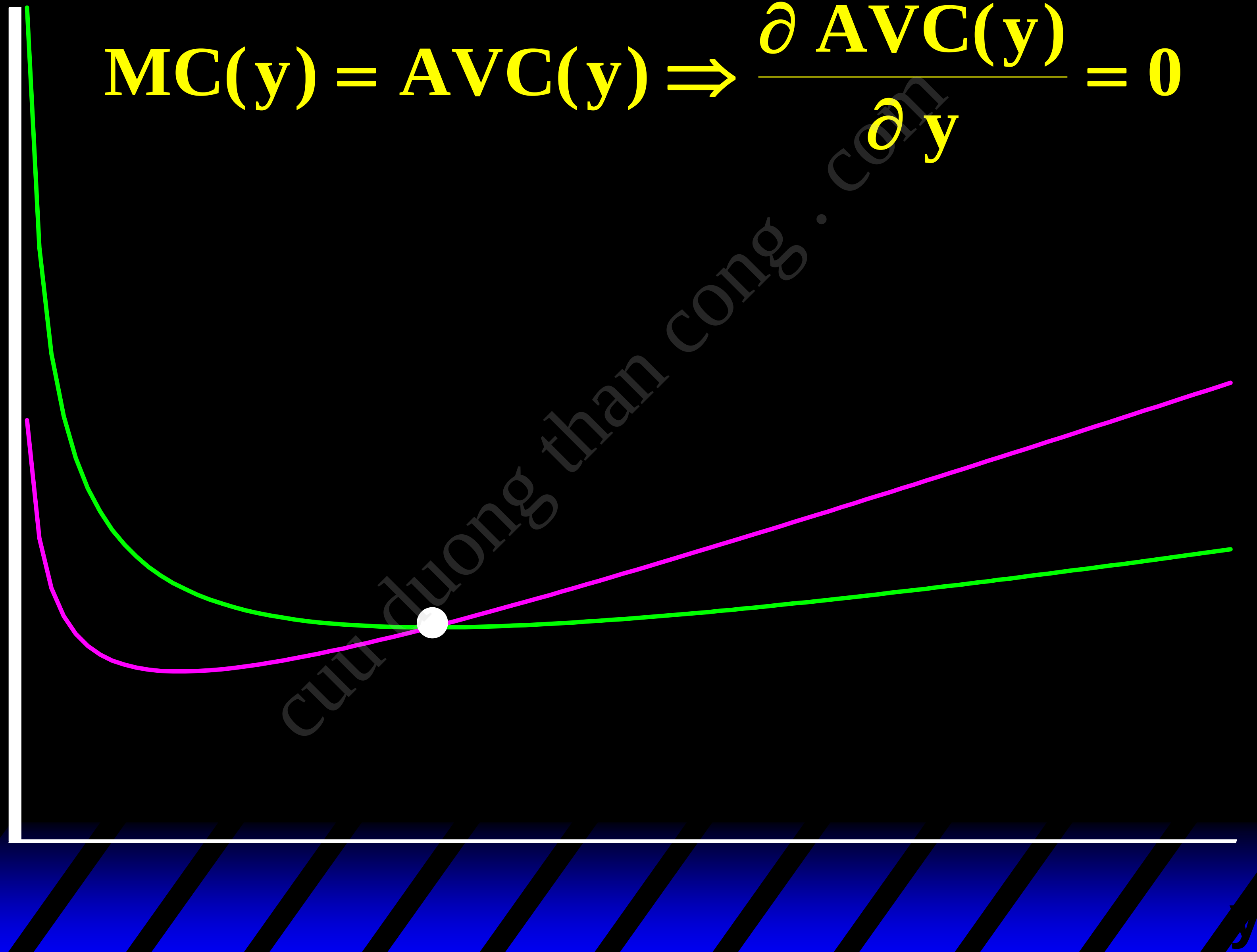
$$MC(y) < AVC(y) \Rightarrow \frac{\partial\, AVC(y)}{\partial\, y} < 0$$



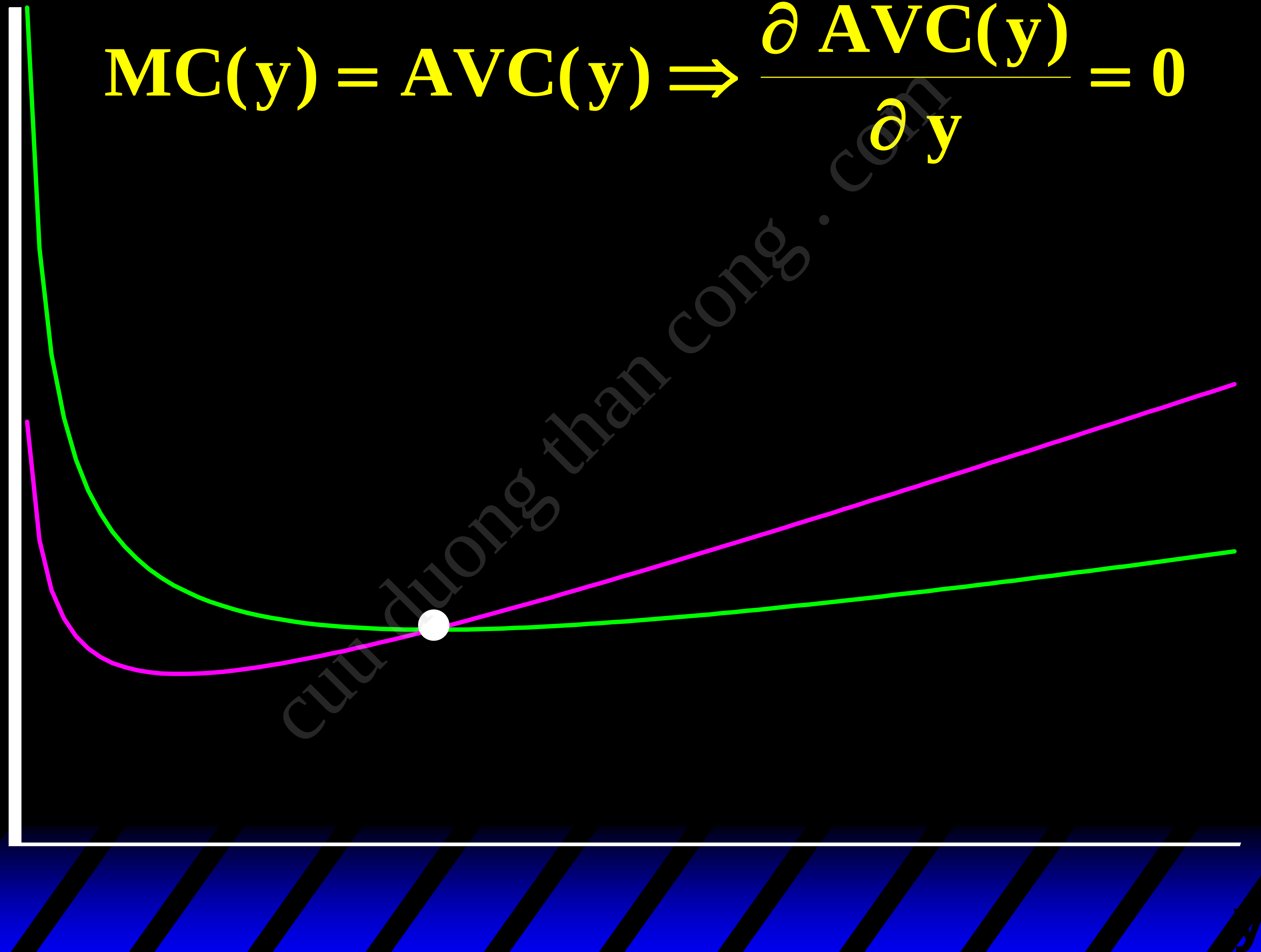
$$MC(y) > AVC(y) \Rightarrow \frac{\partial\, AVC(y)}{\partial\, y} > 0$$



$$MC(y) = AVC(y) \Rightarrow \frac{\partial\, AVC(y)}{\partial\, y} = 0$$



$$\mathbf{MC(y) = AVC(y) \Rightarrow \frac{\partial\, AVC(y)}{\partial\, y} = 0}$$



Marginal & Average Cost Functions

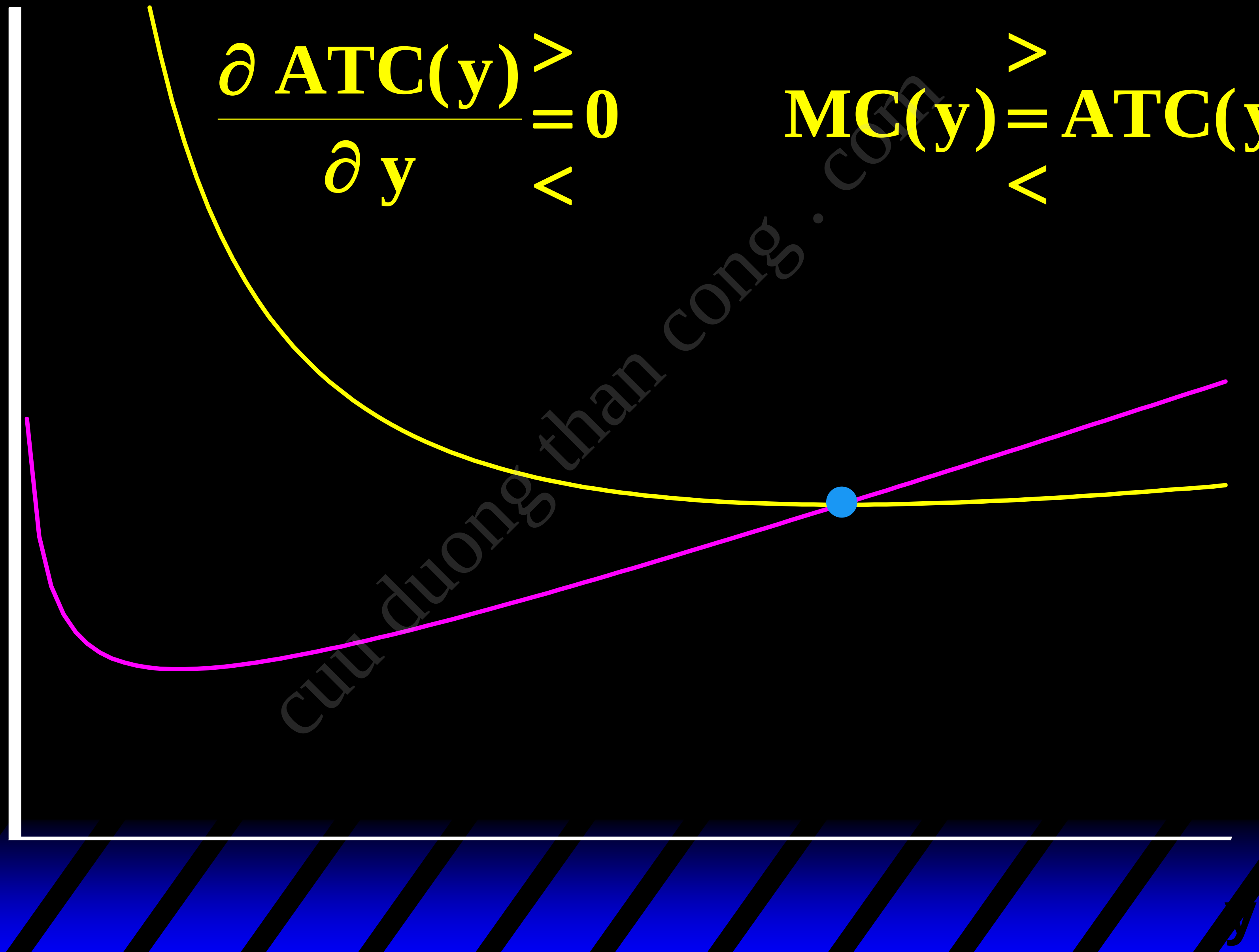
$$\mathrm{ATC}(y) = \frac{c(y)}{y},$$

$$\frac{\partial\, \mathrm{ATC}(y)}{\partial\, y} = \frac{y \times \mathrm{MC}(y) - 1 \times c(y)}{y^2}.$$

$$\frac{\partial\, \mathrm{ATC}(y)}{\partial\, y} \begin{matrix} > \\ = \\ < \end{matrix} 0 \qquad y \times \mathrm{MC}(y) \begin{matrix} > \\ = \\ < \end{matrix} c(y).$$

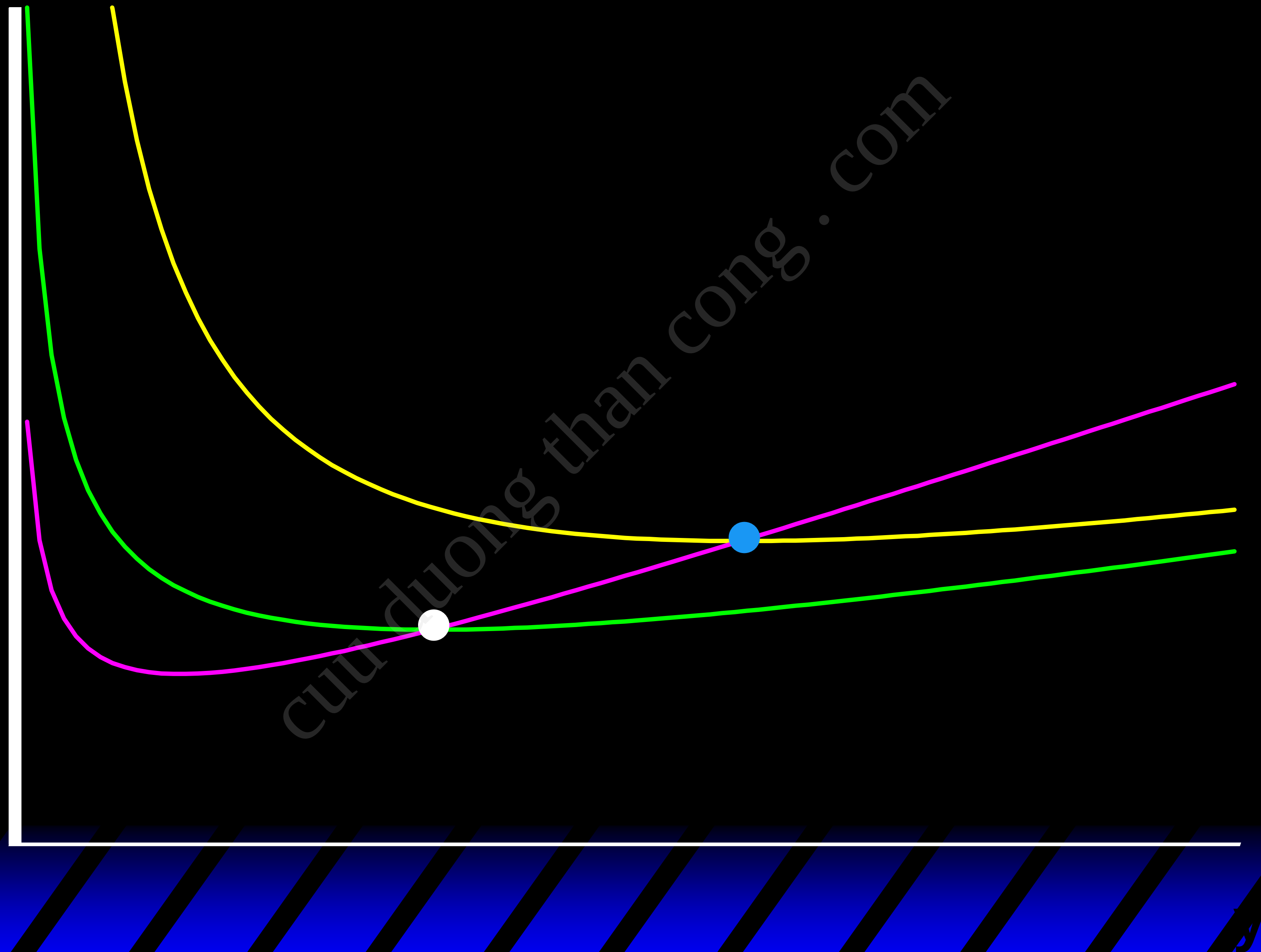
$$\frac{\partial\, \mathrm{ATC}(y)}{\partial\, y} \begin{matrix} > \\ = \\ < \end{matrix} 0 \quad \text{as} \quad \mathrm{MC}(y) \begin{matrix} > \\ = \\ < \end{matrix} \frac{c(y)}{y} = \mathrm{ATC}(y).$$

$$\frac{\partial\, ATC(y)}{\partial y} \gtreqless 0 \qquad MC(y) \gtreqless ATC(y)$$



Marginal & Average Cost Functions

- The short-run MC curve intersects the short-run AVC curve from below at the AVC curve's minimum.
- And, similarly, the short-run MC curve intersects the short-run ATC curve from below at the ATC curve's minimum.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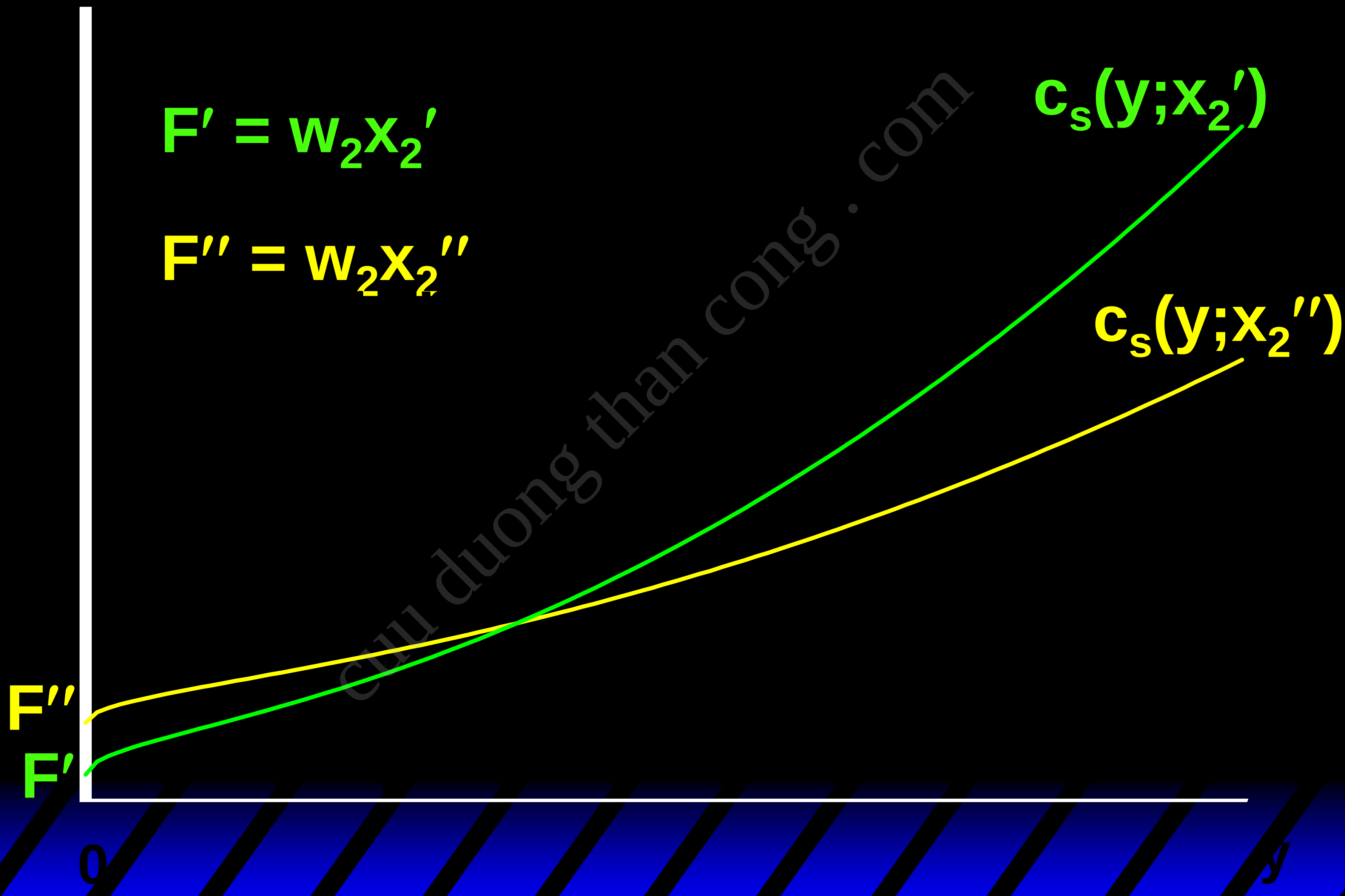
Short-Run & Long-Run Total Cost Curves

- A firm has a different short-run total cost curve for each possible short-run circumstance.
- Suppose the firm can be in one of just three short-runs;

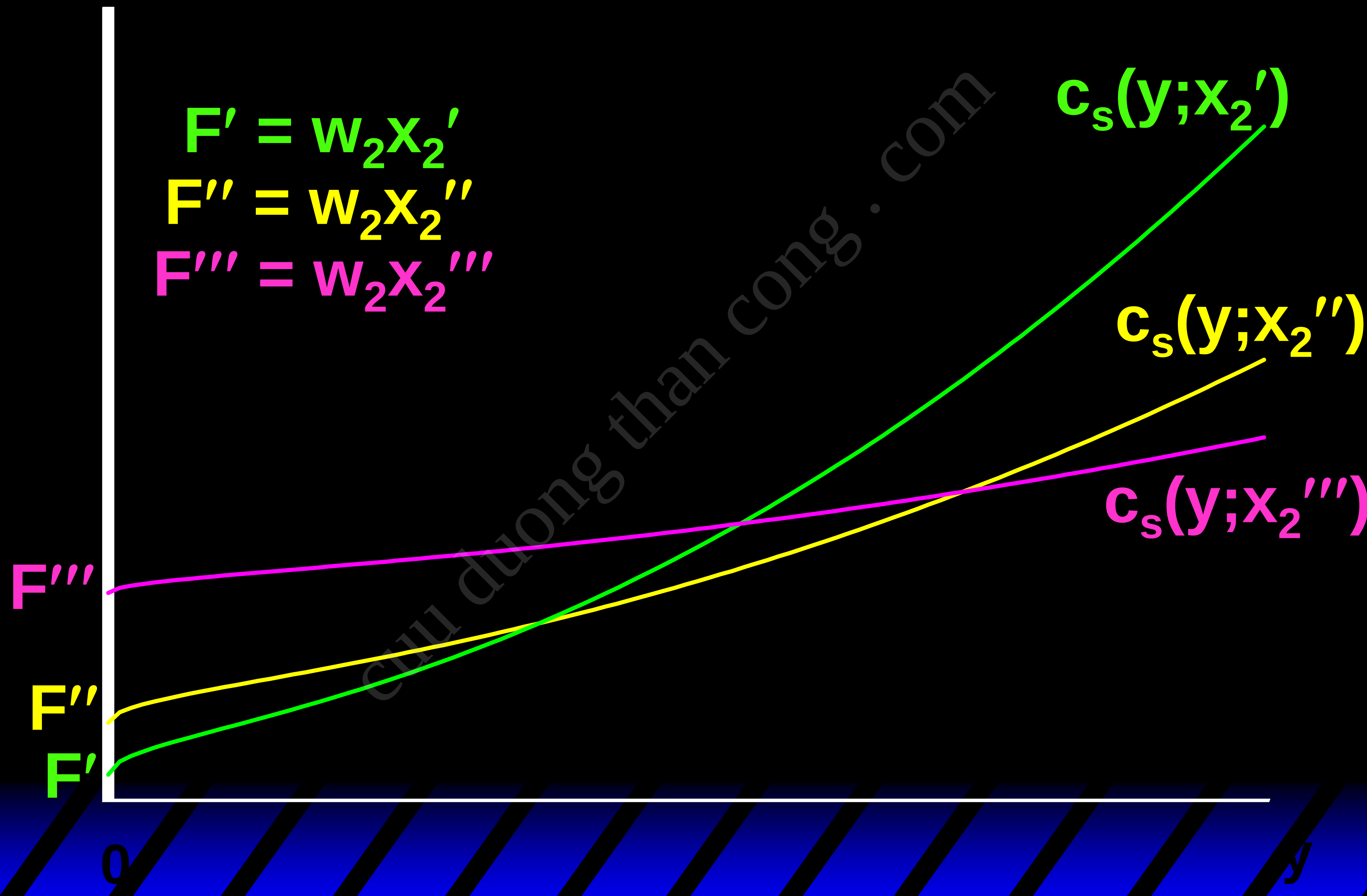
$$x_2 = x_2'$$

or $$x_2 = x_2''$$ $$x_2' < x_2'' < x_2'''.$$

or $$x_2 = x_2'''.$$

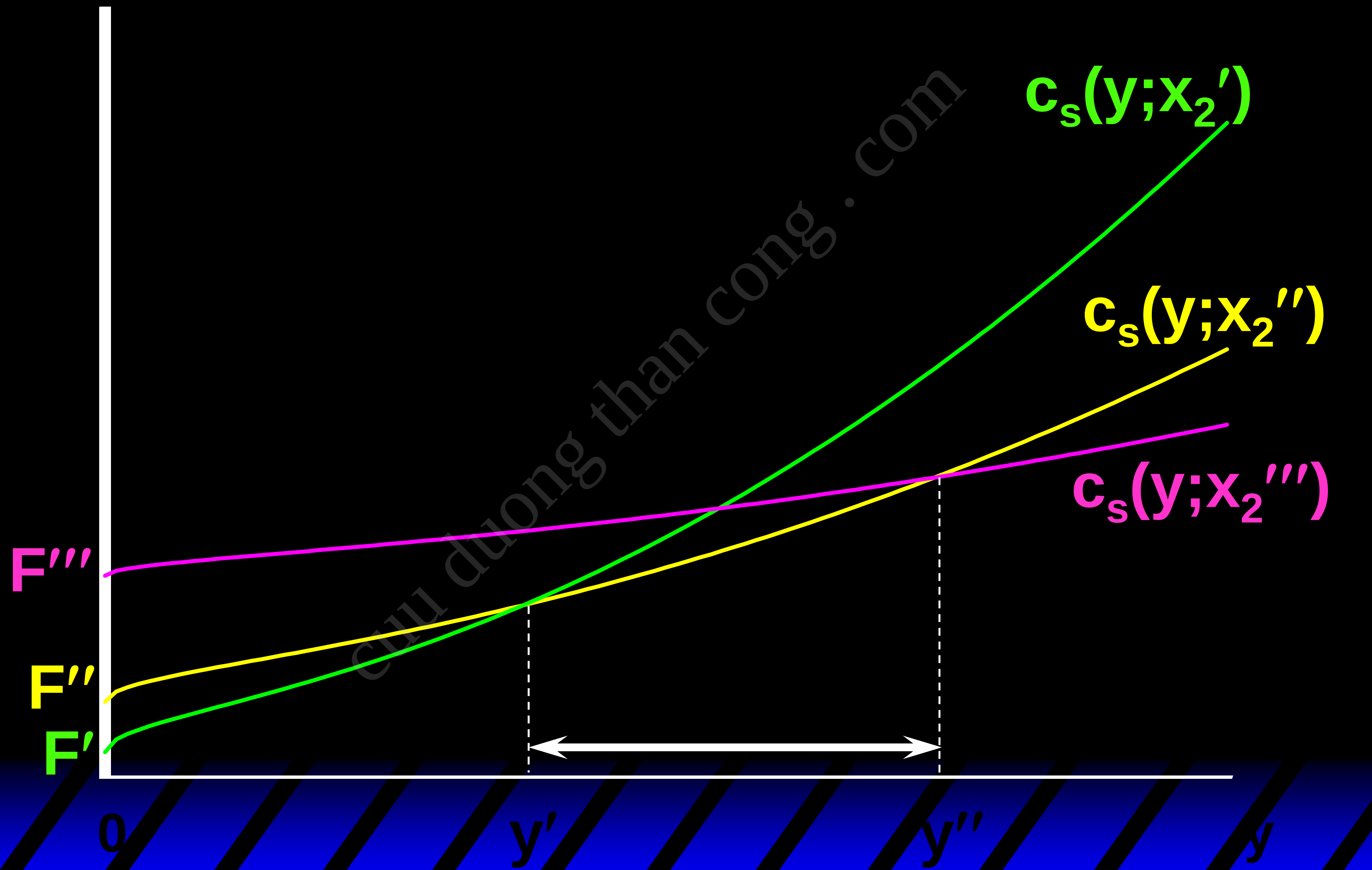


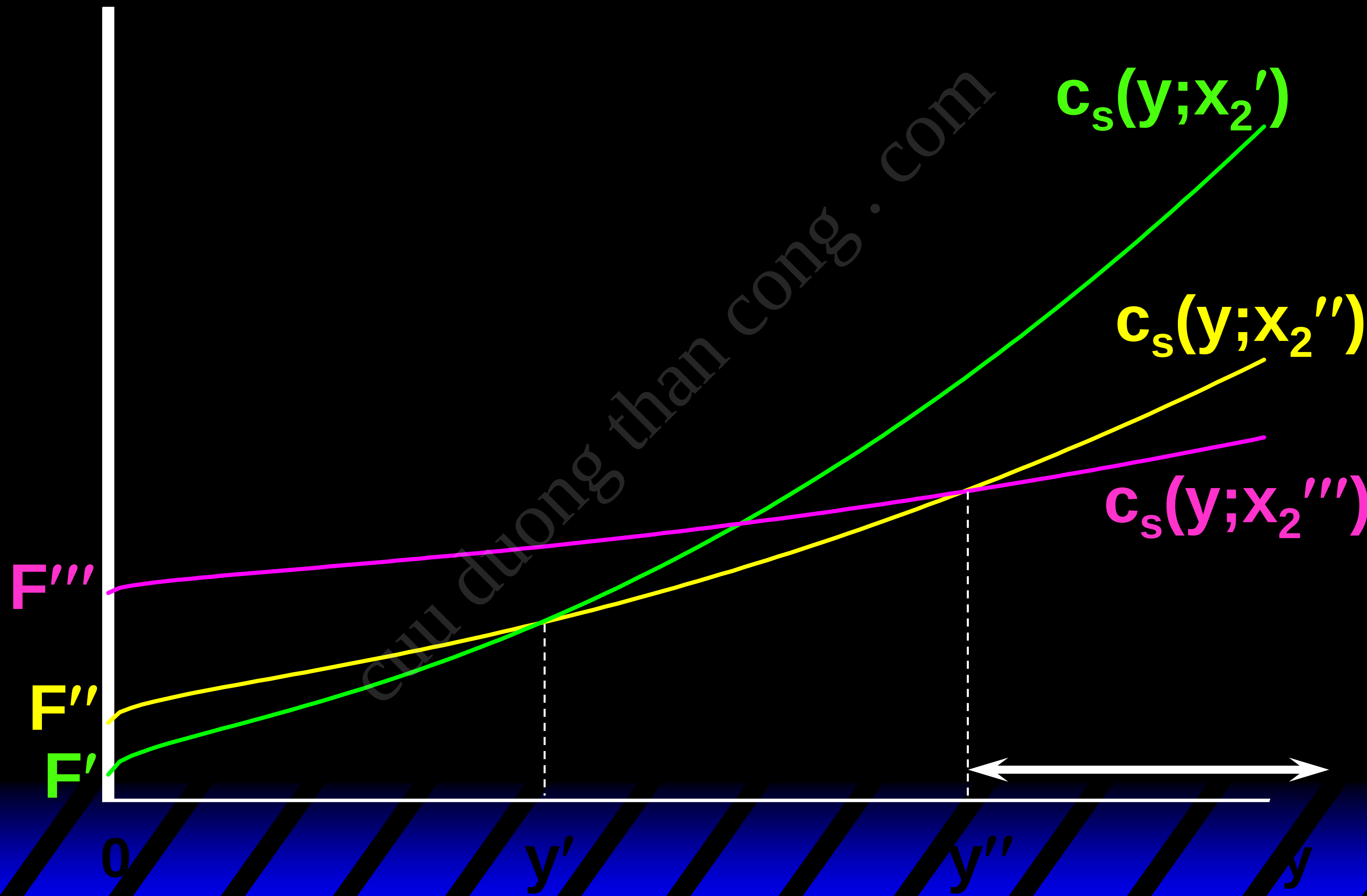
F′ = w2x2′
F″ = w2x2″
cs(y;x2′)
cs(y;x2″)
F″
F′
0
y


F′ = w2x2′
F″ = w2x2″
F‴ = w2x2‴
cs(y;x2′)
cs(y;x2″)
cs(y;x2‴)
F‴
F″
F′
0
y

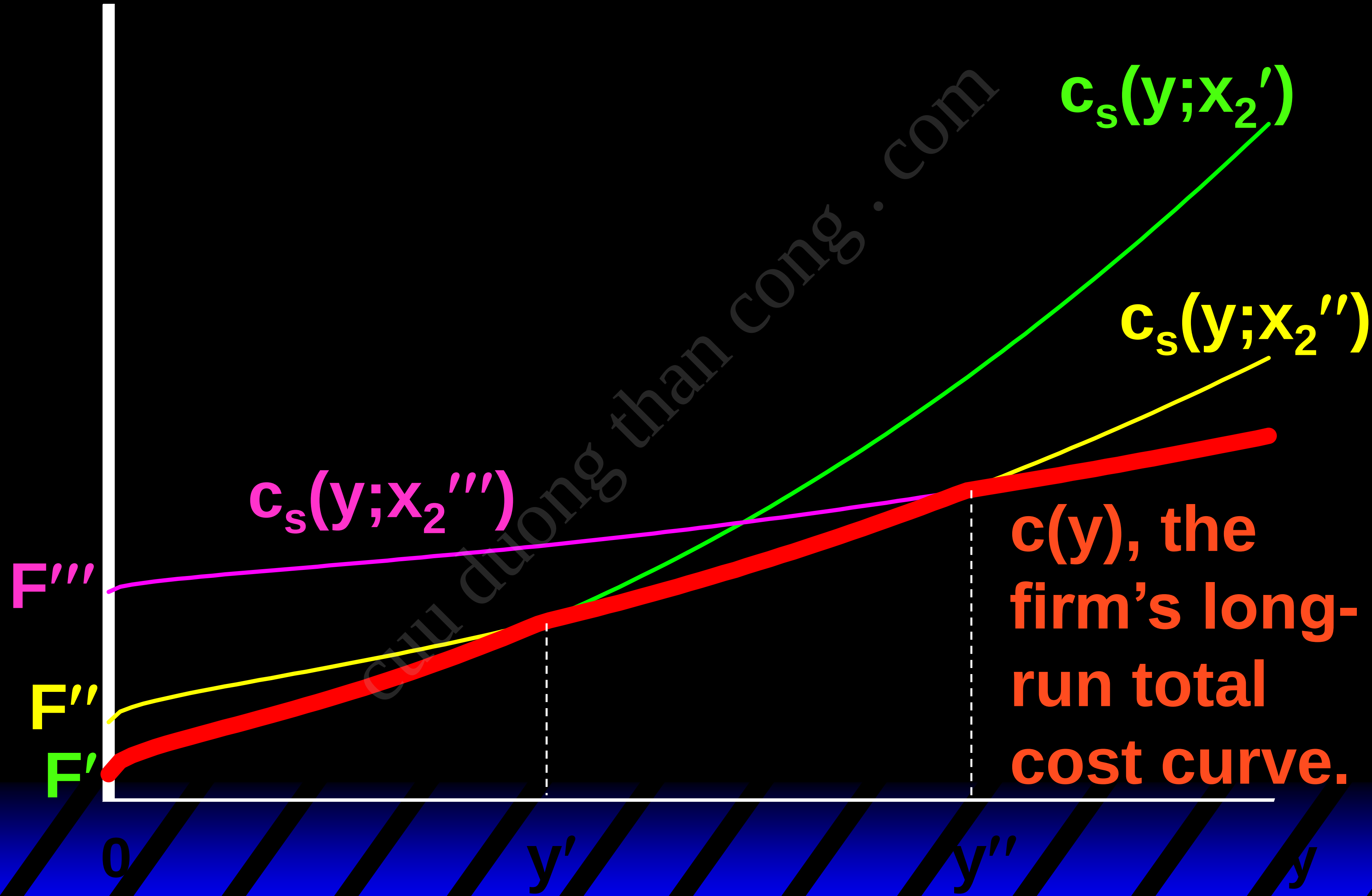
Short-Run & Long-Run Total Cost Curves

- The firm has three short-run total cost curves.
- In the long-run the firm is free to choose amongst these three since it is free to select x_2 equal to any of x_2', x_2'', or x_2'''.
- How does the firm make this choice?


$c_s(y;x_2')$
$c_s(y;x_2'')$
$c_s(y;x_2''')$
F'''
F''
F'
0
y'
y''
y



$c_s(y;x_2')$
$c_s(y;x_2'')$
$c_s(y;x_2''')$
F'''
F''
F'
0
y'
y''
y

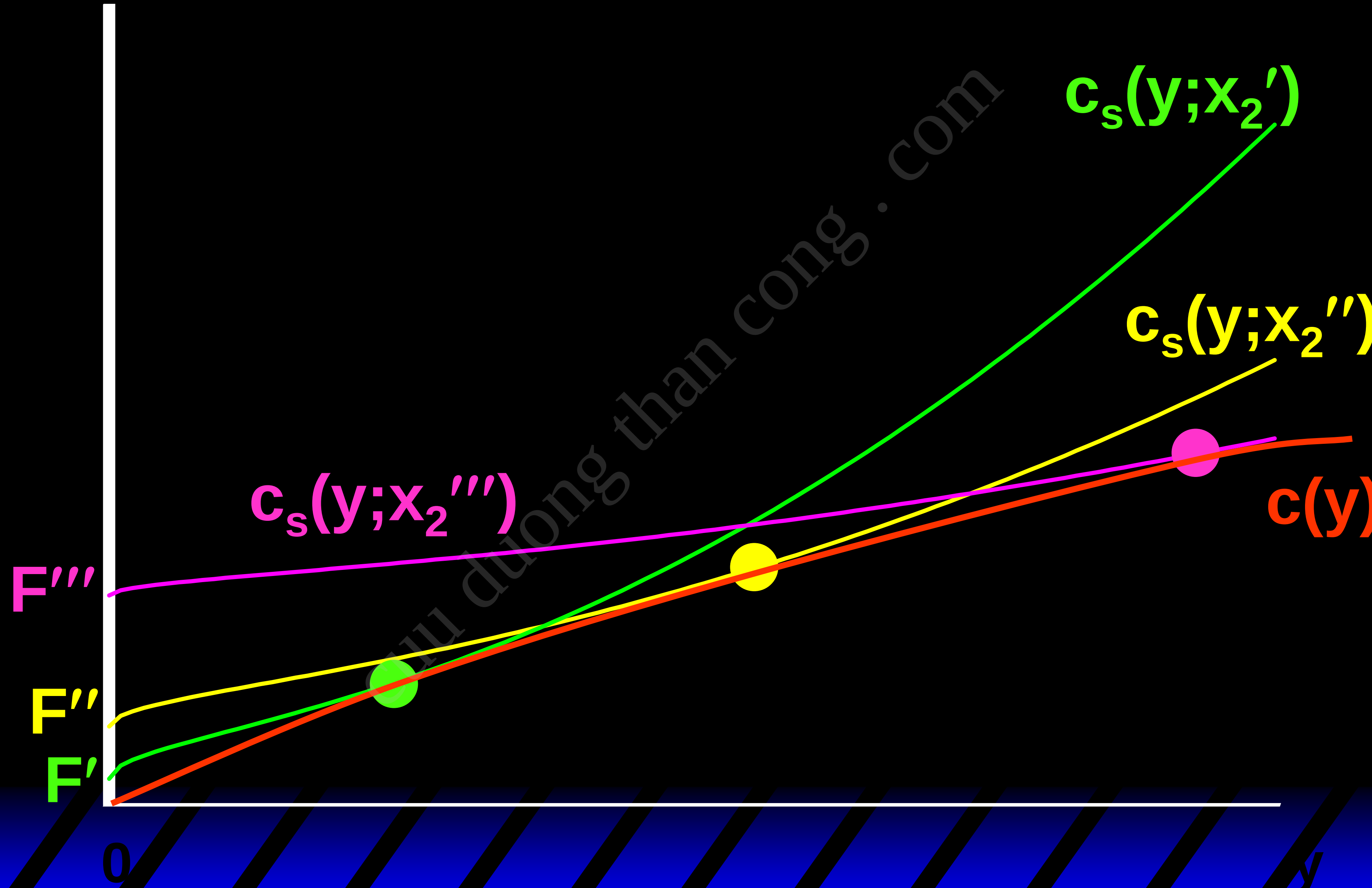

$c_s(y;x_2')$
$c_s(y;x_2'')$
$c_s(y;x_2''')$
F'''
F''
F'
0
y'
y''
y
c(y), the firm's long-run total cost curve.

Short-Run & Long-Run Total Cost Curves

- **The firm's long-run total cost curve consists of the lowest parts of the short-run total cost curves. The long-run total cost curve is the lower envelope of the short-run total cost curves.**

Short-Run & Long-Run Total Cost Curves

- If input 2 is available in continuous amounts then there is an infinity of short-run total cost curves but the long-run total cost curve is still the lower envelope of all of the short-run total cost curves.



$c_s(y;x_2')$
$c_s(y;x_2'')$
$c_s(y;x_2''')$
$c(y)$
F'''
F''
F'
0
y

Short-Run & Long-Run Average Total Cost Curves

- For any output level y, the long-run total cost curve always gives the lowest possible total production cost.
- Therefore, the long-run av. total cost curve must always give the lowest possible av. total production cost.
- The long-run av. total cost curve must be the lower envelope of all of the firm's short-run av. total cost curves.

Short-Run & Long-Run Average Total Cost Curves

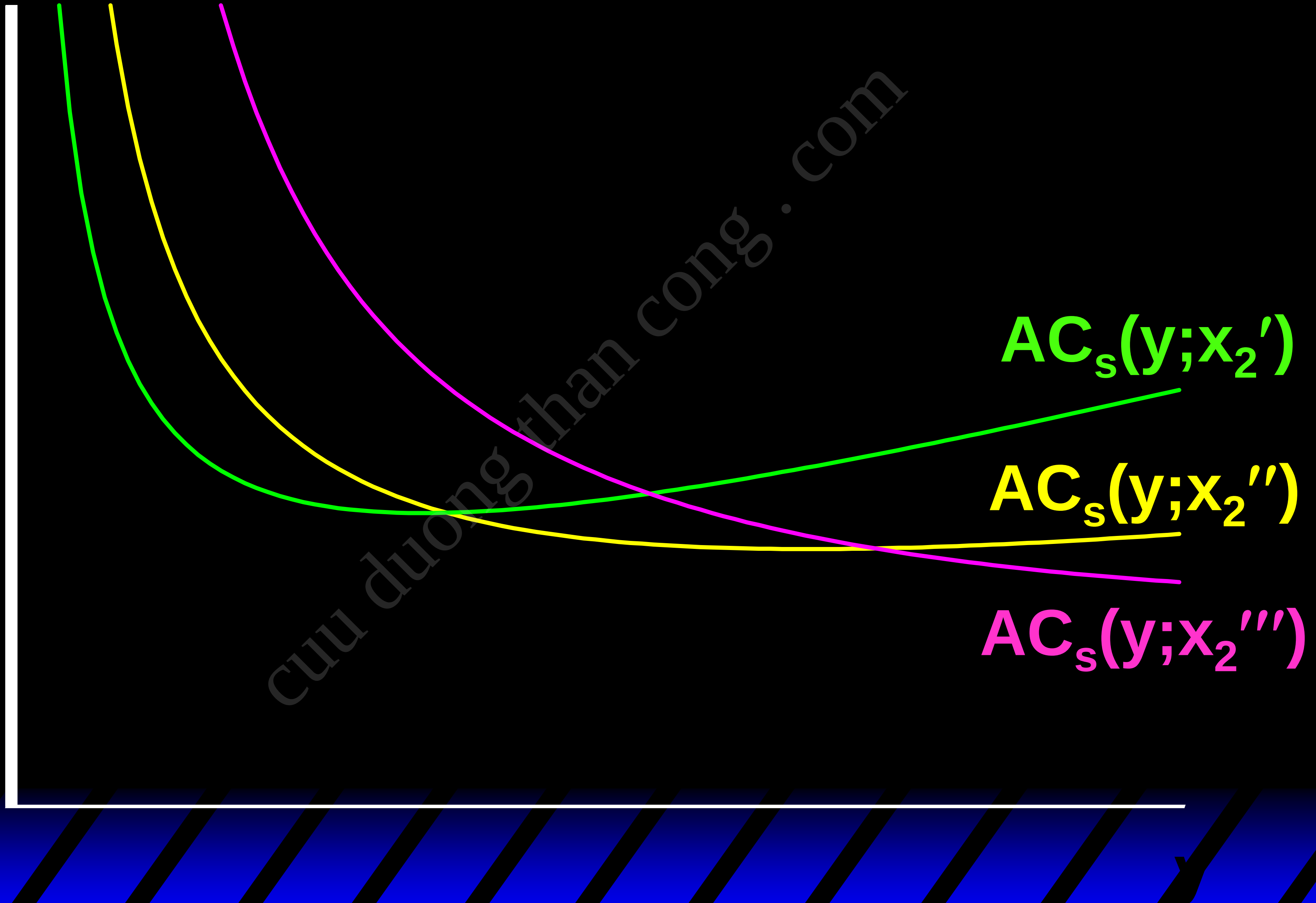
- **E.g. suppose again that the firm can be in one of just three short-runs;**

  $x_2 = x_2'$

  or $x_2 = x_2''$ $(x_2' < x_2'' < x_2''')$

  or $x_2 = x_2'''$

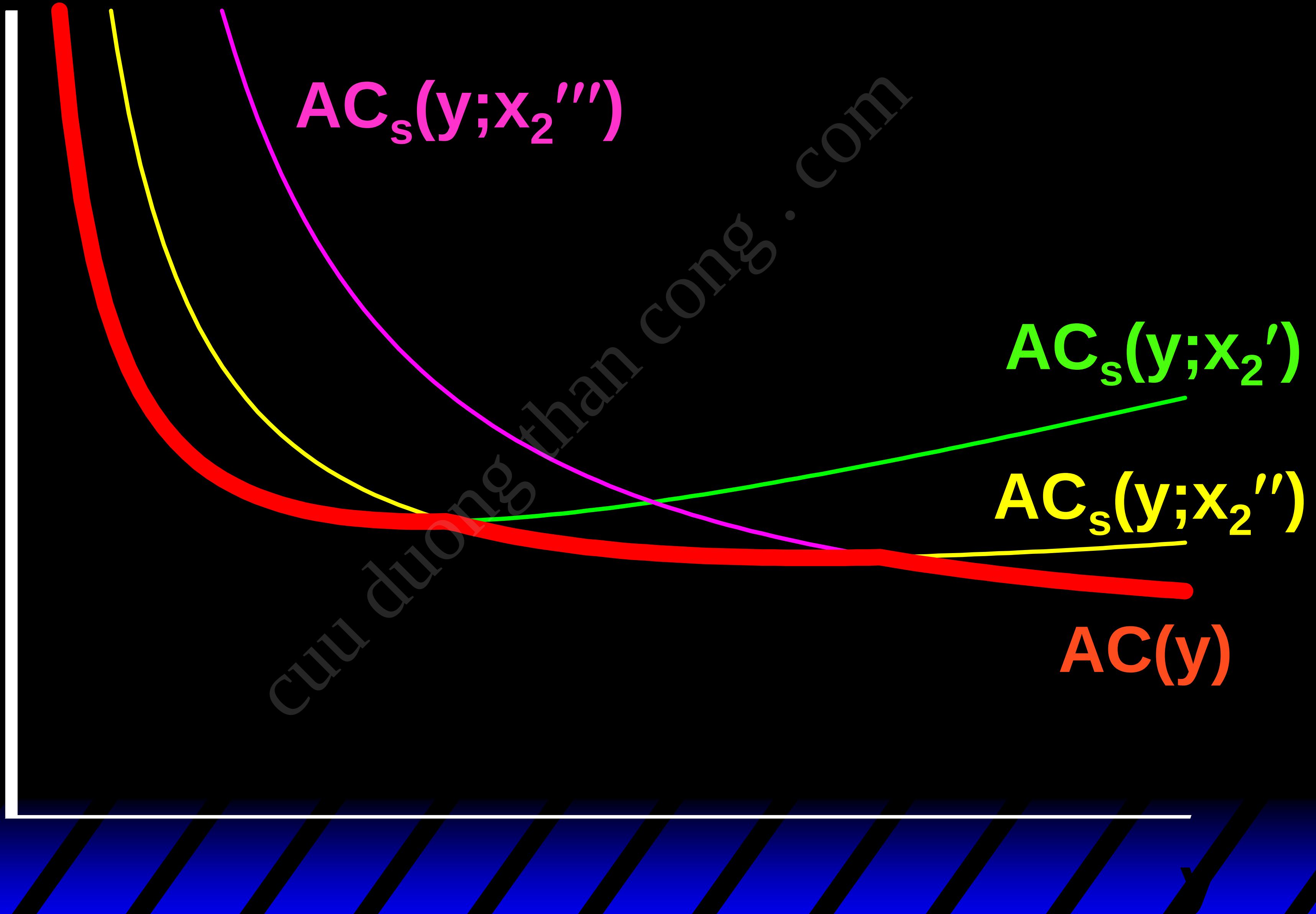
  then the firm's three short-run average total cost curves are ...



$AC_s(y;x_2')$
$AC_s(y;x_2'')$
$AC_s(y;x_2''')$
y

Short-Run & Long-Run Average Total Cost Curves

- **The firm's long-run average total cost curve is the lower envelope of the short-run average total cost curves ...**


$AC_s(y;x_2''')$
$AC_s(y;x_2')$
$AC_s(y;x_2'')$
AC(y)
y

Short-Run & Long-Run Marginal Cost Curves

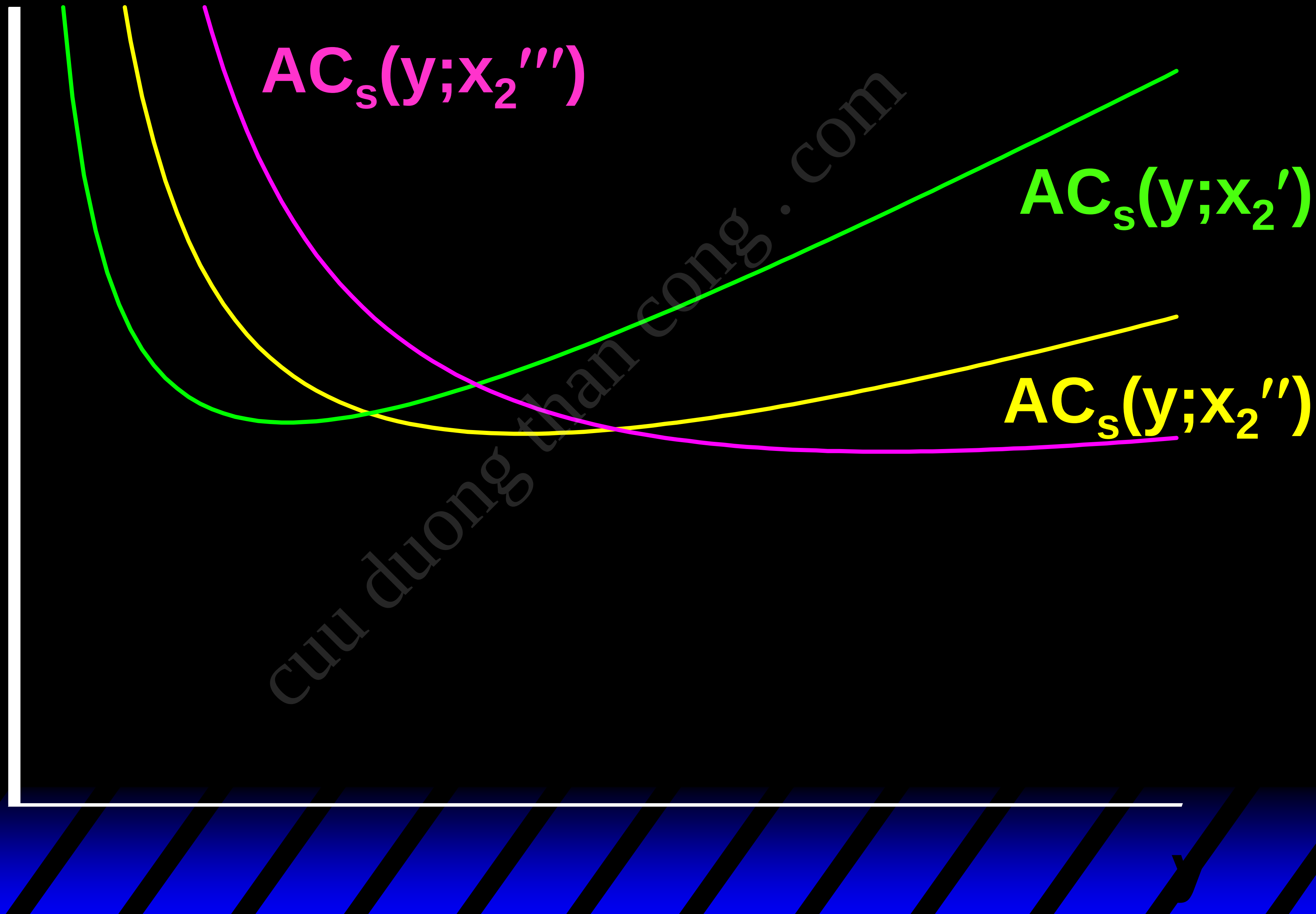
- **Q: Is the long-run marginal cost curve the lower envelope of the firm's short-run marginal cost curves?**

Short-Run & Long-Run Marginal Cost Curves

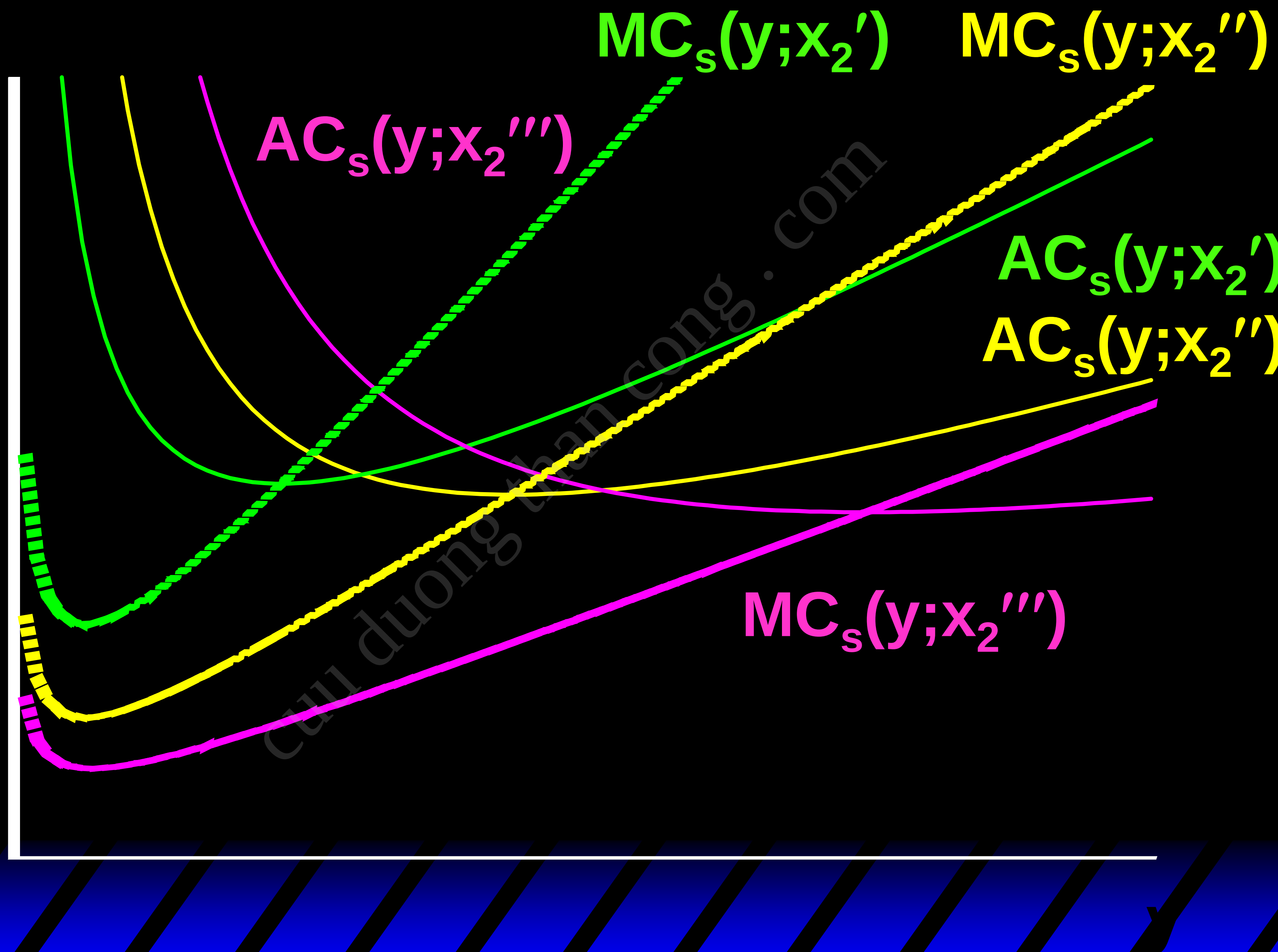
- **Q: Is the long-run marginal cost curve the lower envelope of the firm's short-run marginal cost curves?**
- **A: No.**

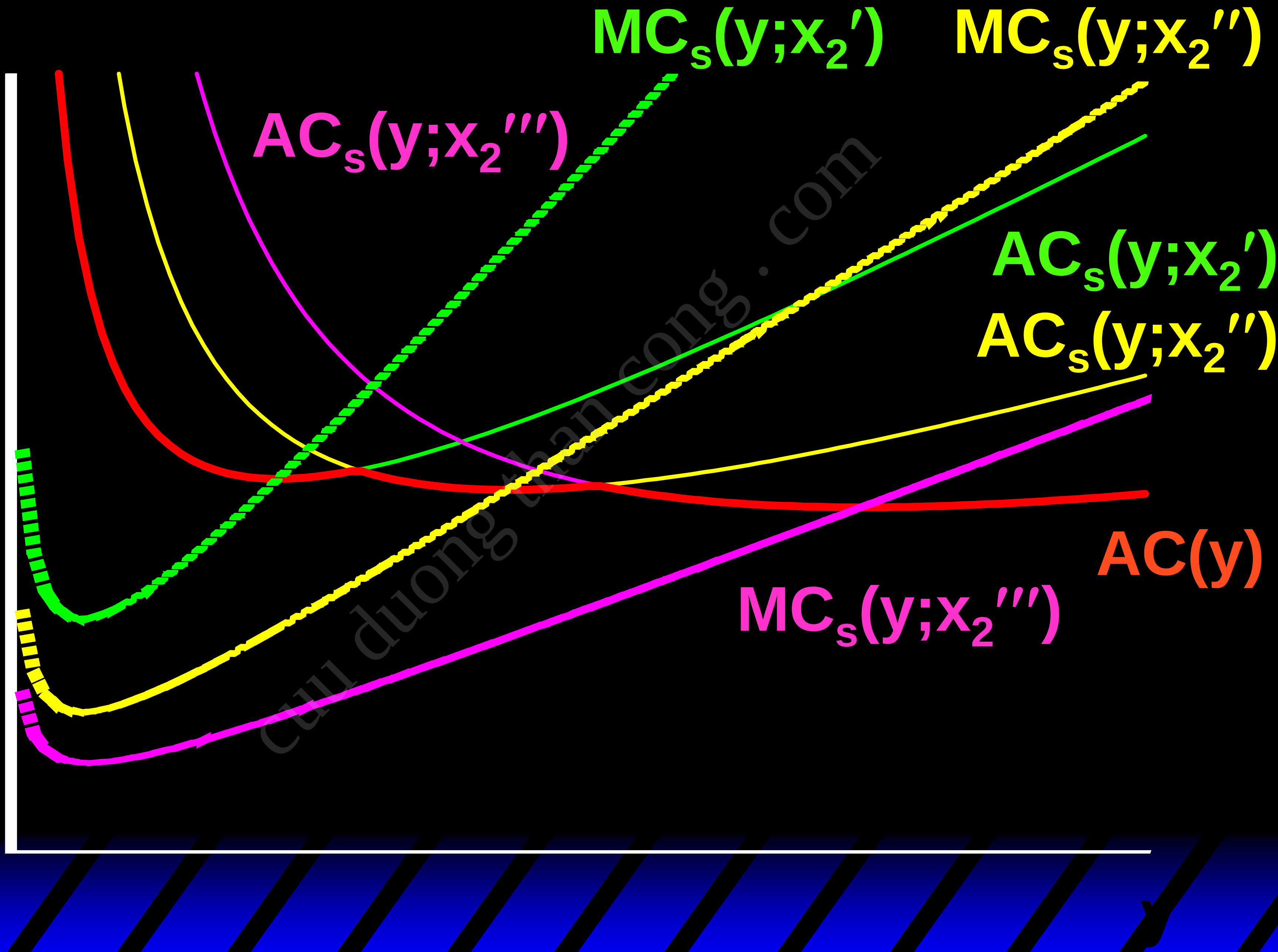
Short-Run & Long-Run Marginal Cost Curves

- The firm's three short-run average total cost curves are ...

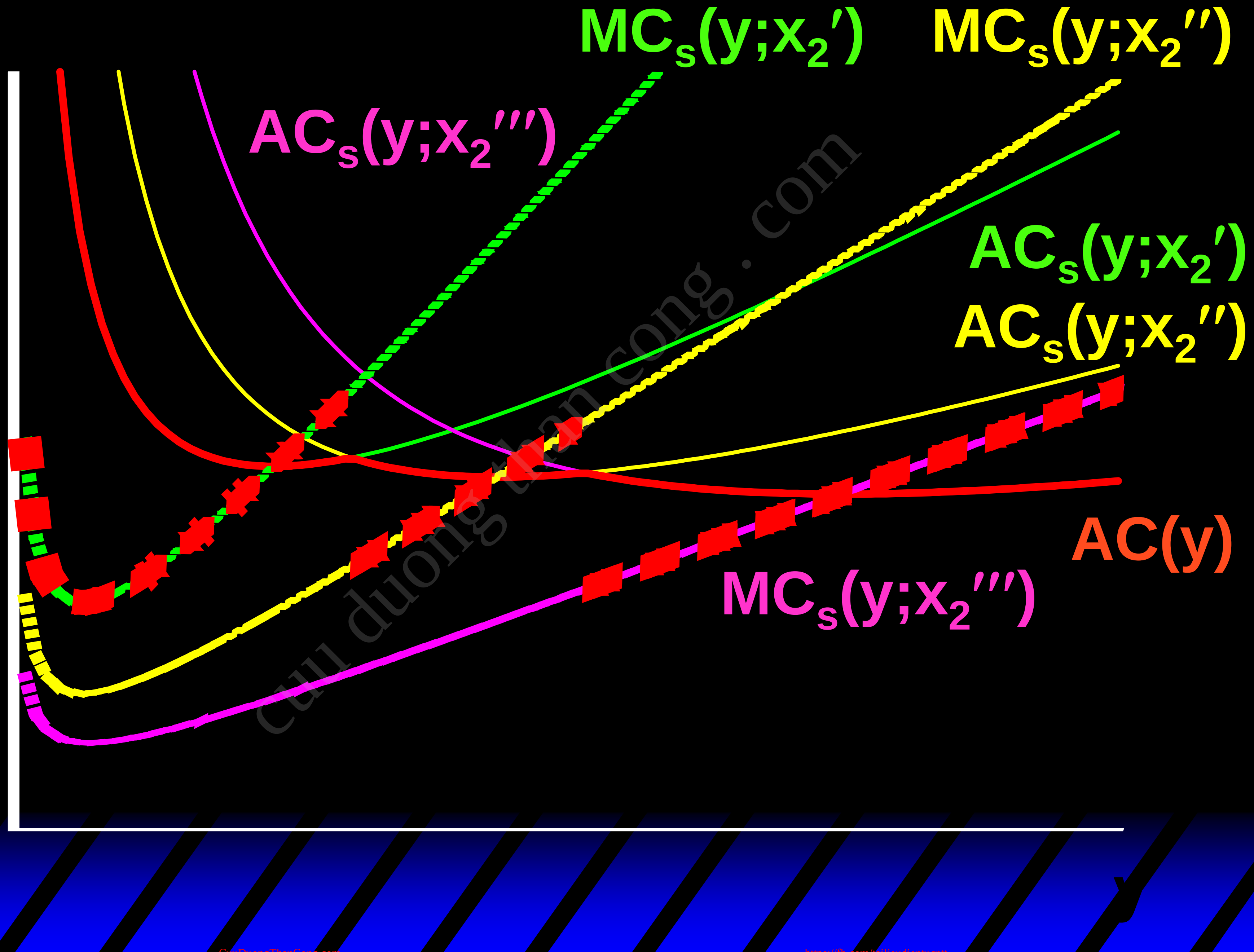


ACs(y;x2''')
ACs(y;x2')
ACs(y;x2'')
y

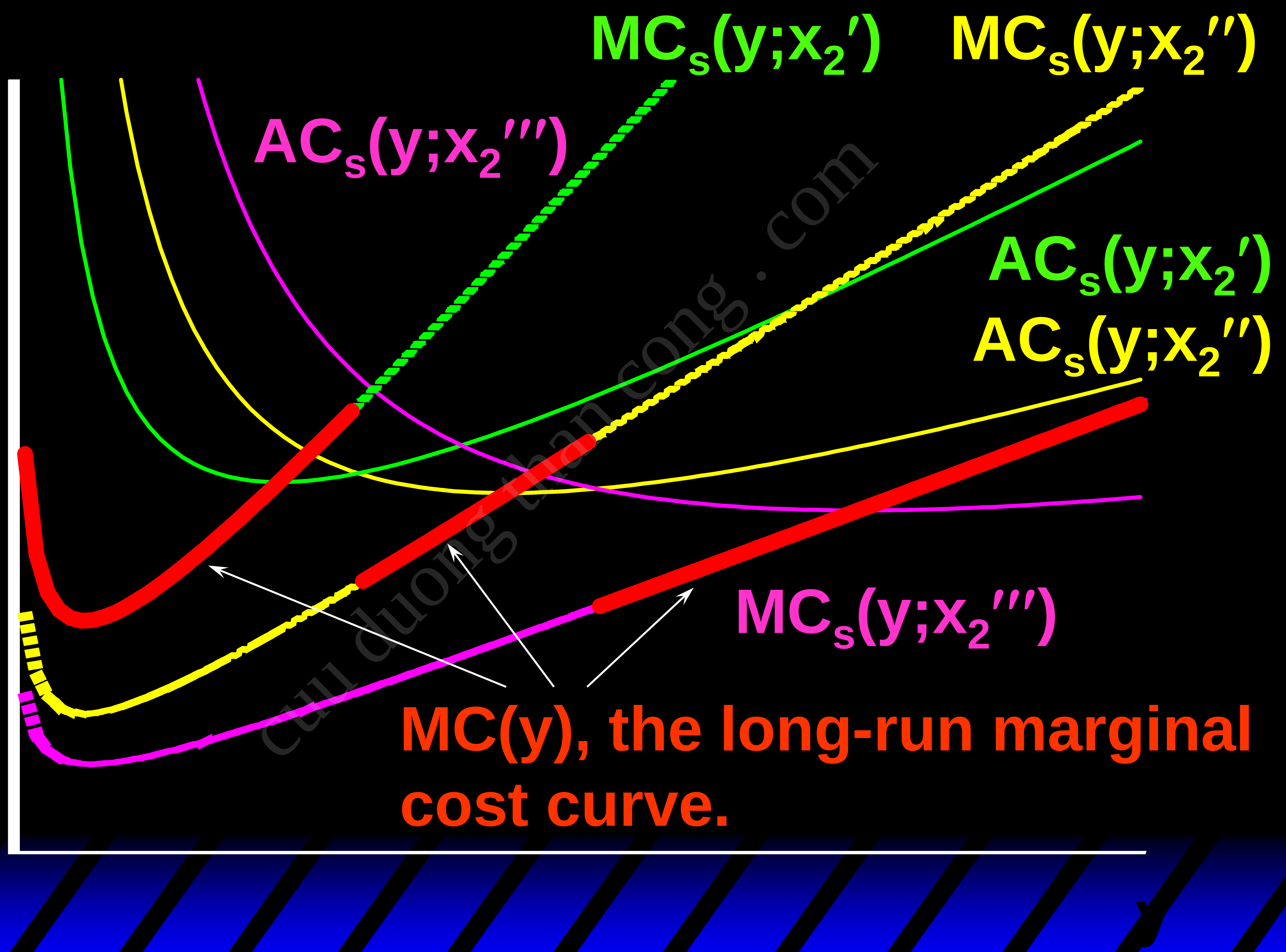

MCs(y;x2′)
MCs(y;x2″)
ACs(y;x2‴)
ACs(y;x2′)
ACs(y;x2″)
MCs(y;x2‴)
y



$MC_s(y;x_2')$
$MC_s(y;x_2'')$
$AC_s(y;x_2''')$
$AC_s(y;x_2')$
$AC_s(y;x_2'')$
$AC(y)$
$MC_s(y;x_2''')$
y

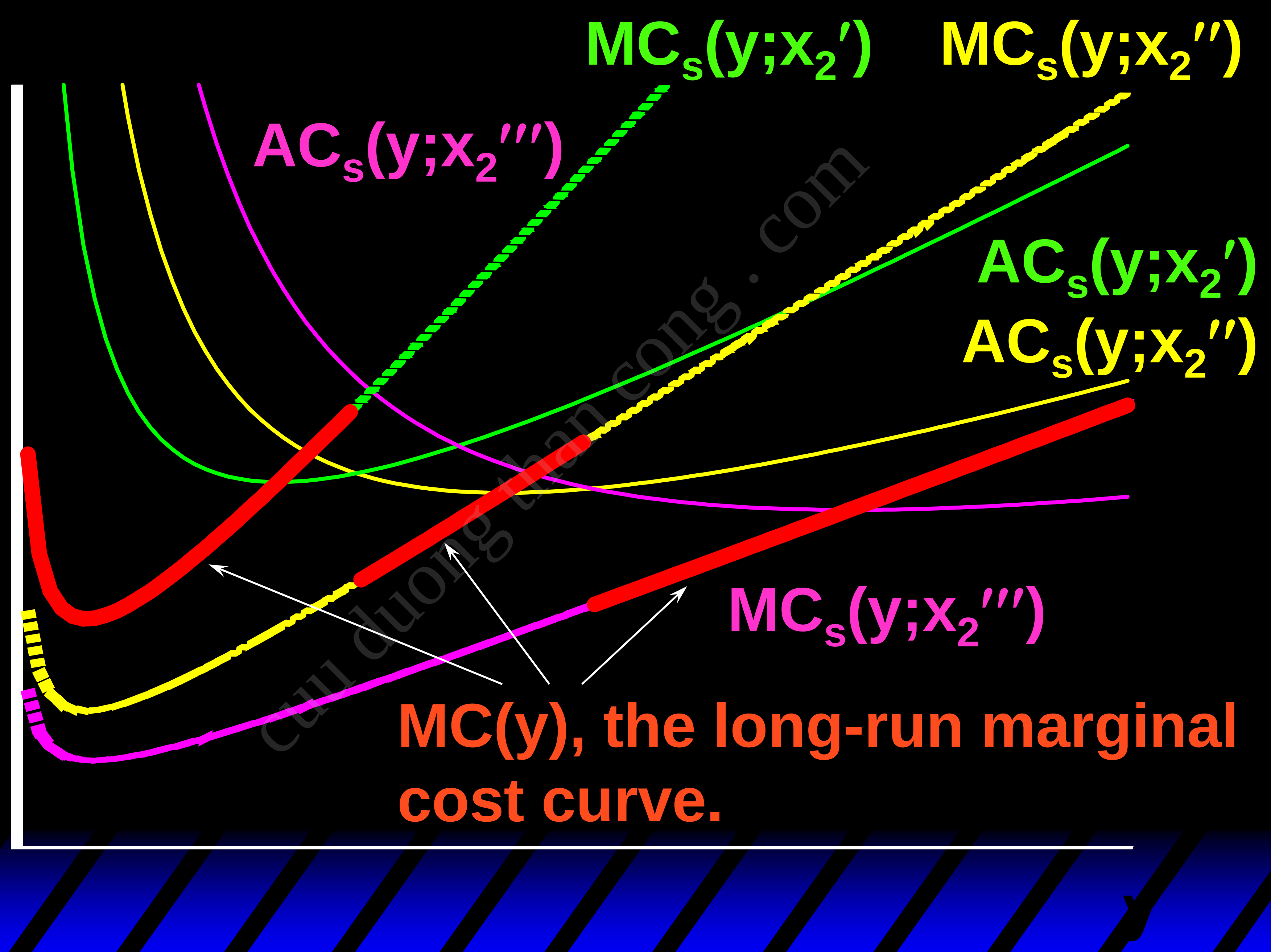


MCs(y;x2′)
MCs(y;x2″)
ACs(y;x2‴)
ACs(y;x2′)
ACs(y;x2″)
AC(y)
MCs(y;x2‴)
y


$MC_s(y;x_2')$
$MC_s(y;x_2'')$
$AC_s(y;x_2''')$
$AC_s(y;x_2')$
$AC_s(y;x_2'')$
$MC_s(y;x_2''')$
MC(y), the long-run marginal cost curve.
y

Short-Run & Long-Run Marginal Cost Curves

- **For any output level y > 0, the long-run marginal cost of production is the marginal cost of production for the short-run chosen by the firm.**


MCs(y;x2′)
MCs(y;x2″)
ACs(y;x2‴)
ACs(y;x2′)
ACs(y;x2″)
MCs(y;x2‴)
MC(y), the long-run marginal cost curve.
y

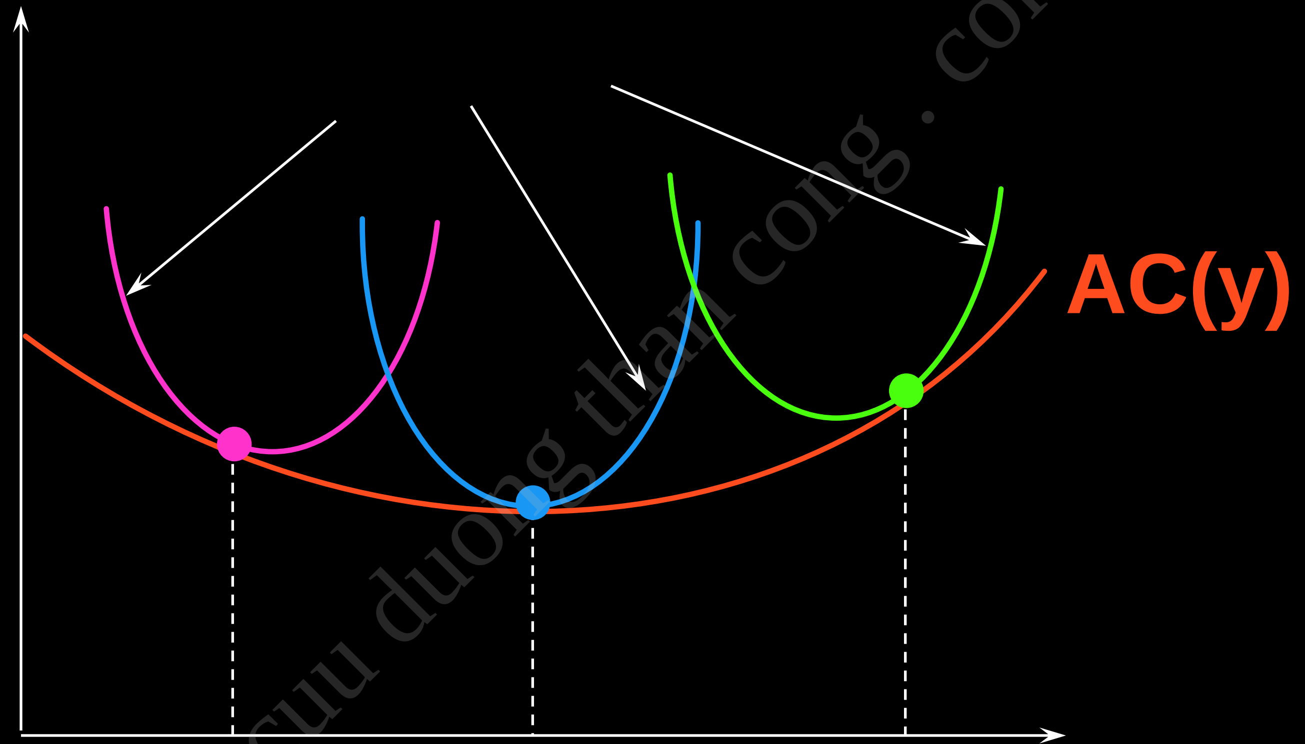
Short-Run & Long-Run Marginal Cost Curves

- **For any output level y > 0, the long-run marginal cost is the marginal cost for the short-run chosen by the firm.**
- **This is always true, no matter how many and which short-run circumstances exist for the firm.**

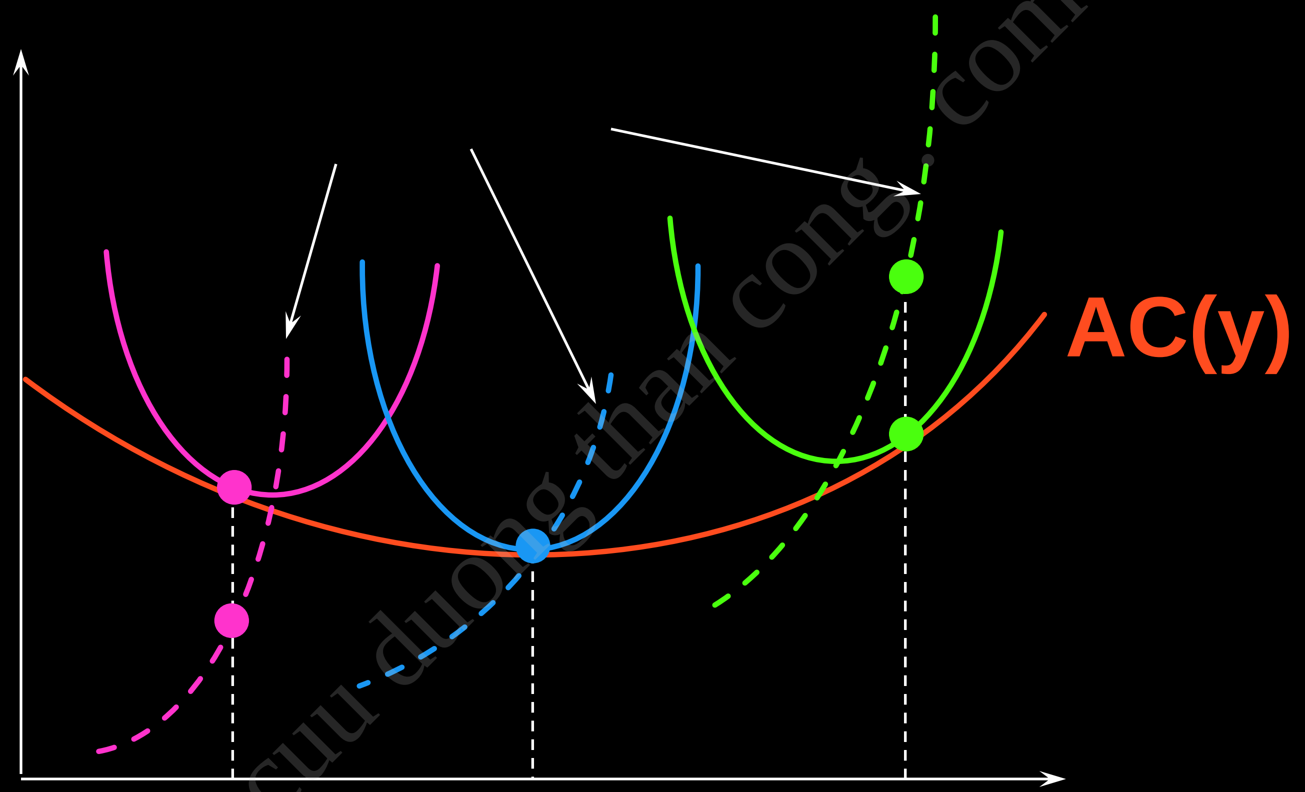
Short-Run & Long-Run Marginal Cost Curves

- For any output level y > 0, the long-run marginal cost is the marginal cost for the short-run chosen by the firm.
- So for the continuous case, where x_2 can be fixed at any value of zero or more, the relationship between the long-run marginal cost and all of the short-run marginal costs is ...

Short-Run & Long-Run Marginal Cost Curves



Short-Run & Long-Run Marginal Cost Curves



3. Cost Minimization

- **A firm is a cost-minimizer if it produces any given output level $y \geq 0$ at smallest possible total cost.**
- **c(y) denotes the firm's smallest possible total cost for producing y units of output.**
- **c(y) is the firm's total cost function.**

Cost Minimization

- When the firm faces given input prices $w = (w_1,w_2,\ldots,w_n)$ the total cost function will be written as

$$c(w_1,\ldots,w_n,y).$$

The Cost-Minimization Problem

- Consider a firm using two inputs to make one output.
- The production function is

$$y = f(x_1,x_2).$$

- Take the output level $y \geq 0$ as given.
- Given the input prices w_1 and w_2, the cost of an input bundle (x_1,x_2) is $w_1x_1 + w_2x_2$.

The Cost-Minimization Problem

- For given w_1, w_2 and y, the firm's cost-minimization problem is to solve $\min_{x_1,x_2 \geq 0} w_1x_1 + w_2x_2$

subject to $f(x_1, x_2) = y.$

The Cost-Minimization Problem

- The levels $x_1^*(w_1,w_2,y)$ and $x_2^*(w_1,w_2,y)$ in the least-costly input bundle are the firm's **conditional demands for inputs 1 and 2**.
- The (smallest possible) total cost for producing y output units is therefore

$$c(w_1,w_2,y) = w_1x_1^*(w_1,w_2,y) + w_2x_2^*(w_1,w_2,y).$$

Conditional Input Demands

- Given w_1, w_2 and y, how is the least costly input bundle located?
- And how is the total cost function computed?

Iso-cost Lines

- A curve that contains all of the input bundles that cost the same amount is an iso-cost curve.
- E.g., given w_1 and w_2, the \$100 iso-cost line has the equation

$$w_1 x_1 + w_2 x_2 = 100.$$

Iso-cost Lines

- Generally, given w_1 and w_2, the equation of the $c iso-cost line is

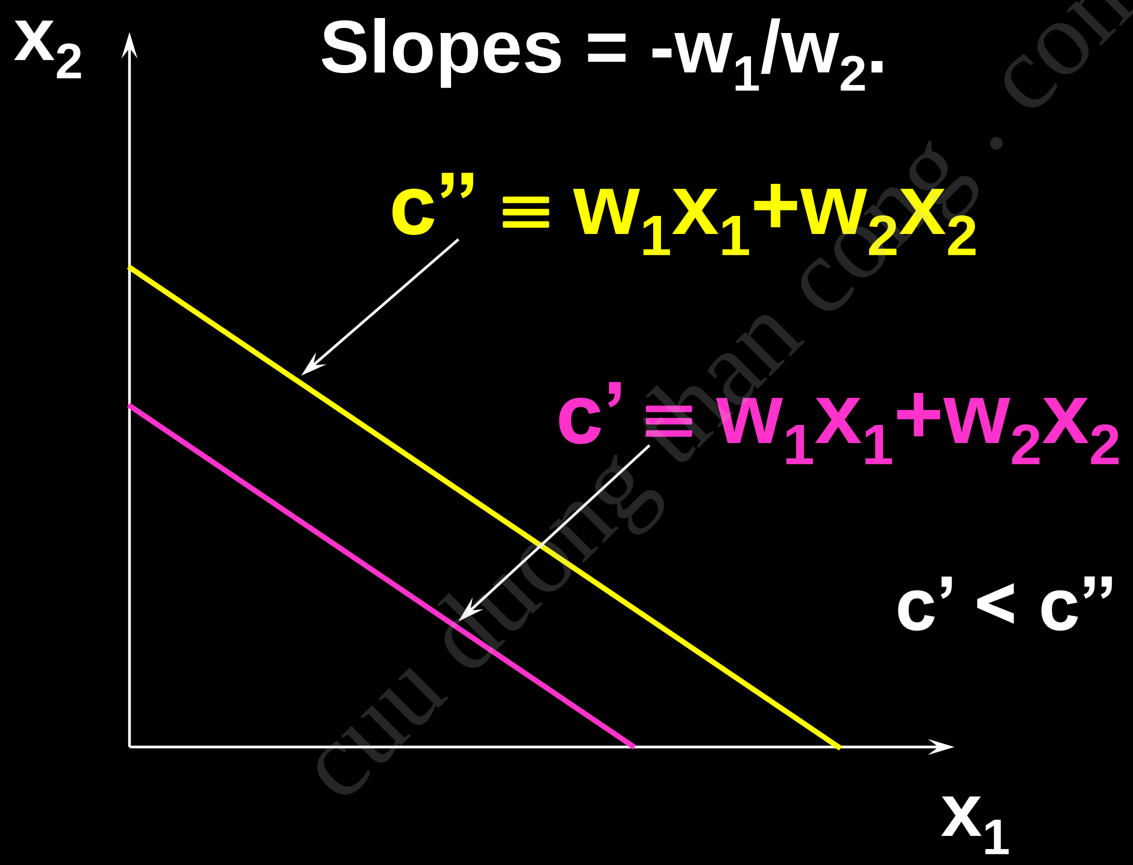
$$w_1 x_1 + w_2 x_2 = c$$

i.e.

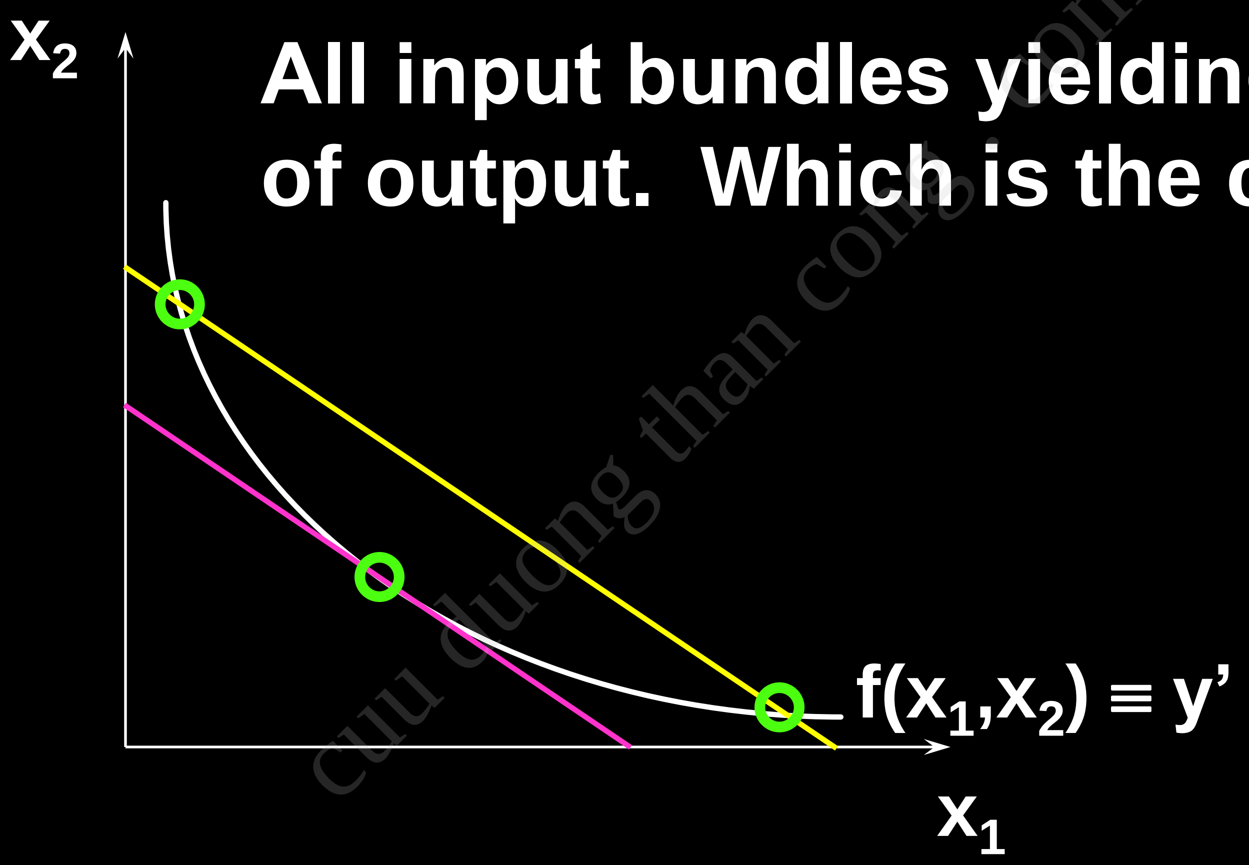
$$x_2 = -\frac{w_1}{w_2} x_1 + \frac{c}{w_2}.$$

- Slope is - w_1/w_2.

Iso-cost Lines

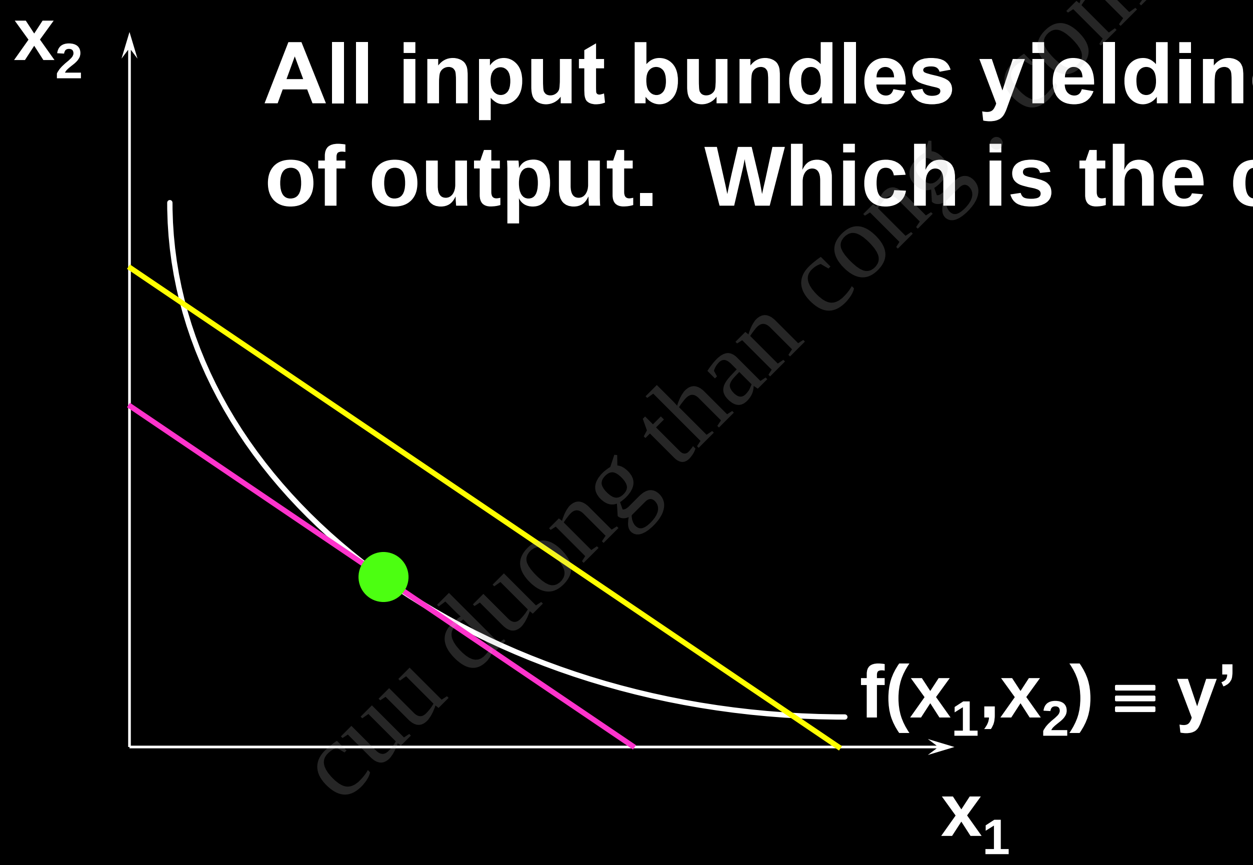


The Cost-Minimization Problem



The Cost-Minimization Problem

All input bundles yielding y' units of output. Which is the cheapest?



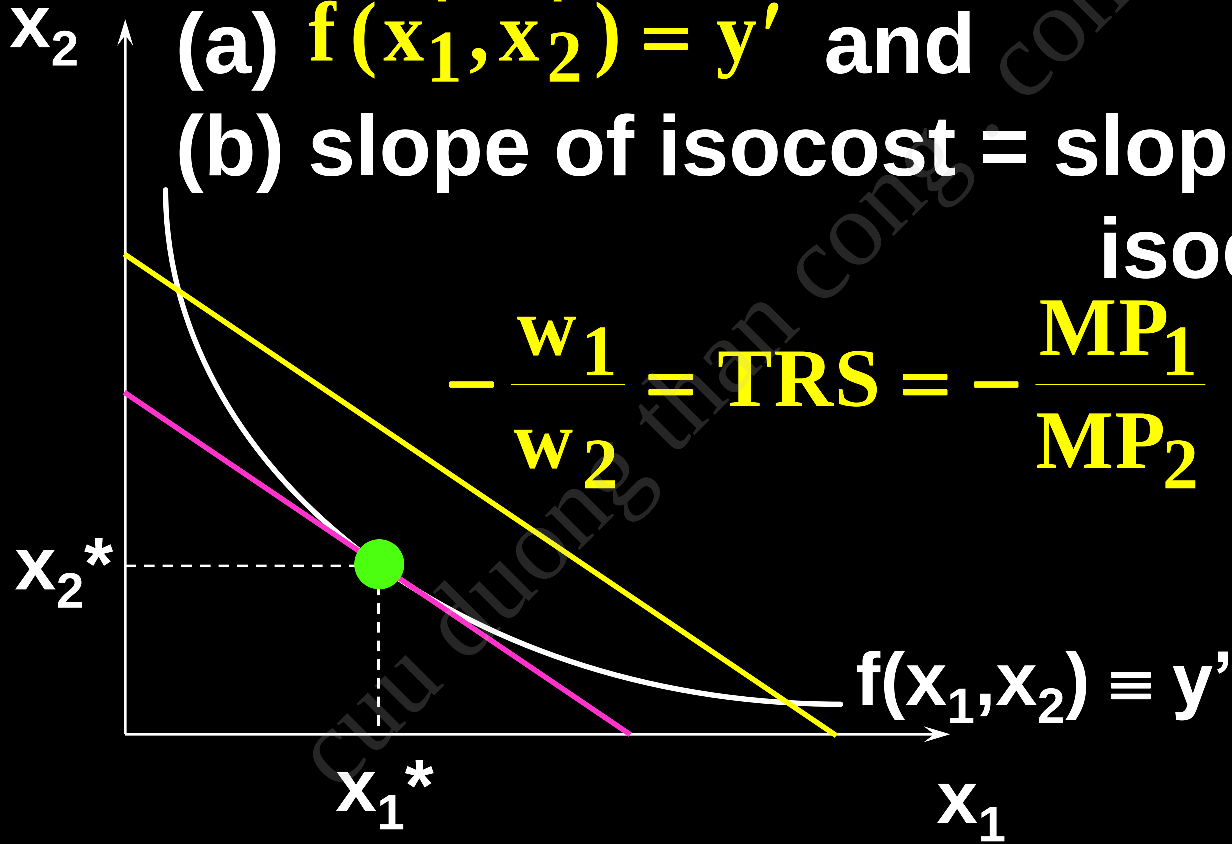
The Cost-Minimization Problem

At an interior cost-min input bundle:

(a) $f(x_1^*, x_2^*) = y'$ and

(b) slope of isocost = slope of isoquant; i.e.

$$-\frac{w_1}{w_2} = TRS = -\frac{MP_1}{MP_2} \text{ at } (x_1^*, x_2^*).$$



A Cobb-Douglas Example of Cost Minimization

- A firm's Cobb-Douglas production function is $y = f(x_1, x_2) = x_1^{1/3} x_2^{2/3}.$
- Input prices are w_1 and w_2.
- What are the firm's conditional input demand functions?

A Cobb-Douglas Example of Cost Minimization

At the input bundle (x_1*,x_2*) which minimizes the cost of producing y output units:

(a) $$y = (x_1^*)^{1/3}(x_2^*)^{2/3}$$ and

(b) $$-\frac{w_1}{w_2} = -\frac{\partial y/\partial x_1}{\partial y/\partial x_2} = -\frac{(1/3)(x_1^*)^{-2/3}(x_2^*)^{2/3}}{(2/3)(x_1^*)^{1/3}(x_2^*)^{-1/3}}$$

$$= -\frac{x_2^*}{2x_1^*}.$$

A Cobb-Douglas Example of Cost Minimization

(a) $y = (x_1^*)^{1/3}(x_2^*)^{2/3}$ (b) $\dfrac{w_1}{w_2} = \dfrac{x_2^*}{2x_1^*}.$

A Cobb-Douglas Example of Cost Minimization

(a) $y = (x_1^*)^{1/3}(x_2^*)^{2/3}$ **(b)** $\dfrac{w_1}{w_2} = \dfrac{x_2^*}{2x_1^*}$.

From (b), $x_2^* = \dfrac{2w_1}{w_2} x_1^*$.

Now substitute into (a) to get

$$y = (x_1^*)^{1/3}\left(\frac{2w_1}{w_2}x_1^*\right)^{2/3} = \left(\frac{2w_1}{w_2}\right)^{2/3} x_1^*.$$

So $x_1^* = \left(\dfrac{w_2}{2w_1}\right)^{2/3} y$ **is the firm's conditional demand for input 1.**

A Cobb-Douglas Example of Cost Minimization

Since $x_2^* = \frac{2w_1}{w_2} x_1^*$ and $x_1^* = \left(\frac{w_2}{2w_1}\right)^{2/3} y$

$$x_2^* = \frac{2w_1}{w_2}\left(\frac{w_2}{2w_1}\right)^{2/3} y = \left(\frac{2w_1}{w_2}\right)^{1/3} y$$

is the firm's conditional demand for input 2.

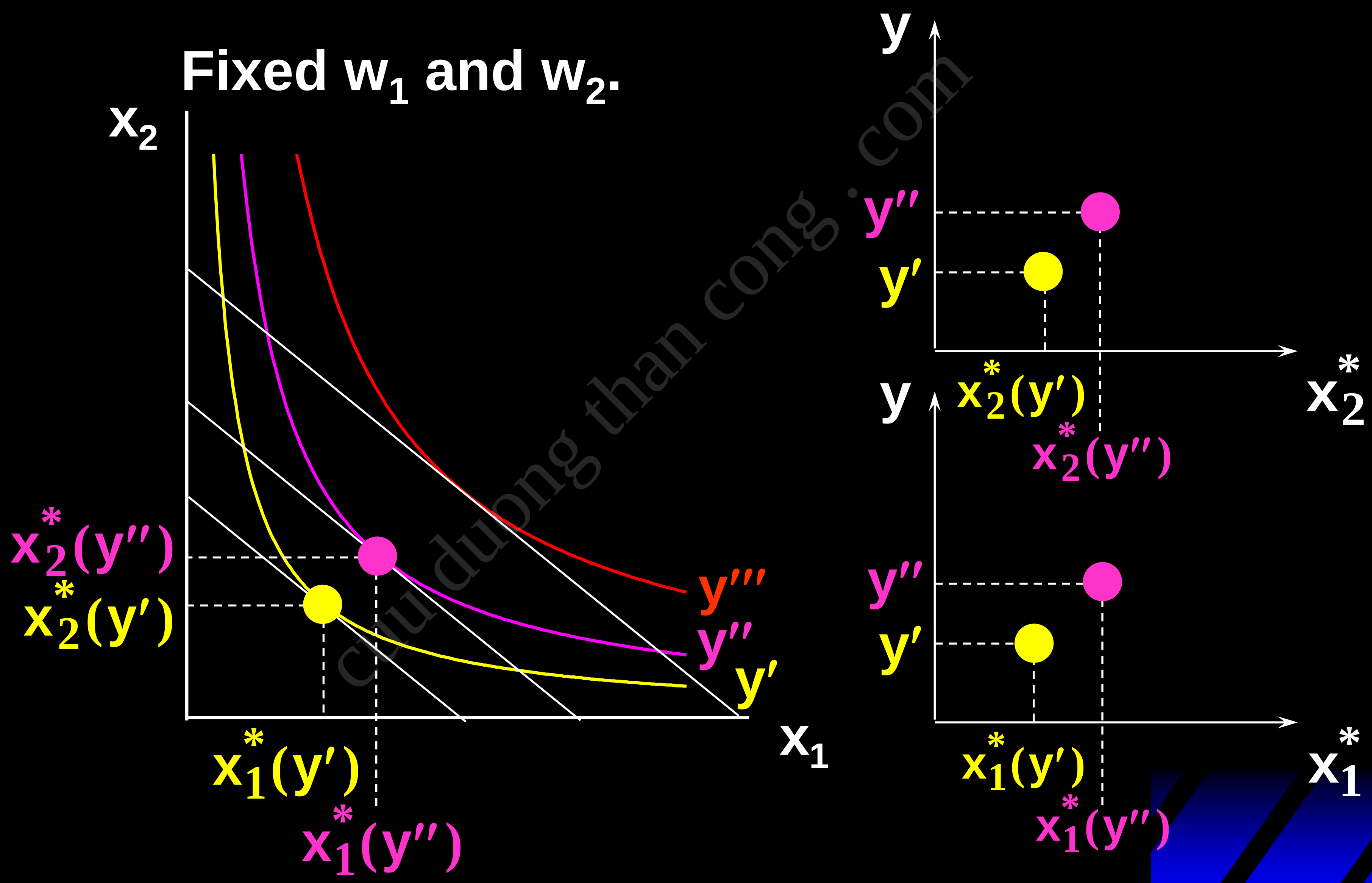
A Cobb-Douglas Example of Cost Minimization

So the cheapest input bundle yielding y output units is

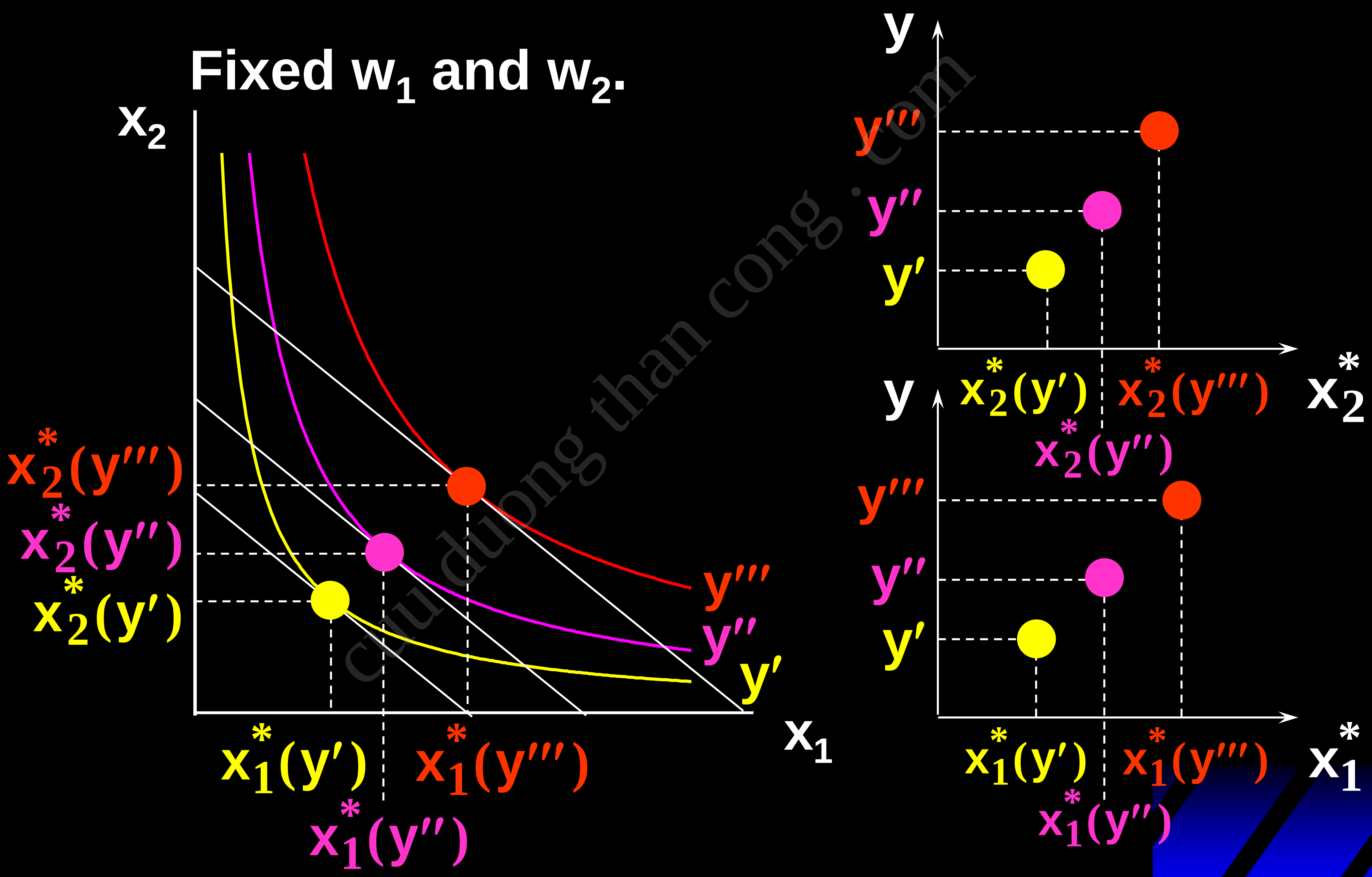
$$\left(x_1^*(w_1,w_2,y),x_2^*(w_1,w_2,y)\right)$$

$$=\left(\left(\frac{w_2}{2w_1}\right)^{2/3}y,\left(\frac{2w_1}{w_2}\right)^{1/3}y\right).$$

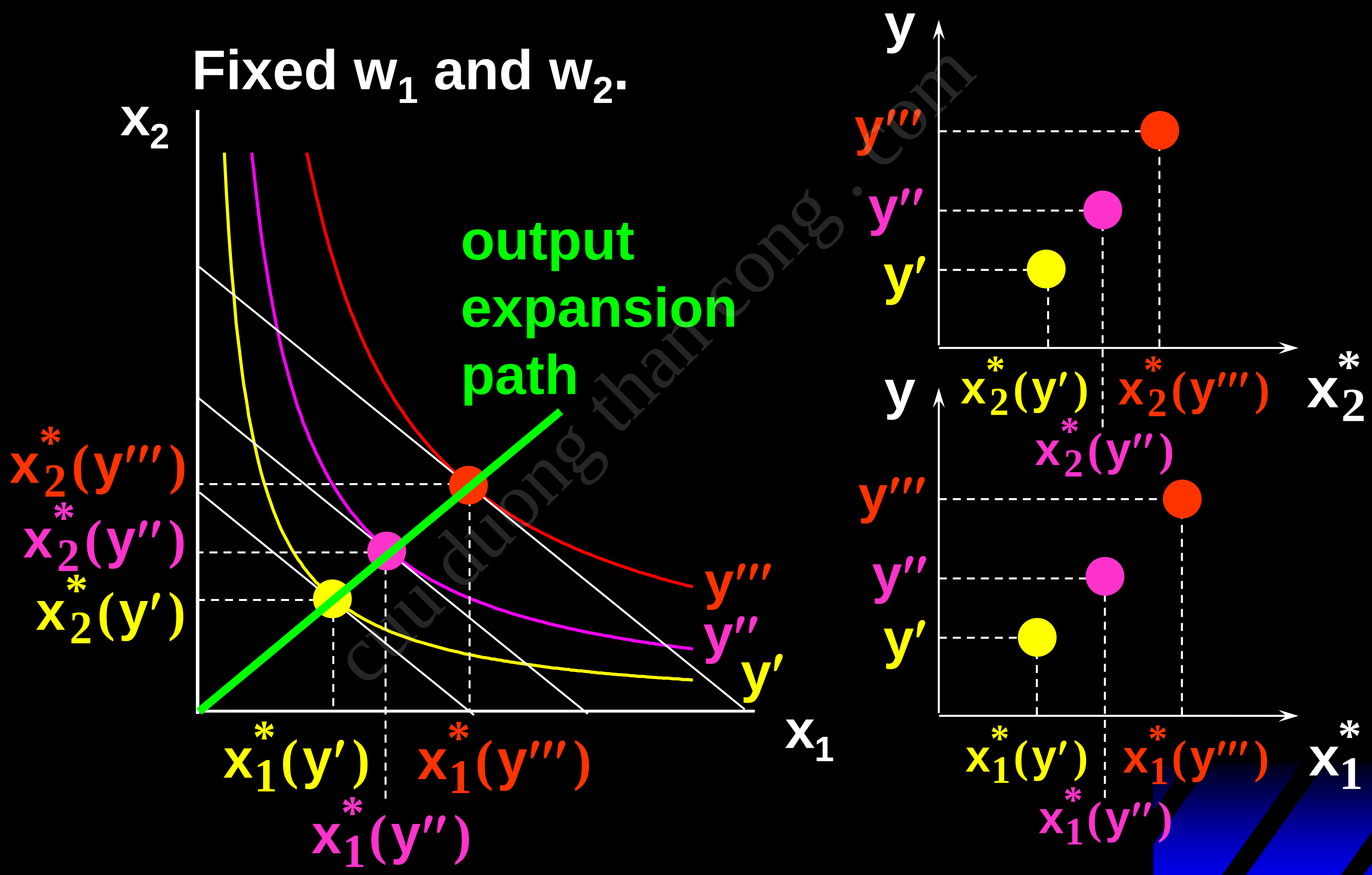
Conditional Input Demand Curves



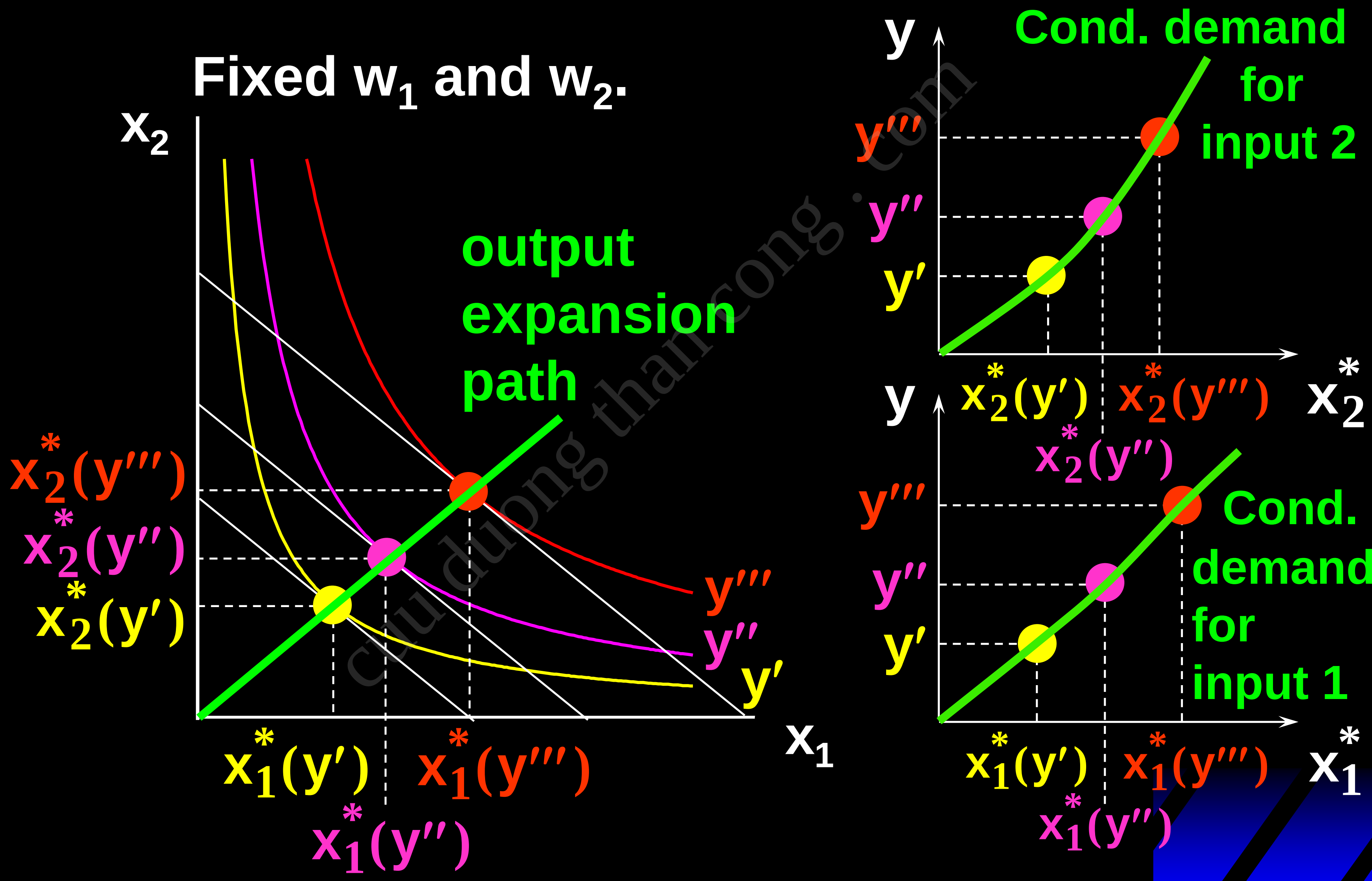
Conditional Input Demand Curves



Conditional Input Demand Curves



Conditional Input Demand Curves



A Cobb-Douglas Example of Cost Minimization

For the production function

$$y = f(x_1, x_2) = x_1^{1/3} x_2^{2/3}$$

the cheapest input bundle yielding y output units is

$$\left(x_1^*(w_1, w_2, y), x_2^*(w_1, w_2, y)\right)$$

$$= \left(\left(\frac{w_2}{2w_1}\right)^{2/3} y, \left(\frac{2w_1}{w_2}\right)^{1/3} y \right).$$

A Cobb-Douglas Example of Cost Minimization

- So the firm's total cost function is

$$c(w_1, w_2, y) = w_1 x_1^*(w_1, w_2, y) + w_2 x_2^*(w_1, w_2, y)$$

A Cobb-Douglas Example of Cost Minimization

So the firm's total cost function is

$$c(w_1, w_2, y) = w_1 x_1^*(w_1, w_2, y) + w_2 x_2^*(w_1, w_2, y)$$

$$= w_1 \left(\frac{w_2}{2w_1}\right)^{2/3} y + w_2 \left(\frac{2w_1}{w_2}\right)^{1/3} y$$

A Cobb-Douglas Example of Cost Minimization

So the firm's total cost function is

$$c(w_1, w_2, y) = w_1 x_1^*(w_1, w_2, y) + w_2 x_2^*(w_1, w_2, y)$$

$$= w_1 \left(\frac{w_2}{2w_1}\right)^{2/3} y + w_2 \left(\frac{2w_1}{w_2}\right)^{1/3} y$$

$$= \left(\frac{1}{2}\right)^{2/3} w_1^{1/3} w_2^{2/3} y + 2^{1/3} w_1^{1/3} w_2^{2/3} y$$

A Cobb-Douglas Example of Cost Minimization

So the firm's total cost function is

$$c(w_1, w_2, y) = w_1 x_1^*(w_1, w_2, y) + w_2 x_2^*(w_1, w_2, y)$$

$$= w_1 \left(\frac{w_2}{2w_1}\right)^{2/3} y + w_2 \left(\frac{2w_1}{w_2}\right)^{1/3} y$$

$$= \left(\frac{1}{2}\right)^{2/3} w_1^{1/3} w_2^{2/3} y + 2^{1/3} w_1^{1/3} w_2^{2/3} y$$

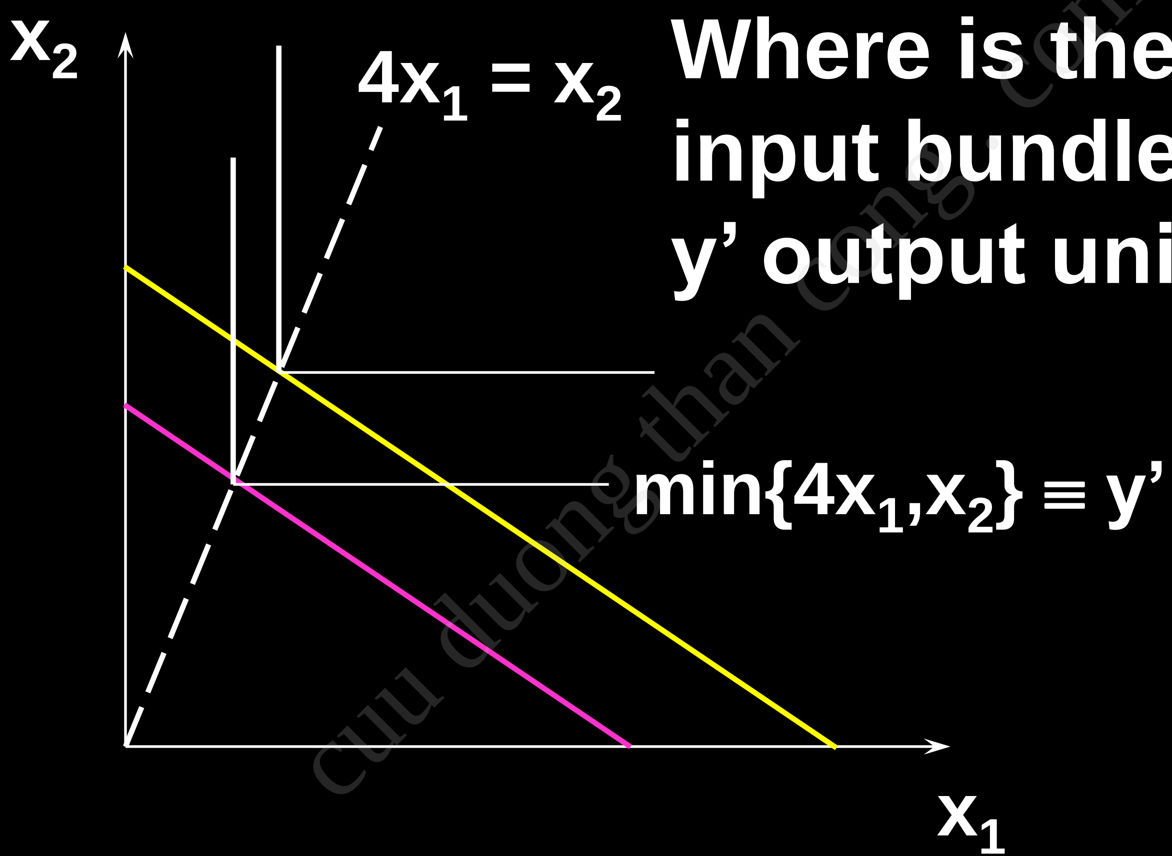
$$= 3\left(\frac{w_1 w_2^2}{4}\right)^{1/3} y.$$

A Perfect Complements Example of Cost Minimization

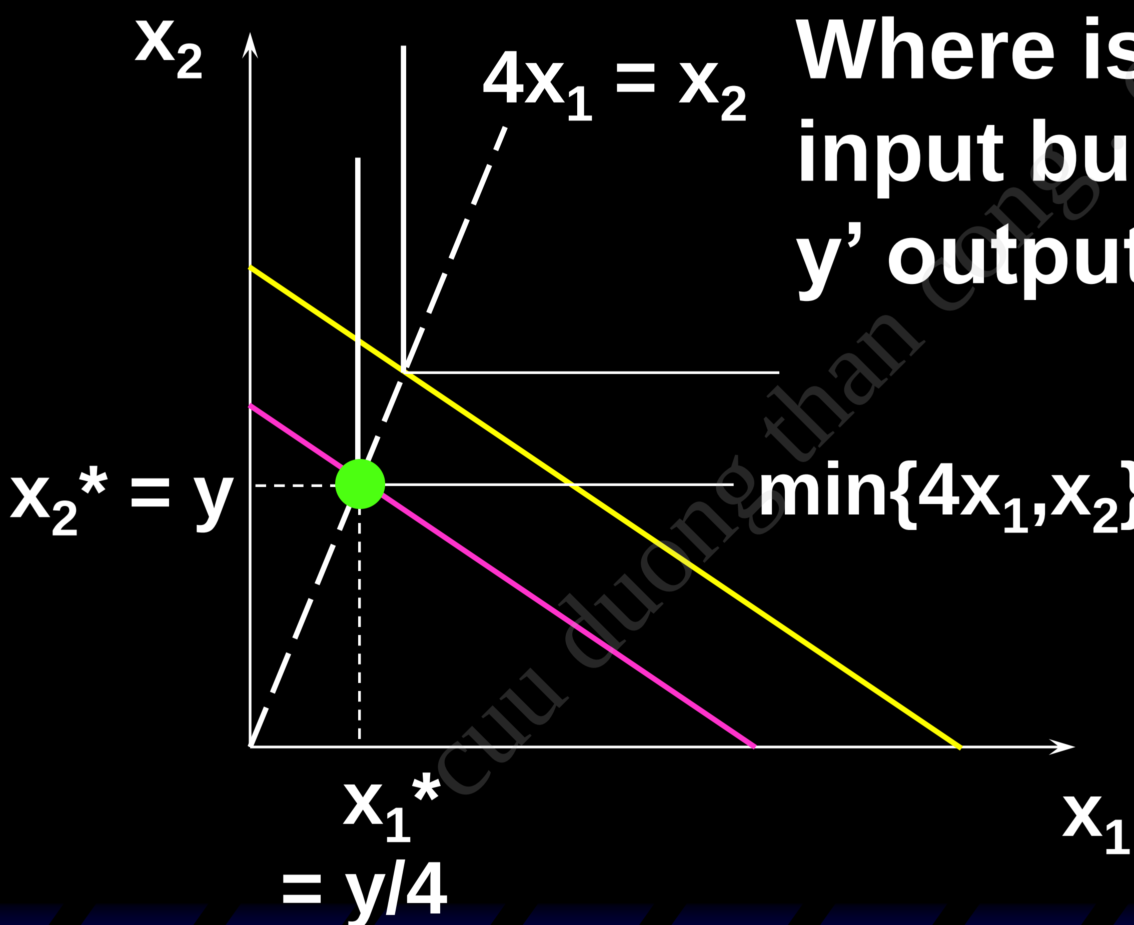
- The firm's production function is $y = \min\{4x_1, x_2\}$.
- Input prices w_1 and w_2 are given.
- What are the firm's conditional demands for inputs 1 and 2?
- What is the firm's total cost function?

A Perfect Complements Example of Cost Minimization

Where is the least costly input bundle yielding y’ output units?



A Perfect Complements Example of Cost Minimization



Where is the least costly input bundle yielding y' output units?

A Perfect Complements Example of Cost Minimization

The firm's production function is

$$y = \min\{4x_1, x_2\}$$

and the conditional input demands are

$$x_1^*(w_1, w_2, y) = \frac{y}{4} \quad \text{and} \quad x_2^*(w_1, w_2, y) = y.$$

A Perfect Complements Example of Cost Minimization

The firm's production function is

$$y = \min\{4x_1, x_2\}$$

and the conditional input demands are

$$x_1^*(w_1, w_2, y) = \frac{y}{4} \quad \text{and} \quad x_2^*(w_1, w_2, y) = y.$$

So the firm's total cost function is

$$c(w_1, w_2, y) = w_1 x_1^*(w_1, w_2, y) + w_2 x_2^*(w_1, w_2, y)$$

A Perfect Complements Example of Cost Minimization

The firm's production function is

$$y = \min\{4x_1, x_2\}$$

and the conditional input demands are

$$x_1^*(w_1, w_2, y) = \frac{y}{4} \quad \text{and} \quad x_2^*(w_1, w_2, y) = y.$$

So the firm's total cost function is

$$c(w_1, w_2, y) = w_1 x_1^*(w_1, w_2, y) + w_2 x_2^*(w_1, w_2, y)$$

$$= w_1 \frac{y}{4} + w_2 y = \left(\frac{w_1}{4} + w_2\right) y.$$

Average Total Production Costs

- For positive output levels y, a firm's average total cost of producing y units is

$$AC(w_1, w_2, y) = \frac{c(w_1, w_2, y)}{y}.$$

Returns-to-Scale and Av. Total Costs

- The returns-to-scale properties of a firm's technology determine how average production costs change with output level.
- Our firm is presently producing y' output units.
- How does the firm's average production cost change if it instead produces 2y' units of output?

Constant Returns-to-Scale and Average Total Costs

- **If a firm's technology exhibits constant returns-to-scale then doubling its output level from y' to 2y' requires doubling all input levels.**
- **Total production cost doubles.**
- **Average production cost does not change.**

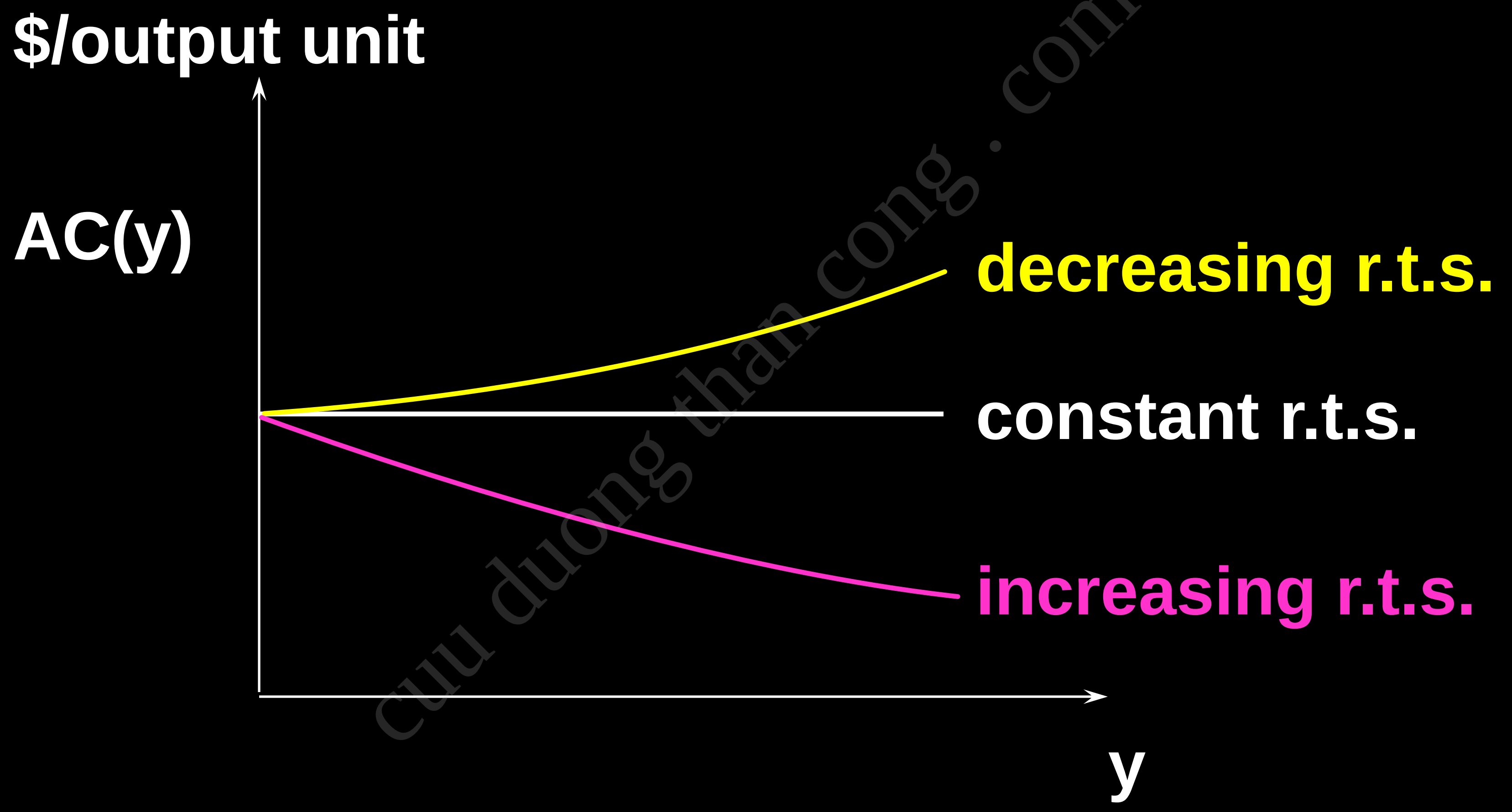
Decreasing Returns-to-Scale and Average Total Costs

- If a firm’s technology exhibits decreasing returns-to-scale then doubling its output level from y’ to 2y’ requires more than doubling all input levels.
- Total production cost more than doubles.
- Average production cost increases.

Increasing Returns-to-Scale and Average Total Costs

- If a firm's technology exhibits increasing returns-to-scale then doubling its output level from y' to 2y' requires less than doubling all input levels.
- Total production cost less than doubles.
- Average production cost decreases.

Returns-to-Scale and Av. Total Costs

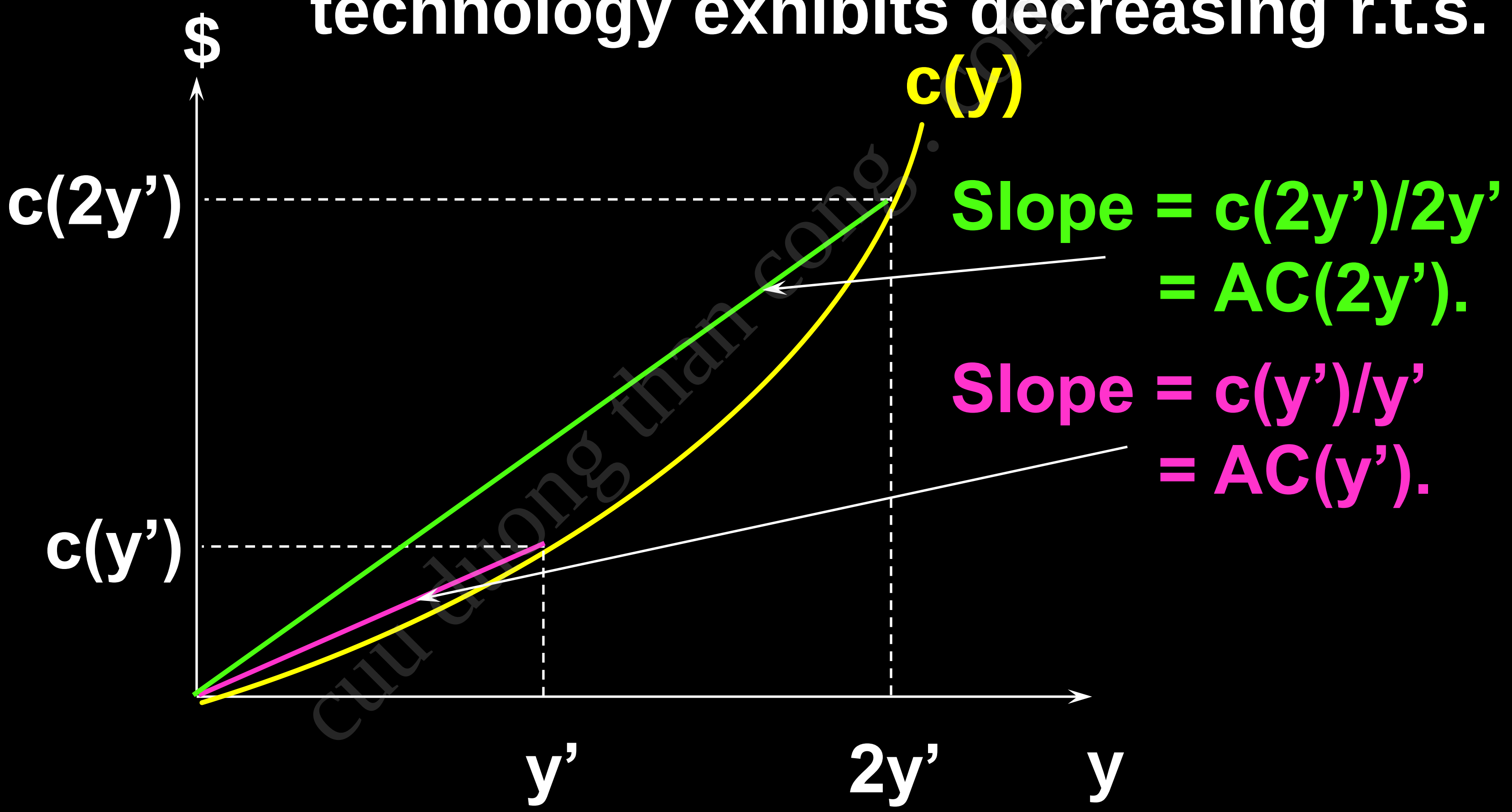


Returns-to-Scale and Total Costs

- **What does this imply for the shapes of total cost functions?**

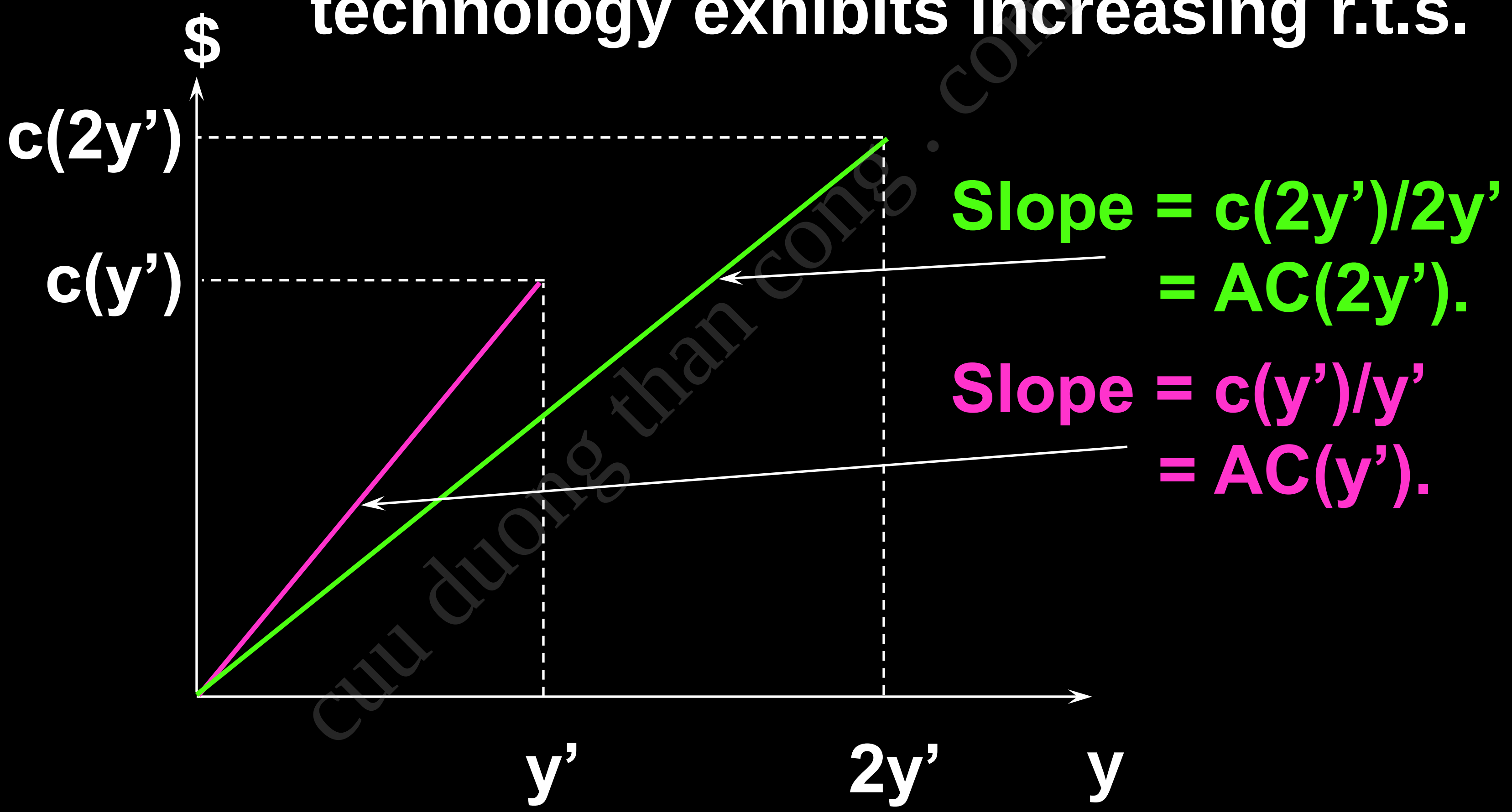
Returns-to-Scale and Total Costs

Av. cost increases with y if the firm's technology exhibits decreasing r.t.s.



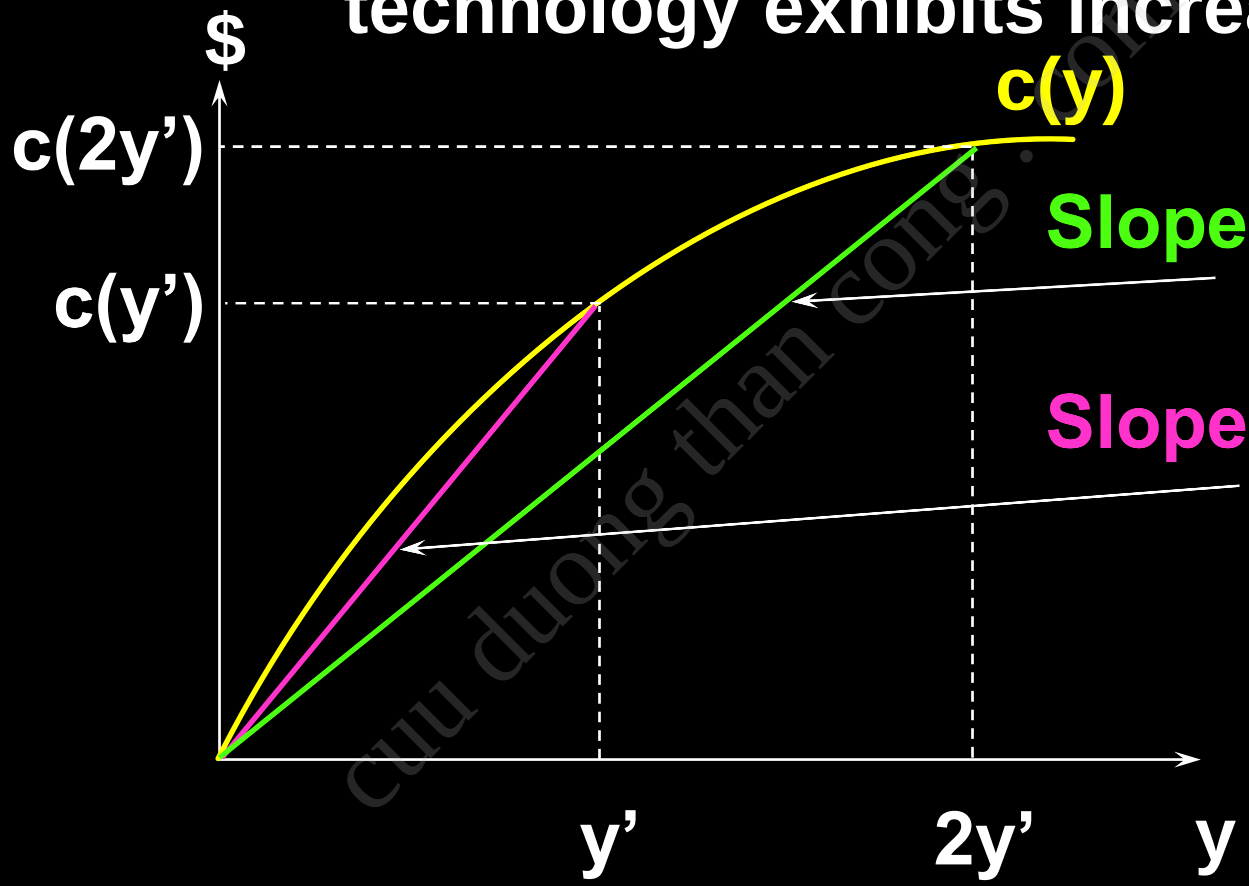
Returns-to-Scale and Total Costs

Av. cost decreases with y if the firm's technology exhibits increasing r.t.s.



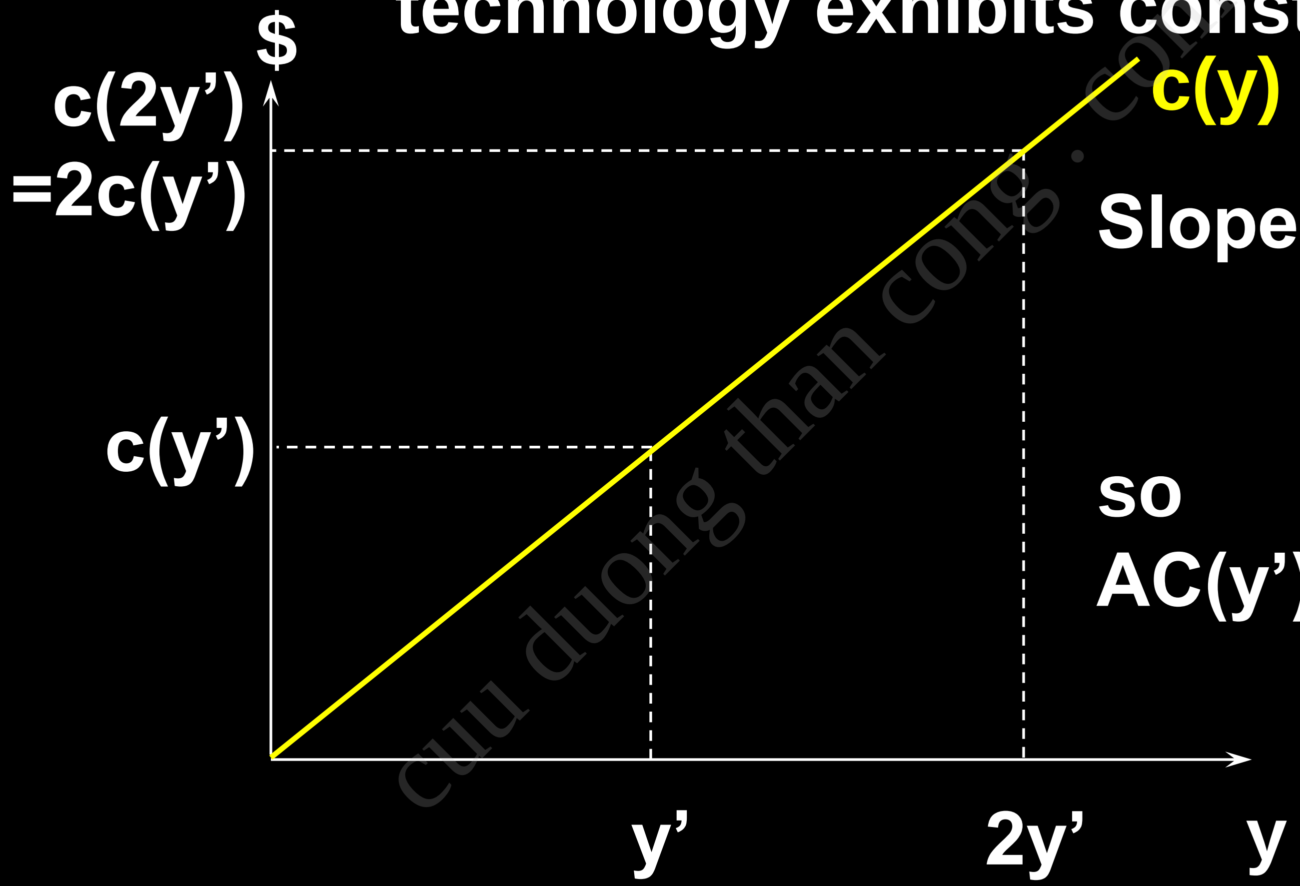
Returns-to-Scale and Total Costs

Av. cost decreases with y if the firm's technology exhibits increasing r.t.s.



Returns-to-Scale and Total Costs

Av. cost is constant when the firm's technology exhibits constant r.t.s.



Slope = c(2y')/2y'
= 2c(y')/2y'
= c(y')/y'

so

AC(y') = AC(2y').

Short-Run & Long-Run Total Costs

- In the long-run a firm can vary all of its input levels.
- Consider a firm that cannot change its input 2 level from x_2' units.
- How does the short-run total cost of producing y output units compare to the long-run total cost of producing y units of output?

Short-Run & Long-Run Total Costs

- The long-run cost-minimization problem is $$\min_{x_1,x_2\geq 0} w_1x_1 + w_2x_2$$ subject to $f(x_1,x_2) = y.$
- The short-run cost-minimization problem is $$\min_{x_1\geq 0} w_1x_1 + w_2x_2'$$ subject to $f(x_1,x_2') = y.$

Short-Run & Long-Run Total Costs

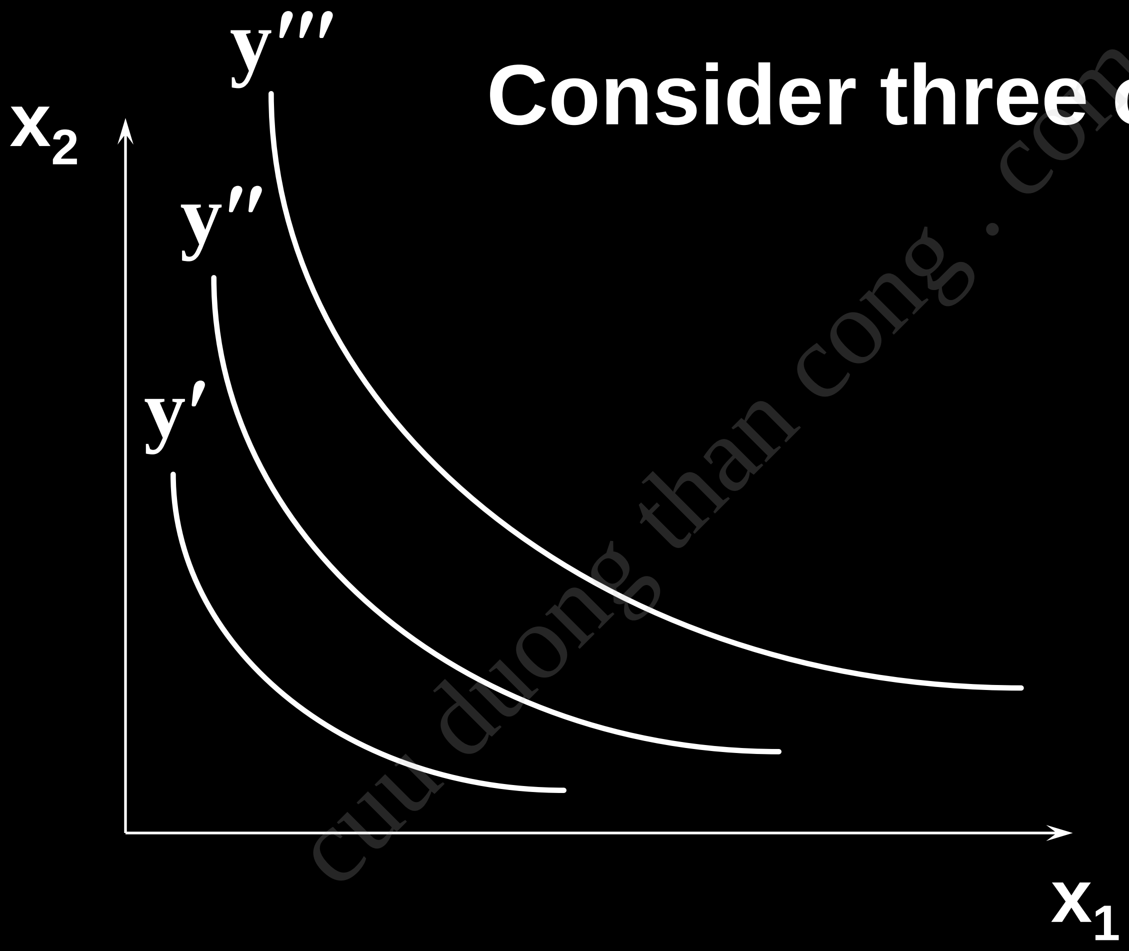
- The short-run cost-min. problem is the long-run problem subject to the extra constraint that $x_2 = x_2'$.
- If the long-run choice for x_2 was x_2' then the extra constraint $x_2 = x_2'$ is not really a constraint at all and so the long-run and short-run total costs of producing y output units are the same.

Short-Run & Long-Run Total Costs

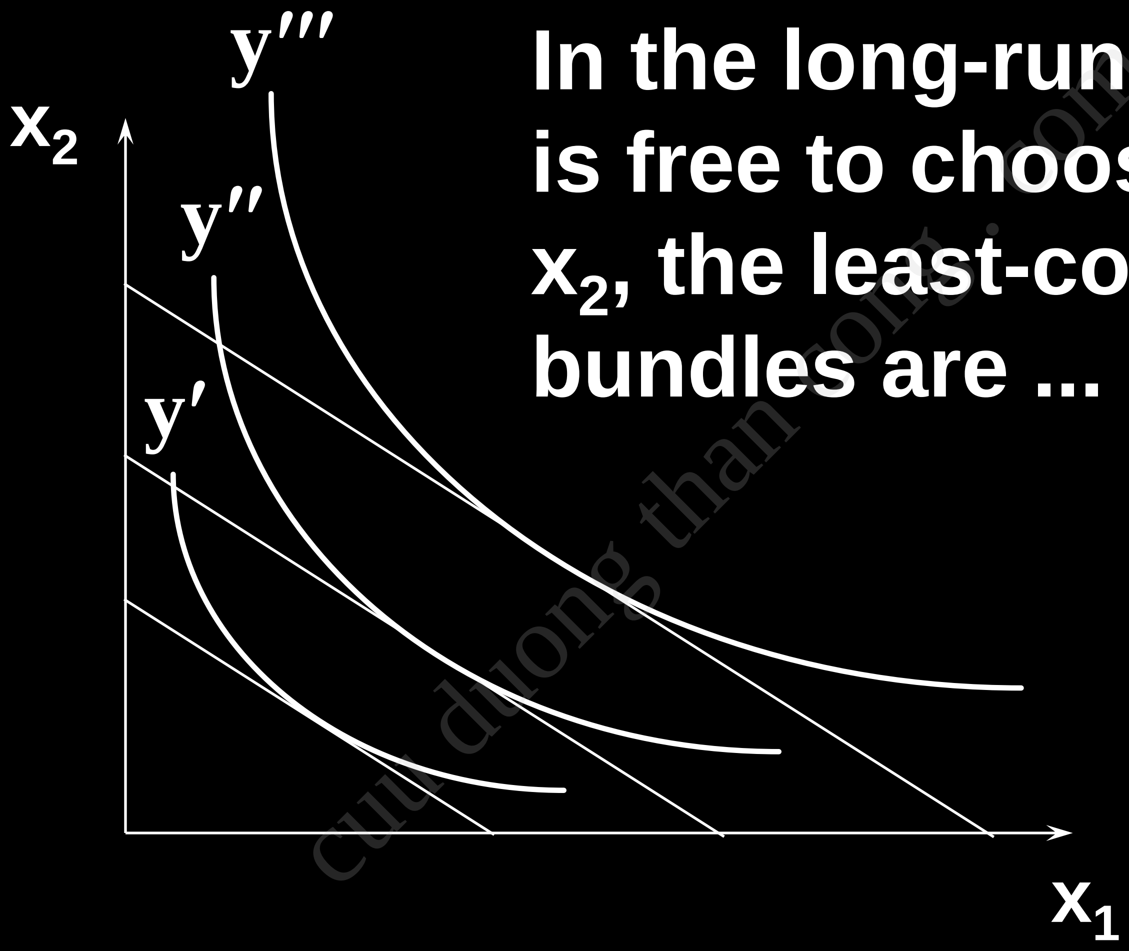
- The short-run cost-min. problem is therefore the long-run problem subject to the extra constraint that $x_2 = x_2''$.
- But, if the long-run choice for $x_2 \neq x_2''$ then the extra constraint $x_2 = x_2''$ prevents the firm in this short-run from achieving its long-run production cost, causing the short-run total cost to exceed the long-run total cost of producing y output units.

Short-Run & Long-Run Total Costs

Consider three output levels.



Short-Run & Long-Run Total Costs

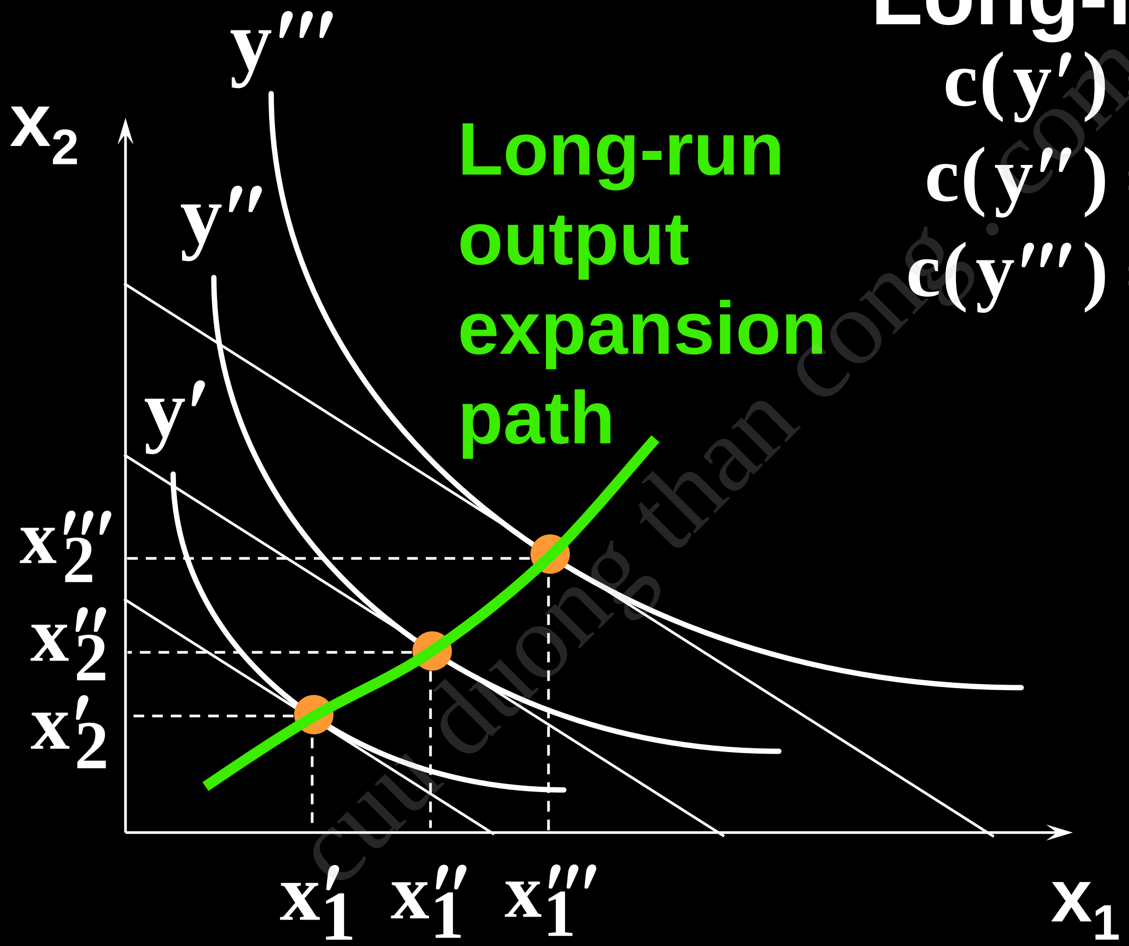


In the long-run when the firm is free to choose both x_1 and x_2, the least-costly input bundles are ...

Short-Run & Long-Run Total Costs

Long-run costs are:

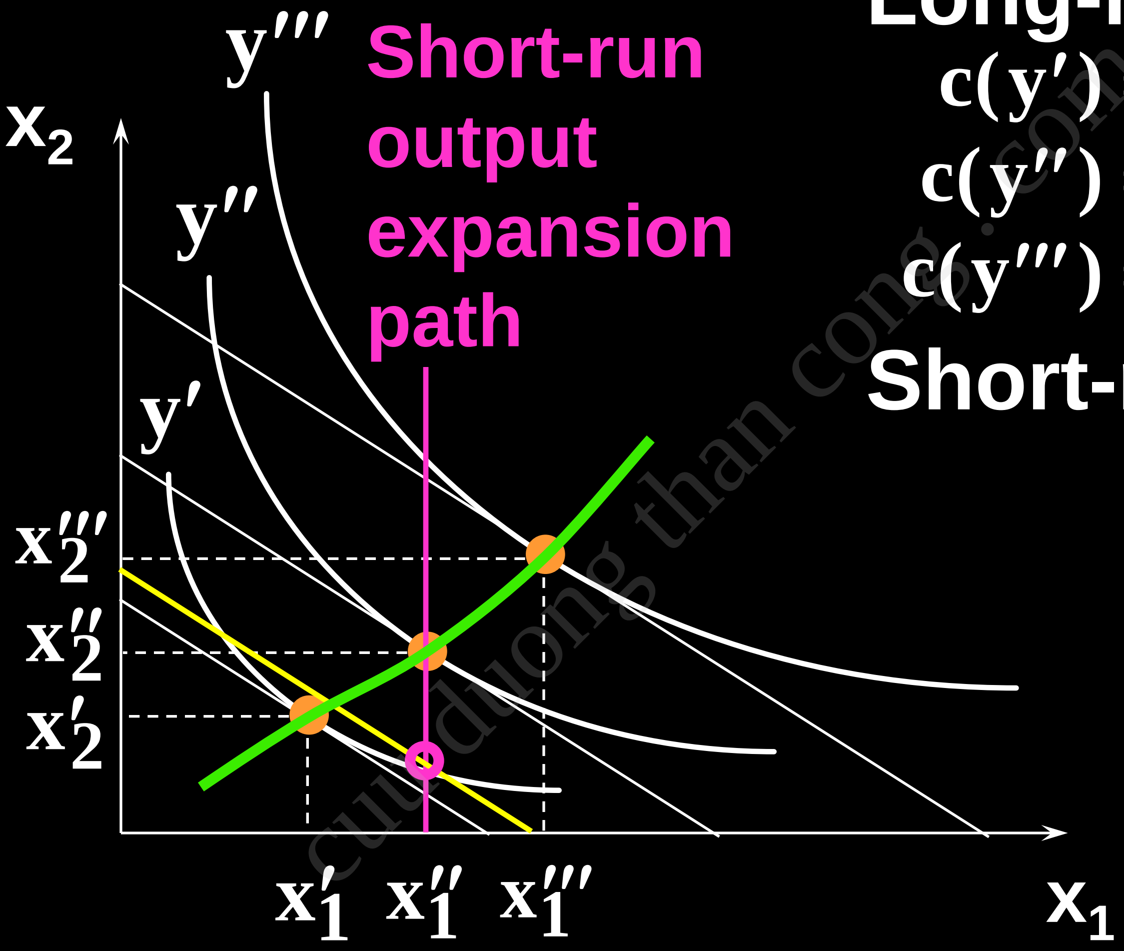
$$c(y') = w_1x_1' + w_2x_2'$$
$$c(y'') = w_1x_1'' + w_2x_2''$$
$$c(y''') = w_1x_1''' + w_2x_2'''$$



Short-Run & Long-Run Total Costs

- **Now suppose the firm becomes subject to the short-run constraint that $x_2 = x_2$".**

Short-Run & Long-Run Total Costs



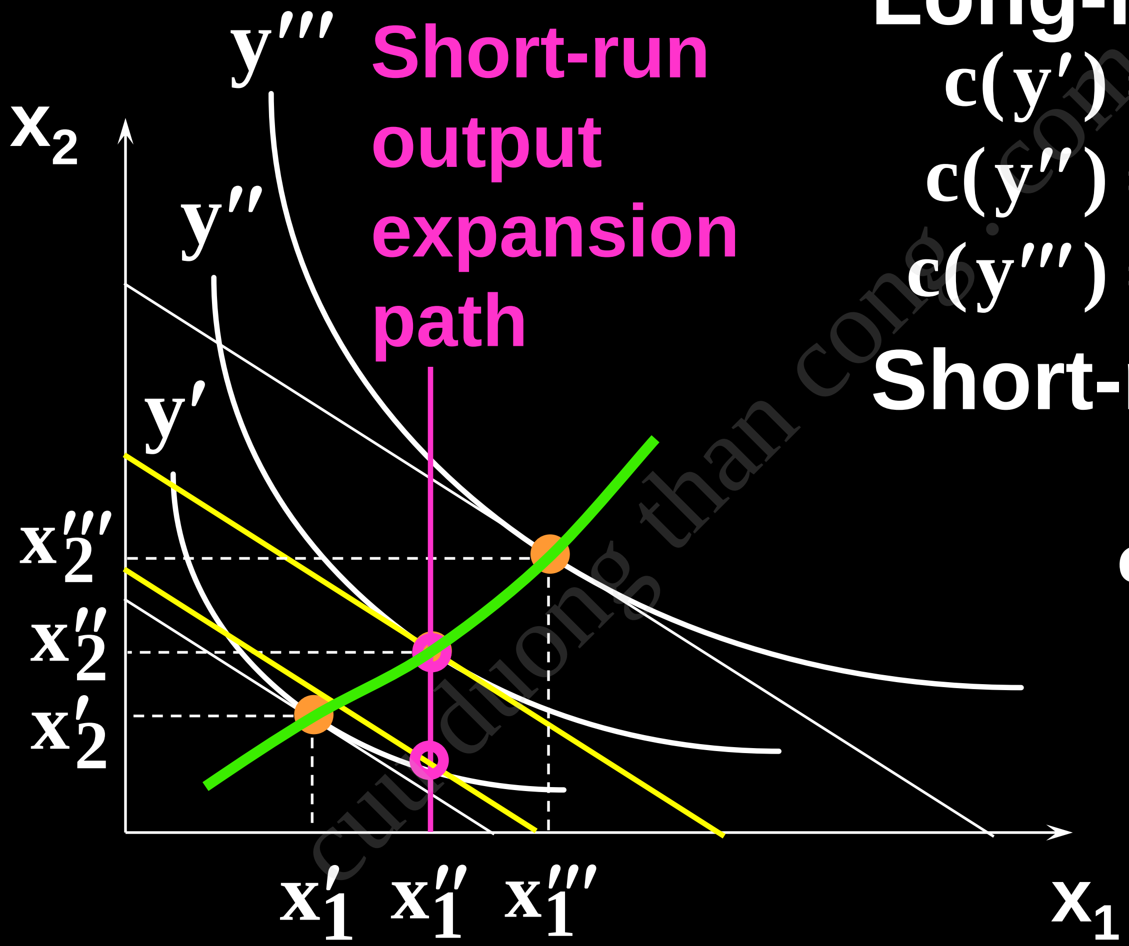
Long-run costs are:

$$c(y') = w_1x_1' + w_2x_2'$$
$$c(y'') = w_1x_1'' + w_2x_2''$$
$$c(y''') = w_1x_1''' + w_2x_2'''$$

Short-run costs are:

$$c_s(y') > c(y')$$

Short-Run & Long-Run Total Costs



Long-run costs are:

$$c(y') = w_1 x_1' + w_2 x_2'$$

$$c(y'') = w_1 x_1'' + w_2 x_2''$$

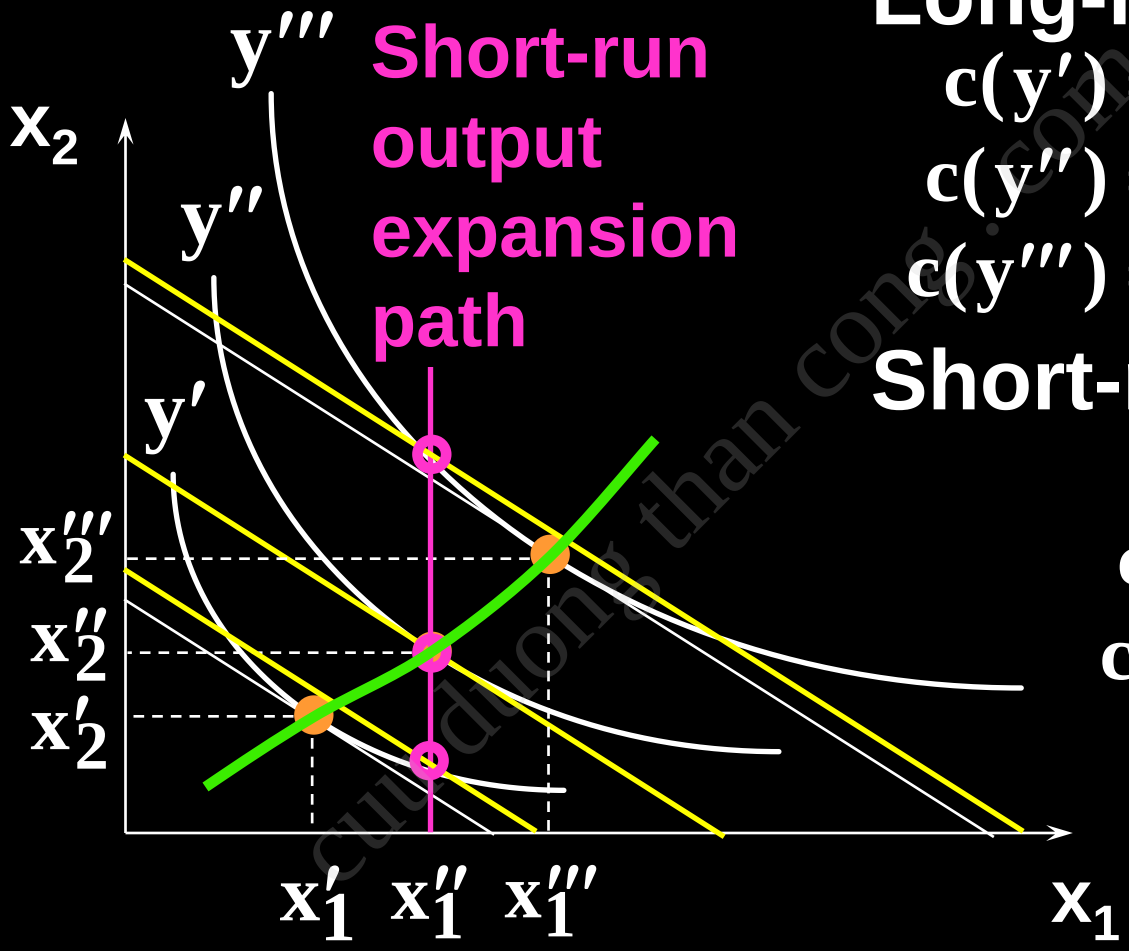
$$c(y''') = w_1 x_1''' + w_2 x_2'''$$

Short-run costs are:

$$c_s(y') > c(y')$$

$$c_s(y'') = c(y'')$$

Short-Run & Long-Run Total Costs



Long-run costs are:

$$c(y') = w_1x_1' + w_2x_2'$$

$$c(y'') = w_1x_1'' + w_2x_2''$$

$$c(y''') = w_1x_1''' + w_2x_2'''$$

Short-run costs are:

$$c_s(y') > c(y')$$

$$c_s(y'') = c(y'')$$

$$c_s(y''') > c(y''')$$

Short-Run & Long-Run Total Costs

- **Short-run total cost exceeds long-run total cost except for the output level where the short-run input level restriction is the long-run input level choice.**
- **This says that the long-run total cost curve always has one point in common with any particular short-run total cost curve.**

Short-Run & Long-Run Total Costs

A short-run total cost curve always has one point in common with the long-run total cost curve, and is elsewhere higher than the long-run total cost curve.