Chapter 3: The Z – Transform and its Application to the Analysis of LTI System

The ***z-transform*** of the discrete-time system *x(n)* is defined as the power series

$$x(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad (3.1.1)$$

Where *z* - complex variable.

It sometimes called ***the direct z-transform***.

The inverse procedure is called the ***inverse z-transform***.

$$X(z) \equiv Z\{x(n)\} \qquad (3.1.2)$$

$$x(n) \overset{z}{\leftrightarrow} X(z) \qquad (3.1.3)$$

The ***region of convergence*** (***ROC***) of *X(z)* is the set of all values z for which *X(z)* attains a finite value .

3.1 The z-transform

Let us express the *complex variable z* in polar form as

$$z = r e^{j\theta} \qquad (3.1.4)$$

$r = |z|$ and $\theta = \measuredangle z$, Then

$$x(z)|_{z=re^{j\theta}}| = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\theta n}$$

In the *ROC* of *X(z)*, | *x (z)* | < ∞ , then

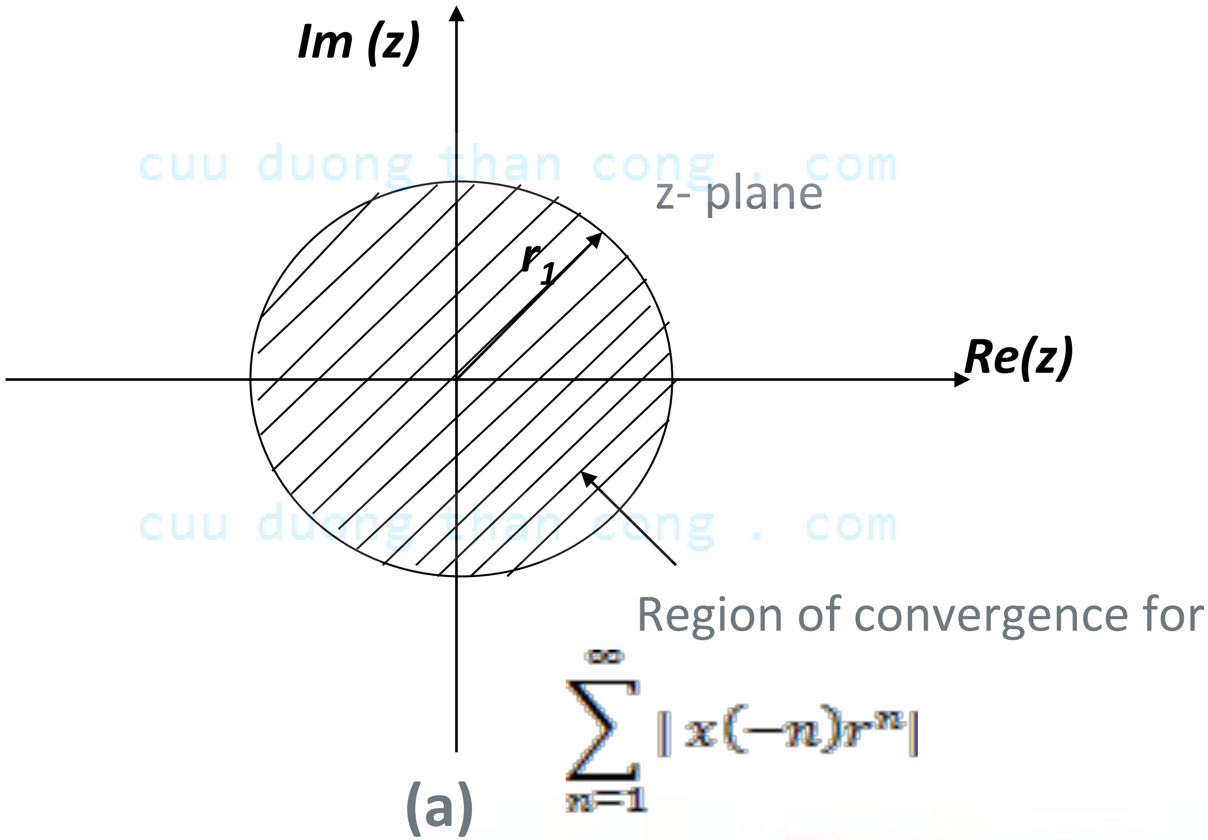
$$|X(z)| \le \sum_{n=-\infty}^{\infty} |x(n) r^{-n}| \qquad (3.1.5)$$

$$|X(z)| \le \sum_{n=1}^{\infty} |x(-n) r^{n}| + \sum_{n=0}^{\infty} \left|\frac{x(n)}{r^{n}}\right| \qquad (3.1.6)$$

3.1.1 The direct z-transform

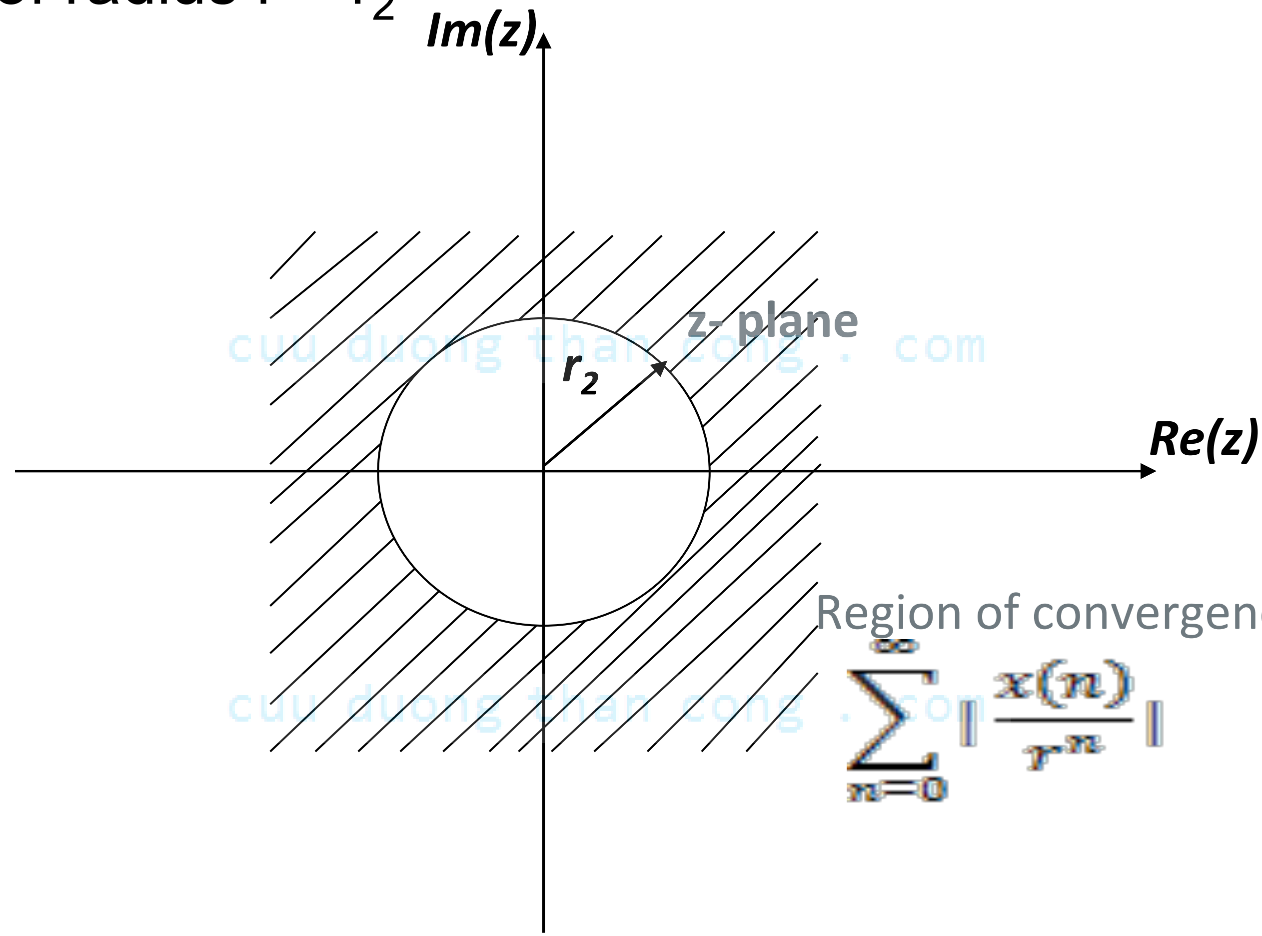
Figure 3.1 *Region of convergence* for *X(z)* and its corresponding causal and anticausal components.

ROC for the first sum consists of all points in a circle of some radius $r_1 < \infty$.



3.1.1 The direct z-transform

ROC for the second sum consists of all points outside a circle of radius $r > r_2$

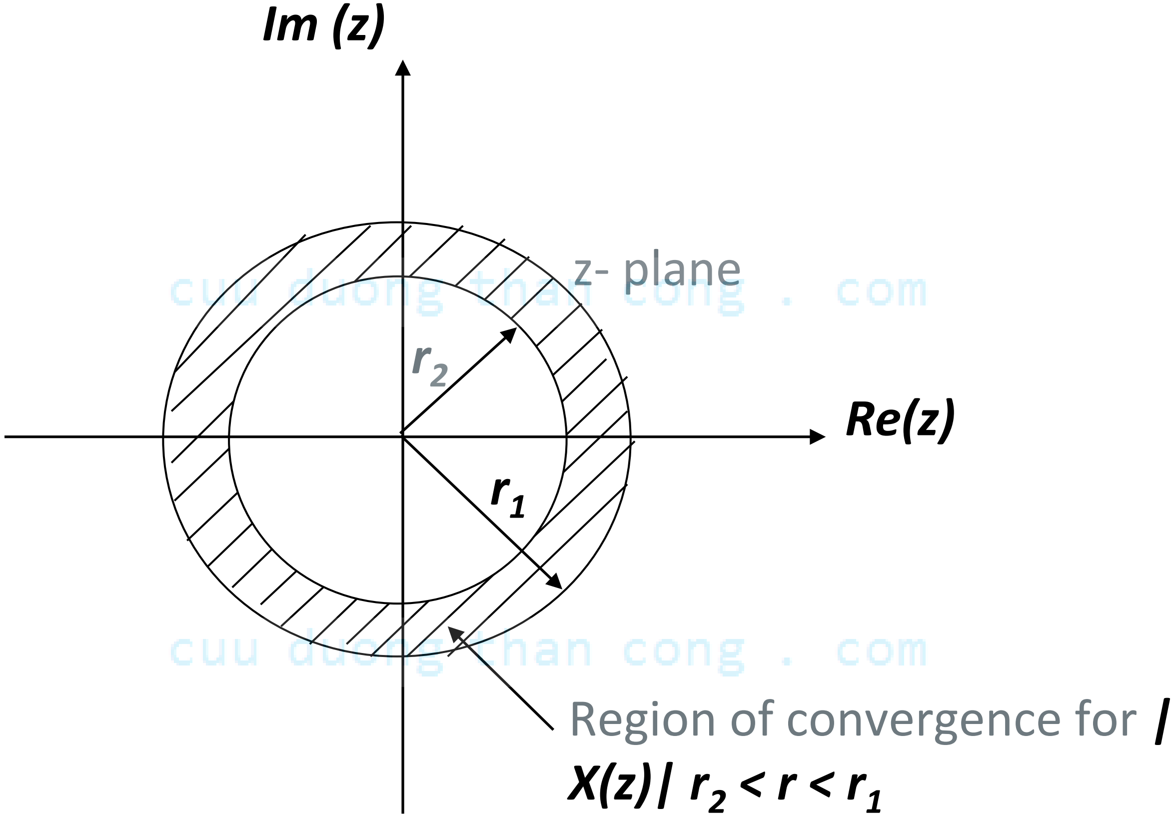


$$\sum_{n=0}^{\infty} \left| \frac{x(n)}{r^n} \right|$$

(b)

3.1.1 The direct z-transform

ROC of *X(z)* is generally specified as the annular region in the *z*-plane , $r_2 < r < r_1$,

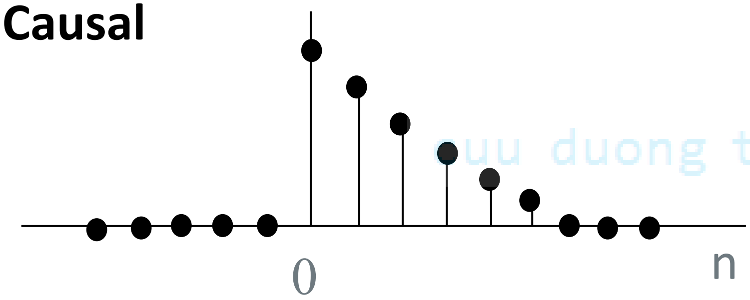
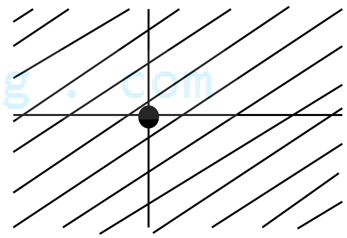


(c)

3.1.1 The direct z-transform

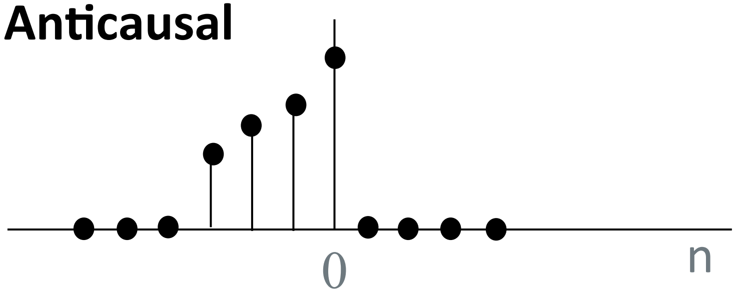
A discrete–time *x(n)* is ***uniquely*** determined by *its z-transform x(z)* and the *region of convergence of x(z)*.

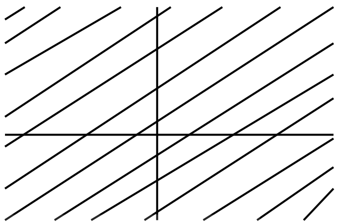
Table 3.1Characteristic Families of signal with their corresponding ROC.

Signal		ROC	
	Finite- Duration Signal		
Causal			**Entire z-plane except z=0**

3.1.1 The direct z-transform

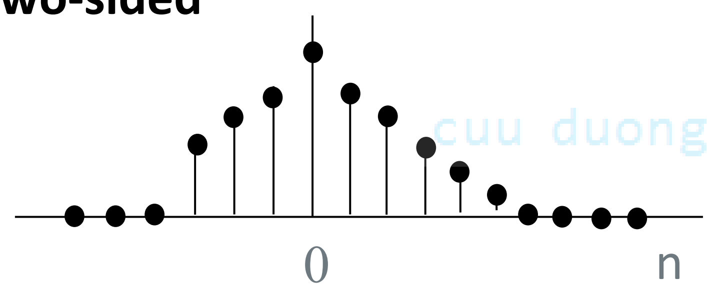
Anticausal

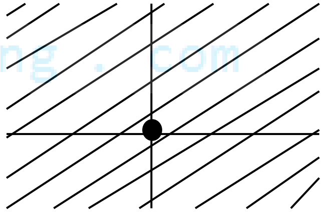




Entire z-plane except $z = \infty$

Two-sided



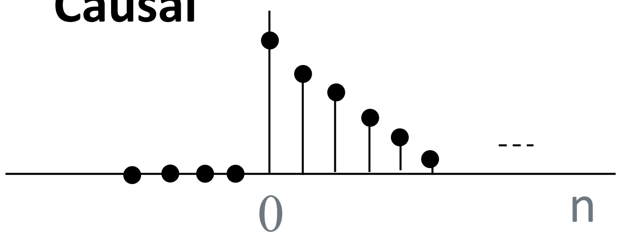


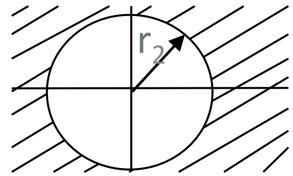
Entire z-plane except $z = 0$ and $z = \infty$

3.1.1 The direct z-transform

Infinite – Duration Signals

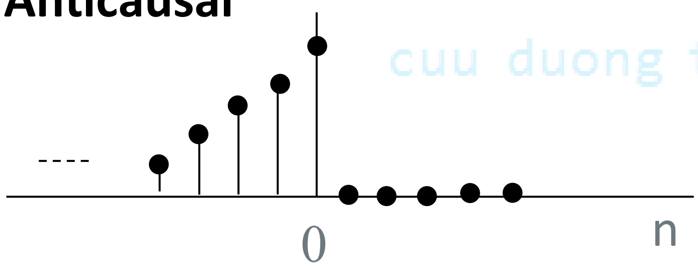
Causal

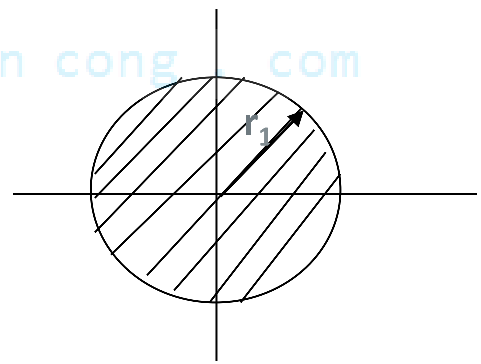




$|z| > r_2$

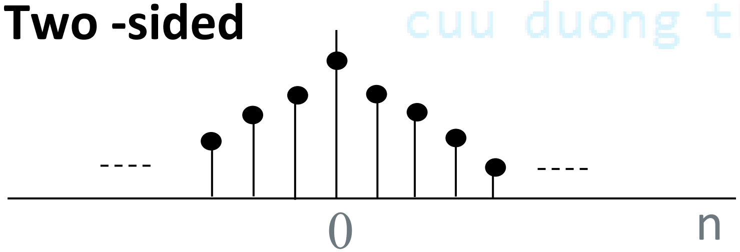
Anticausal

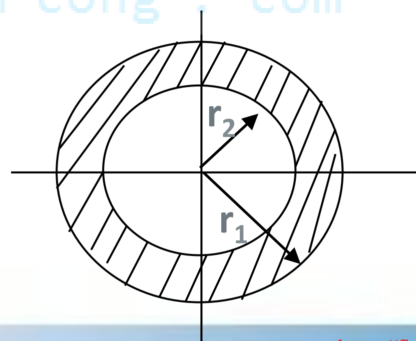




$|z| < r_1$

Two -sided





$r_2 < |z| < r_1$

3.1.1 The direct z-transform

These types of signal are called ***right-sided***, ***left-sided***, and ***finite-duration two-sided***, signals.

*If there is a ROC for an infinite duration two-sided signal, it is a **ring** (**annular region**) in the z-plane.*

The ***one-sided*** or ***unilateral z-transform*** given by

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n} \qquad (3.1.11)$$

3.1.2 The Inverse z-Transform

The procedure for transform from the z-domain to the time domain is called ***the inversion z-transform***.

Cauchy integral theorem.

We have

$$X(z) = \sum_{k=-\infty}^{\infty} x(k) z^{-k} \qquad (3.1.12)$$

then

$$\oint_C X(z) z^{n-1} dz = \oint_C \sum_{k=-\infty}^{\infty} x(k) z^{n-1-k} dz \qquad (3.1.13)$$

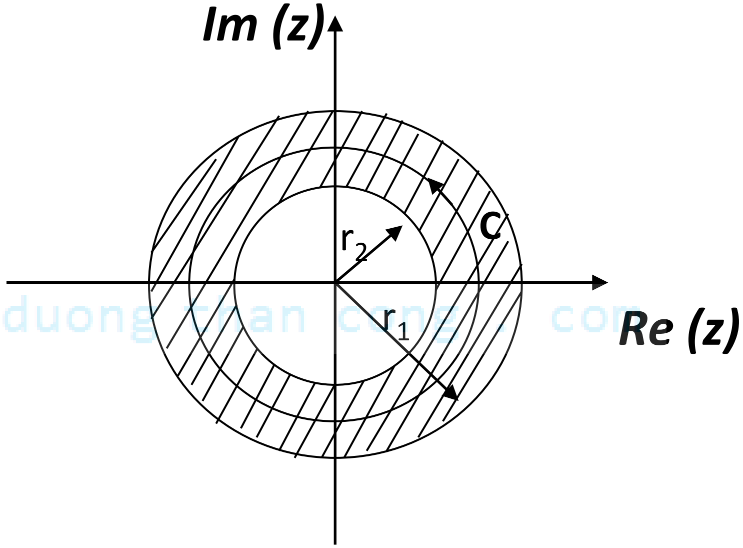
Where *C* the closed contour in the *ROC* of *X(z)*.

3.1.2 The Inverse z-Transform

or

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz \qquad (3.1.16)$$

Figure 3.1.5 Contour *C* for integral in (3.1.13)



3.2 Properties of the z–Transform

+ Linearity

if $x_1(n) \overset{z}{\leftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\leftrightarrow} X_2(z)$

then

$$x(n) = a_1x_1(n) + a_2x_2(n) \overset{z}{\leftrightarrow} X(z) = a_1X_1(z) + a_2X_2(z) \qquad (3.2.1)$$

+ Time shifting

if $x(n) \overset{z}{\leftrightarrow} X(z)$

then $x(n-k) \overset{z}{\leftrightarrow} z^{-k} X(z) \qquad (3.2.5)$

+ Scaling in the z-domain

If $x(n) \overset{z}{\leftrightarrow} X(z)$, ROC: $r_1 < |z| < r_2$

then $a^n x(n) \overset{z}{\leftrightarrow} X(a^{-1}z)$, ROC: $|a|r_1 < |z| < |a|r_2 \qquad (3.2.9)$

for any constant *a*, real or complex.

3.2 Properties of the z–Transform

+ Time reversal

if $x(n) \overset{z}{\leftrightarrow} X(z)$, ROC: $r_1 < |z| < r_2$

then $x(-n) \overset{z}{\leftrightarrow} X(z^{-1})$, ROC: $1/r_2 < |z| < 1/r_1$ (3.2.12)

+ Differentiation in the z-domain

if $x(n) \overset{z}{\leftrightarrow} X(z)$

then $nx(n) \overset{z}{\leftrightarrow} -z\, dX(z)/dz$ (3.2.14)

+ Convolution of two sequences

if $x_1(n) \overset{z}{\leftrightarrow} X_1(z)$, $x_2(n) \overset{z}{\leftrightarrow} X_2(z)$,

then $x(n) = x_1(n) * x_2(n) \overset{z}{\leftrightarrow} X(z) = X_1(z)\, X_2(z)$ (3.2.17)

3.2 Properties of the z–Transform

+ Correlation of two sequences

if $x_1(n) \overset{z}{\leftrightarrow} X_1(z)$, and $x_2(n) \overset{z}{\leftrightarrow} X_2(z)$

then

$$r_{x1x2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n-l) \leftrightarrow R_{x1x2}(z) = X_1(z)X_2(z^{-1}) \qquad (3.2.18)$$

+ Multiplication of two sequences

if $x_1(n) \overset{z}{\leftrightarrow} X_1(z)$, $x_2(n) \overset{z}{\leftrightarrow} X_2(z)$

then

$$x(n) = x_1(n)x_2(n) \overset{z}{\leftrightarrow} X(z) = \frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right) v^{-1}dv \qquad (3.2.19)$$

C – closed contour that encloses the origin and lies within the region of convergence common to both $X_1(v)$ and $X_2(1/v)$

3.2 Properties of the z–Transform

+ Parseval's relation

If $x_1(n)$ and $x_2(n)$ are complex-valued sequences , then

$$\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(v) X_2^*\left(\frac{1}{v^*}\right) v^{-1} dv \quad (3.2.22)$$

provided that $r_{1l} r_{2l} < 1 < r_{1u} r_{2u}$,

where $r_{1l} < |z| < r_{1u}$ and $r_{2l} < |z| < r_{2u}$ are the ROC of $X_1(z)$ and $X_2(z)$.

+ The initial value theorem

If $x(n)$ is *causal* then

$$x(0) = \lim_{z \to \infty} X(z) \qquad (3.2.23)$$

3.2 Properties of the z–Transform

+ Table 3.2 *Properties of the z-transform*

Property	Time Domain	z- Domain	ROC
Notation	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_2(z)$	ROC_2
Linearity	$a_1x_1(n)+ a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least intersection of ROC $_1$ and ROC_2
Time shifting	$x(n-k)$	$z^{-k}X(z)$	That of X (z), except $z=0$ if $k=0$ and $z = \infty$ if $k < 0$
Scaling in the z -domain	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\| r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$1/r_1 < \|z\| < 1/r_2$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC

3.2 Properties of the z–Transform

Real part	$Re\{x(n)\}$	$1/2[X(z)+X^*(z^*)]$	Includes ROC
Imaginary part	$Im\{x(n)\}$	$1/2j[X(z) - X^*(z^*)]$	Includes ROC
Differentiation in	$nx(n)$	$-z\dfrac{dX(z)}{dz}$	$r_2 < \lvert z \rvert < r_1$
Convolution	$x_1(n) * x_2(n)$	$X_1(z) X_2(z)$	At least, the intersection of ROC_1 and ROC_2
Correlation	$r_{x1x2}(l) = x_1(l) * x_2(-l)$	$R_{x1x2}(z) = X_1(z) X_2(z^{-1})$	At least, the intersection of ROC_1 of $X_1(z)$ and $X_2(z^{-1})$
Initial value theorem	If $x(n)$ causal	$x(0) = \lim X(z)$	
Multiplication	$x_1(n)x_2(n)$	$\dfrac{1}{2\pi j}\oint_C X_1(v)X_2\left(\dfrac{z}{v}\right)v^{-1}\,dv$	At least, $r_{1l}r_{2l} < \lvert z \rvert < r_{1u}r_{2u}$

Parseval's relation $\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \dfrac{1}{2\pi j}\oint_C X_1(v)X_2^*\left(\dfrac{1}{v^*}\right)v^{-1}\,dv$

3.2 Properties of the z–Transform

Table 3.3 *Some common z-transform pairs*

	Signal, *x(n)*	z- transform, *x(z)*	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
7	$(\cos\omega_0 n)u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$

3.2 Properties of the z–Transform

	Signal, *x(n)*	z- transform, *x(z)*	ROC
8	$(\sin\omega_0 n)u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
9	$(a^n\cos\omega_0 n)u(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$
10	$(a^n\sin\omega_0 n)u(n)$	$\frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$

3.3 Rational z- transforms

+ Poles and Zeros

The ***zeros*** of a z-transform *X(z)* are the values of *z* for which ***X(z) = 0***

The ***pole*** of a z-transform are value of *z* for which ***X(z) = ∞***

+ If *X(z)* is a rational function, then

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{K=0}^{N} a_k z^{-k}} \quad (3.3.1)$$

3.3.1 Poles and Zeros

+ If $a_0 \neq 0$, $b_0 \neq 0$

$$X(z) = \frac{N(z)}{M(z)} = \frac{b_0}{a_0} z^{-M+N} \frac{(z - z_1(z - z_2)\dots(z - z_M)}{(z - p_1)(z - p_2)\dots(z - p_N)}$$

$$X(z) = G z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k} \qquad (3.3.2)$$

Where: $G = \frac{b_0}{a_0}$

+ *X(z)* has ***M finite zeros*** at $z = z_1, z_2, \dots, z_M$

N finite poles at $z = r_1, r_2, \dots, r_N$

3.3.1 Poles and Zeros

+ We can represent *X(z)* graphically by a *pole-zero plot* in the complex plane, which shows the location of ***poles*** by ***crosses*** (**x**) and the location of ***zeros*** by ***circles*** (**o**).

+ The z-transform *X(z)* is a complex function of the complex variable $z = Re(z) + jIm(z)$

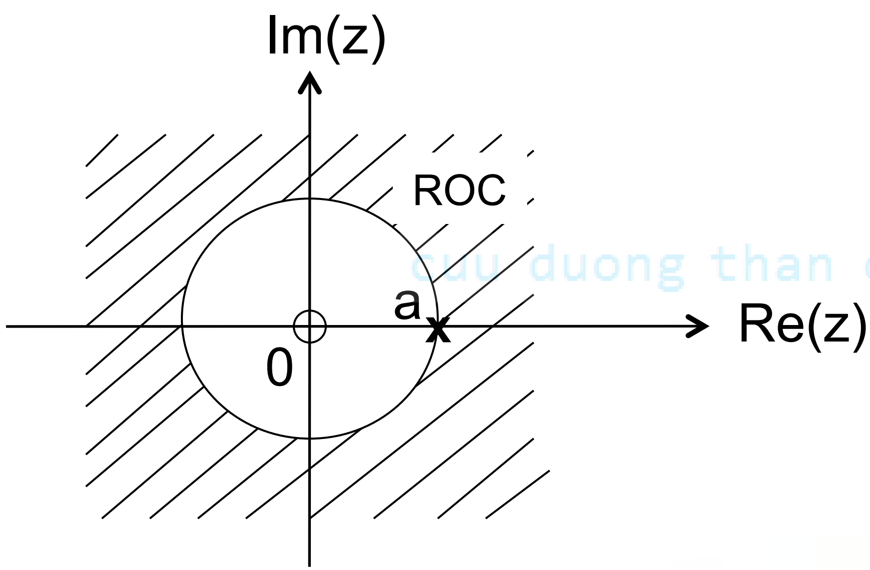


Figure 3.7 Pole-zero plot for the causal exponential signal $x(n) = a^n u(n)$ $a>0$

$$X(z) = \frac{z}{z-a}$$

ROC: |z| > a

3.3.1 Poles and Zeros

+ $|X(z)|$ is a real and positive function of *z*. Since *z* represents a point in the complex plane, $|X(z)|$ is a *two-dimensional function* and describes a "surface".

For example the z-transform

$$X(z) = \frac{z^{-1} - z^{-2}}{1 - 1.2732z^{-1} + 0.81z^{-2}} \quad (3.3.3)$$

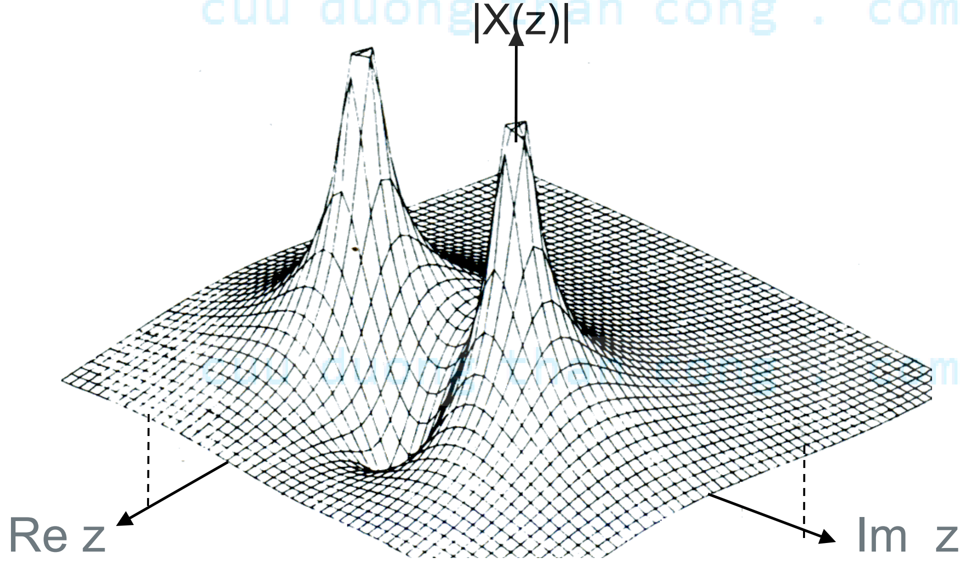


Figure 3.3.4 Graph of $|X(z)|$ for the z-transform in (3.3.3)

3.3.2 Pole location and time- domain behavior for causal signals

+ We consider the relation between *the z-plane location* of a pole pair and the *form of the corresponding signal* in the time domain.

+ The ***circle*** |z| = *1* has a radius of *1*, it is called *the unit circle*.

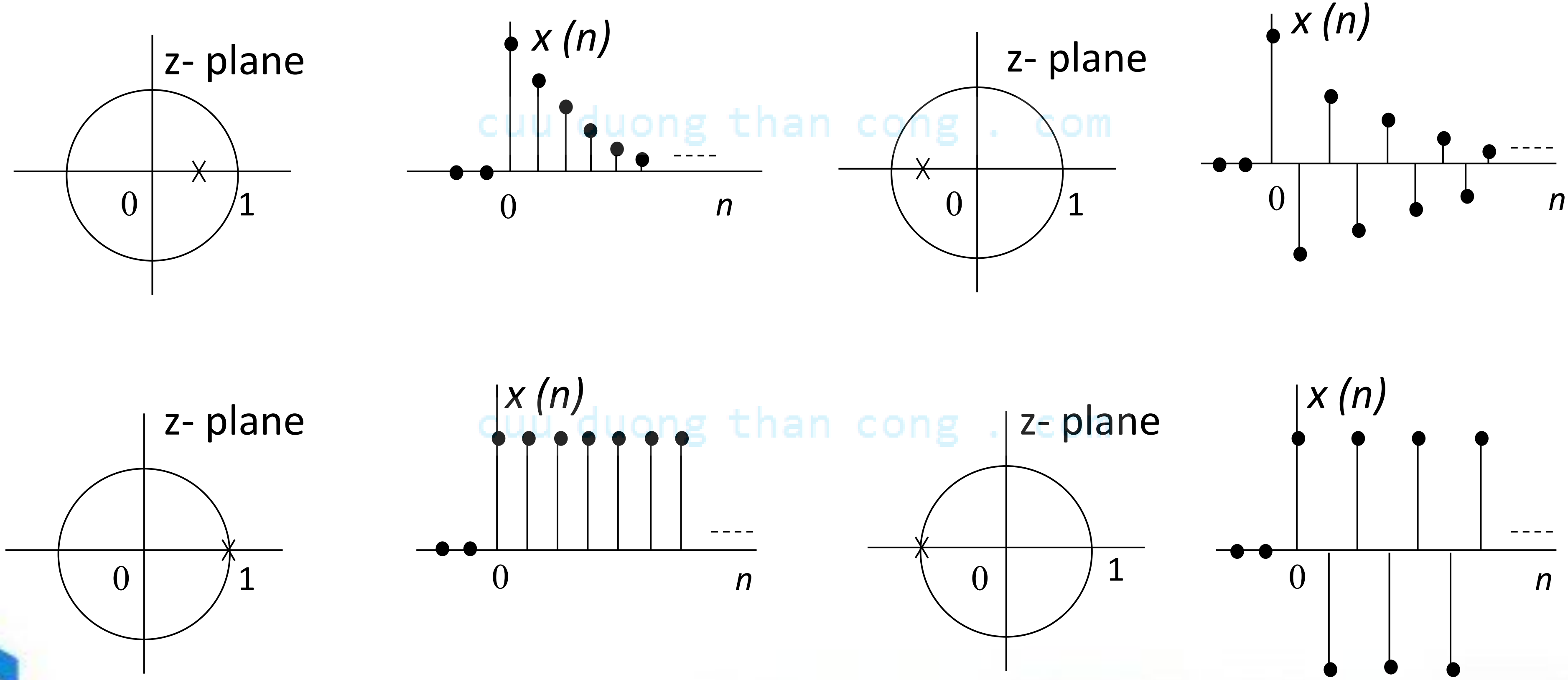
+ For example

$$x(n) = a^n u(n) \overset{z}{\leftrightarrow} X(z) = \frac{1}{1 - az^{-1}}, \quad ROC: |z| > |a|$$

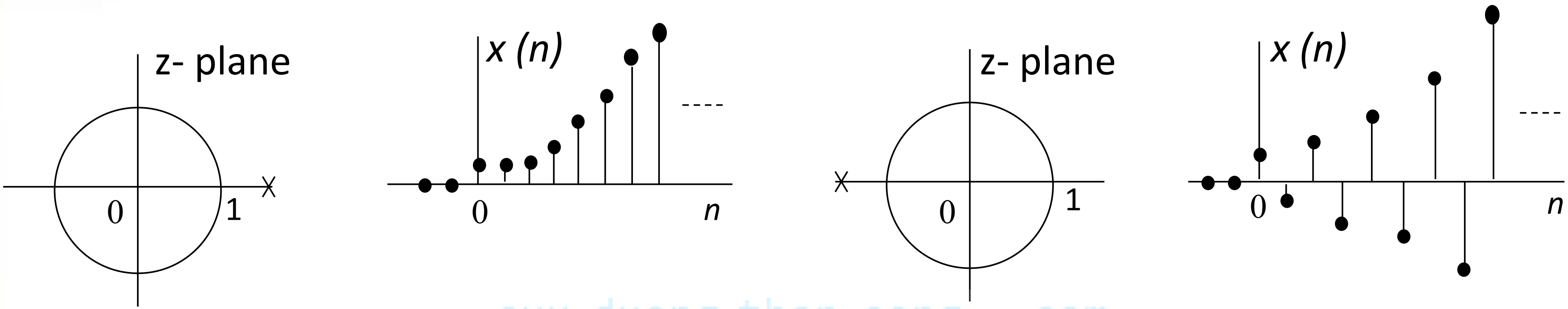
having one *zero* at $z_1 = 0$ and the *pole* at $p_1 = a$ on the real axis, (see fig. 3.7, fig.3.11)

3.3.2 Pole location and time- domain behavior for causal signals

Figure 3.11 Time-domain behavior of a *single-real pole causal signal* as a function of the location of the pole with respect to the *unit circle*.



3.3.2 Pole location and time- domain behavior for causal signals



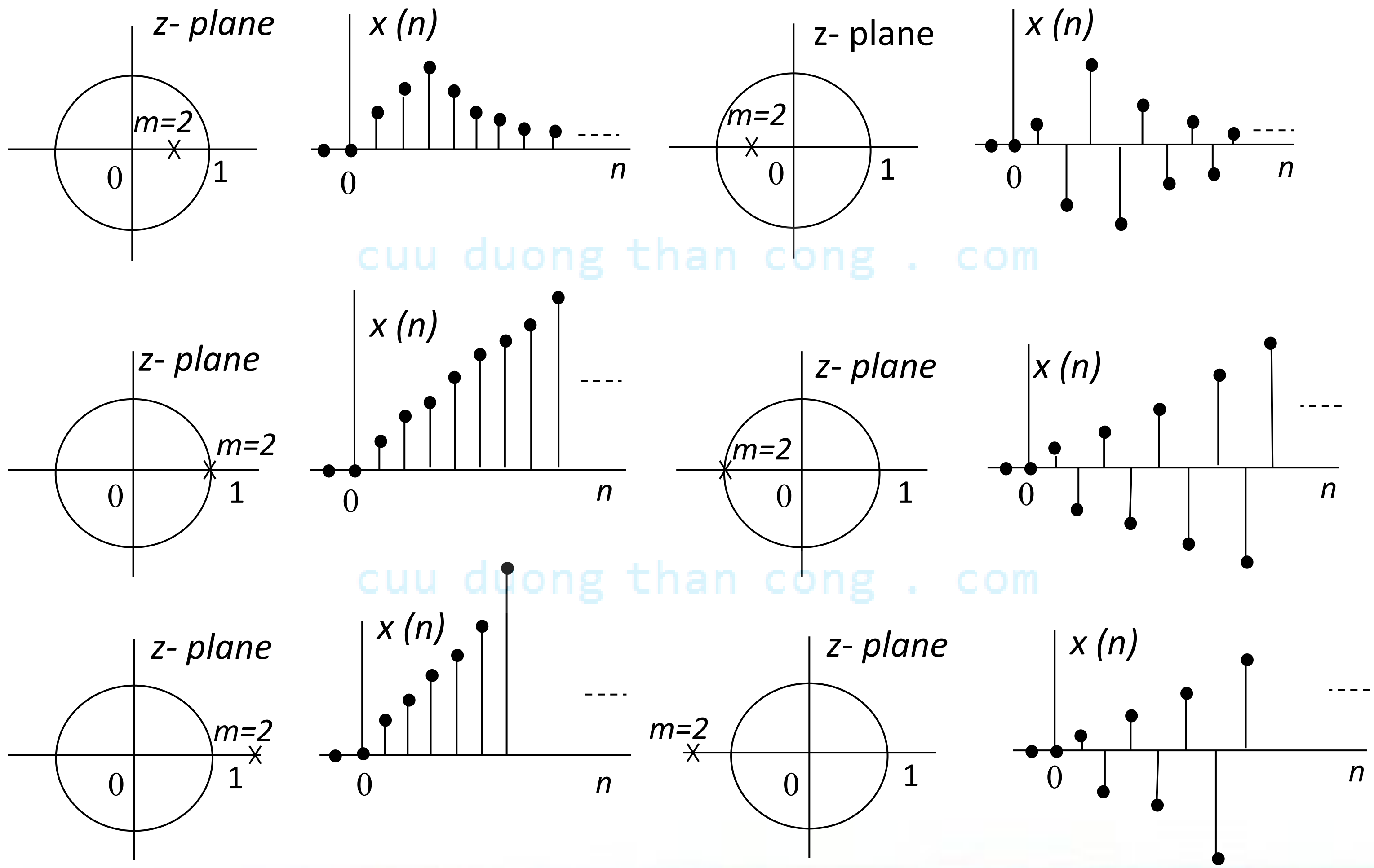
The ***signal is decaying*** if the *pole is inside the init circle*, ***fixed*** if the *pole is on the unit circle*, and ***growing*** if the *pole is outside the unit circle* .

A *causal real signal* with a ***double real pole*** has the form : $X(n) = na^n u(n)$ (see table 3.3)

A *double real pole on the unit circle* results in an ***unbound signal*** (see Fig 3.3.6)

3.3.2 Pole location and time- domain behavior for causal signals

Figure 3.12 Time- domain behavior of *causal signal* corresponding to a ***double (m=2) real pole***, as a function of the pole location.



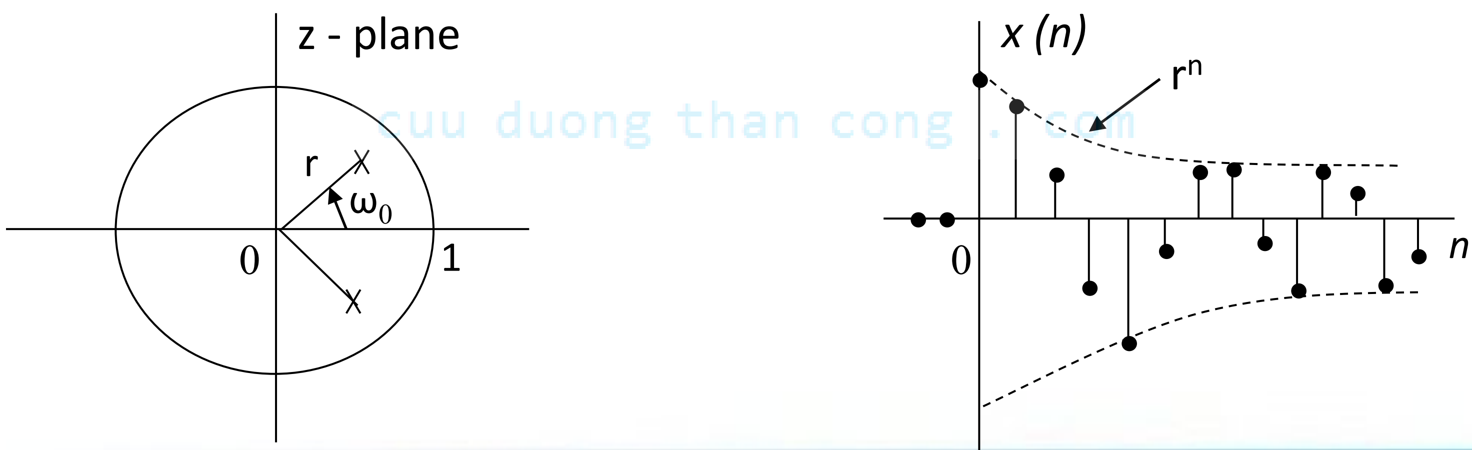
3.3.2 Pole location and time- domain behavior for causal signals

Figure 3.13 illustrates the case of *a pair of complex – conjugate poles*.

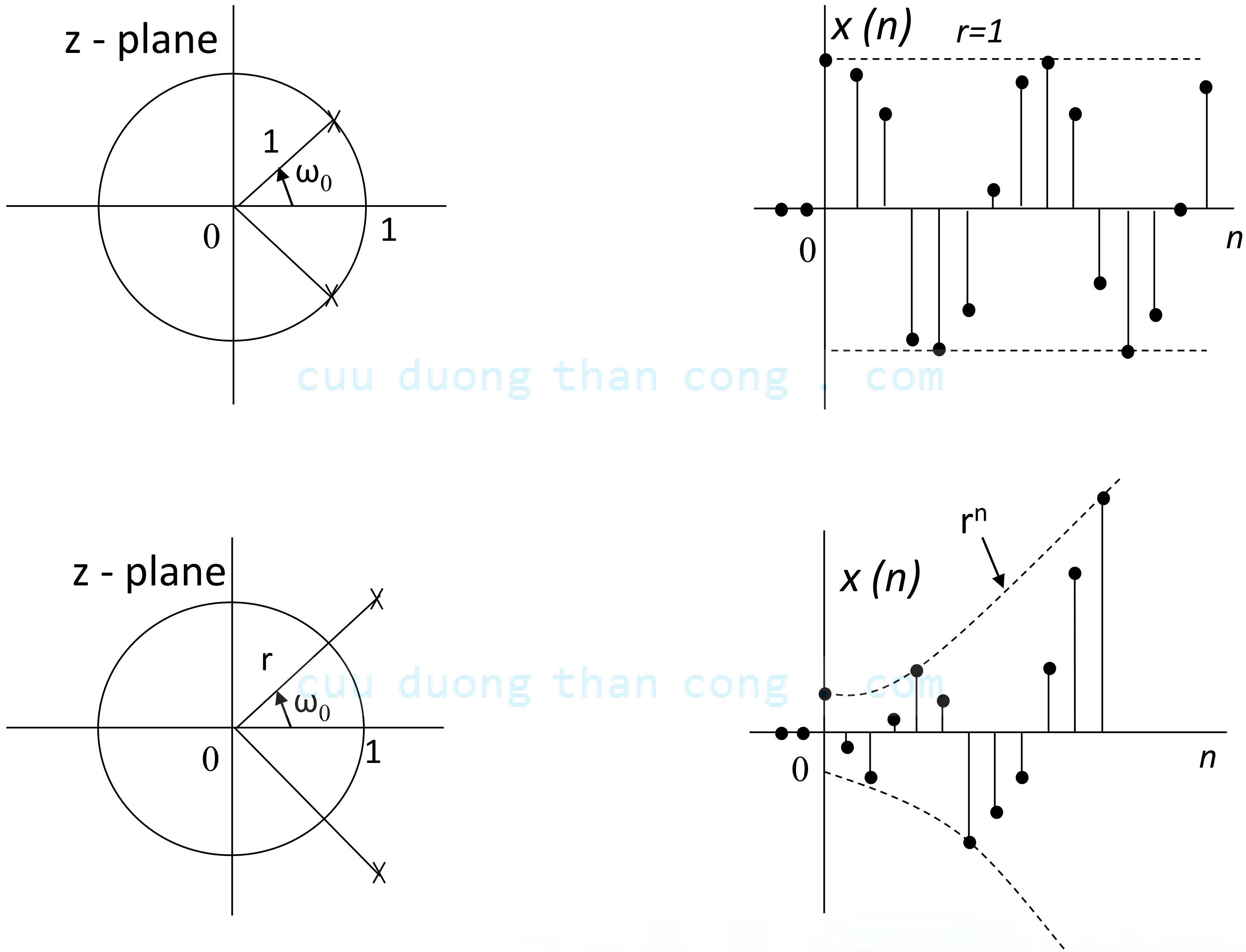
This configuration of poles results in an *exponentially weighted sinusoidal signal*.

The *amplitude* of the signal is ***growing*** if $r > 1$, ***constant*** if $r = 1$ (sinusoidal signals), and ***decaying*** if $r < 1$.

Figure 3.13 A pair of complex- conjugate poles corresponds to causal signals with oscillatory behavior.



3.3.2 Pole location and time- domain behavior for causal signals



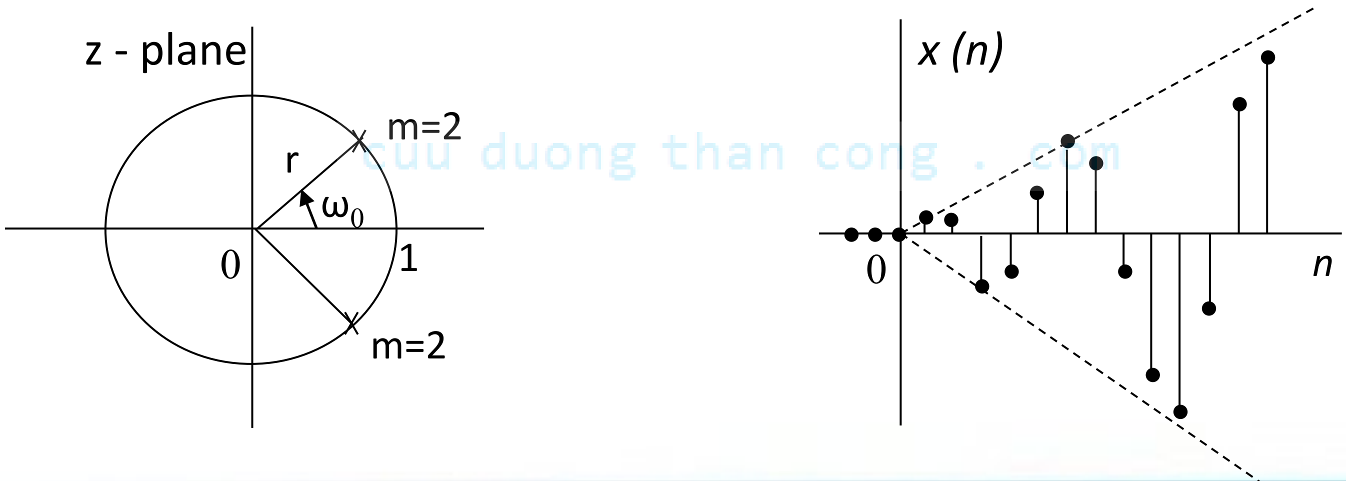
3.3.2 Pole location and time- domain behavior for causal signals

Fig 3.14 shows the behavior of a causal signal with a double pair of poles on the unit circle.

A signal with a *pole near the origin decays more rapidly than* one associated with a *pole near the unit circle*.

Everything we have said about causal signals applies as well to *causal LTI systems*.

Figure 3.14 Causal signal corresponding to a double pair of complex – conjugate poles on the unit circle.



3.3.3 The system Function of a Linear Time-Invariant System

$$Y(z) = H(z)X(z) \qquad (3.3.4)$$

Y (z) - the z-transform of the output sequence *y(n)*.

X (z) - the z-transform of the input sequence *x(n)*.

H (z) - the z-transform of the unit sample response *h(n)*.

$$H(z) = \frac{Y(z)}{X(z)} \qquad (3.3.5)$$

$$H(z) = \sum_{n-\infty}^{\infty} h(n)z^{-n} \qquad (3.3.6)$$

H(z) is called the ***systems function***.

We have linear constant- coefficient difference equation:

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k (n-k) \qquad (3.3.7)$$

3.3.3 The system Function of a Linear Time-Invariant System

By applying the time-shifting property, we obtain.

$$Y(z) = -\sum_{k=1}^{N} a_k Y(z) z^{-k} + \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{\infty} a_k z^{-k}}$$

Or

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \qquad (3.3.8)$$

+ If $a_k = 0$ for $1 \le k \le N$, then

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \frac{1}{z^M} \sum_{k=0}^{M} b_k z^{M-k} \qquad (3.3.9)$$

H(z) contains *M zeros*, *M th-order* at the origin $z = 0$

It is called an ***all-zero system***.

3.3.3 The system Function of a Linear Time-Invariant System

System has a finite-duration impulse response (FIR), and it is called an ***FIR system*** or ***moving average*** (***MA***) ***system***.

+ If $b_k = 0$ for $1 \le k \le M$, then

$$H(z) = \frac{b_0 z^N}{\sum_{k=0}^{N} a_k z^{N-k}} \qquad a_0 \equiv 1 \qquad (3.3.10)$$

H(z) consists of *N pole*, and an *N th-order zero* at the origin *z = 0.* This system is called an ***all-pole system***.

The impulse response of such a system is infinite in duration, and hence it is an *IIR system*.

The general form of the system by (3.3.8) is called *a pole-zero system*, with *N poles* and *M zeros* and in ***an IIR system***.

3.4 Inversion of the z-transform

The inverse *z*-transform is formally given by

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)\, z^{n-1}\, dz \qquad (3.4.1)$$

C - a circle in the ROC of *x(z)* in the z-plane

\+ ***There are 3 methods*** for the evaluation

1. *Direct evaluation* of (3.4.1), by contour integration.
2. *Expansion into a series of terms*, in the variables *z*, and z^{-1}
3. *Partial-fraction expansion* and table lookup.

3.4.1 Inverse z-transform by contour Integration

+ *Cauchy residue theorem*

Let *f(z)* be a function of the complex variable *z*, and *C* be a closed path in the z-plane.

If the derivative *df(z)/dz* exists on and inside the contour *C* and if *f(z)* has no poles at $z = z_0$, then

$$\frac{1}{2\pi j}\oint_C \frac{f(z)}{z-z_0}dz = \begin{cases} f(z_0, \text{ if } z_0 \text{ is inside } C \\ 0, \quad \text{if } z_0 \text{ is outside } C \end{cases} \qquad (3.4.2)$$

3.4.1 Inverse z-transform by contour Integration

If the *(k+1)* order derivative of *f(z)* exists and *f(z)* has no poles at $z = z_0$, then

$$\frac{1}{2\pi j}\oint_C \frac{f(z)}{(z-z_0)^k}dz = \begin{cases} \frac{1}{(k-1)!}\frac{d^{k-1}f(z)}{dz^{k-1}}\Big|_{z=z_0} \text{ if } z_0 \text{ is outside } C \\ 0, \end{cases} \quad (3.4.3)$$

Suppose that $p(z) = \frac{f(z)}{g(z)}$

f(z) has no pole inside the contour *C*

g(z) is a polynomial with distinct roots $z_1, z_2, \ldots, z_n$ inside *C* then

$$\frac{1}{2\pi j}\oint_C \frac{f(z)}{g(z)}dz = \sum_{i=1}^{n} A_i \quad (3.4.4)$$

Chapter 3: Inverse z-transform by contour Integration

Where $A_i(z) = (z - z_i)p(z) = (z-z_i)f(z)/g(z)$ (3.4.5)

$\{A_i(z_i)\}$ are residues of the corresponding poles at $z = z_i$, $i = 1,2,\ldots,n$.

In the case the inverse z-transform we have

$$x(n) = \sum_{i}^{\text{all poles}\{z_i\}\text{ inside } C} (z-z_i)X(z)z^{n-1}\big|_{z=z_i}$$

If $X(z)z^{n-1}$ has no poles inside the contour C for one or more values of n, then $x(n) = 0$ for these values

3.4.2 Inverse z-transform by Power Series Expansion

Given a z-transform *X(z)* with its corresponding ROC , we can expand *X(z)* into a power series of the form.

$$X(z)=\sum_{n-\infty}^{\infty} C_n z^{-n} \qquad (3.4.7)$$

Which converges in the given ROC. Then, by uniqueness of the z-transform, $\boldsymbol{X(n) = C_n}$ for all *n*. When *X(z)* is rational, the expansion can be performed by *long division*.

3.4.2 Inverse z-transform by Power Series Expansion

For example

Determine the inverse *z*-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

Where ROC: |z| > 1

3.4.2 Inverse z-transform by Power Series Expansion

Solution.

Since the ROC is the exterior of a circle , we expect *x(n)* to be a causal signal.

Thus, we seek a power series expansion in negative powers of *z*.

$$X(z) = \frac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = 1 + \frac{3}{2}z^{-1} + \frac{7}{4}z^{-2} + \frac{15}{8}z^{-3} + \frac{31}{16}z^{4} + \cdots$$

by comparing this relation with (3.11) we conclude that

$$x(n) = \left\{\underset{\uparrow}{1}, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \ldots\right\}$$

3.4.3 The inverse z-Transform by Partial-Fraction Expansion

The function $X(z)$ as a linear combination.

$$X(z) = \alpha_1 X_1(z) + \alpha_2 X_2(z) + \ldots + \alpha_K X_K(z) \qquad (3.4.8)$$

Where $X_1(z), \ldots X_k(z)$ are expressions with inverse transform $x_1(n), \ldots, x_k(n)$ available in a table of z-transform pairs. Then,

$$x(n) = \alpha_1 x_1(n) + \alpha_2 x_2(n) + \ldots + \alpha_K x_K(n) \qquad (3.4.9)$$

We assume that $a_0 = 1$, so that (3.3.1) can by expressed as

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} \qquad (3.4.10)$$

3.4.3 The inverse z-Transform by Partial-Fraction Expansion

Any *improper rational function* ($M \geq N$) can be expressed as.

$$X(z) = \frac{N(z)}{D(z)} = c_0 c_1 z^{-1} + \cdots + c_{M-N} z^{-(M-N)} + \frac{N_1(z)}{D_1(z)} \quad (3.4.11)$$

We perform a partial fraction expansion of the proper rational function. From (3.4.10) with $a_N \neq 0$ and $M < N$

Then, we invert each of the terms.

$$X(z) = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M Z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_N} \quad (3.4.13)$$

$$\Rightarrow \quad \frac{X(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_N} \quad (3.4.14)$$

3.4.3 The inverse z-Transform by Partial-Fraction Expansion

Distinct poles

Suppose that the poles $p_1, p_2, \ldots p_N$ are all different . Then we seek an expansion of the form

$$\frac{X(z)}{z} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \cdots + \frac{A_N}{z-p_N} \qquad (3.4.15)$$

$$\Leftrightarrow \quad \frac{(z-p_k)X(z)}{z} = \frac{(z-p_k)A_1}{z-p_1} + \cdots + A_k + \cdots + \frac{(z-p_k)A_N}{z-p_N} \qquad (3.4.20)$$

with $z = p_k$,

$$A_k = \frac{(z-p_k)X(z)}{z}\Big|z = p_k, \quad k = 1,2,\ldots,N \qquad (3.4.21)$$

3.4.3 The inverse z-Transform by Partial-Fraction Expansion

Multiple- order poles

If *X(z)* has of multiplicity *l*, that is, it contains in its denominator the factor $(z - p_k)^l$, then the expansion (3.4.15) is no longer true.

The partial factor expansion must contain the terms.

$$\frac{A_{1k}}{z-p_k}+\frac{A_{2k}}{(z-p_k)^2}+\cdots+\frac{A_{lk}}{(z-p_k)^l}$$

3.4.3 The inverse z-Transform by Partial-Fraction Expansion

Now, first *X(z)* contains distinct poles.

$$X(z) = A_1 \frac{1}{1 - p_1 z^{-1}} + A_2 \frac{1}{1 - p_2 z^{-1}} + \cdots + A_N \frac{1}{1 - p_N z^{-1}} \qquad (3.4.27)$$

From $x(n) = Z^{-1}\{ X(z) \}$, then

$$Z^{-1}\left\{\frac{1}{1 - P_k z^{-1}}\right\} = \begin{cases} (p_k)^n u(n), & \text{if } ROC: |z| > |p_k| \text{ (causal signals)} \\ -(p_k)^n u(-n-1), & \text{if } ROC: |z| < |p_k| \text{ (anticausal signals)} \end{cases} \qquad (3.4.28)$$

With $|z| > p_{max}$ where $p_{max} = max\ \{|p_1|\}$

Then

$$x(n) = (A_1 p_1^n + A_2 p_2^n + \cdots + A_N p_N^n) u(n) \qquad (3.4.29)$$

If all poles are ***real***, (3.4.29) is a linear combination of real exponential signals.

3.4.3 The inverse z-Transform by Partial-Fraction Expansion

If all poles are distinct but some of then are ***complex***,

$$x_k(n) = [A_k(p_k)^n + A_k^*(p_k^*)^n]u(n) \qquad (3.4.30)$$

$$A_k = |A_k|e^{j\alpha_k} \qquad (3.4.31)$$

$$P_k = |A_k|e^{j\beta_k} \qquad (3.4.32)$$

$$x_k(n) = |A_k|r_k^n\left[e^{j(\beta_k n+\alpha_k)} + e^{-j(\beta_k n+\alpha_k)}\right]u(n)$$

or

$$x_k(n) = 2|A_k|r_k^n\cos(\beta_k n + \alpha_k)u(n) \qquad (3.4.33)$$

3.4.3 The inverse z-Transform by Partial-Fraction Expansion

$$Z^{-1}\left\{\frac{A_k}{1-p_k z^{-1}}+\frac{A_k^*}{1-p_k^* z^{-1}}\right\}=2|A_k|r_k^n\cos(\beta_k n+\alpha_k)u(n) \quad (3.4.34)$$

if the ROC: is $|z| > |p| = r_k$

X(z) has ***multiple poles***.

$$Z^{-1}\left\{\frac{pz^{-1}}{(1-pz^{-1})^2}\right\}=np^n u(n) \quad (3.4.35)$$

provided that the ROC is $|z| > |p|$

3.4.4 Decomposition of Rational z-Transforms

If $X(z)$ expressed as.

$$X(z) = \frac{\sum_{k=1}^{M}(1 - z_k z^{-1})}{1 + \sum_{k=1}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})} \quad (3.4.40)$$

If $M \geq N$, $a_0 \equiv 1$, then

$$X(z) = \sum_{k=0}^{M-N} c_k z^{-k} + X_{pr}(z) \quad (3.4.41)$$

$$X_{pr}(z) = A_1 \frac{1}{1 - p_1 z^{-1}} + A_2 \frac{1}{1 - p_2 z^{-1}} + \cdots + A_N \frac{1}{1 - p_N z^{-1}} \quad (3.4.42)$$

$$\frac{A_1}{1 - pz^{-1}} + \frac{A_1^*}{1 - p^k z^{-1}} = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}} \quad (3.4.43)$$

3.4.4 Decomposition of Rational z-Transforms

Where $b_0 = 2\,Re(A)$ $a_1 = -\,2\,Re(p)$ (3.4.44)

$b_1 = 2\,Re(A_P^*)$ $a_2 = |p|^2$

$$\Rightarrow \quad X(z) = \sum_{k=0}^{M-N} c_k z^{-k} + \sum_{k=1}^{K_1} \frac{b_k}{1 + a_k z^{-1}} + \sum_{k=1}^{K_2} \frac{b_{0k} + b_{1k} z^{-1}}{1 + a_{1k} z^{-1} + a_{2k} z^{-2}} \quad (3.4.45)$$

Where $k_1 + 2k_2 = N$

Assuming for simplicity that $M = N$

$$X(z) = b_0 \prod_{k=1}^{K_1} \frac{1 + b_k z^{-1}}{1 + a_k z^{-1}} \prod_{k=1}^{K_2} \frac{1 + b_{1k} z^{-1} + b_{2k} z^{-2}}{1 + a_{1k} z^{-1} + a_{2k} z^{-2}} \quad (3.4.48)$$

Where $b_{1k} = -\,2\,Re(z_k)$, $a_{1k} = -\,2\,Re(p_k)$

$b_{2k^*} = |z_k|^2$ $a_{2k} = |p_k|^2$ (3.4.47)

3.5 The one-side z-Transform

Definition and properties

The *one-sided* or *unilateral* z-transform of a signal *x(n)* is defined

$$X^+(z) \equiv \sum_{n=0}^{\infty} x(n) z^{-n} \qquad (3.5.1)$$

notations $Z^+\{x(n)\}$ and $x(n) \overset{z^+}{\longleftrightarrow} X^+(z)$

3.5.1 Definition and properties

The one-side z-transform has the following ***characteristics***:

1. It does not contain information about the signal *x(n)* for negative values of time.
2. It is unique only for causal signals, because only these signals are zero for $n < 0$
3. The ROC of $X^+(z)$, is always the exterior of a circle.

3.5.1 Definition and properties

Shifting Property

Case1: Time delay if $x(n) \overset{z^+}{\longleftrightarrow} X^+(z)$

then

$$x(n-k) \overset{z^+}{\leftrightarrow} z^{-k}[X^+(z) + \sum_{n=1}^{k} x(-n)z^{n}] \quad k > 0 \qquad (3.5.2)$$

In case *x(n)* is causal, then

$$x(n-k) \overset{z^+}{\longleftrightarrow} z^{-k}X^+(z) \qquad (3.5.3)$$

$$Z^+\{x(n-k)\} = [x(-k)+x(-k+1)z^{-1} + \ldots + x(-1)z^{-k+1}] + z^{-k}X^+(z) \;\; k > 0 \qquad (3.5.4)$$

3.5.1 Definition and properties

Case 2: Time advance if $x(n) \overset{z^+}{\longleftrightarrow} X^+(z)$

then

$$x(n+k) \overset{z^+}{\longleftrightarrow} z^k \left[X^+(z) - \sum_{n=0}^{k-1} x(n) z^{-n} \right] k > 0 \qquad (3.5.5)$$

Final value theorem

if $x(n) \overset{z^+}{\longleftrightarrow} X^+(z)$

then

$$\lim_{n\to\infty} x(n) = \lim_{z\to 1} (z-1) X^+(z) \qquad (3.5.6)$$

The limit in (3.5.6) exists if the ROC of *(z-1)* $X^+(z)$ includes the unit circle.

3.5.2 Solution of Difference Equations

By *reducing the difference equation* relating the two time-domain signals to an equivalent algebraic equation relating their one-sided z-transforms.

This equation can be easily solved *to obtain the transform of the desired signal*.

The signal in the time domain is obtained by inverting the resulting z- transform.

3.6 Analysis of Linear Time – invariant System in the z-Domain

Response of system with rational system functions

If *x(n)* has a rational z-transform *X(z)* of the form

$$X(z) = \frac{N(z)}{Q(z)} \qquad (3.6.1)$$

we represent

$$H(z) = \frac{B(z)}{A(z)}$$

If the system is initially relaxed, that is

y(-1) = y(-2) = …= y(-N) = 0, then

$$Y(z) = H(z)X(z) = \frac{B(z)N(z)}{A(z)Q(z)} \qquad (3.6.2)$$

3.6.1 Response of system with rational system functions

If system contains simple poles $p_1,p_2,\dots p_N$ and z-transform of the input signal contains poles $q_1,q_2,\dots q_L$. Where $p_k \neq q_m$, then

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^{L} \frac{Q_k}{1 - q_k z^{-1}} \qquad (3.6.3)$$

The inverse transform of Y(z) yields

$$y(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n) + \sum_{k=1}^{L} Q_k (q_k)^n u(n) \qquad (3.6.4)$$

Scale factors $\{A_k\}$ and $\{ Q_k \}$ are functions of both sets of poles $\{ p_k \}$ and $\{ q_k \}$.

The first part called ***the natural response*** *of the system*. The second part is called ***forced response*** *of the system.*

3.6.2 Response of poles – zero system with Nonzero Initial conditions

Suppose that *X(n)* is applied to the pole-zero system at *n = 0*. (*x(n)* is causal)

Since the input *x(n)* is causal and output *y(n)* for *n ≥ 0*

$$Y^+(z) = -\sum_{k=1}^{N} a_k z^{-k}\left[Y^+(z) + \sum_{n=1}^{k} y(-n) z^{n}\right] + \sum_{k=0}^{M} b_k z^{-k} X^+(z) \qquad (3.6.5)$$

Or

$$Y^+(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}} X(z) - \frac{\sum_{K=1}^{N} a_k z^{-k} \sum_{n=1}^{k} y(-n) z^{k}}{1+\sum_{K=1}^{N} a_k z^{-k}} \qquad (3.6.6)$$

$$= H(z)X(z) + \frac{N_0(z)}{A(z)}$$

3.6.2 Response of poles – zero system with Nonzero Initial conditions

Where

$$N_0(z) = -\sum_{k=1}^{N} a_k z^{-k} \sum_{n=1}^{k} y(-n) z^n \qquad (3.6.7)$$

$$Y_{zs}(z) = H(z)X(z) \qquad (3.6.8)$$

$$Y_{zi}^{+}(z) = \frac{N_0(z)}{A(z)} \qquad (3.6.9)$$

Thus, $y(n) = y_{zs}(n) + y_{zi}(n)$ (3.6.10)

Since $A(z)$ has poles as $p_1, p_2, \ldots, p_N$, then

$$y_{zi}(n) = \sum_{k=1}^{N} D_k (p_k)^n u(n) \qquad (3.6.11)$$

3.6.2 Response of poles – zero system with Nonzero Initial conditions

This can be added to (3.6.4)

$$y(n) = \sum_{k=1}^{N} A'_k (p_k)^n u(n) + \sum_{k=1}^{L} Q_k (q_k)^n u(n) \qquad (3.6.12)$$

Where

$$A'_k = A_k + D_k \qquad (3.6.13)$$

The effect of the initial conditions is to alter the natural response of the system though modifications of the *scale factors* $\{A_k\}$

There are ***no new poles*** introduced by the nonzero initial conditions.

3.6.3 Transient and Steady – state Responses

The ***natural response*** of a causal system has the form

$$y_{nr}(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n) \qquad (3.6.14)$$

Where $\{p_k\}$, $k = 1, 2 \ldots, N$ are the poles of the system.

$\{A_k\}$ are scale factors

The ***forced response*** of the system has the form

$$y_{fr}(n) = \sum_{k=1}^{L} Q_k (q_k)^n u(n) \qquad (3.6.15)$$

Where $\{q_k\}$, $K = 1, 2, \ldots, L$ are the poles

$\{Q_k\}$ are scale factors

3.6.3 Transient and Steady – state Responses

when the *causal input signal* is a *sinusoid the poles fall on the unit circle*, consequently *the forced response* is also a *sinusoid*.

It is called the ***steady-state response*** of the system.

3.6.4 Causality and Stability

A linear time- invariant system is causal if and only if the ROC of the system function is the exterior of the circle of radius r < ∞ , including the point z = ∞.

A necessary and sufficient conditions for a linear time-invariant system to be ***BIBO stable*** is

$$\sum_{n=-\infty}^{\infty} | H(n)| < \infty$$

3.6.4 Causality and Stability

H(z) must contain the unit circle within its ROC .

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$|H(z)| \le \sum_{n=-\infty}^{\infty} |h(n) z^{-n}| = \sum_{n=-\infty}^{\infty} |h(n)| |z^{-n}|$$

when on the unit circle (|*z*| = *1*)

$$|H(z)| \le \sum_{n=-\infty}^{\infty} |h(n)|$$

A linear time-invariant system is BIBO stable if and only if the ROC of the system function includes the unit circle.

A causal linear time-invariant system is BIBO stable if and only if the poles of H(z) are inside the unit circle.

3.6.5 Pole – zero cancellations

When a z-transform has *a pole* that is at the *same location as a zero*, ***the pole is canceled by the zero***.

Pole-zero cancellations can occur either *in the system function itself* or in the *product of the system function with the z-transform* of the input signal.

3.6.6 Multiple–Order poles and stability

The input is bounded if its z-transform contains pole $\{ q_k \}$, $k = 1, 2, \ldots, L$

Which satisfy the condition $| q_k | \leq 1$ for all k.

Thus, the *forced response* of the systems is also ***bounded*** , even when the input signal contains one or more distinct poles on the unit circle.

3.6.8 Stability of Second–Order Systems

A *causal two-pole system* described by the *second-order difference equation*.

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) + b_0 x(n) \qquad (3.6.26)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{b_0 z^2}{z^2 + a_1 z + a_2} \qquad (3.6.27)$$

$$p_1, p_2 = -\frac{a_1}{2} \pm \frac{\sqrt{a_1^2 - 4a_2}}{4} \qquad (3.6.28)$$

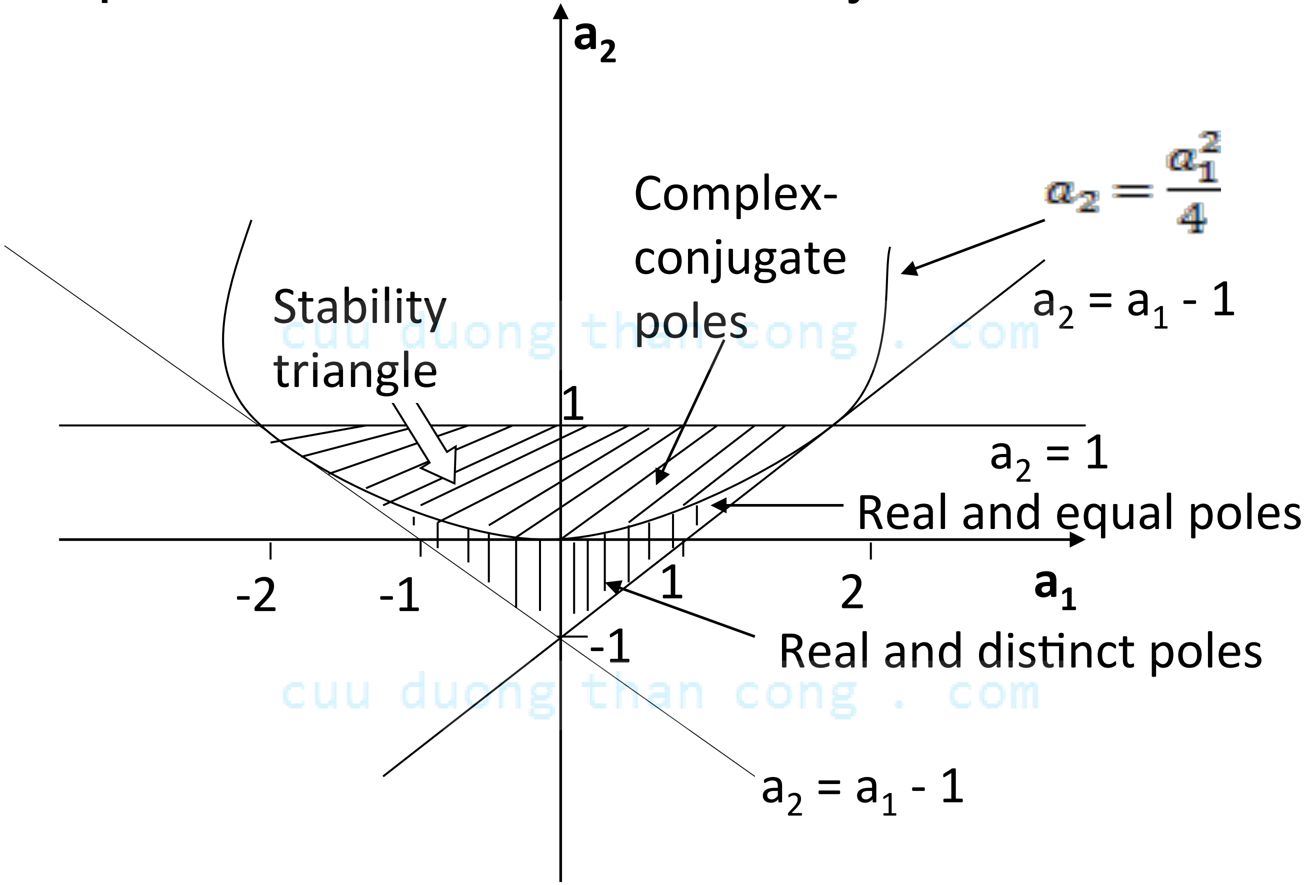
for stability

$$|a_2| = |p_1 p_2| = |p_1|\,|p_2| < 1 \qquad (3.6.31)$$

$$|a_1| < 1 + a_2 \qquad (3.6.32)$$

3.6.8 Stability of Second–Order Systems

Figure 3.15 Region of stability(stability triangle) in the (a_1, a_2) coefficient plane for a second-order system.



Problems : 3.1, 3.2, 3.5, 3.11, 3.15, 3.43, 3.47.