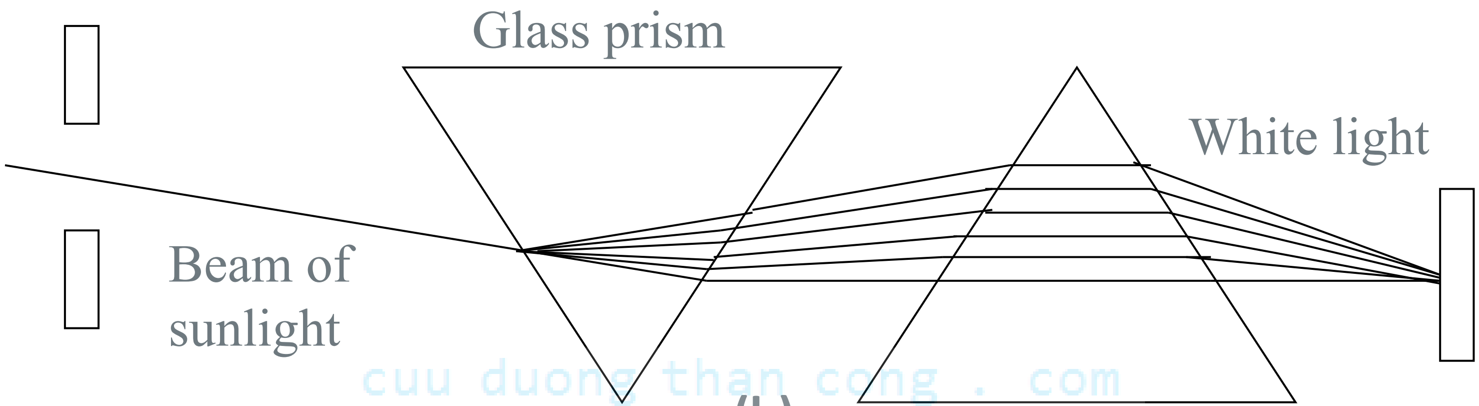
CHAPTER 4 : Frequency Analysis of Signals

The decomposition of the signal in terms of sinusoidal components. A signal is said to ***be represented in the frequency domain***.

For *the class of periodic signals*, such a decomposition ***is called a Fourier series***.

For *the class of finite energy signals*, the decomposition ***is called the Fourier transform***.

Figure 4.1.1(b) synthesis of the white light(sunlight) using glass prisms



(b)

The process of determine the spectrum of a signal in practice, based on measurements of the signals, ***is called spectrum estimation***

The *instruments* or *software programs* used so obtain spectral estimates of such signals ***are know as spectrum analyzers***.

4.1.1 : The Fourier Series for Continuous-Time Periodic Signals

A linear combination of harmonically related *complex exponentials* of the form :

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t} \qquad (4.1.1)$$

Fundamental period $T_p = 1/F_0$, $k = 0, \pm 1, \pm 2, \ldots.$

Coefficients $\{c_k\}$

The periodic signal by the series (4.1.1), is called ***a Fourier series*** or ***a Synthesis equation***.

4.1.1 : The Fourier Series for Continuous-Time Periodic Signals

The so- called ***Dirichlet conditions*** guarantee that the (4.1.1) is true if

1. The signal *x(t)* has *a finite number of discontinuities* in any period.
2. The signal *x(t)* contains a *finite number of maxima and minima* during any period.
3. The signal *x(t)* is *absolutely integrable* in any period, that is,

$$\int_{T_p} |x(t)|dt < \infty \qquad (4.1.6)$$

All periodic signals of practical interest satisfy these conditions

4.1.1 : The Fourier Series for Continuous-Time Periodic Signals

Analysis equation

$$c_k = \frac{1}{T_p}\int_{T_p} x(t)e^{-j2\pi kF_0 t}dt \quad (4.1.9)$$

If $c_k = |c_k|\, e^{j\theta k}$ then

$$x(t) = c_0 + 2\sum_{k=1}^{\infty}|c_k|\cos(2\pi kF_0 t + \theta_k) \quad (4.1.10)$$

Or

$$x(t) = a_0 + \sum_{k=1}^{\infty}(a_k\cos 2\pi kF_0 t - b_k \sin 2\pi kF_0 t) \quad (4.1.11)$$

Where

$a_0 = c_0$ $a_k = 2\,|c_k|\cos\theta_k$ $b_k = 2\,|c_k|\sin\theta_k$

4.1.2 : Power Density Spectrum of Periodic signals

A periodic signal has infinite energy and a finite average power:

$$P_x = \frac{1}{T_p}\int_{T_p} |x(t)|^2 dt \qquad (4.1.12)$$

and

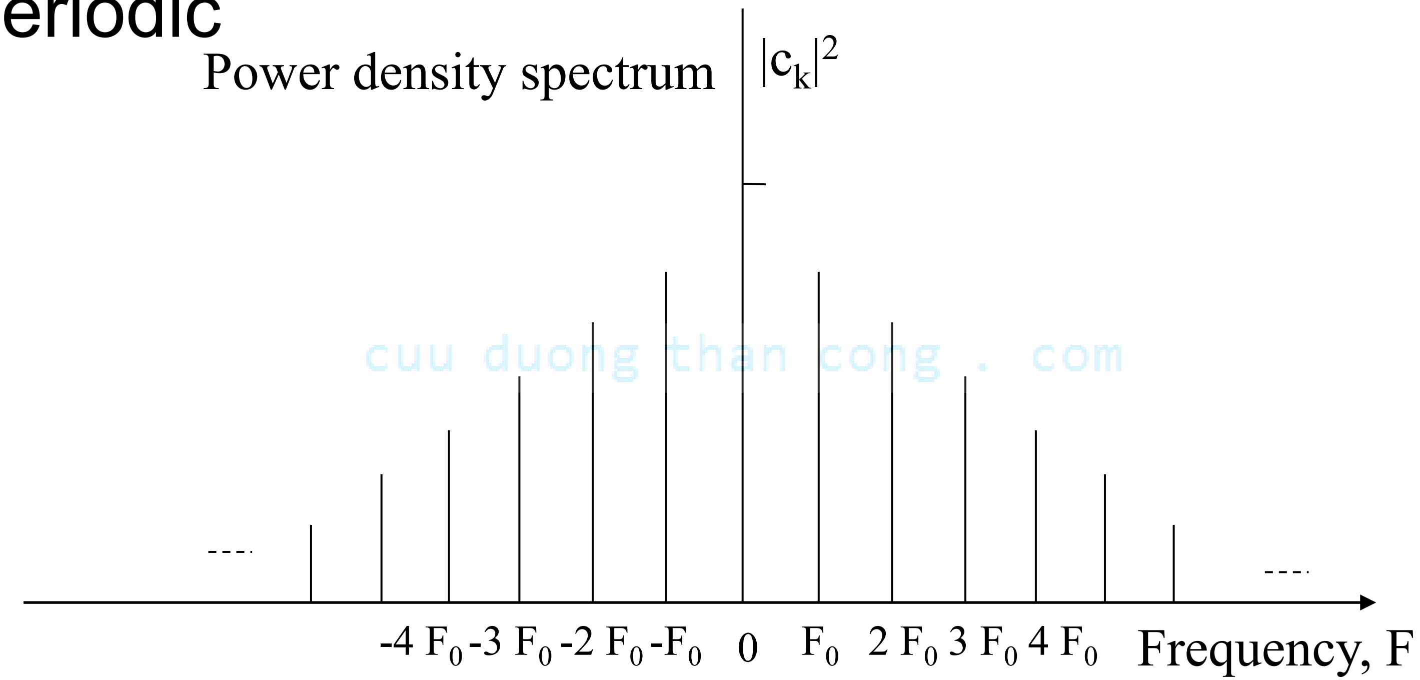
$$P_x = \sum_{k=-\infty}^{\infty} |c_k|^2 \qquad (4.1.14)$$

Which is called ***Parseval's relation*** for power signals.

Fig 4.1.2 is called *the power density spectrum* of the periodic signal *x(t)*.

4.1.2 : Power Density Spectrum of Periodic signals

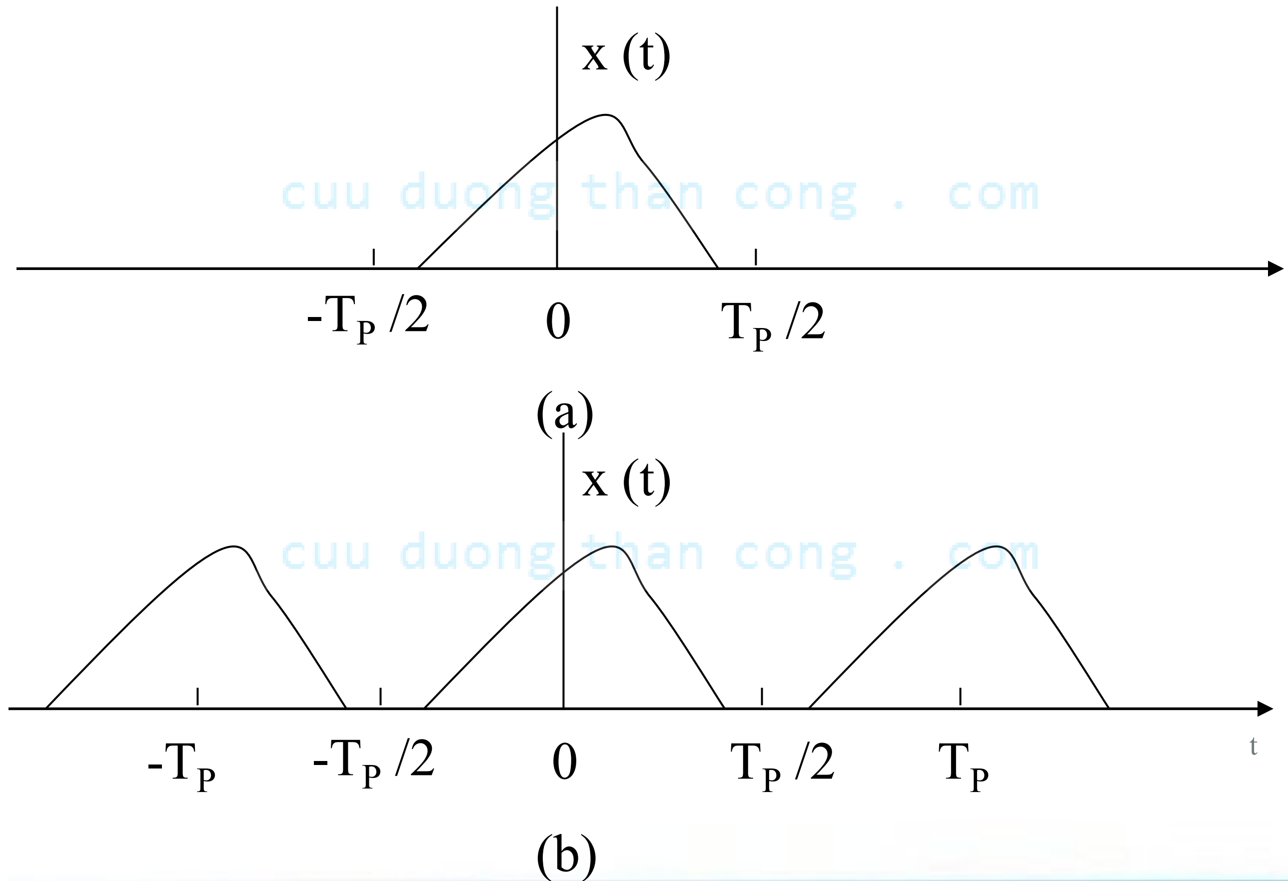
Figure 4.1.2 Power density spectrum of a continuous – time periodic



Since the power in a periodic signal exits only at discrete values of frequencies, the signal is said to have ***a line spectrum***.

4.1.3 : The Fourier Transform for Continuous – Time a periodic Signals

Figure 4.1.7 (a) ***Aperiodic signal*** *x(t)* and (b) *periodic signal* $x_p(t)$ constructed by repeating *x(t)* with a period T_p



4.1.3 : The Fourier Transform for Continuous – Time a periodic Signals

In Fig 4.1.7, that is

$$x(t) = \lim_{T_p \to \infty} x_p(t)$$

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}, \qquad F_0 = \frac{1}{T_p} \qquad (4.1.20)$$

where

$$c_k = \frac{1}{T_p} \int_{-\frac{T_p}{2}}^{\frac{T_p}{2}} x_p(t) e^{-j2\pi k F_0 t} dt \qquad (4.1.21)$$

$$c_k = \frac{1}{T_p} \int_{-\infty}^{\infty} x(t) e^{-j2\pi k F_0 t} dt \qquad (4.1.23)$$

4.1.3 : The Fourier Transform for Continuous – Time a periodic Signals

A function ***X(F)*** is called ***the Fourier transform of x(t)*** or ***analysis equation direct transform***, as

$$X(F)=\int_{-\infty}^{\infty} x(t)\,e^{-j2\pi Ft}\,dt \qquad (4.1.24)$$

$$x(t)=\int_{-\infty}^{\infty} X(F)\,e^{j2\pi Ft}\,dF \qquad (4.1.28)$$

(4.1.28) is called ***the inverse Fourier transform*** or ***Synthesis equation inverse transform***

4.1.4 : Energy Density Spectrum of Aperiodic Signals

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(F)|^2 dF \qquad (4.1.38)$$

This is ***Parseval's relation*** for aperiodic, finite energy signals.

X(F) is usually expressed in polar form as

$$X(F) = |X(F)| e^{j\Theta(F)}$$

where $|X(F)|$ is the magnitude spectrum

and $\theta(F)$ is the phase spectrum

$$\Theta(F) = \angle X(F)$$

4.2 : Frequency Analysis of Discrete- Time Signals

synthesis equation

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \qquad (4.2.1)$$

Where $\{ c_k \}$ are the coefficients in the series representation.

(4.2.1) is called ***the discrete – time Fourier series*** (***DTFS***).

Analysis equation

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \qquad (4.2.8)$$

c_k represents the amplitude and phase associated with the frequency component

4.2.1:The Fourier Series for Discrete- Time Periodic Signals

$$C_{k+N} = C_k \qquad (4.2.9)$$

$\{C_k\}$ is *a periodic sequence* with fundamental period N.

Thus, ***the spectrum*** of a signal $x(n)$ which is periodic with period N, is a ***periodic sequence with period N***.

We *will focus our attention* ***on the single period*** with range $k = 0,1,\ldots, N-1$.

If we use a *sampling frequency* F_S, the range $0 \le k \le N-1$ corresponds to the *frequency range* $0 \le F \le F_S$.

4.2.2 : Power Density Spectrum of Periodic Signals

The *average power* of a discrete –time periodic signal with period *N*.

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2 \qquad (4.2.10)$$

or

$$P_x = \sum_{n=0}^{N-1} |c_k|^2 \qquad (4.2.11)$$

The sequence $|c_k|^2$ *for* $k = 0, 1, \ldots, N-1$ is the *distribution of power as a function of frequency* and is called ***the power density spectrum*** of the periodic signal.

4.2.2 : Power Density Spectrum of Periodic Signals

Energy of the sequence *x(n)* over a signal period:

$$E_N = \sum_{n=0}^{N-1} |x(n)|^2 = N \sum_{k=0}^{N-1} |c_k|^2 \qquad (4.2.12)$$

In the case of continuous-time signals, the power density spectrum $|c_k|^2$ ***does not contain any phase information***.

The spectrum is discrete and periodic with a *fundamental period equal to that of the signal itself*.

4.2.3 : The Fourier Transform of Discrete – Time Aperiodic Signals

The frequency is *unique over the frequency interval* of (*0, 2π*)

Synthesis equation inverse transform

$$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega \qquad (4.2.28)$$

Analysis equation direct transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \qquad (4.2.29)$$

4.2.4 : Convergence of the Fourier Transform

Uniform convergence is guaranteed if *x(n) is absolutely summable*. Indeed, if

$$\sum_{n-\infty}^{\infty} |x(n)| < \infty \qquad (4.2.32)$$

Then

$$|X(\omega)| = \left|\sum_{n-\infty}^{\infty} x(n)e^{-j\omega n}\right| \leq \sum_{n-\infty}^{\infty} |x(n)| < \infty$$

(4.2.32) is a sufficient condition for the exitstence of the discrete-time Fourier transform

4.2.4 : Convergence of the Fourier Transform

Let us consider an example from the *class of finite energy signals*. Suppose that

$$X(\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases} \qquad (4.2.35)$$

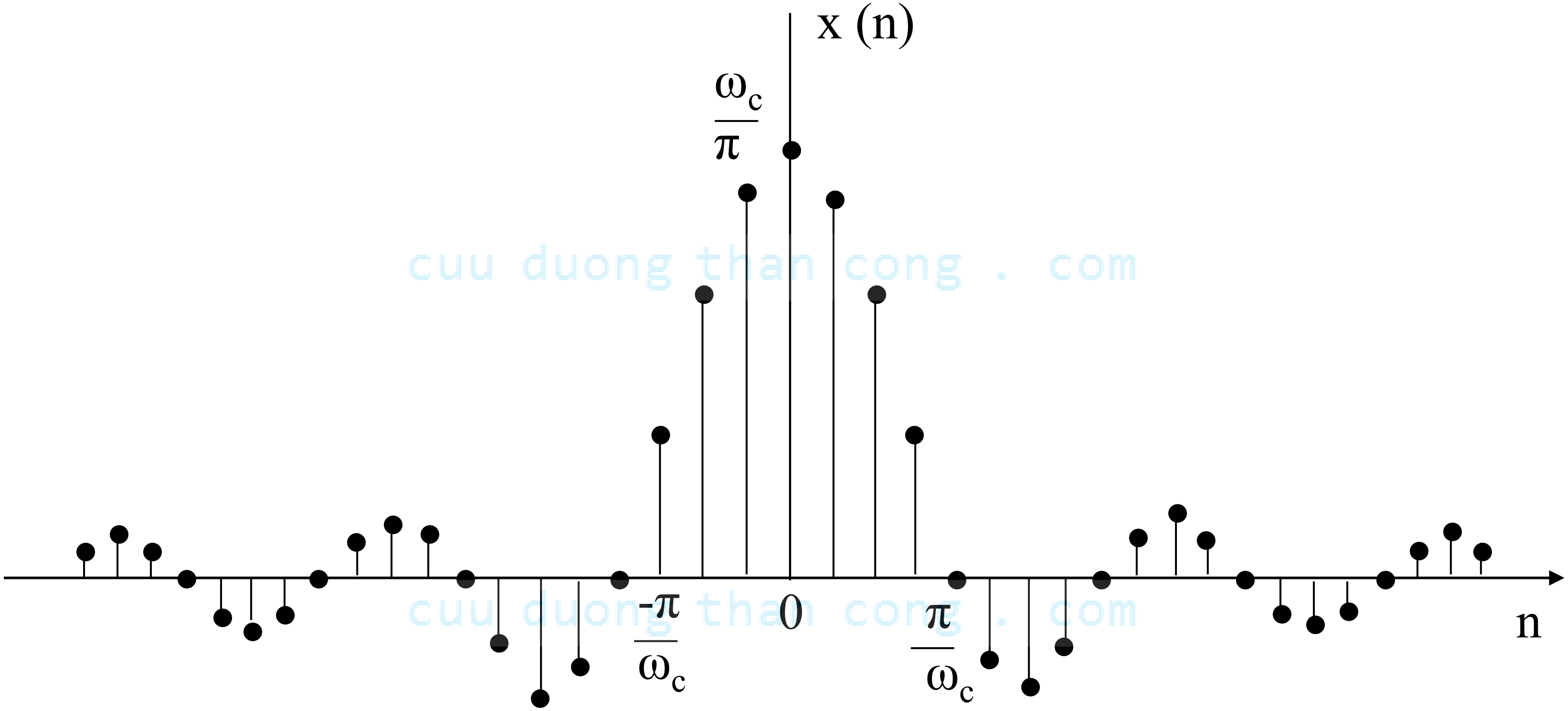
Hence

$$x(n) = \begin{cases} \dfrac{\omega_c}{\pi}, & n = 0 \\ \dfrac{\omega_c}{\pi}\dfrac{\sin\omega_c}{\omega_c n}, & n \ne 0 \end{cases} \qquad (4.2.36)$$

Sometimes,

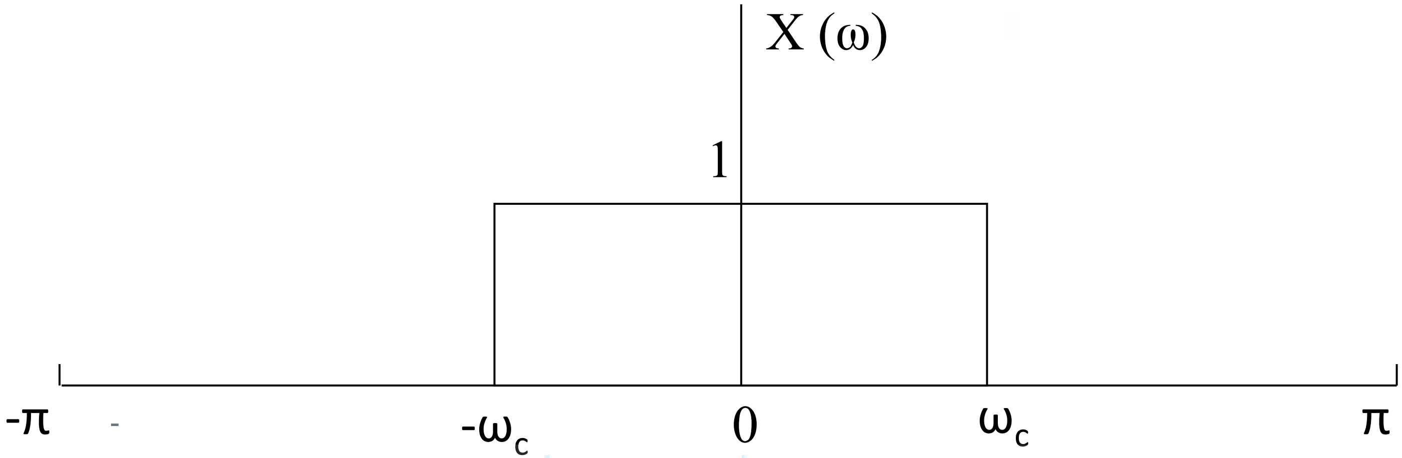
$$x(n) = \frac{\sin\omega_c n}{\pi n}, \quad -\infty < n < \infty \qquad (4.2.37)$$

Figure 4.2.4 Fourier transform pair in (4.2.35) and (4.2.36)



(a)

4.2.4 : Convergence of the Fourier Transform



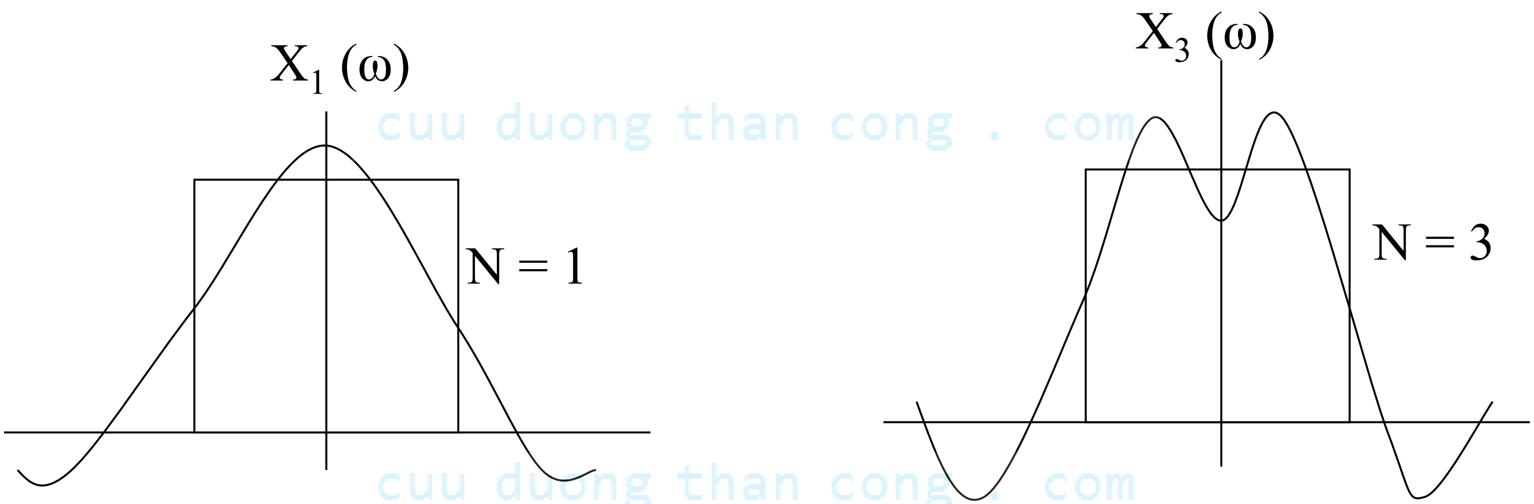
(b)

The finite sum.

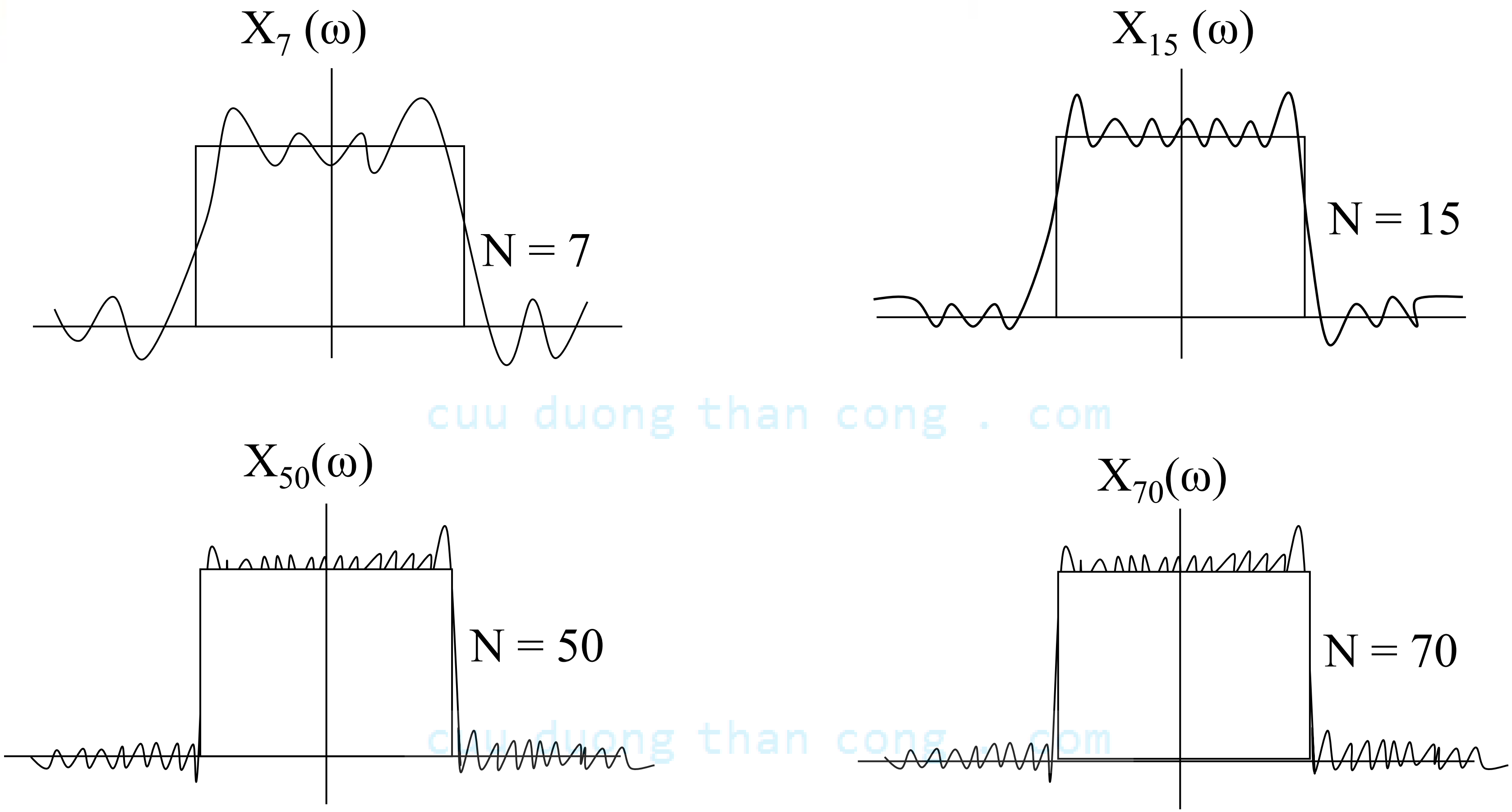
$$X_N(\omega) = \sum_{n=-N}^{N} \frac{\sin\omega_c n}{\pi n} e^{-j\omega n} \qquad (4.2.39)$$

4.2.4 : Convergence of the Fourier Transform

Figure 4.2.5 Illustration of convergence of the Fourier transform and the Gibbs phenomenon at the point of discontinuity



4.2.4 : Convergence of the Fourier Transform



$\omega_N(\omega)$ converges to $X(\omega)$ when $N \to \infty$ is called *Gibbs phenomenon*.

4.2.5 : Energy Density Spectrum of Aperiodic Signals

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega \qquad (4.2.41)$$

This is ***Parseval's relation*** for discrete – time a aperiodic signals with finite energy.

$$S_{xx}(\omega) = |X(\omega)|^2 \qquad (4.2.43)$$

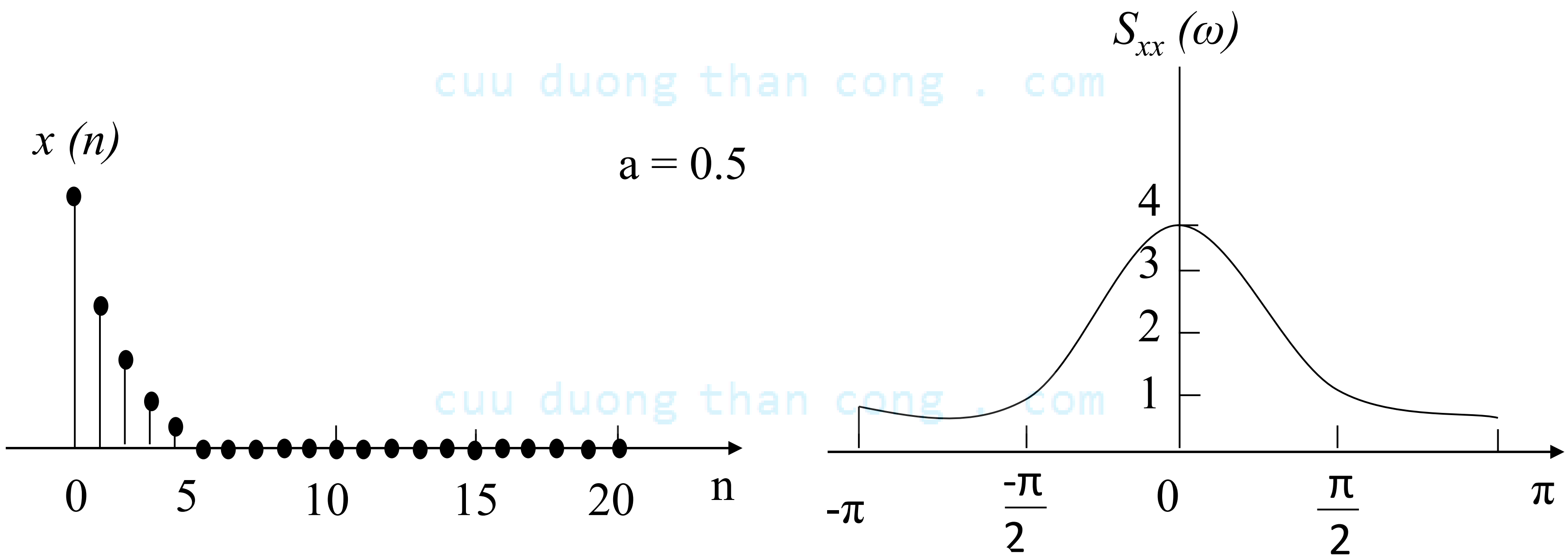
represents the distribution of energy as a function of frequency, and it is called the ***energy density spectrum of x(n).***

4.2.5 : Energy Density Spectrum of Aperiodic Signals

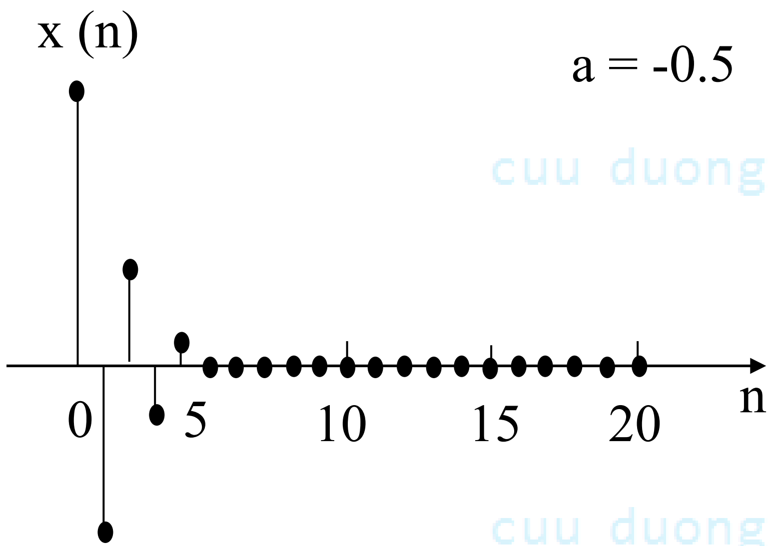
Figure 4.15

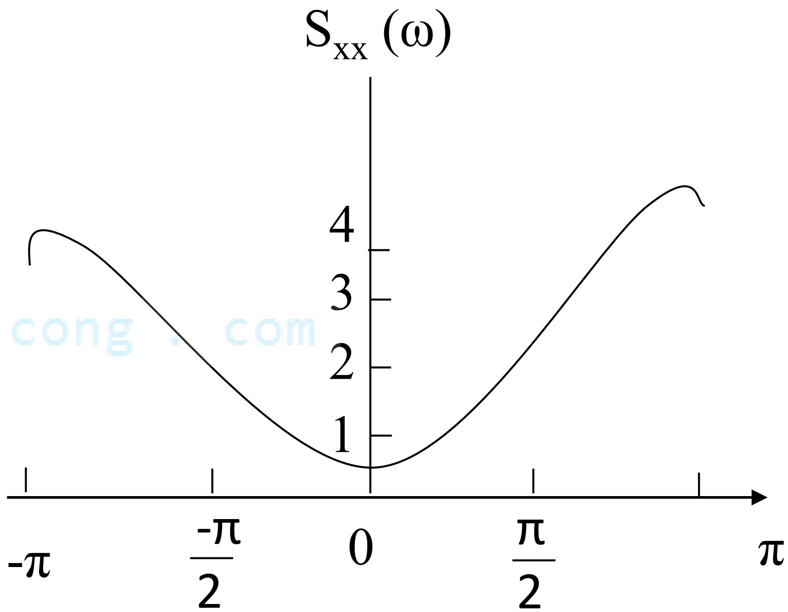
(a) Sequence $x(n) = (1/2)^2 u(n)$ and $x(n) = (-1/2)^n u(n)$;

(b) their energy density spectra



4.2.5 : Energy Density Spectrum of Aperiodic Signals

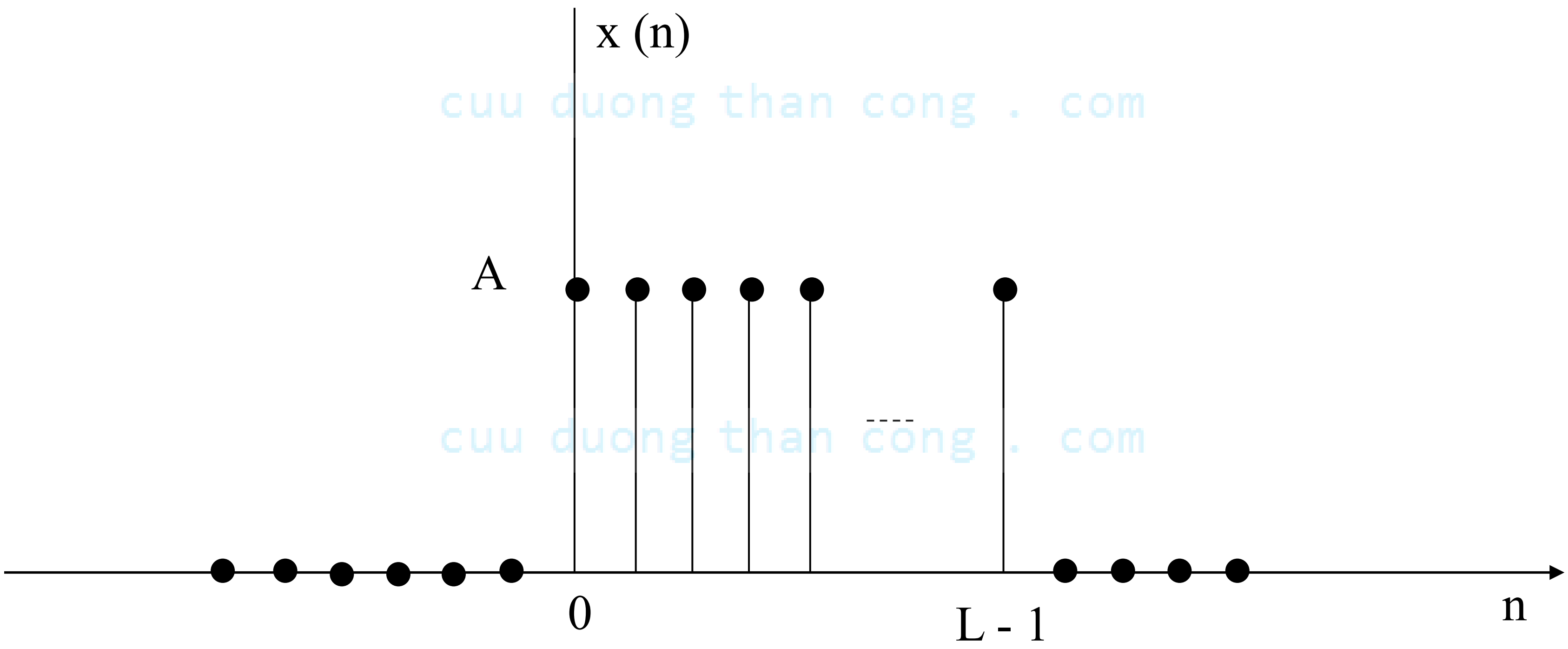




4.2.5 : Energy Density Spectrum of Aperiodic Signals

For example the sequence $$x(n) = \begin{cases} A, & 0 \le n \le L-1 \\ 0, & otherwise \end{cases} \qquad (4.2.8)$$

Figure 4.2.7 Discrete – time rectangular pulse



4.2.5 : Energy Density Spectrum of Aperiodic Signals

$$X(\omega) = \sum_{n=0}^{L-1} Ae^{-j\omega n} = Ae^{-j(\omega/2)(L-1)} \frac{\sin(L/2)}{\sin(\omega/2)} \qquad (4.2.49)$$

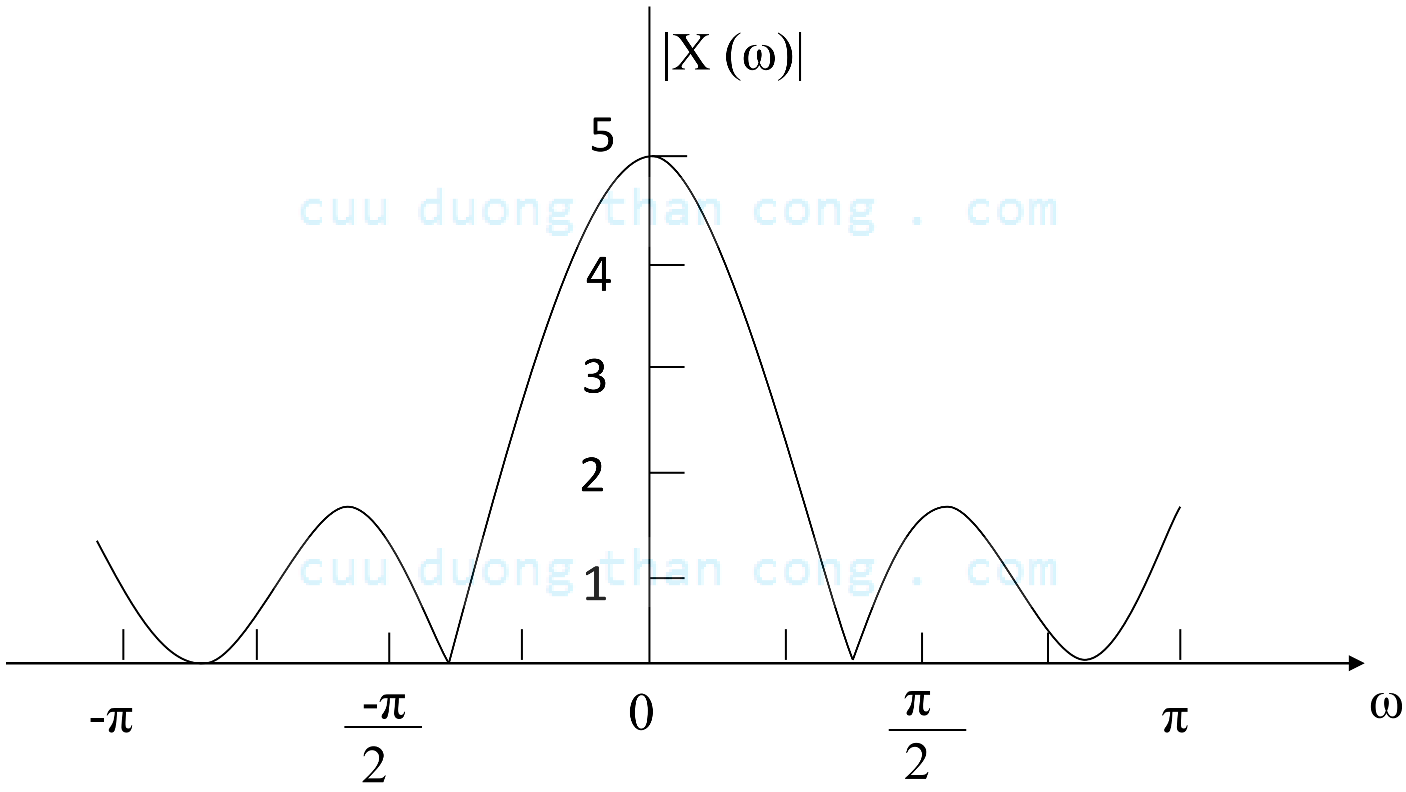
The magnitude and phase spectra of *x(n)* are.

$$|X(\omega)| = \begin{cases} |A|L, & \omega = 0 \\ |A| \left| \dfrac{\sin(\omega L/2)}{\sin(\omega/2)} \right|, & otherwise \end{cases} \qquad (4.2.50)$$

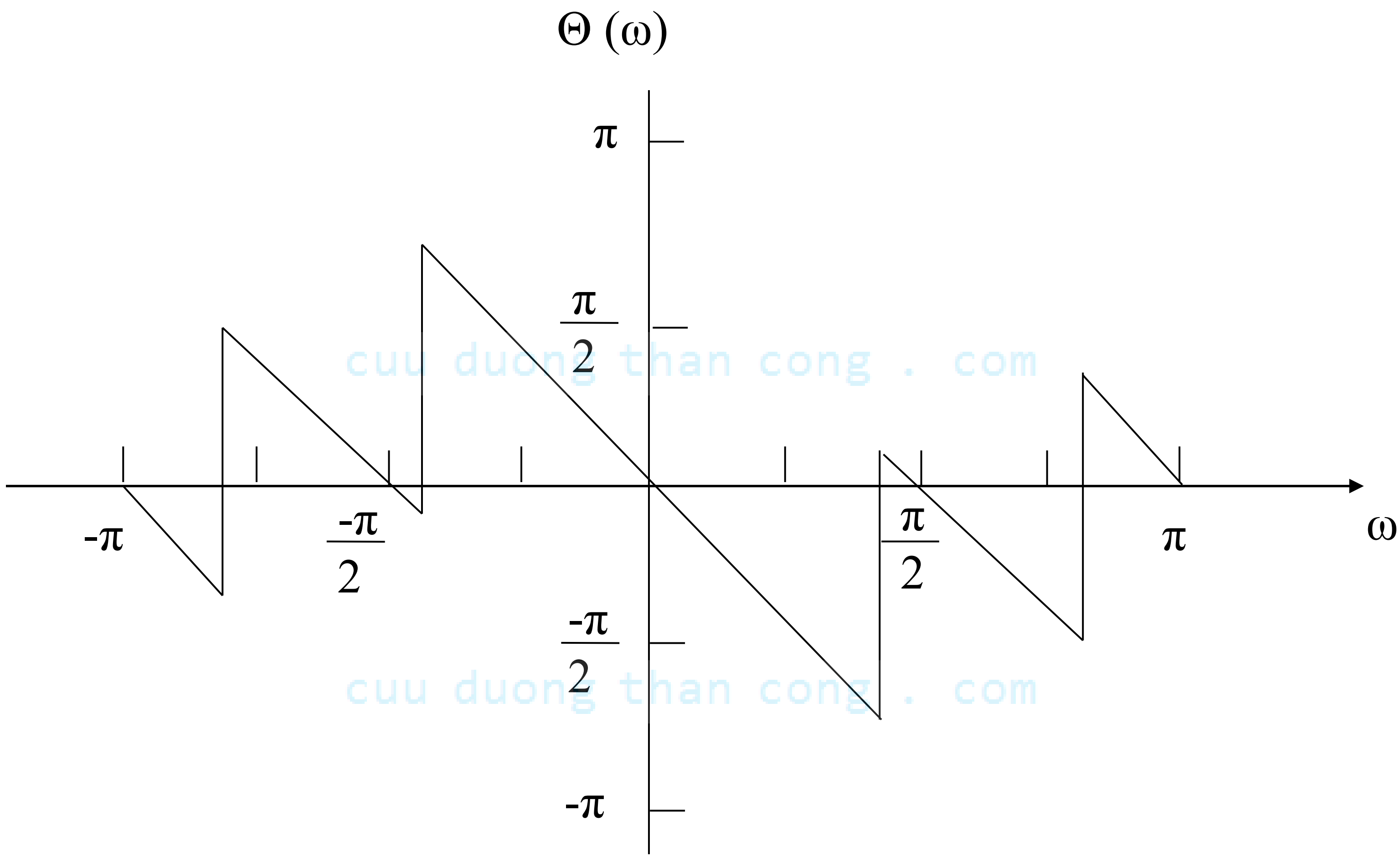
and $\measuredangle X(\omega) = \measuredangle A - \frac{\omega}{2}(L-1) + \measuredangle \frac{\sin(\omega L/2)}{\sin(\omega/2)} \qquad (4.2.51)$

4.2.5 : Energy Density Spectrum of Aperiodic Signals

Figure 4.2.8 Magnitude and phase of Fourier transform of *the discrete-time rectangular pulse* in Fig 4.2.7.



4.2.5 : Energy Density Spectrum of Aperiodic Signals



4.2.6 : Relationship of the Fourier Transform to the z-transform

$$X(z) = \sum_{n-\infty}^{\infty} x(n) z^{-n}, \quad ROC: r_2 < |z| < r_1 \qquad (4.2.54)$$

$$Z = re^{j\omega} \qquad (4.2.55)$$

This yields

$$X(z)|_{z=re^{j\omega}} = \sum_{n-\infty}^{\infty} [x(n) r^{-n}] e^{-j\omega n} \qquad (4.2.56)$$

If *X(z)* converges for |z| = *1* then

$$X(z)|_{z=e^{j\omega}} \equiv X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad (4.2.57)$$

4.2.6 : Relationship of the Fourier Transform to the z-transform

Therefore, ***the Fourier transform*** can be viewed ***as the z-transform*** of the sequence evaluated ***on the unit circle***.

The existence of the z-transform requires that the sequence $\{x(n)r^{-n}\}$ be *absolutely summable* for some value of *r*, that is.

$$\sum_{n=-\infty}^{\infty} |x(n)r^{-n}| < \infty \qquad (4.2.57)$$

4.2.6 : Relationship of the Fourier Transform to the z-transform

***If the region of convergence contains the unit circle, the Fourier transform X(ω) exists*.**

The existence of ***the Fourier transform***, which is defined *for* ***finite energy signals***, dose *not necessarily ensure the existence of the z-transform*.

4.2.7 : The Cepstrum

$$\{x(n)\} \overset{z}{\leftrightarrow} X(z)$$

{ x(n) } – stable sequence

X(z) converges on the unit circle.

The complex cepstrum of the sequence *{ x(n) }* is defined as *the sequence* ***{ $c_x(n)$ }***, which is *the inverse z-transform of $C_x(z)$*, where

$$C_x(z) = \ln X(z) \qquad (4.2.61)$$

The complex cepstrum exists if $C_x(z)$ converges in the annular region $r_1 < |z| < r_2$.

Where $0 < r_1 < 1$ and $r_2 > 1$.

4.2.7 : The Cepstrum

$$C_x(n) = \frac{1}{2\pi j}\int_C \ln X(z)\, z^{n-1} dz \qquad (4.2.63)$$

C – is a closed contour about the origin and lies within the region of convergence.

$$C_x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [\ln|X(\omega)| + j\theta(\omega)] e^{j\omega n} d\omega \qquad (4.2.68)$$

The *complex cepstrum* is used in practice *to separate signals, that are convolved* .

The process of separating two convolved signals is called ***deconvolution***.

4.2.8 : The Fourier Transform of signals with Poles on the Unit Circle

The Fourier transform of a sequence x(n) can be determined by evaluating its z–transform *X(z)* ***on the unit circle*** , provided that *the unit circle lies within the region of convergence* of *X(z)*.

There are some aperiodic sequences that are *neither absolutely summable nor square summable*.

For example

$$x_1(n) = u(n) \qquad X_1(z) = \frac{1}{1 - z^{-1}}$$

4.2.8 : The Fourier Transform of signals with Poles on the Unit Circle

$x_2(n) = (\cos\omega_0 n)u(n)$

$$X_2(z) = \frac{1 - z^{-1}\cos\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}$$

Note that both of these sequences have *poles on the unit circle*.

By allowing impulses in the spectrum of a signal, it is *possible to extent the Fourier transform representation* to **some signal sequence** that are *neither absolutely summable nor square summable*.

4.2.9 : The Sampling Theorem Revisited

$x(n) = x_a(nT) \qquad -\infty < n < \infty \qquad (4.2.71)$

The Fourier transform relation

$$X_a(F) = \int_{-\infty}^{\infty} X_a(t) e^{-j2\pi Ft} dt \qquad (4.2.72)$$

The inverse Fourier transform

$$x_a(t) = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi Ft} dt \qquad (4.2.73)$$

The spectrum of *x(n)*

$$X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi fn} \qquad (4.2.75)$$

dce

4.2.9 : The Sampling Theorem Revisited

The reconstruction formula

$$x_a(t) = \sum_{n-\infty}^{\infty} x_a(nT) \frac{\sin(\pi/T)(1-nT)}{(\pi/T)(t-nT)} \qquad (4.2.90)$$

$$x(n) = x_a(nT); \qquad T = \frac{1}{F_S}; \qquad F_S = \frac{1}{2B}$$

$$g(t) = \frac{\sin(\pi/T)t}{(\pi/T)t} = \frac{\sin 2\pi Bt}{2\pi Bt} \qquad (4.2.91)$$

The formula in (4.2.90) for reconstructing the analog signal $x_a(t)$ from its samples is called the ***ideal interpolation formula***.

4.2.9 : The Sampling Theorem Revisited

A bandlimited continuous-time signal, with highest frequency (bandwidth) *B* hertz, can be ***uniquely recovered*** from it samples provided that *the sampling rate* $F_S \geq 2B$ *samples per second*.

$F_S / 2$ is called the ***folding frequency***

4.2.10 :Frequency- Domain Classification of Signals: The Concept of Bandwidth

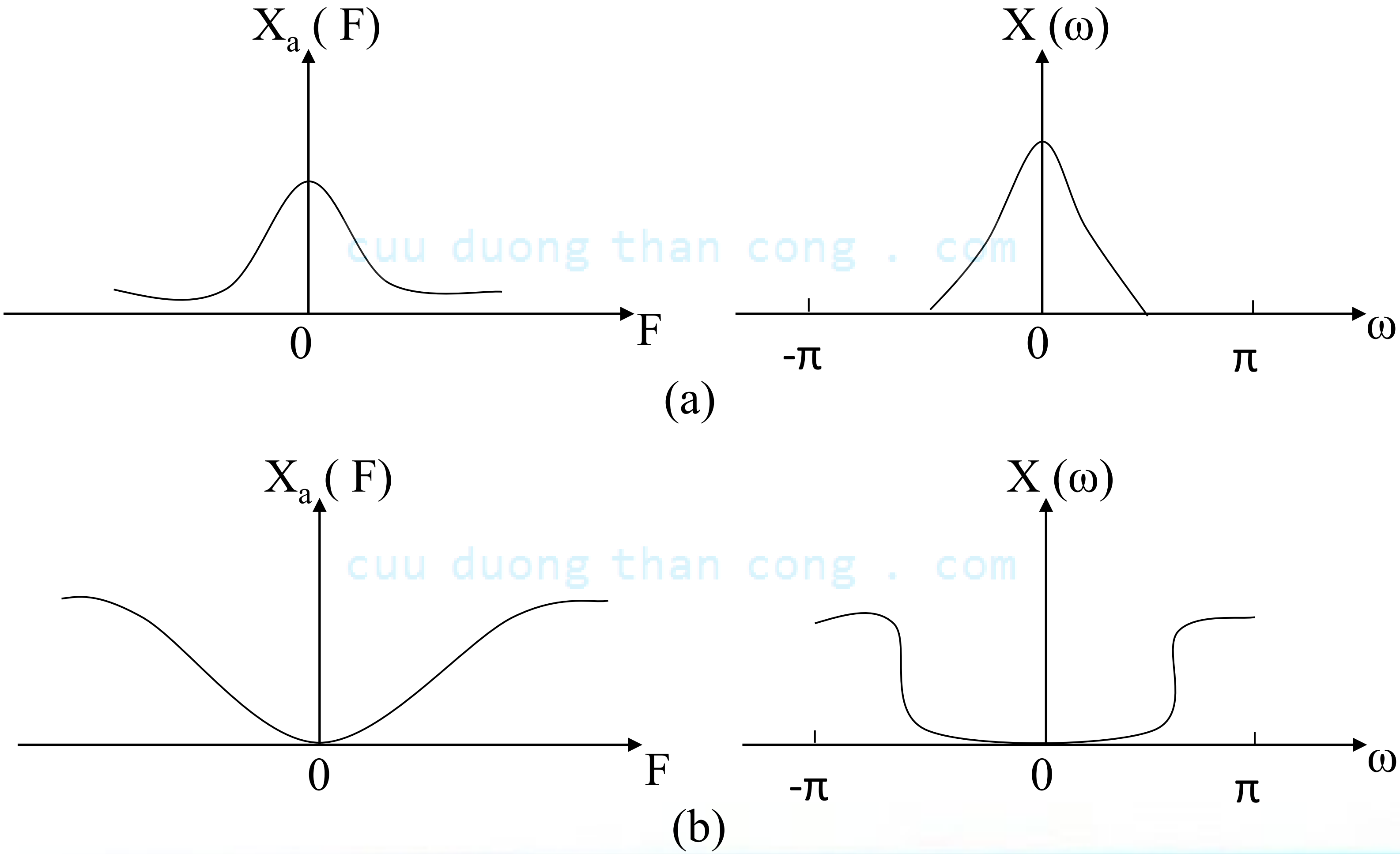
If a power signal has its *power density spectrum* concentrated about *zero frequency*, such a signal is called ***a low-frequency signal***.

If the signal power density spectrum is concentrated at *high frequencies* , the signal is called ***a high-frequency signal***.

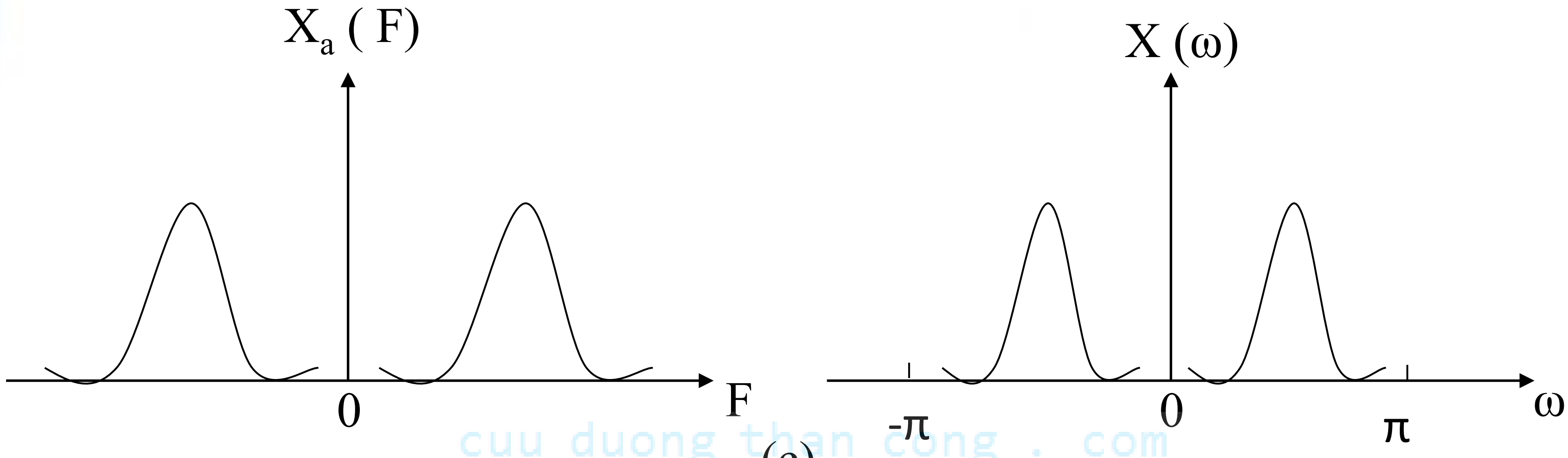
A signal having a power density spectrum concentrated *somewhere in the broad frequency range* between low frequencies and high frequencies is called ***a medium-frequency signal*** or ***a bandpass signal***.

4.2.10 :Frequency- Domain Classification of Signals: The Concept of Bandwidth

Figure 4.25 (a) Low-frequency, (b) high-frequency , and (c) medium-frequency signals



4.2.10 :Frequency- Domain Classification of Signals: The Concept of Bandwidth



(c)

The quantitative measure of the range of frequencies is called the ***bandwidth*** of a signal.

The term ***narrowband*** is used to describe the signal if its bandwidth ($F_2 - F_1$) is much smaller than *the median frequency* $(F_2+F_1)/2$. Otherwise, the signal is called ***wideband***.

A signal is ***bandlimited*** (***periodically***) if *its spectrum* is zero outside the frequency *range* $|F| \geq B$.

4.2.10 :Frequency- Domain Classification of Signals: The Concept of Bandwidth

A discrete-time finite-energy signal $x(n)$ is said to be (***periodically***) ***bandlimited*** if

$$|X(\omega)| = 0, \qquad \text{for } \omega_0 < |\omega| < \pi$$

A signal $x(t)$ will be called ***time-limited*** if $x(t) = 0 \qquad |t| > \tau$

If the signal is periodic with period T_p, it will be called ***periodically time-limited*** if

$$x_p(t) = 0, \qquad \tau < |t| < T_p / 2$$

A discrete-time signal $x(n)$ of finite duration $x(n) = 0 \quad |n| < N$

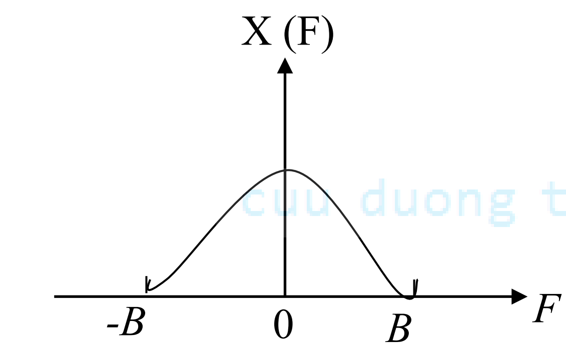
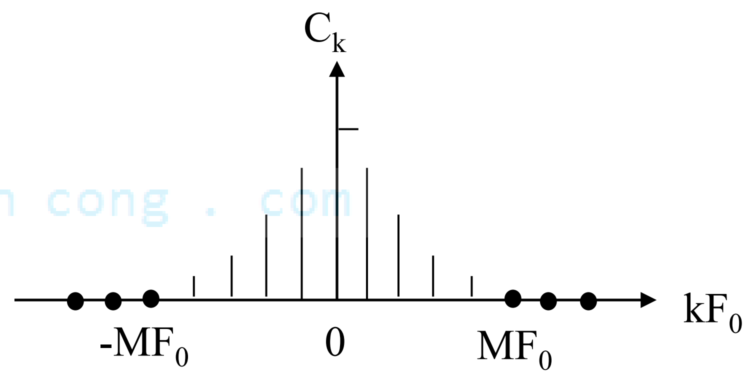
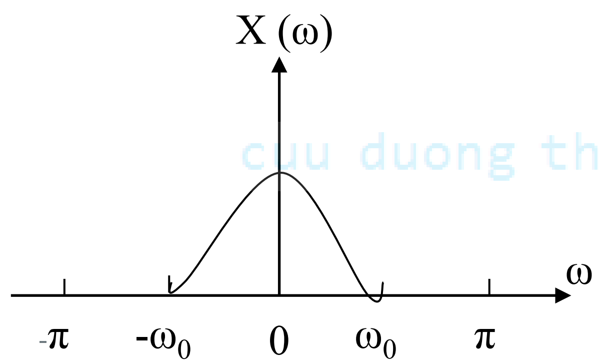
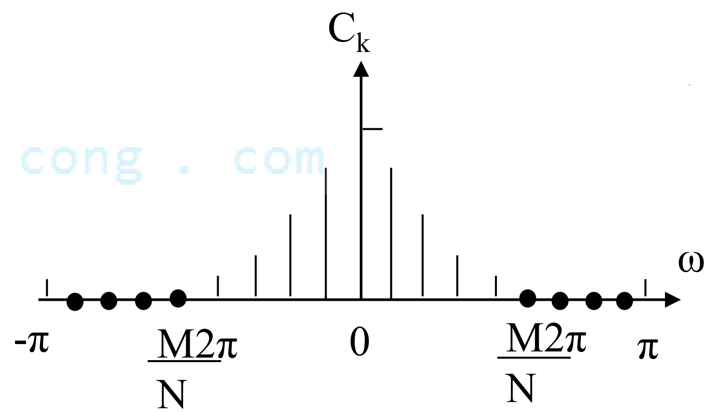
It is also called ***time-limited.***

No signal can be time-limited and bandlimited ***simultaneously.***

A short-duration signal has *a large bandwidth* and *a small bandwidth signal* has *a long duration*.

4.2.10 :Frequency- Domain Classification of Signals: The Concept of Bandwidth

Figure 4.2.11 Some examples of bandlimited signals

	A periodic signals	Periodic signals
Continuous - time		
Discrete - time		

4.2.11 The Frequency Ranges of Some Natural Signals.

Table 4.1 Frequency ranges of some biological signals

Type of Signal	Frequency Range (Hz)
Electroretinogram[a]	0 – 20
Electronystagmogram[b]	0 – 20
Pneumogram[c]	0 – 40
Electrocardiogram (ECG)	0 – 100
Electroencephalogram (EEG)	0 – 100
Electromyogram[d]	10 – 100
Sphygmomanogram[e]	0 – 200
Speech	100 – 4000

4.2.11 The Frequency Ranges of Some Natural Signals

- a A graphic recording of retina characteristics.
- b A graphic recording of in voluntary movement of the eyes.
- c A graphic recording of respiratory activity
- d A graphic recording of muscular action, such as muscular contraction
- e A recording of blood pressure.

Table 4.2 Frequency ranges of some seiamic signals

Type of Signal	Frequency Range (Hz)
Wind noise	100 – 1000
Seismic exploration signals	10 – 100
Earthquake and nuclear explosion signals	0.01 - 10
Seismic noise	0.1- 1

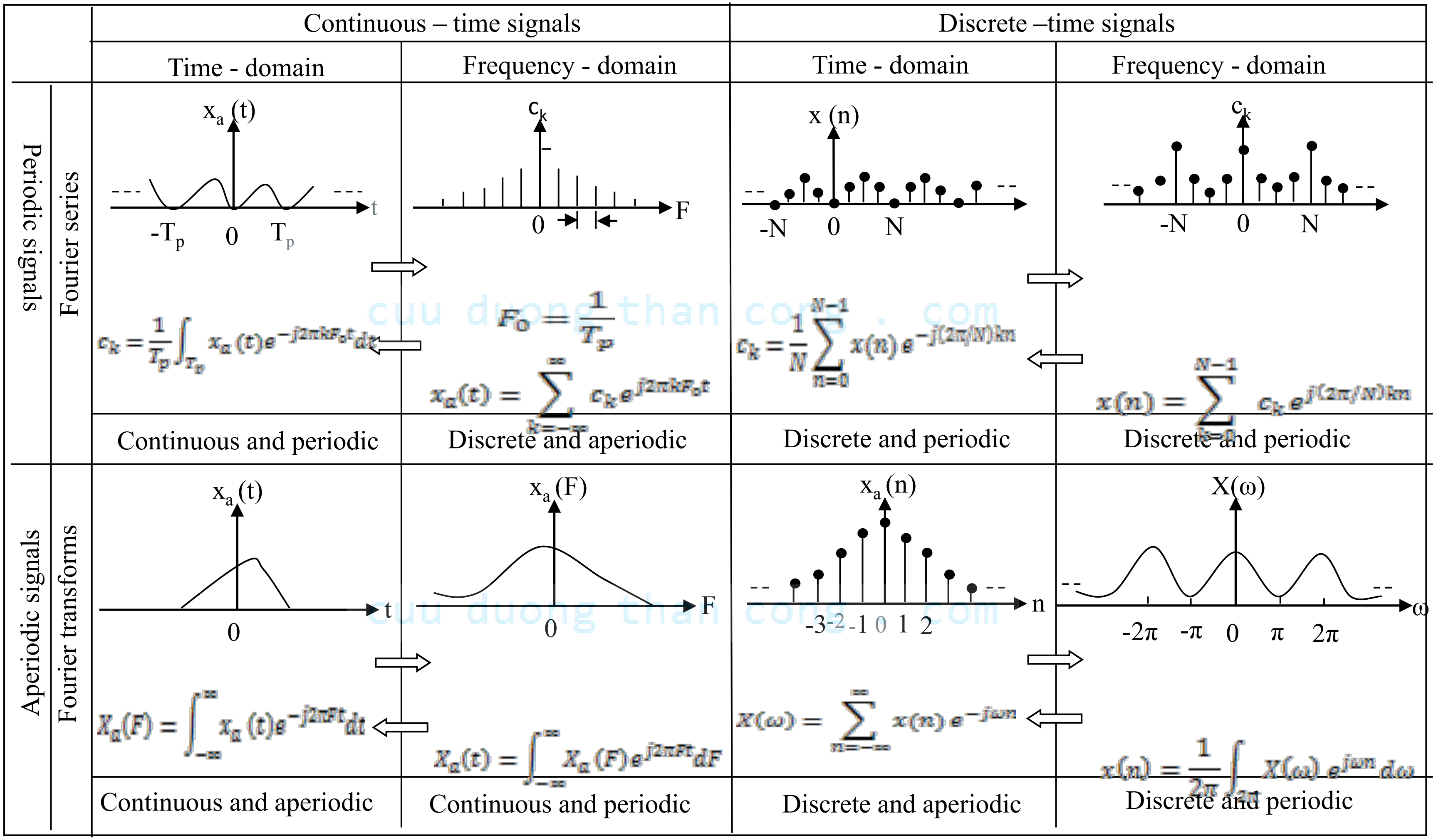
4.2.11 The Frequency Ranges of Some Natural Signals

Table 4.3 Frequency ranges of Electromagnetic signals

Type of Signal	Wavelength (m)	Frequency Range (Hz)
Radio broadcast	10^4 - 10^2	3×10^4 -3×10^6
Shortwave radio signals	$10^2 – 10^{-2}$	3×10^6 -3×10^{10}
Radar, satellite communications,		
space communications,		
common- carrier microwave	$1 – 10^{-2}$	3×10^8 -3×10^{10}
Infrared	$10^{-3} – 10^{-6}$	3×10^{11} -3×10^{14}
Visible light	$3.9 \times 10^{-7} – 8.1 \times 10^{--7}$	3.7×10^{14} -7.7×10^{14}
Ultraviolet	$10^{-7} – 10^{--8}$	3×10^{15} - 3×10^{16}
Gamma rays and X rays	$10^{-9} – 10^{--10}$	3×10^{17} - 3×10^{18}

4.2.12 : Physical and Mathematical Dualities

Figure 4.27 Summary of analysis and synthesis formulas



4.2.12 : Physical and Mathematical Dualities

Continuous-time signal have a periodic spectra

The frequency range of continuous-time signal extends from $F = 0$ to $F = \infty$.

Discrete-time signals have periodic spectra

The frequency range of discrete-time signals is finite and extends from $\omega = -\pi$ to $\omega = \pi$ radians.

Periodic signals have discrete spectra

The linear spacing is equal to the inverse of the period :

$\Delta F = 1/T_p$ for continuous-time periodic signal

$\Delta f = 1/N$ for discrete-time signals.

Aperiodic finite energy signal have continuous spectra.

$X(F)$ and $X(\omega)$ are continuous function of the variables F and ω.

4.3 Properties of the Fourier transform for Discrete – Time signals

4.3.1 *Symmetry Properties of the Fourier Transform*.

Suppose that both the signal *x(n)* and its transform *X(ω)* are complex- valued functions. Then

$$x(n) = x_R(n) + j\, x_I(n) \qquad (4.3.4)$$

$$X(\omega) = X_R(\omega) + jX_I(\omega) \qquad (4.3.5)$$

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} [x_R(n)\cos\omega n + x_I(n)\sin\omega n] \qquad (4.3.6)$$

$$X_I(\omega) = -\sum_{n=-\infty}^{\infty} [x_R(n)\sin\omega n - x_I(n)\cos\omega n] \qquad (4.3.7)$$

$$x_R(n) = \frac{1}{2\pi}\int_{2\pi} [X_R(\omega)\cos\omega n - X_I(\omega)\sin\omega n]\, d\omega \qquad (4.3.8)$$

4.3.1 Symmetry Properties of the Fourier Transform

$$x_I(n) = \frac{1}{2\pi}\int_{2\pi}[X_R(\omega)\sin\omega n + X_I(\omega)\cos\omega n]\,d\omega \qquad (4.3.9)$$

Real signals: $x_R(n) = x(n)$ *and* $x_I(n) = 0$

$X_R(-\omega) = X_R(\omega)$, (even) (4.3.12)

$X_I(-\omega) = X_I(\omega)$, (odd) (4.3.13)

We have $X^*(\omega) = - X(-\omega)$ (4.3.14)

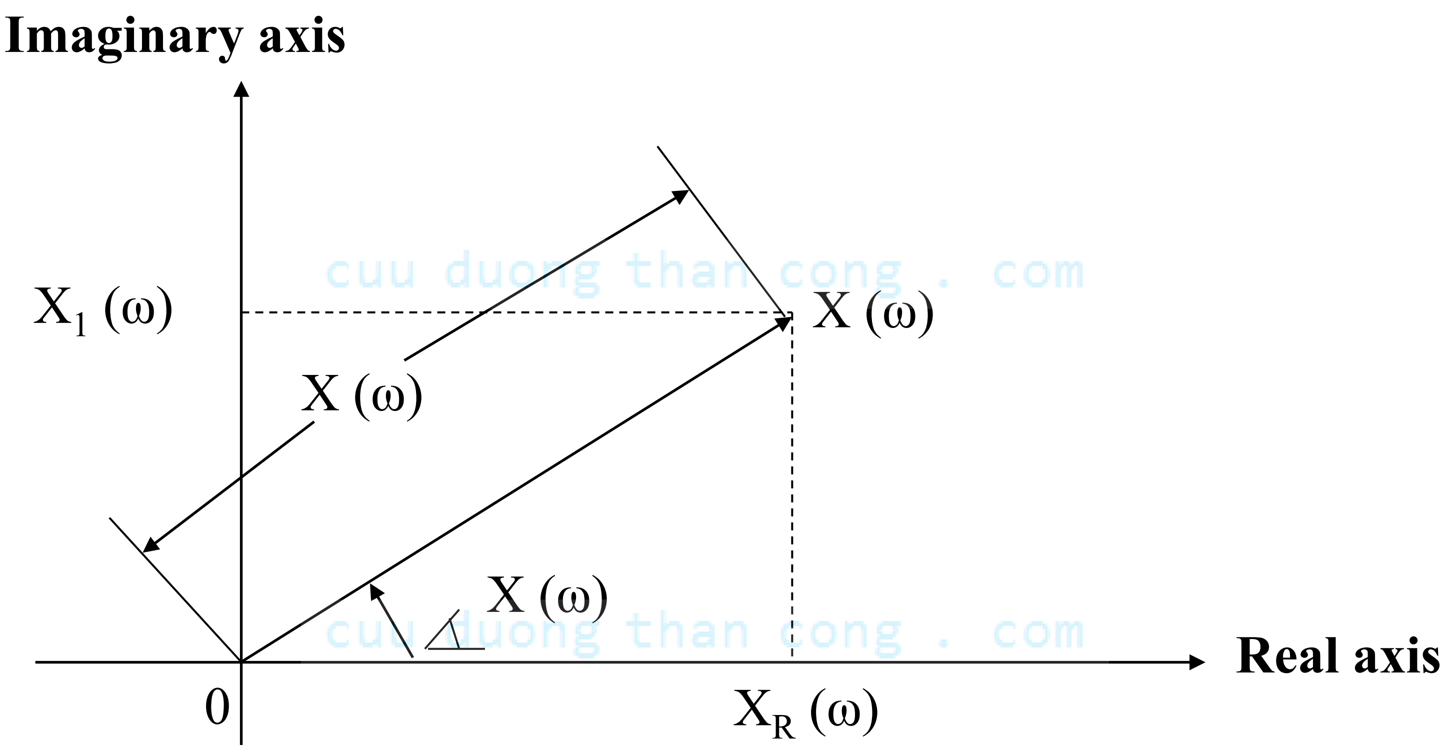
The spectrum of a real signal has ***Hermitian symmstry***.

cuu duong than cong . com

$$x(n) = \frac{1}{\pi}\int_0^{\pi}[X_R(\omega)\cos\omega n - X_I(\omega)\sin\omega n]\,d\omega \qquad (4.3.20)$$

4.3.1 Symmetry Properties of the Fourier Transform

Figure 4.28 Magnitude and phase functions



4.3.1 Symmetry Properties of the Fourier Transform

Real and even signals.

If *x(n)* is real and even, then *x(n)cosωn* is even and *x(n)sinωn* is odd .We obtain.

$$X_R(\omega) = x(0) + 2\sum_{n=1}^{\infty} x(n)\cos\omega n, \qquad (even) \qquad (4.3.21)$$

$$X_I(\omega) = 0 \qquad 4.3.22)$$

$$x(n) = \frac{1}{\pi}\int_0^{\pi} X_R(\omega)\cos\omega n \, d\omega \qquad (4.3.23)$$

4.3.1 Symmetry Properties of the Fourier Transform

***Real and odd signals*.**

If *x(n)* is real and odd, then *x(n)cosωn* is odd and *x(n)sinωn* is even .

$$X_I(\omega) = 0 \qquad (4.3.24)$$

$$X_I(\omega) = -2\sum_{n=1}^{\infty} x(n)\, sin\omega n, \qquad (odd) \qquad (4.3.25)$$

$$x(n) = -\frac{1}{\pi}\int_0^{\pi} I(\omega) sin\omega n\, d\omega \qquad (4.3.26)$$

4.3.1 Symmetry Properties of the Fourier Transform

Purely imaginary signals.

$x_R(n) = 0$ and $x(n) = j\, x_I(n)$

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} x_I(n) \sin \omega n, \qquad (odd) \qquad (4.3.27)$$

$$X_I(\omega) = \sum_{n=-\infty}^{\infty} x_I(n) \cos \omega n, \qquad (even) \qquad (4.3.28)$$

$$x_I(n) = \frac{1}{\pi} \int_0^{\pi} [X_R(\omega) \sin \omega n + X_I(\omega) \cos \omega n]\, d\omega \qquad (4.3.29)$$

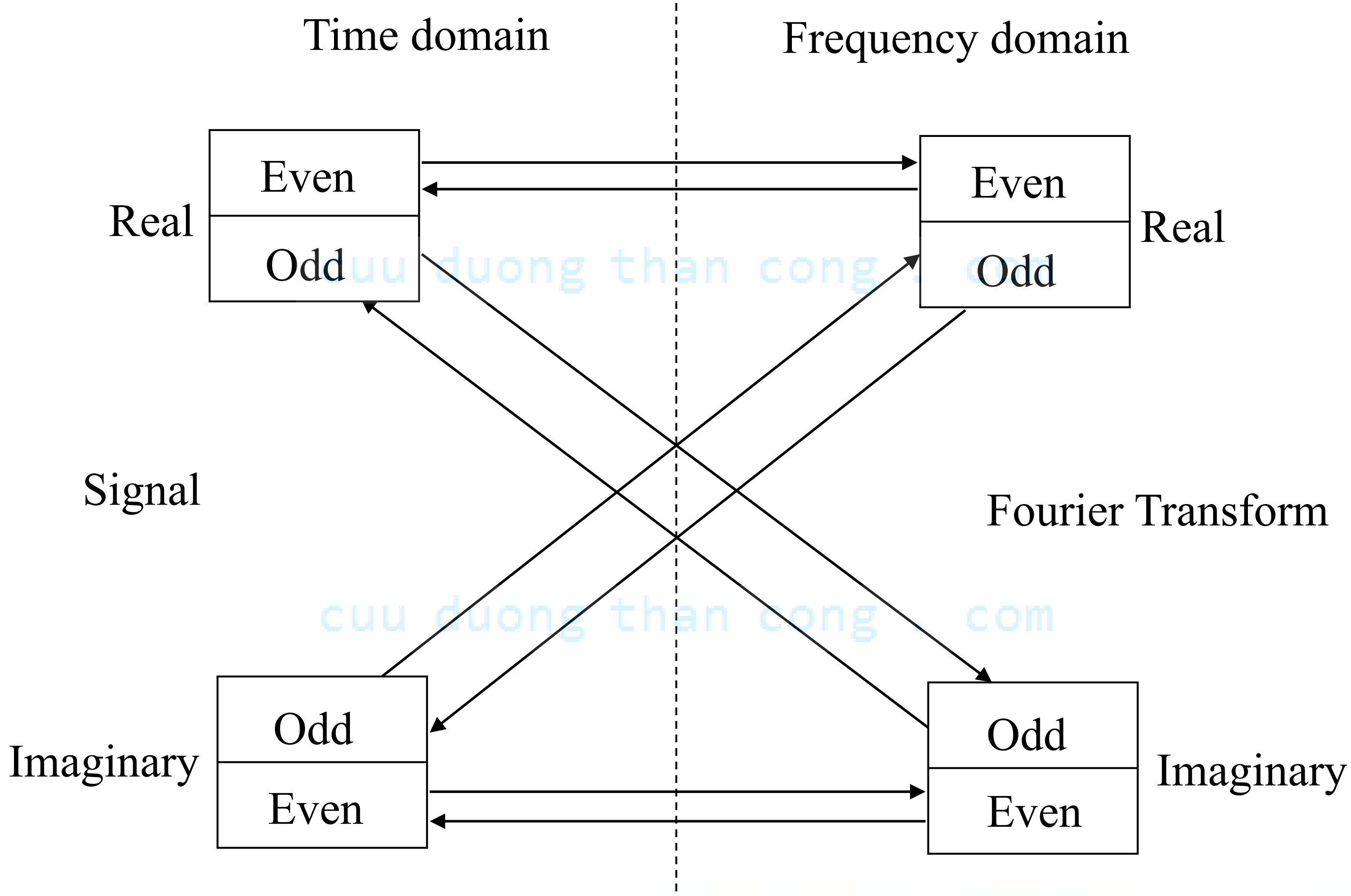
4.3.1 Symmetry Properties of the Fourier Transform

Table 4.4 Symmetry Properties of the Discrete-Time Fourier Transform

Sequence	DTFT
$x(n)$	$X(\omega)$
$x^*(n)$	$X^*(-\omega)$
$x^*(-n)$	$X^*(\omega)$
$x_R(n)$	$X_e(\omega) = \frac{1}{2}[X(\omega) + X^*(-\omega)]$
$j x_I(n)$	$X_0(\omega) = \frac{1}{2}[X(\omega) - X^*(-\omega)]$
$x_e(n) = \frac{1}{2}[x(n) + x^*(-n)]$	$X_R(\omega)$
Real Signals	
Any real signals $x(n)$	$X(\omega) = X^*(-\omega)$ $X_R(\omega) = X_R(-\omega)$ $X_I(\omega) = -X_I(-\omega)$ $\lvert X(\omega)\rvert = \lvert X(-\omega)\rvert$ $\measuredangle X(\omega) = -\measuredangle X(-\omega)$
$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$ (real and even)	$X_R(\omega)$ (real and even)
$x_0(n) = \frac{1}{2}[x(n) - x(-n)]$ (real and odd)	$j X_I(\omega)$ (imaginary and odd)

4.3.1 Symmetry Properties of the Fourier Transform

Figure 4.29 Summary of symmetry properties for the Fourier transform



4.3.2 Fourier Transform Theorems and Properties.

Linearity. If $x_1(n) \overset{F}{\leftrightarrow} X_1(\omega)$ and $x_2(n) \overset{F}{\leftrightarrow} X_2(\omega)$

then

$$a_1x_1(n) + a_2x_2(n) \overset{F}{\leftrightarrow} a_1X_1(\omega) + a_2X_2(\omega) \qquad (4.3.44)$$

Time shifting. If $x(n) \overset{F}{\leftrightarrow} X(\omega)$

then

$$x(n-k) \overset{F}{\leftrightarrow} e^{-j\omega k}X(\omega) \qquad (4.3.47)$$

Time reversal. If $x(n) \overset{F}{\leftrightarrow} X(\omega)$

then

$$x(n-k) \overset{F}{\leftrightarrow} X(\omega) \qquad (4.3.48)$$

Convolution theorem. If $x_1(n) \overset{F}{\leftrightarrow} X_1(\omega)$ and $x_2(n) \overset{F}{\leftrightarrow} X_2(\omega)$

then

$$x(n) = x_1(n) * x_2(n) \overset{F}{\leftrightarrow} X(\omega) = X_1(\omega)X_2(\omega) \qquad (4.3.49)$$

4.3.2 Fourier Transform Theorems and Properties.

The correlation theorem.

If $x_1(n) \overset{F}{\leftrightarrow} X_1(\omega)$ and $x_2(n) \overset{F}{\leftrightarrow} X_2(\omega)$

then $$r_{x_1x_2}(n) \overset{F}{\leftrightarrow} S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega) \qquad (4.3.50)$$

$S_{x1x2}(\omega)$ is called the ***cross-energy density spectrum*** of $x_1(n),x_2(n)$

The Wiener – Khintchine theorem.

Let $x(n)$ be a real signal. Then

$$r_{xx}(l) \overset{F}{\leftrightarrow} S_{xx}(\omega) \qquad (4.3.51)$$

4.3.2 Fourier Transform Theorems and Properties.

Frequency shifting.

if $x(n) \overset{F}{\leftrightarrow} X(\omega)$

then $e^{j\omega_0 n} x(n) \overset{F}{\leftrightarrow} X(\omega - \omega_0)$ (4.3.52)

Parseval's theorem

If $x_1(n) \overset{F}{\leftrightarrow} X_1(\omega)$ and $x_2(n) \overset{F}{\leftrightarrow} X_2(\omega)$

then $\sum_{n-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega$ (4.3.54)

4.3.2 Fourier Transform Theorems and Properties.

Multiplication of two sequences (Windowing theorem).

If $x_1(n) \overset{F}{\leftrightarrow} X_1(\omega)$ and $x_2(n) \overset{F}{\leftrightarrow} X_2(\omega)$

then $$x_3(n) \equiv x_1(n)x_2(n) \overset{F}{\leftrightarrow} X_3(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)\,d\lambda \qquad (4.3.57)$$

Differentiation in the frequency domain

if $$x(n) \overset{F}{\leftrightarrow} X(\omega)$$

then $$nx(n) \overset{F}{\leftrightarrow} j\frac{dX(\omega)}{d\omega} \qquad (4.3.58)$$

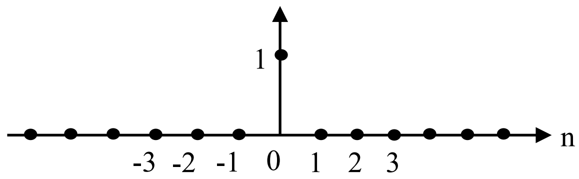
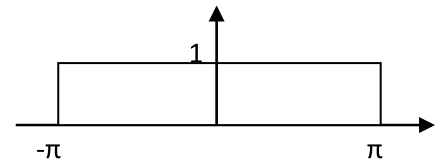
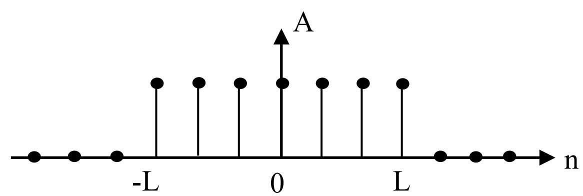
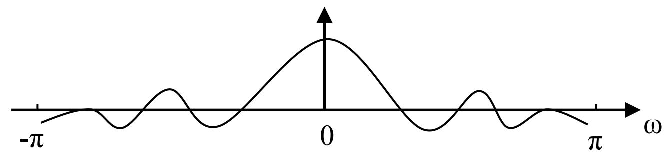
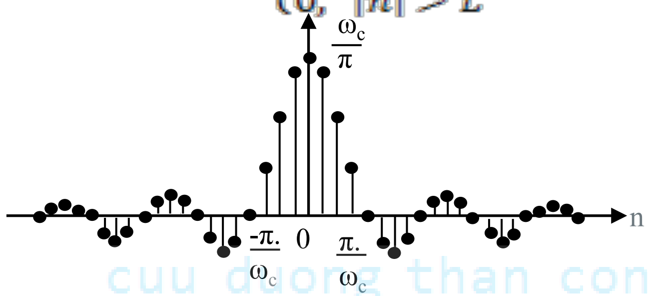
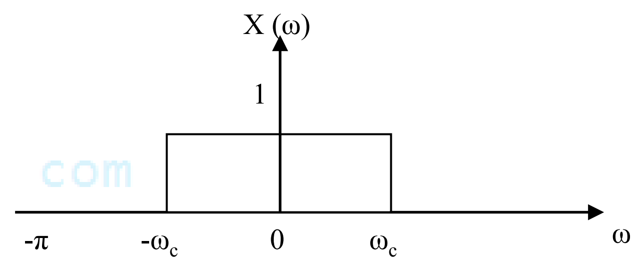
4.3.2 Fourier Transform Theorems and Properties.

Table 4.5 Properties of the Fourier Transform for Discrete –Time signals

Property	Time Domain	Frequency Domain
Notation $x(n)$	$X(\omega)$	
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_2(\omega)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation	$r_{x1x2}(l) = x_1(l) * x_2(-l)$	$S_{x1x2}(\omega) = X_1(\omega)\,X_2(-\omega)$ $X_1(\omega) = X_2^*(\omega)$ if [$x_2(n)$ is real]
Wiener- Khintchine theorem	$r_{xx}(l)$	$S_{xx}(\omega)$
Frequency shifting	$e^{j\omega 0n}x(n)$	$X(\omega - \omega_0)$
Modulation	$x(n)\cos\omega_0 n$	$½X(\omega + \omega_0) + ½X(\omega - \omega_0)$
Multiplication	$x_1(n)\,x_2(n)$	
Differentiation in the frequency domain	$nx(n)$	
Conjugation	$x^*(n)$	$X^*(-\omega)$
Parseval's theorem		

4.3.2 Fourier Transform Theorems and Properties.

Table 4.6 Some Useful Fourier transform pairs for discrete-time a periodic signals

Signal x (n)	Spectrum X (ω)
 x (n) = δ (n)	 X (π) = 1
 $x(n) = \begin{cases} A, & \lvert n\rvert \le L \\ 0, & \lvert n\rvert > L \end{cases}$	 $X(\omega) = A\dfrac{\sin\left(L+\frac{1}{2}\right)\omega}{\sin\frac{\omega}{2}}$
 $x(n)\begin{cases} \dfrac{\omega_c}{\pi}, & n = 0 \\ \dfrac{\sin\omega_c n}{\pi n}, & n \ne 0 \end{cases}$	 $x(n)\begin{cases} 1, & \lvert\omega\rvert < \omega_c \\ 0, & \omega_c \le \lvert\omega\rvert \le \pi \end{cases}$
$x(n) = \begin{cases} a^n, & n \ge 0 \\ 0, & n > 0 \end{cases}$	$X(\omega) = \dfrac{1}{1 - ae^{-j\omega}}$

4.4 Frequency –Domain Characteristics of Linear Time-invariant system.

The frequency response H(ω) is the Fourier transform of *the impulse response h(n)* of the system.

4.4.1 ***Response to complex Exponential and sinusoidal signal:***

With any relaxed linear time-invariant system.

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \qquad (4.4.1)$$

and $x(n) = Ae^{j\omega n}, -\infty < n < \infty$ (4.4.2)

then $$y(n) = A\left[\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}\right]e^{j\omega n} \qquad (4.4.3)$$

The frequency response function

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \qquad (4.4.4)$$

4.4.1 Response to complex Exponential and sinusoidal signal

H(ω) exits if the system is BIBO stable, that is if

$$\sum_{k=-\infty}^{\infty} |h(n)| < \infty$$

thus, $y(n) = A\, H(\omega)\, e^{j\omega n}$ (4.4.5)

(4.4.2) is called as *eigenfunction* of the system.

The multiplicative factor is called *eigenvalue* of the system.

In general:

$$h(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega) e^{j\omega k}\, d\omega \qquad (4.4.12)$$

$$H(\omega) = H_R(\omega) + H_I(\omega) \qquad (4.4.13)$$

4.4.1 Response to complex Exponential and sinusoidal signal

$$H_R(\omega) = \sum_{k=-\infty}^{\infty} h(k)\cos\omega k$$

$$H_I(\omega) = \sum_{k=-\infty}^{\infty} h(k)\sin\omega k \qquad (4.4.14)$$

$$|H(\omega)| = \sqrt{H_R^2(\omega) + H_I^2(\omega)}$$

$$\theta(\omega) = \tan^{-1}\frac{H_I(\omega)}{H_R(\omega)} \qquad (4.4.15)$$

4.4.1 Response to complex Exponential and sinusoidal signal

If $x(n) = A \cos \omega n$

then $y(n) = A\,|H(\omega)| \cos [\, \omega n + \theta(\omega)\,]$ (4.4.17)

If $x(n) = A \sin \omega n$

then $y(n) = A\,|H(\omega)| \sin [\, \omega n + \theta(\omega)\,]$ (4.4.18)

By knowing $H(\omega)$, we are able to determine ***the response of the system*** to any sinusoidal input signal.

$H(\omega)$ is called ***the frequency response*** of the system

$|H(\omega)|$ is called the ***magnitude response*** and $\theta(\omega)$ is called the ***phase response*** of the system

4.4.1 Response to complex Exponential and sinusoidal signal

In the must general case

If

$$x(n) = \sum_{i=1}^{L} A_i cos\,(\omega_i n + \emptyset i), \qquad -\infty < n < \infty$$

then

$$y(n) = \sum_{i=1}^{L} A_i |H(\omega_i)| cos[\omega_i n + \emptyset i + \Theta(\omega_i)] \qquad (4.4.19)$$

4.4.2 Steady – State and Transient Response to sinusoidal Input signal

We determined the response of a linear time-invariant system to *exponential and sinusoidal input signals* applied to the system at $n = -\infty$.

The response at the output of the system is ***the steady-state response***. There is ***no transient response*** in this case.

When excited by a complex exponential, or by a sinusoid at $n = 0$ or at some finite time instant , *the response* consists of two terms, ***the transient response*** and ***the steady-state response***.

In many practical applications, ***the transient response*** is ***unimportant and usually ignored***.

4.4.3 Steady – State Response to Periodic input signals

$$x(n) = \sum_{k=0}^{N-1} C_k e^{j2\pi kn/N}, \qquad k = 0,1 \ldots, N-1 \qquad (4.4.26)$$

$$Y_K(n) = C_K H\left(\frac{2\pi}{N}\right) e^{j2\pi kn/N}, \qquad k = 0,1 \ldots, N-1 \qquad (4.4.27)$$

where $H\left(\frac{2\pi}{N}\right) = H(\omega)\big|_{\omega=2\pi k/N}$

$$y(n) = \sum_{k=0}^{N-1} C_k H\left(\frac{2\pi k}{N}\right) e^{j2kn/N} \qquad -\infty < n < \infty \qquad (4.4.28)$$

The response of the system to the periodic input signal *x(n)* is ***also periodic with the same period N***

4.4.4 Response to A periodic input signals

From the convolution theorem, we have

$$Y(\omega) = H(\omega)\, X(\omega) \qquad (4.4.30)$$

where $Y(\omega)$, $X(\omega)$ and $H(\omega)$ are the corresponding Fourier transform of {y(n)}, {x(n)} and {h(n)}.

$$|Y(\omega)| = |H(\omega)|\, |X(\omega)| \qquad (4.4.31)$$

$$\angle Y(\omega) = \angle X(\omega) + \angle H(\omega) \qquad (4.4.32)$$

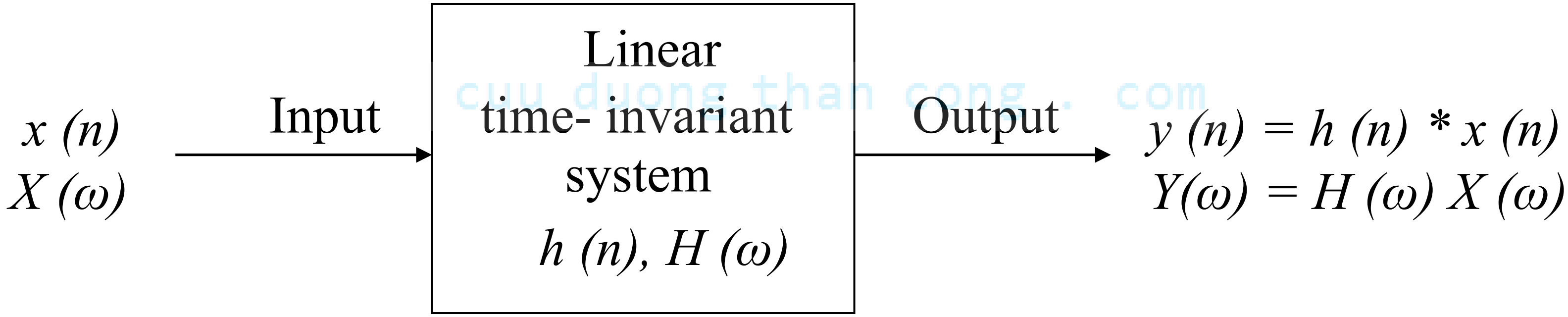
Where $|H(\omega)|$ and $\angle H(\omega)$ ***are the magnitude and phase response*** of the system.

If *the input signal spectrum is changed* by the system in as undesirable way then ***the system has caused magnitude and phase distortion***.

4.4.4 Response to Aperiodic Input Signals

The output of a linear time-invariant system cannot contain frequency components that are not contained in the input signal.

Figure 4.39 Time- and frequency-domain input - output relationships in LTI systems



$$|Y(\omega)|^2 = |H(\omega)|^2 |X(\omega)|^2 \qquad (4.4.34)$$

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$$

where $S_{xx}(\omega)$, $S_{yy}(\omega)$ are ***the energy density spectra*** of the input and output signals.

4.4.5 Relationships Between the System Function and the Frequency Response Function

If the system function *H(z)* ***converges on the unit circle***, thus

$$H(\omega) = H(z)|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n} \qquad (4.4.36)$$

If *H(z) = B(z) / A(z)*,

then

$$H(\omega) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}} = b_0 \frac{\prod_{k=1}^{M}\left(1 - z_k e^{-j\omega k}\right)}{\prod_{k=1}^{N}\left(1 - p_k e^{-j\omega}\right)} \qquad (4.4.38)$$

4.4.5 Relationships Between the System Function and the Frequency Response Function

$$c_l = \sum_{k=0}^{N-|l|} a_k a_{k+l}, \qquad -N \le l \le N \qquad (4.4.42)$$

$$d_l = \sum_{k=0}^{M-|l|} b_k b_{k+l}, \qquad -M \le l \le M \qquad (4.4.43)$$

$$|H(\omega)|^2 = \frac{d_0 + 2\sum_{k=1}^{M} d_k \cos k\omega}{c_0 + 2\sum_{k=1}^{N} c_k \cos k\omega} \qquad (4.4.44)$$

We note *H(ω)* in factored form as

$$H(\omega) = b_0 \frac{\prod_{k=1}^{M}\left(1 - z_k e^{-j\omega k}\right)}{\prod_{k=1}^{N}\left(1 - p_k e^{-j\omega k}\right)} \qquad (4.4.46)$$

or

$$H(\omega) = b_0 e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}\left(e^{j\omega} - z_k\right)}{\prod_{k=1}^{N}\left(e^{j\omega} - p_k\right)} \qquad (4.4.47)$$

In pole form as

$$e^{j\omega} - z_k = V_k(\omega)\, e^{j\theta k(\omega)} \qquad (4.4.48)$$

and

$$e^{j\omega} - p_k = U_k(\omega)\, e^{j\varnothing k(\omega)} \qquad (4.4.49)$$

where

$$V_k(\omega) = |\, e^{j\omega} - z_k \,|\ ,\ \theta_k(\omega) = \angle\, (e^{j\omega} - z_k) \qquad (4.4.50)$$

and

$$U_k(\omega) = |\, e^{j\omega} - p_k \,|\ ,\ \varnothing_k(\omega) = \angle\, (e^{j\omega} - p_k) \qquad (4.4.51)$$

We obtain

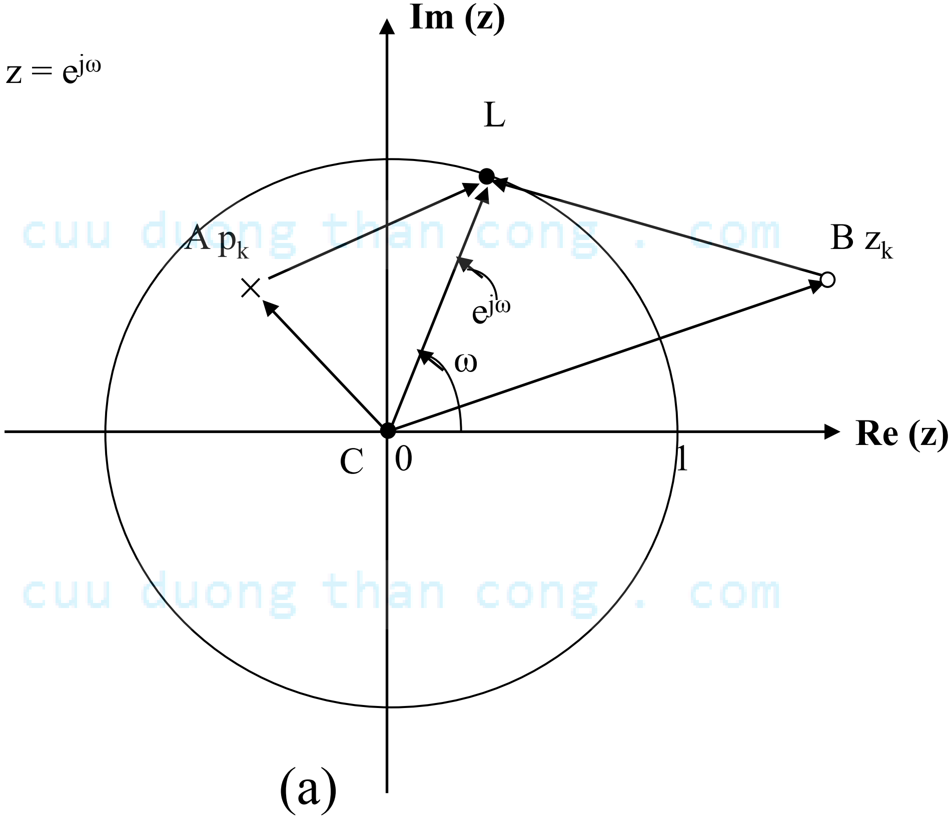
$$|H(\omega)| = |b_0| \frac{V_1(\omega) \ldots V_M(\omega)}{U_1(\omega) \ldots U_N(\omega)} \qquad (4.4.52)$$

$$\angle\, H(\omega) = \angle b_0 + \omega(N\text{-}M) + \theta_1(\omega) + \ldots\ \theta_M(\omega) \qquad (4.4.53)$$

$$-\ [\varnothing(\omega) + \varnothing_2(\omega) + \ldots\ \varnothing_N(\omega)]$$

Let us consider a **pole** $\boldsymbol{p_k}$ and **zero** $\boldsymbol{z_k}$ located at *Point A* and *B* of the *z*–place . Fig 4.40a.

Figure 4.40 Geometric interpretation of the contribution of a pole and a zero to the Fourier transform (1) magnitude : the factor V_k /U_k, (2) phase the factor $\Theta_k - \Phi_k$



4.4.6 Computation of the Frequency Response Function

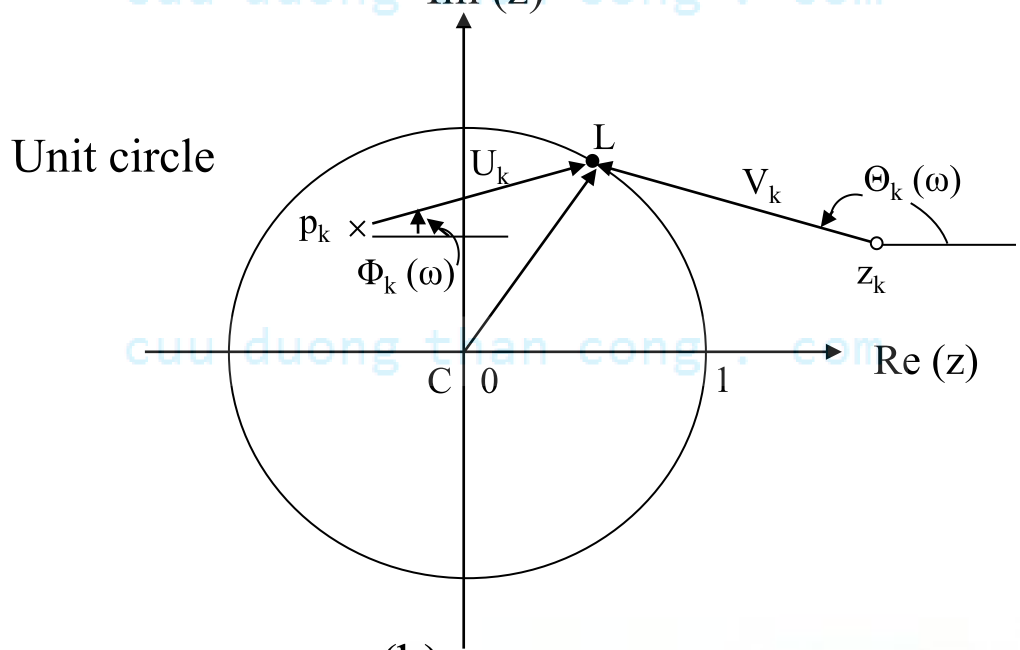
The giver value of ω determines the angle of $e^{j\omega}$ with the positive real axis ***point L, on the unit circle***.

vectors: $CL = CA + AL$, and $CL = CB + BL$.

$$AL = e^{j\omega} - p_k = U_k\,(\omega)\,e^{\varnothing k\,(\omega)}$$

$$BL = e^{j\omega} - z_k = V_k\,(\omega)\,e^{j\theta k\,(\omega)}$$

in Fig 4.40b

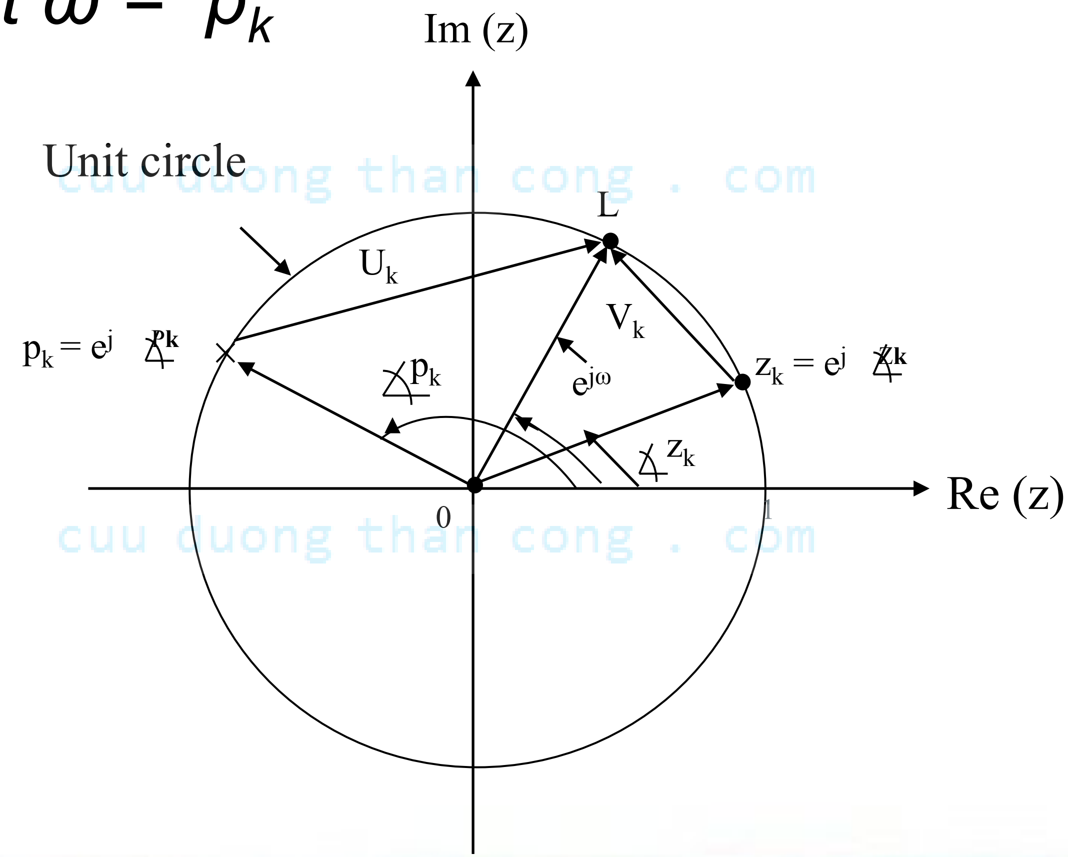


(b)

4.4.6 Computation of the Frequency Response Function

Suppose that z_k, p_k are on the unit circle Fig 4.41

Figure 4.41 A zero on the unit circle causes $|H(\omega)| = 0$ and $\omega = z_k$. In contrast, a pole on the unit circle results in $|H(\omega)| = \infty$ at $\omega = p_k$



4.4.6 Computation of the Frequency Response Function

The presence of a zero close to the unit circle causes the magnitude of the frequency response, at frequencies that correspond to *point of the unit circle close to that point*, to be ***small***.

The presence of a pole close to the unit circle causes *the magnitude of the frequency response* to be ***large*** at frequencies *close to that point*.

Self-study

4.4.7 Input – Output Correlation Functions and Spectra.

4.4.8 Correlation Functions and Power Spectra for Random Input

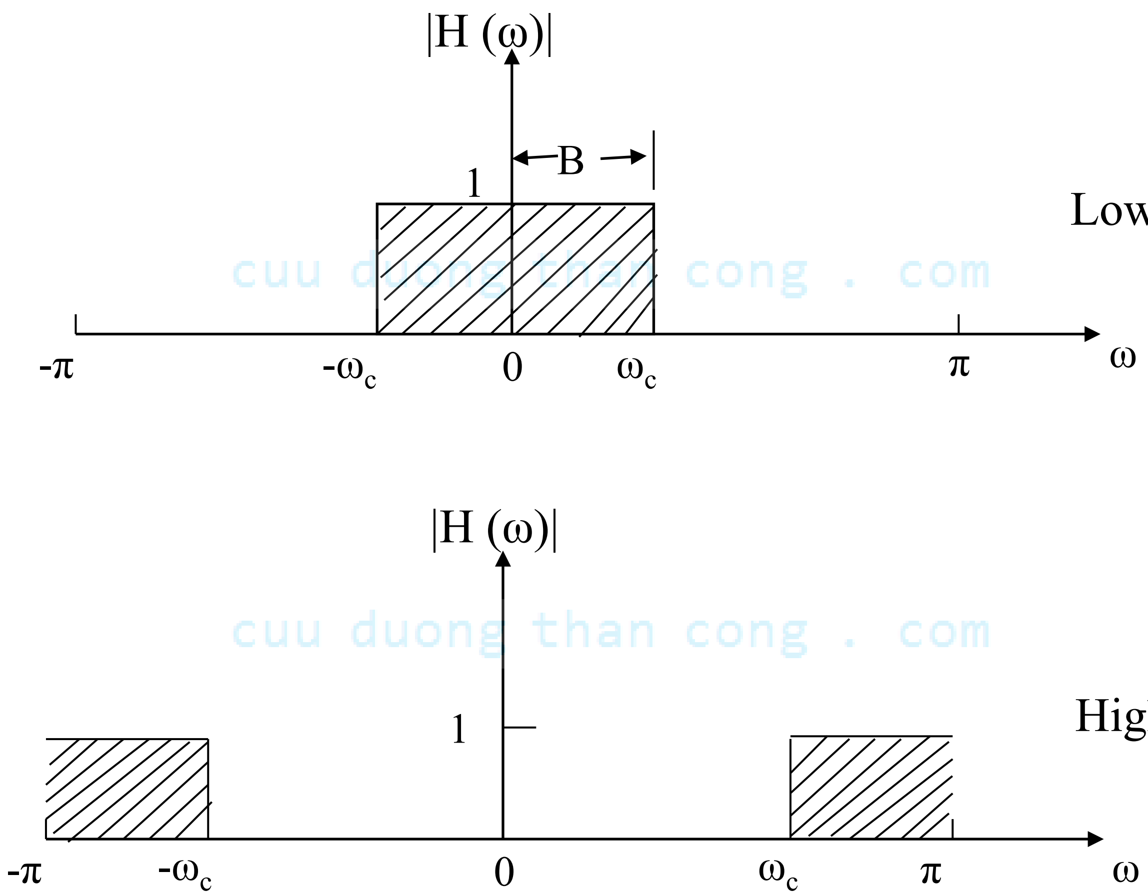
4.5 Linear Time – Invariant Systems as Frequency – Selective Filters

H(ω) acts as a ***weighting function*** or a ***spectral shaping function*** to the different frequency components in the input signal.

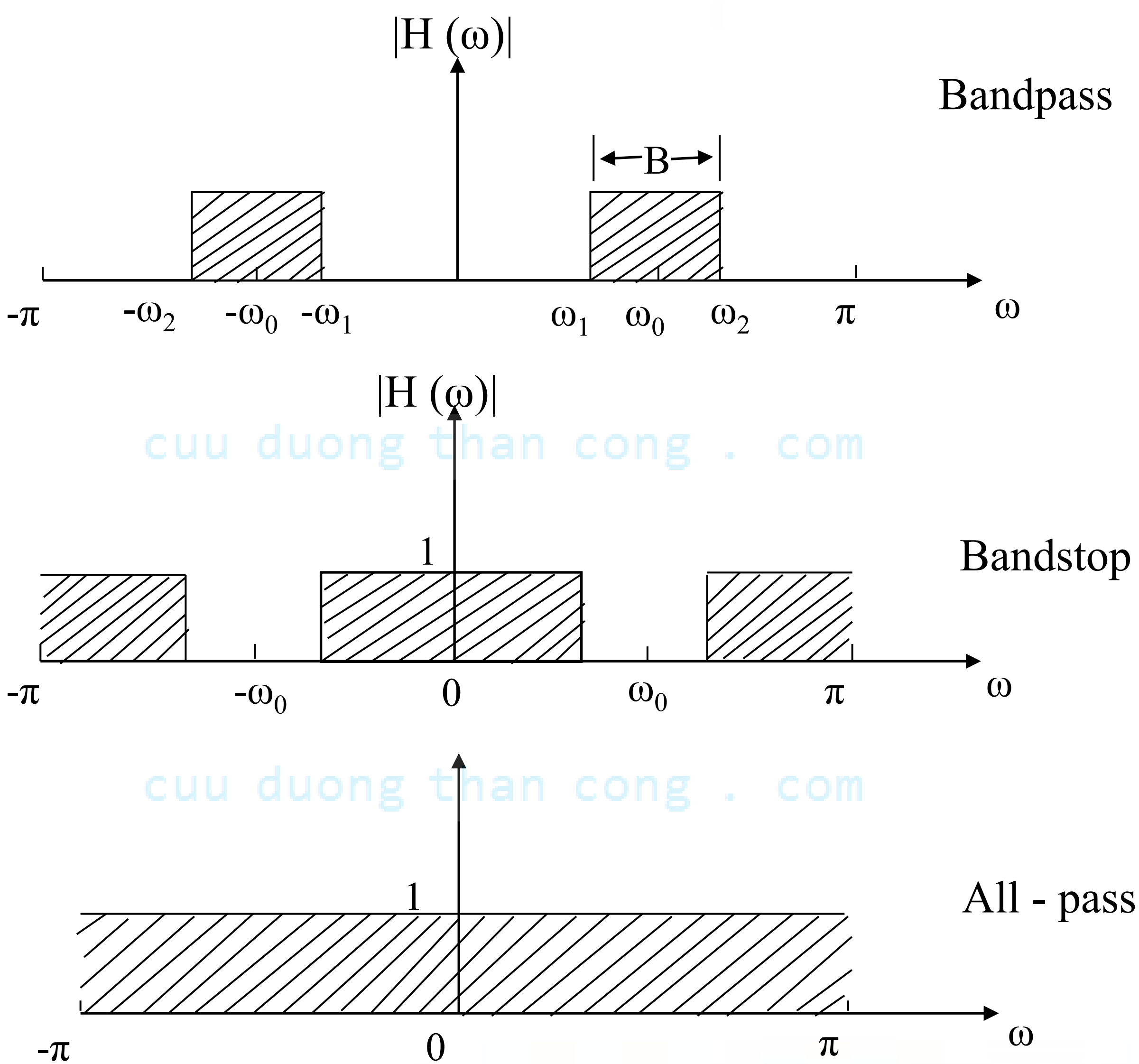
We use the term ***filter*** to describe a linear time – invariant system used to perform ***spectral shaping*** or ***frequency – selective filtering***.

4.5.1 Ideal Filter Characteristics

Figure 4.43 Magnitude responses for some ideal frequency-selective discrete- time filters



4.5.1 Ideal Filter Characteristics



4.5.1 Ideal Filter Characteristics

The ***ideal filters*** have a constant- gain (unity- gain, zero gain).

The *ideal filter* have a linear phase response. To demonstrate this point, we assume

$$H(\omega) = \begin{cases} Ce^{-j\omega n_0}, & \omega_1 < \omega < \omega_2 \\ 0, & \text{otherwise} \end{cases} \quad (4.5.1)$$

$$Y(\omega) = X(\omega)\, H(\omega)$$

$$= C\, X(\omega)\, e^{-j\omega n_0}, \qquad \omega_1 < \omega < \omega_2 \qquad (4.5.2)$$

We obtain

$$y(n) = C\, x(n-n_0) \qquad (4.5.3)$$

$$\Theta(\omega) = -\omega n_0 \qquad (4.5.4)$$

$$\tau_g(\omega) = -\frac{d\Theta(\omega)}{d\omega} = n_0 = constant$$

$\tau_g(\omega)$ called the ***envelope delay*** or the ***group delay*** of the ***filter***

4.5.1 Ideal Filter Characteristics

The *basic principle* underlying *the pole-zero placement method* is *to locate poles* ***near points*** *of the unit circler corresponding to frequencies to be emphasized*, and *to place zeros* ***near the frequencies*** *to be deemphasized*.

The following ***constraints*** must be imposed.

1. ***All poles*** should be placed ***inside*** *the unit circle* in order for the filter to be *stable*. zero can be placed *anywhere* in the z- plane.

2. ***All complex zeros*** and ***poles*** must occur *in* ***complex-conjugate pairs*** in order for the filter coefficients to be real.

4.5.1 Ideal Filter Characteristics

The system function

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M}(1 - z_k z^{-1})}{\prod_{k=1}^{N}(1 - p_k z^{-1})} \qquad (4.5.7)$$

b_0 is selected such that

$$|H(\omega_0)| = 1 \qquad (4.5.8)$$

where ω_0 – a frequency is the passband of the filter.

4.5.2 Lowpass, Highpass, and Bandpass Filters

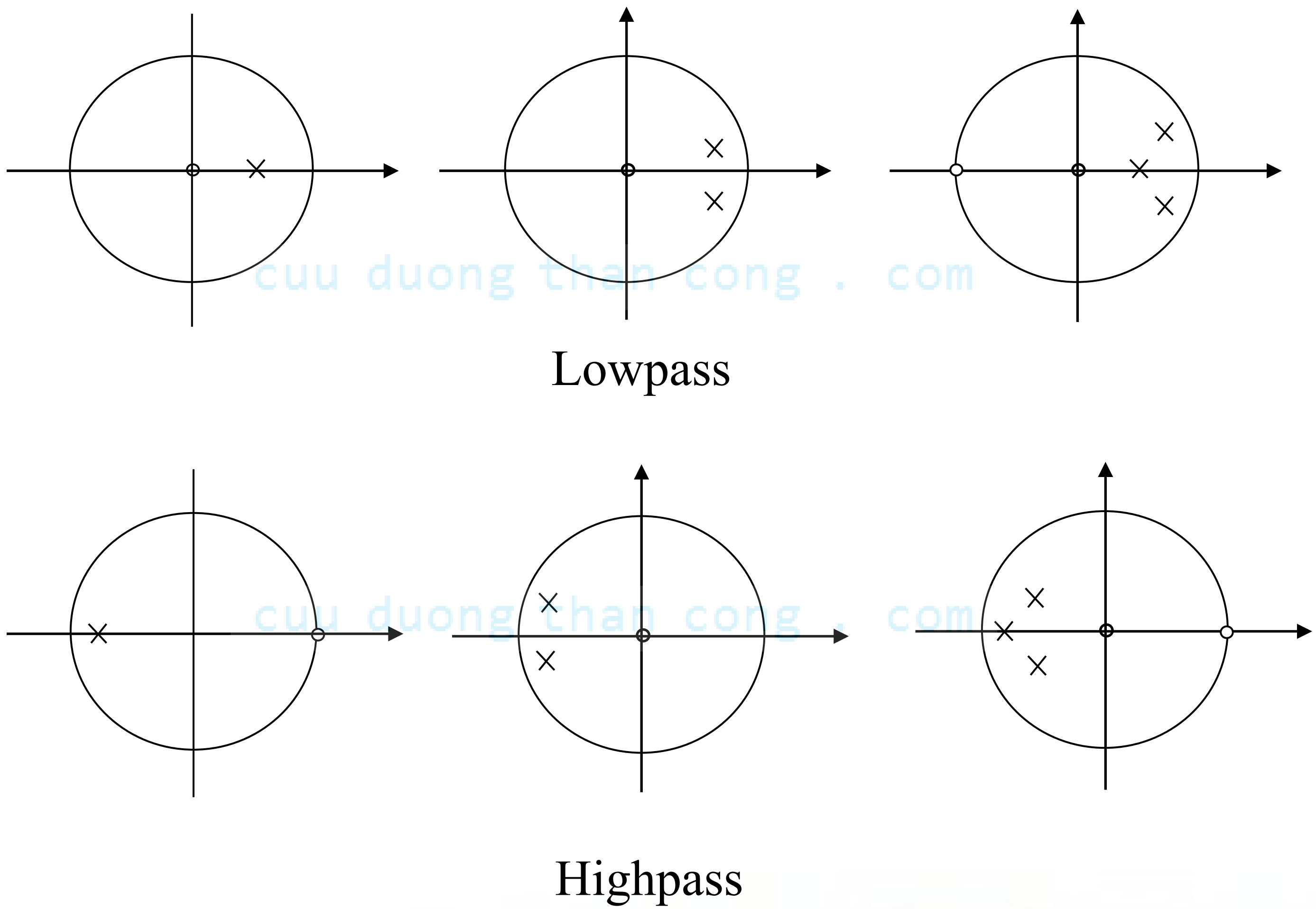
In the design of ***lowpass digital filters***, ***the poles*** should be placed *near the unit circle* at point corresponding to *low frequencies* (***near*** $\boldsymbol{\omega = 0}$), and ***zero*** should be placed *near or on the unit circle* at point corresponding to *high frequencies* (***near*** $\boldsymbol{\omega = \pi}$).

The ***opposite*** hold true for ***highpass filter***.

In Fig 4.44

4.5.2 Lowpass, Highpass, and Bandpass Filters

Figure 4.44 Pole – zero patterns for several low pass and high pass filters.



4.5.2 Lowpass, Highpass, and Bandpass Filters

The ***magnitude*** and ***phase responses*** for the single-pole filter with system function

$$H_1(z) = \frac{1-a}{1-az^{-1}} \qquad (4.5.9)$$

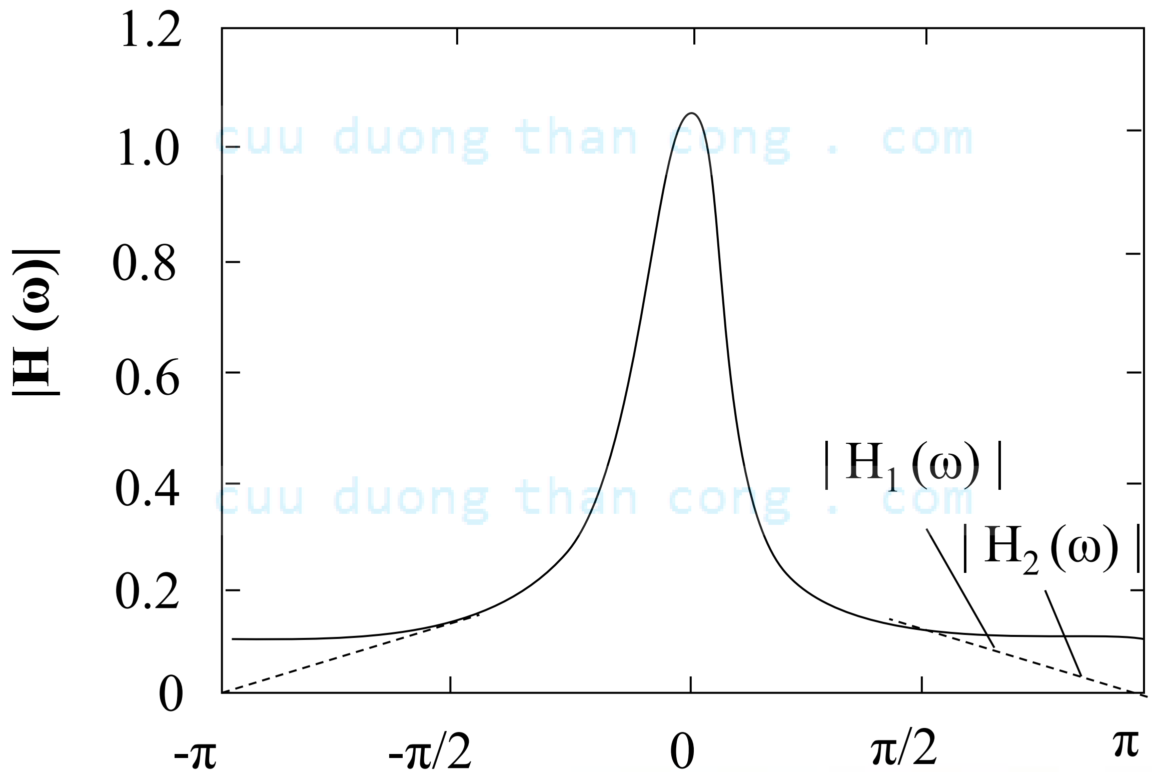
The addition of ***a zero*** at $z = -1$ further *attenuates* the response of the filter at ***high frequencies***.

$$H_2(z) = \frac{1-a}{2}\frac{1+z^{-1}}{1-az^{-1}} \qquad (4.5.10)$$

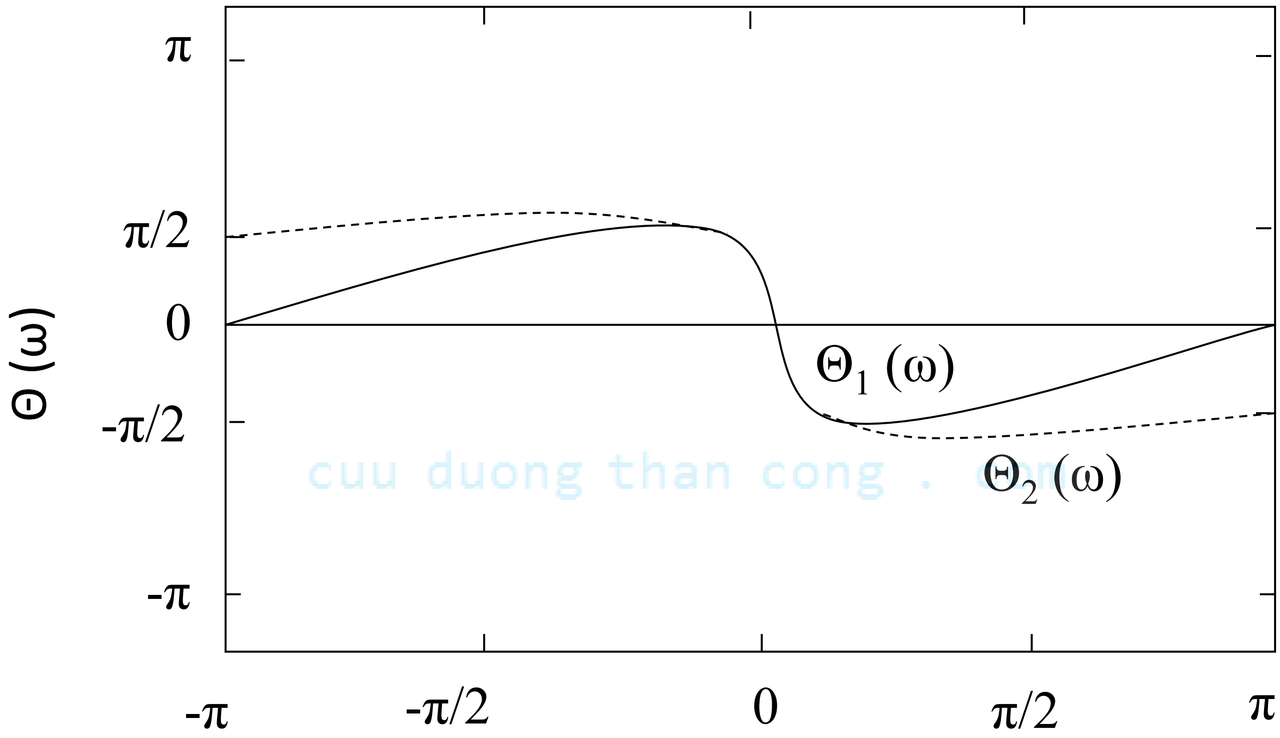
a = 0.9 in Fig 4.45 a.b

4.5.2 Lowpass, Highpass, and Bandpass Filters

Figure 4.45 Magnitude and phase responses for (1) a single – pole filter and (2) a one- pole, one-zero filter; $H_1(z) = (1- a) / (1- az^{-1})$, $H_2(z) = [(1- a) / 2] [(1 + z^{-1}) / (1 – az^{-1})]$ and $a = 0.9$



4.5.2 Lowpass, Highpass, and Bandpass Filters

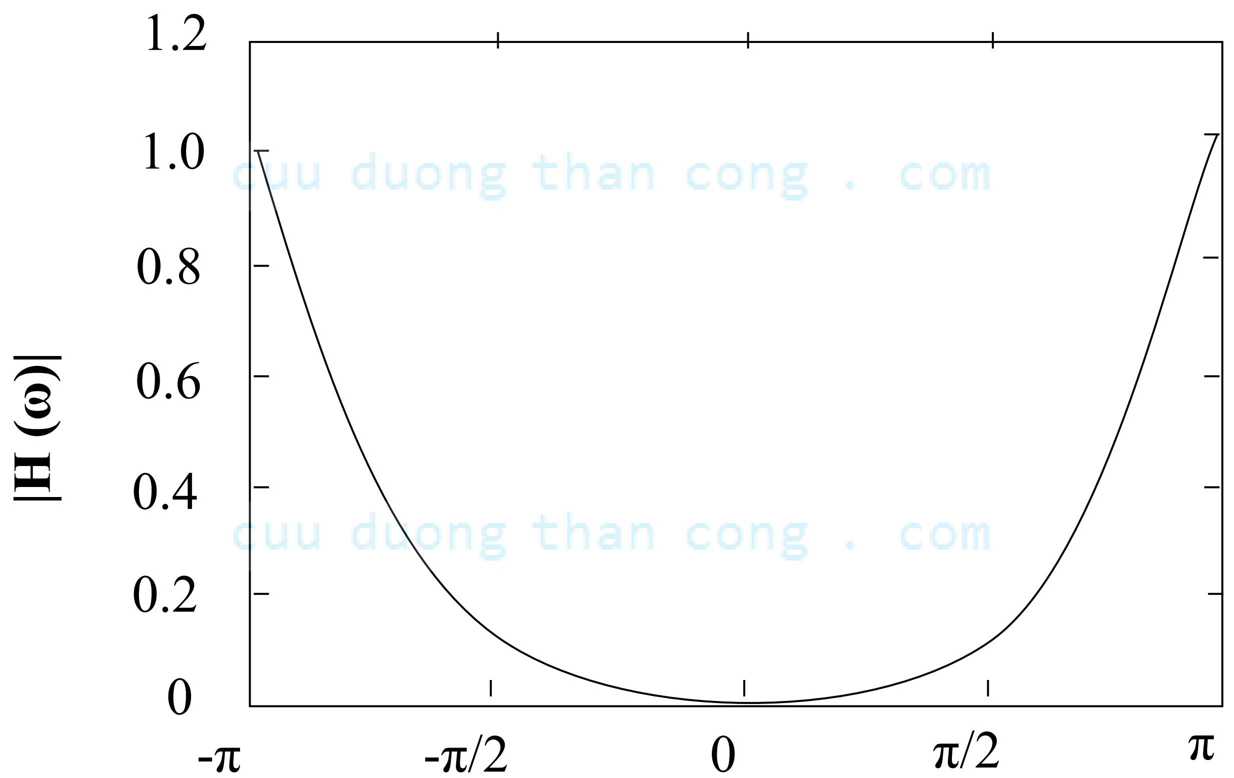


We can obtain simple *highpass filters* by *reflecting* the ***pole-zero locations*** of the *lowpass filter*

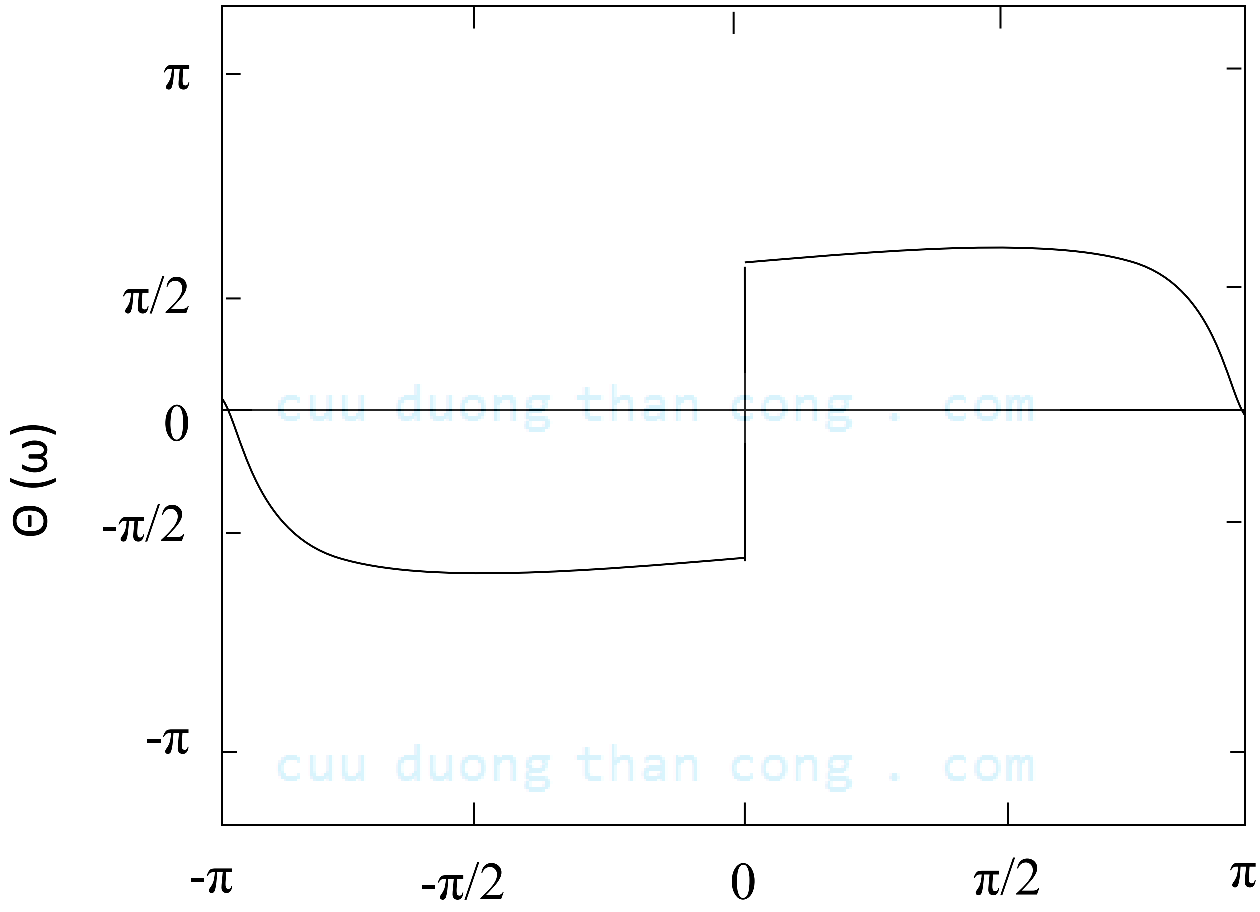
$$H_3(z) = \frac{1-\alpha}{2}\frac{1+z^{-1}}{1+\alpha z^{-1}} \quad (4.5.11)$$

4.5.2 Lowpass, Highpass, and Bandpass Filters

Figure 4.46 Magnitude and phase responses for a sample high pass filter; $H_3(z) = [(1- a) / 2] [(1 - z^{-1}) / (1 – az^{-1})]$ with $a = 0.9$



4.5.2 Lowpass, Highpass, and Bandpass Filters



4.5.2 Lowpass, Highpass, and Bandpass Filters

A simple low pass to high pass filter transformation

Suppose that: $h_{lp}(n)$ the impulse response of the low pass filter.

$H_{lp}(\omega)$ the frequency translation of $h_{lp}(n)$.

Thus, $$H_{hp}(\omega) = H_{lp}(\omega - \pi) \qquad (4.5.12)$$

$$h_{hp}(n) = (e^{j\pi}) h_{lp}(n) = (-1)^{n} h_{hp}(n) \qquad (4.5.13)$$

$h_{hp}(n)$ – The impulse response of the high pass filter

$H_{hp}(n)$ – The frequency translation of $h_{hp}(n)$.

4.5.2 Lowpass, Highpass, and Bandpass Filters

If the lowpass filter is

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \qquad (4.5.15)$$

Its frequency response is

$$H_{lp}(\omega) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}} \qquad (4.5.16)$$

Then

$$H_{hp}(\omega) = \frac{\sum_{k=0}^{M} (-1)^k b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} (-1)^k a_k e^{-j\omega k}} \qquad (4.5.17)$$

and

$$y(n) = -\sum_{k=1}^{N} (-1)^k a_k y(n-k) + \sum_{k=0}^{M} (-1)^k b_k x(n-k) \qquad (4.5.18)$$

4.5.3 Digital Resonators

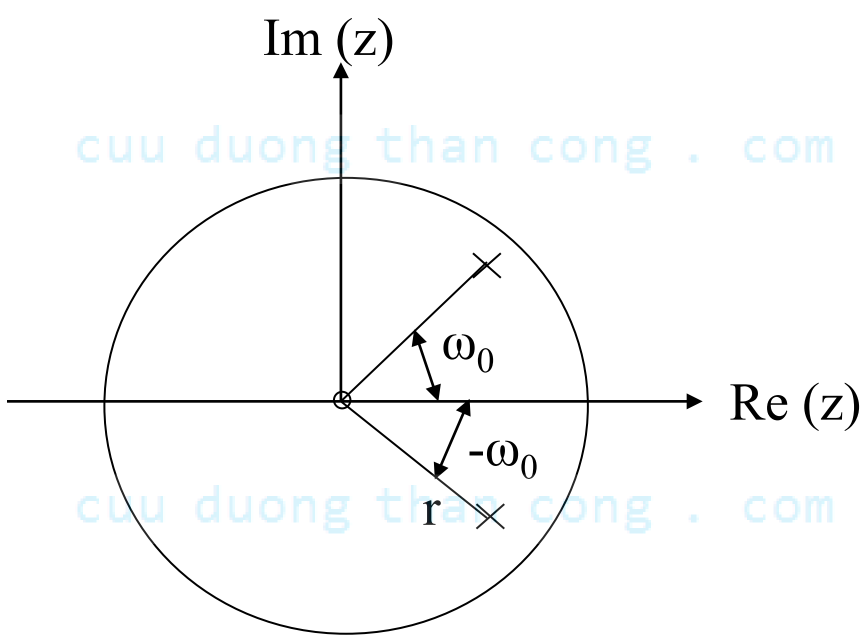
A digital resonator is a *special two-pole bandpass filter* with the *pair of complex-conjugate poles* located ***near the unit circle*** as Fig 4.48a .

The *magnitude* of the frequency response of the filter as Fig4.48b

$$p_{1,2} = re^{\pm j\omega_0}, \qquad 0 < r < 1$$

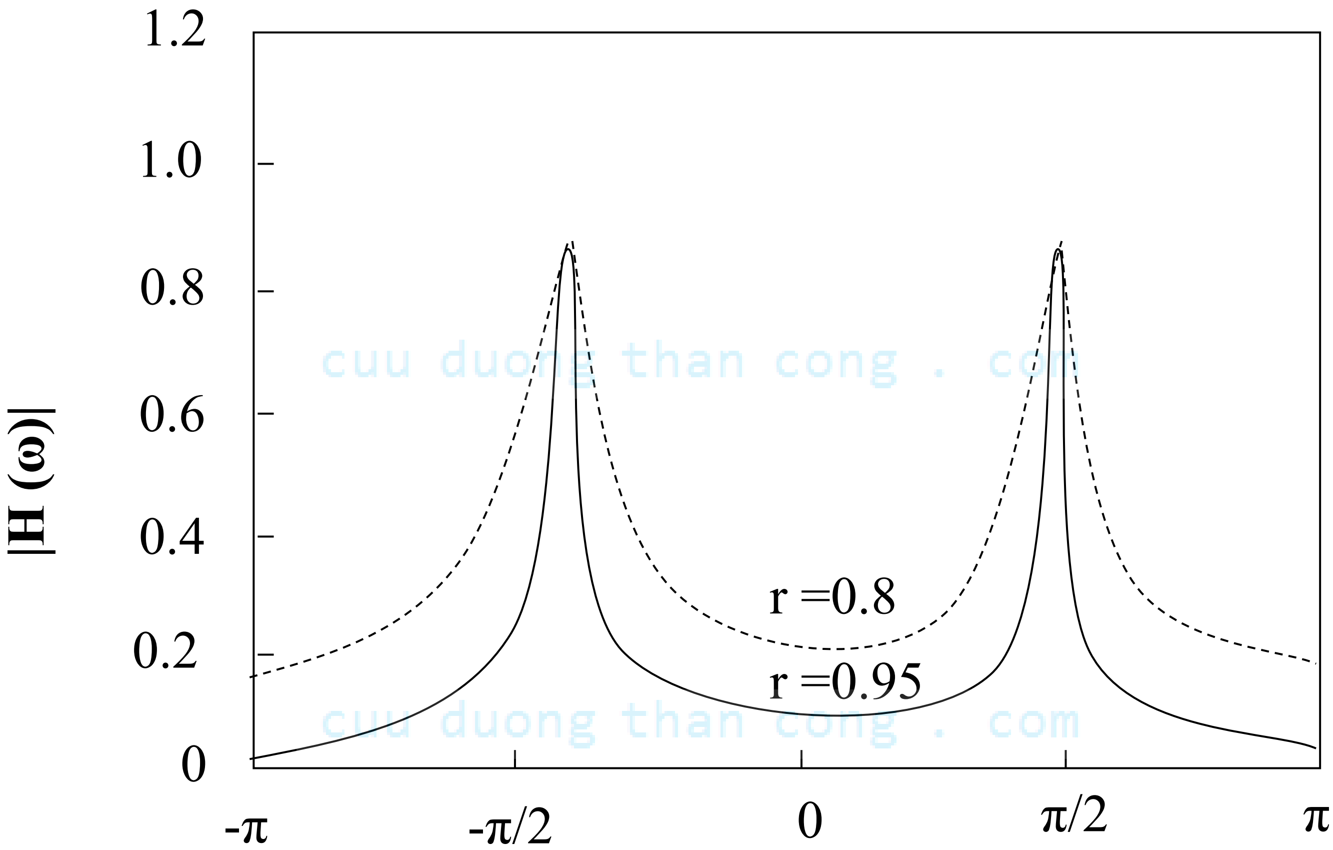
4.5.3 Digital Resonators

Figure 4.48 (a) Pole-zero pattern and (b) the corresponding magnitude and phase response of a digital resonator with (1) $r = 0.8$ and (2) $r = 0.95$



(a)

4.5.3 Digital Resonators



(b)

4.5.3 Digital Resonators

The system function with zero at the origin,

$$H(z) = \frac{b_0}{(1 - re^{j\omega_0}z^{-1})(1 - re^{-j\omega_0}z^{-1})} \quad (4.5.19)$$

$$H(z) = \frac{b_0}{1 - (2r\cos\omega_0)z^{-1} + r^2 z^{-2}} \quad (4.5.20)$$

$$H(\omega_0) = \frac{b_0}{(1 - r)(1 - re^{-j2\omega_0})} \quad (4.5.21)$$

the gain b_0 so that $|H(\omega_0)| = 1$, thus

$$b_0 = (1 - r)\sqrt{1 + r^2 - 2r\cos 2\omega_0} \quad (4.5.22)$$

4.5.3 Digital Resonators

If the *zero of the digital resonator* are placed at ***z* = 1** and ***z* = -1** the resonator has the system function

$$H(z) = G\frac{(1-z^{-1})(1+z^{-1})}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})} \qquad (4.5.27)$$

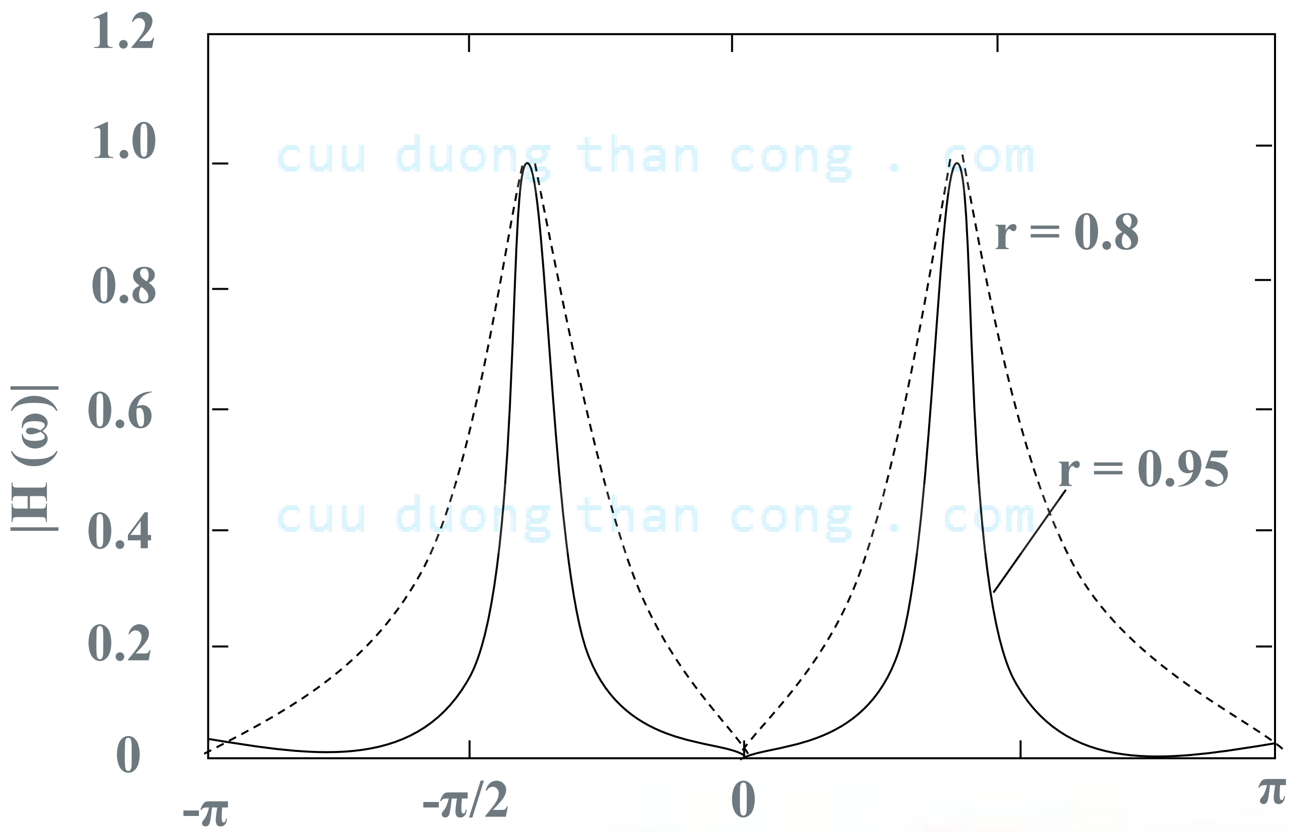
$$= G\frac{1-z^{-2}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$$

$$H(\omega) = b_0\frac{1-e^{-j2\omega}}{[1-re^{j(\omega_0-\omega)}][1-re^{-j(\omega_0+\omega)}]} \qquad (4.5.28)$$

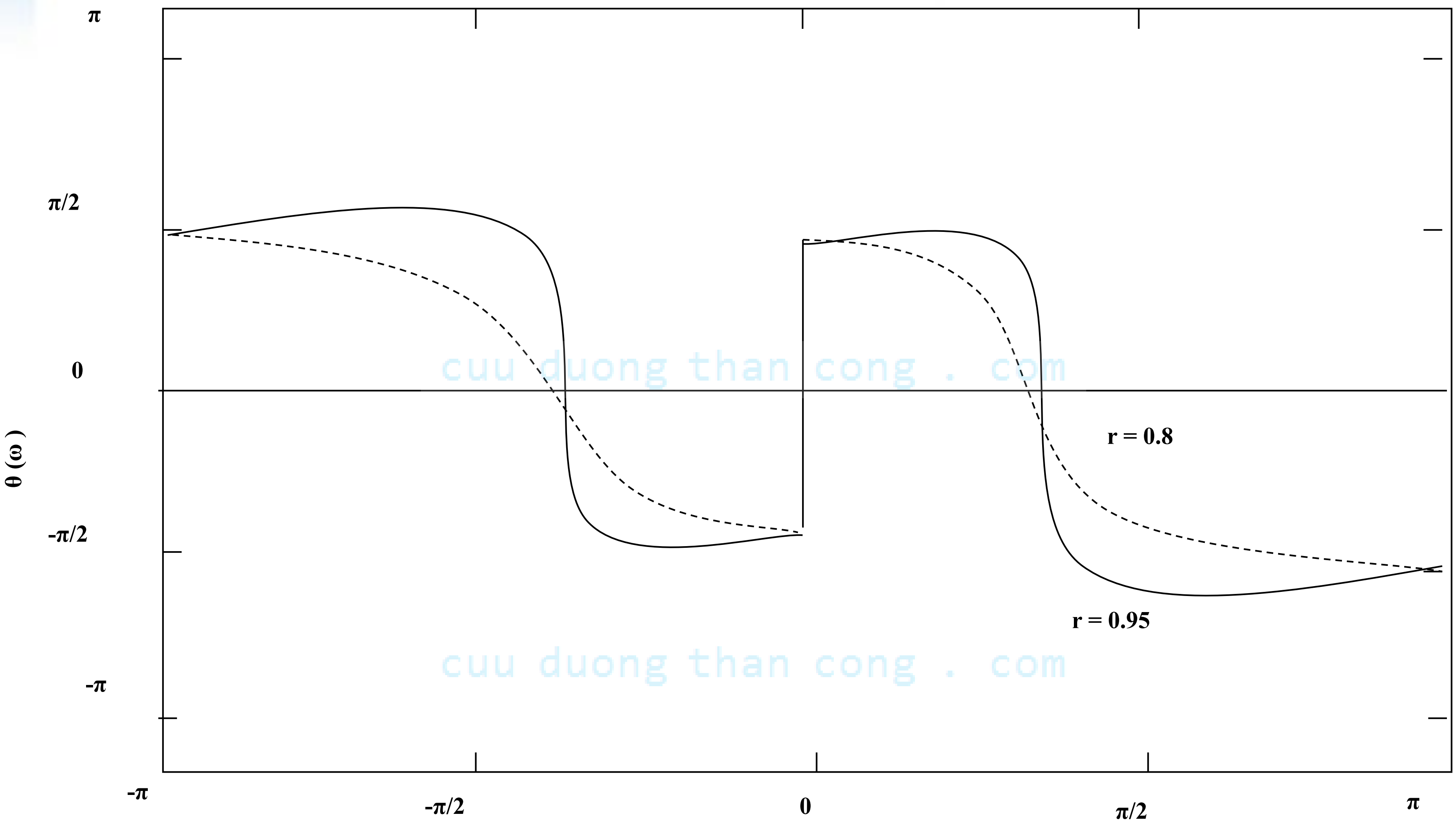
Fig. 4.49 illustrates the *magnitude and phase characteristics* for $\omega_0 = \pi/3$, $r = 0.8$ and $\omega_0 = \pi/3$, $r = 0.95$

4.5.3 Digital Resonators

Figure 4.49 Magnitude and phase responses for digital resonator with zeros at $\omega = 0$ and $\omega = \pi$ and (1) $r = 0.8$ and (2) $r = 0.95$.



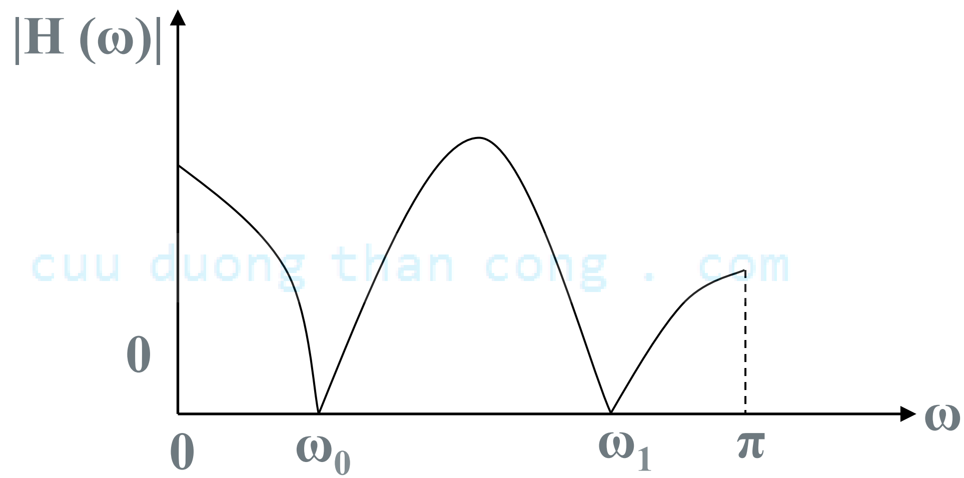
4.5.3 Digital Resonators



4.5.4 Notch Filters

A notch filter is a filter that contains ***one*** or ***more deep notches*** or, ideally, perfect ***null*** in its *frequency response characteristic*.

Figure 4.50 Frequency response characteristic of a notch filter.



4.5.4 Notch Filters

We simply introduce a pair of *complex- conjugate zeros* on the unit circle at an angle ω_0.

$$z_{1,2} = e^{\pm j\omega_0}$$

The system function for an FIR ***notch filter*** is

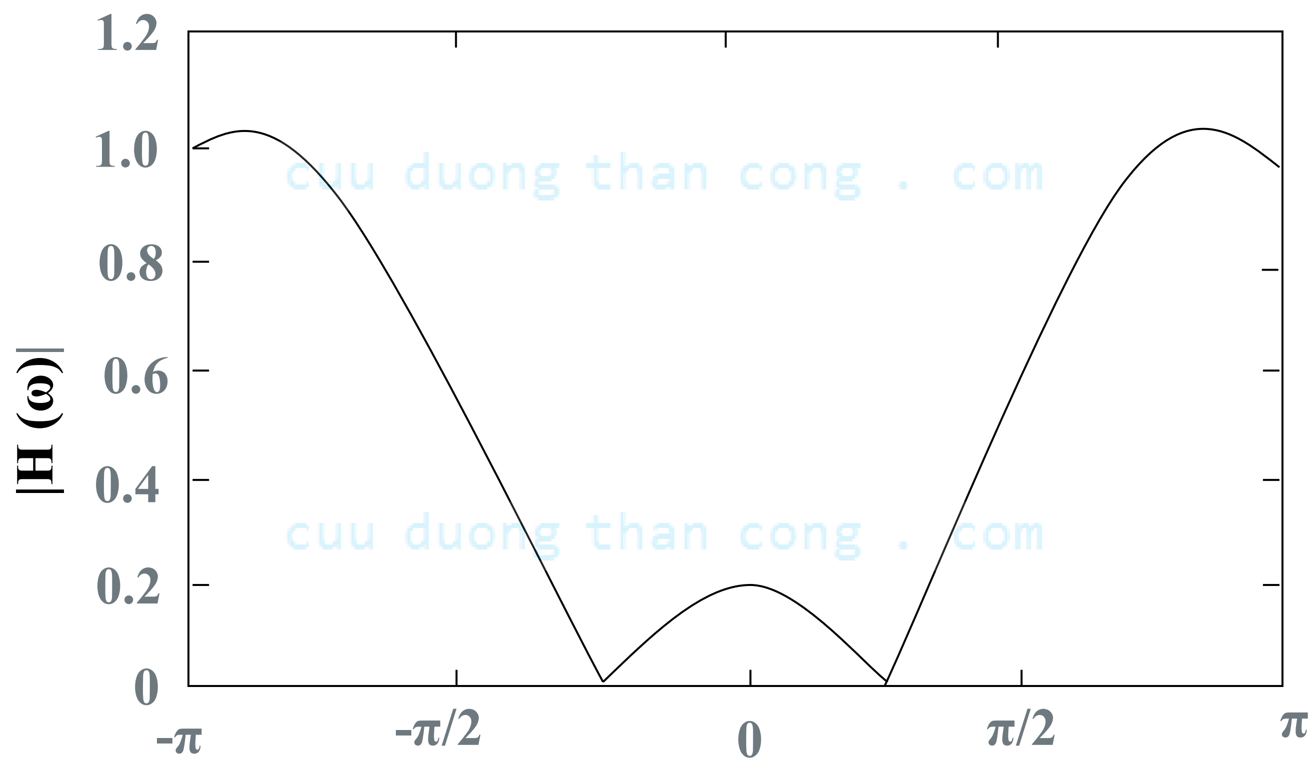
$$H(z) = b_0\left(1 - e^{j\omega_0}z^{-1}\right)\left(1 - e^{-j\omega_0}z^{-1}\right)$$

$$= b_0\left(1 - 2\cos\omega_0 z^{-1} + z^{-2}\right) \qquad (4.5.30)$$

Fig.4.51 shows the magnitude response for a notch filter having a null at $\omega = \pi/4$.

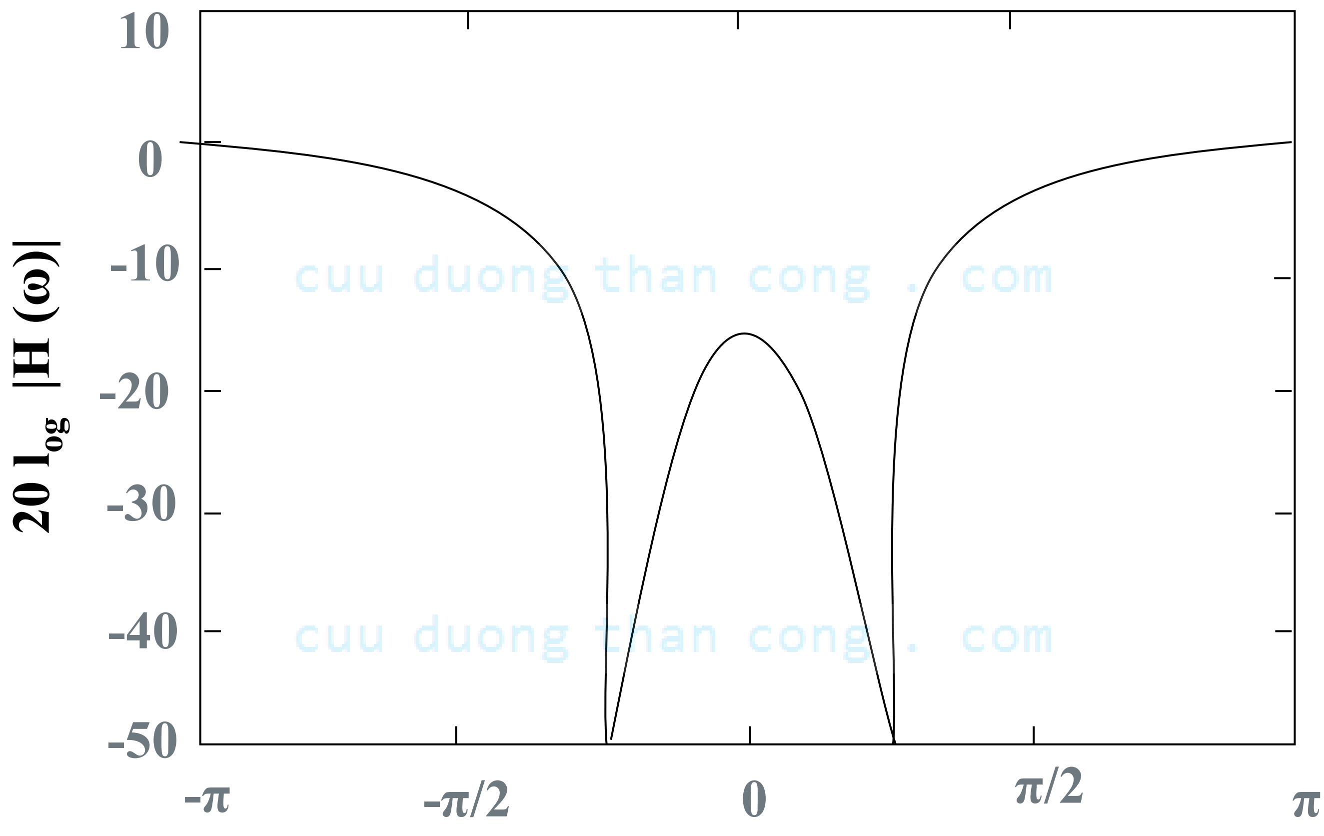
4.5.4 Notch Filters

Figure 4.51 Frequency response characteristic of *a notch filter* with a notch *at* $\omega = \pi/4$ or $f = 1/8$; $H(z) = G\,[1-2\cos\omega_0 z^{-1} + z^{-2}]$.



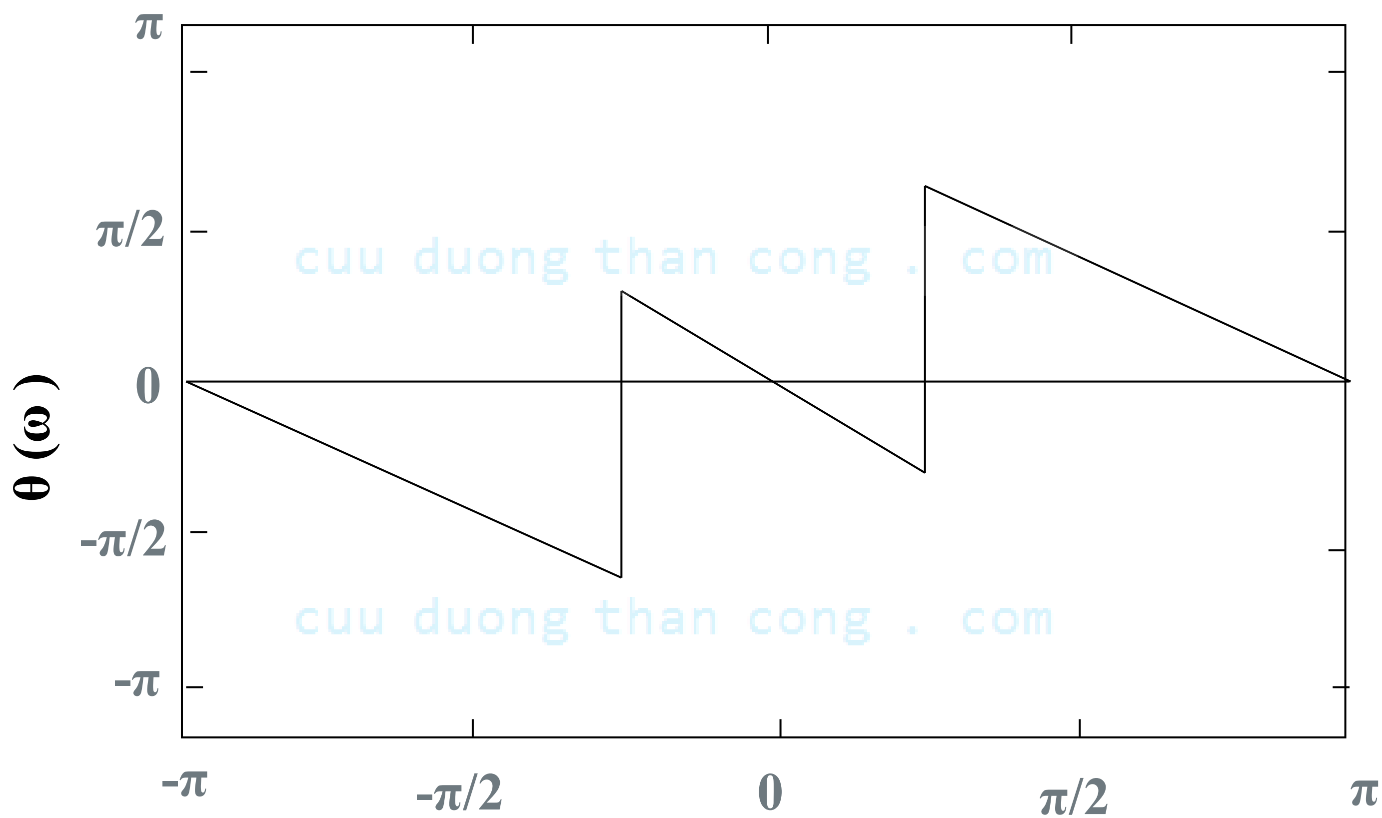
4.5.4 Notch Filters

20 log|H(ω)|



4.5.4 Notch Filters

$\omega = \pi/4$ or $f = 1/8$



4.5.4 Notch Filters

Other frequency components around the desired null are ***severely attenuated***.

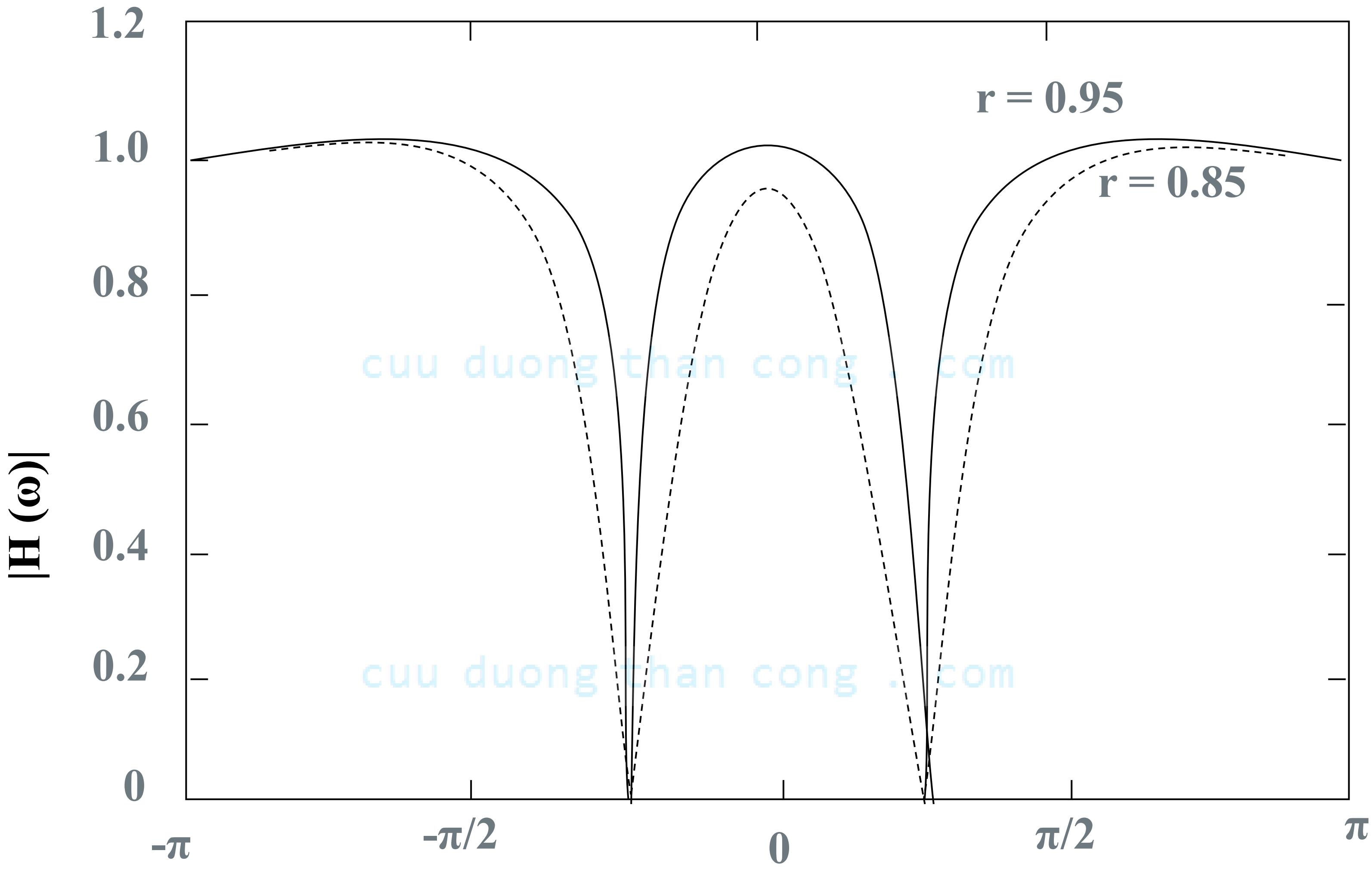
To reduce the band width of the null, we place *a pair of complex-conjugate poles* at

$$p_{1,2} = re^{\pm j\omega_0}$$

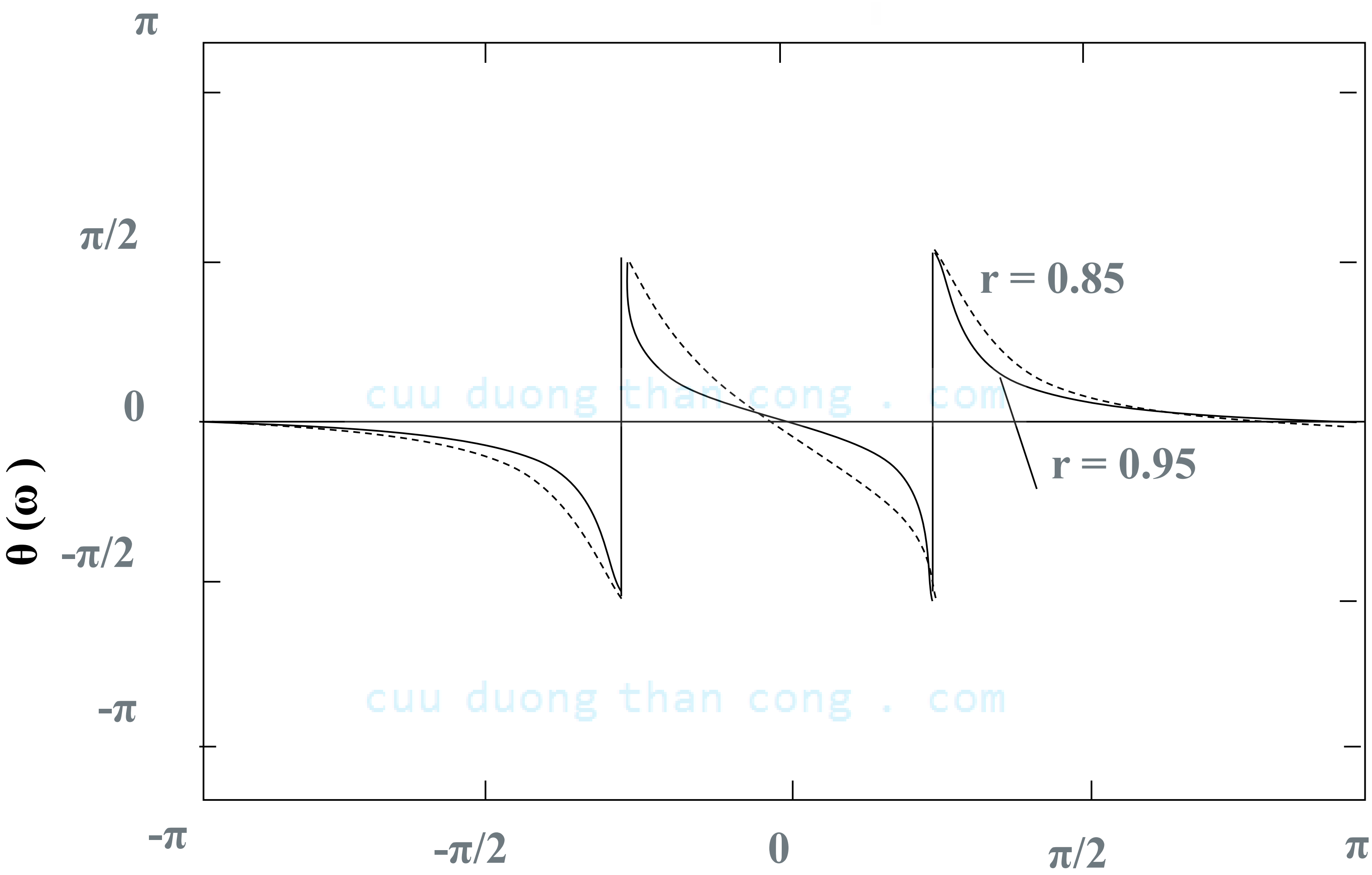
$$H(z) = b_0 \frac{1 - 2cos\omega_0 z^{-1} + z^{-2}}{1 - 2rcos\omega_0 z^{-1} + r^2 z^{-2}} \qquad (4.5.31)$$

Figure 4.52 Frequency response characteristic of a notch filter with a notch at (1) $r = 0.85$ and (2) $r = 0.95$; $H(z) = b_0[\,(1 - 2cos\omega_0 z^{-1} + z^{-2}) / (1 - 2rcos\omega_0 z^{-1} + r^2 z^{-2})\,]$.

4.5.4 Notch Filters



4.5.4 Notch Filters



4.5.4 Comb Filters

A comb filter can be viewed as a *notch filter* in which the *nulls occur periodically* across the frequency band.

Comb filters find applications in a wide range of practical systems such as

- in the rejection of power-line harmonics,
- in the separation of solar and lunar components from ionospheric measurements,
- in the suppression of cluster from fixed objects in moving-target-indicator radar.

4.5.4 Comb Filters

Consider a *moving average (FIR) filter* described by the difference equation

$$y(n) = \frac{1}{M+1}\sum_{k=0}^{M} x(n-k) \qquad (4.5.32)$$

System function

$$H(z) = \frac{1}{M+1} z^{-k} \sum_{k=0}^{M} z^{-k} \qquad (4.5.33)$$

$$= \frac{1}{M+1}\frac{\left[1-z^{-(M+1)}\right]}{\left(1-z^{-1}\right)}$$

Frequency responce

$$H(\omega) = \frac{e^{-j\omega M/2}}{M+1}\frac{\sin\left(\frac{M+1}{2}\right)}{\sin\,(\omega/2)} \qquad (4.5.34)$$

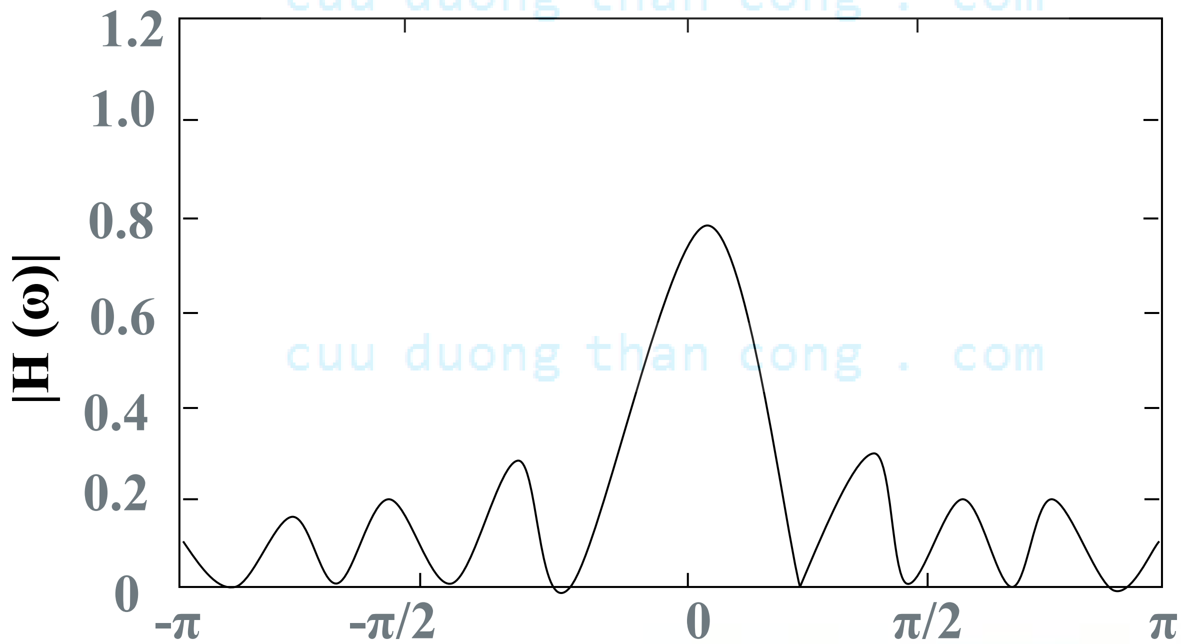
4.5.4 Comb Filters

Filter has zero on the unit circle at

$$z = e^{j2\pi k/(M+1)}, \qquad k = 1, 2, 3, \ldots, M \qquad (4.5.35)$$

$\omega_k = 2\pi k / (M+1)$ for $M = 10$.

Figure 4.53 Magnitude response characteristic of a comb filter given by (5.4.32) with M =10



4.5.4 Comb Filters

In more general term.

$$H(z) = \sum_{k=0}^{M} h(k)\, z^{-k} \qquad (4.5.36)$$

replacing $z = z^L$, where L is a positive integer

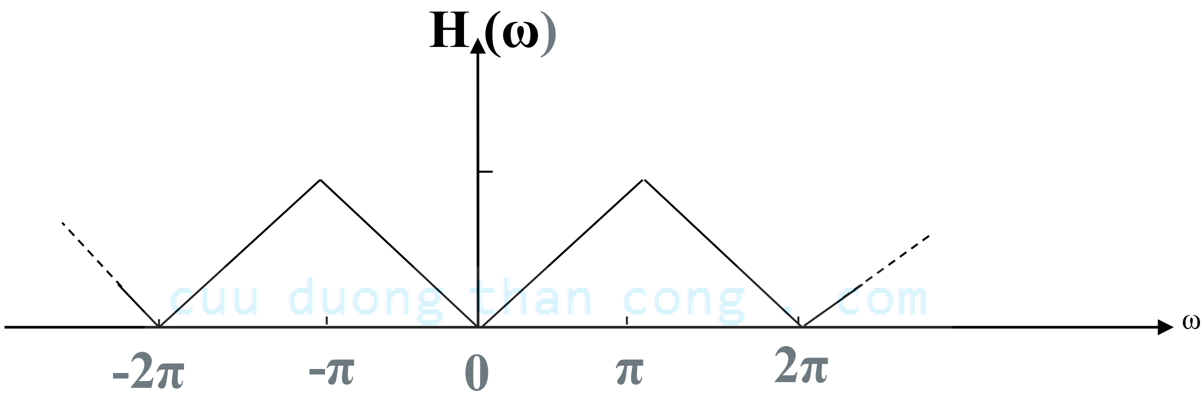
$$H_L(z) = \sum_{k=0}^{M} h(k)\, z^{-kL} \qquad (4.5.37)$$

The frequency response

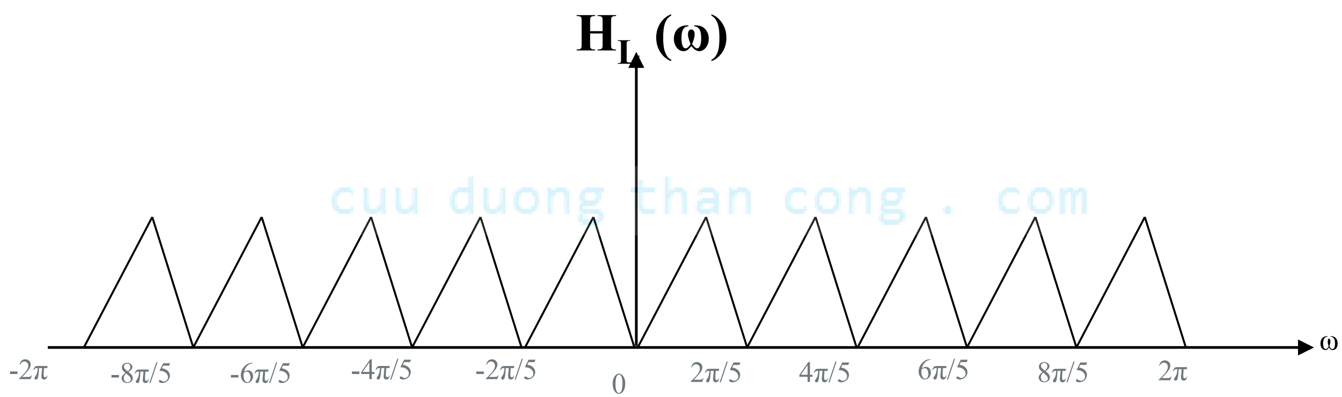
$$H_L(\omega) = \sum_{k=0}^{M} h(k)\, z^{-jkL\omega} = H(L\omega) \qquad (4.5.38)$$

4.5.4 Comb Filters

Figure 4.54 Comb filter with frequency response $H_L(\omega)$ obtained from $H(\omega)$.



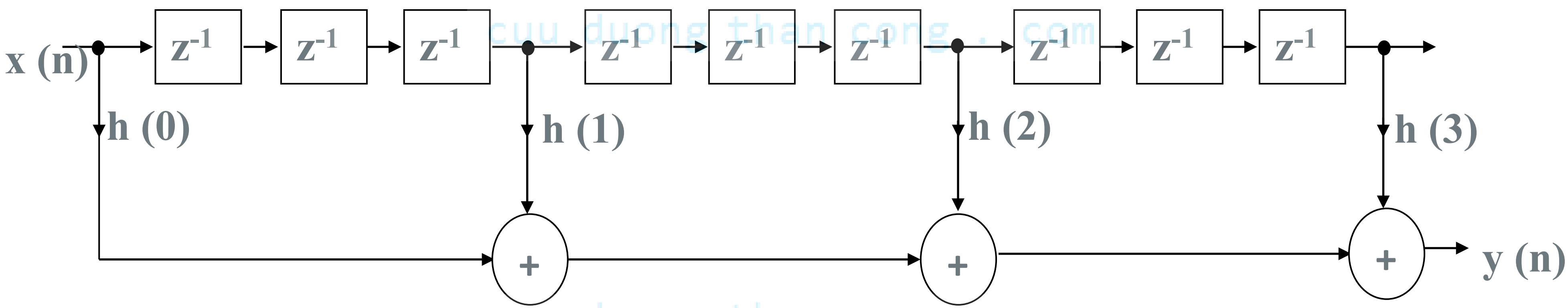
(a)



(b)

4.5.4 Comb Filters

Figure 4.55 Realization of an FIR comb filter having M=3 and L =3.



4.5.6 All-Pass Filters

An all-pass filter is defined as a system that has a constant magnitude response for all frequencies that is,

$$|H(\omega)| = 1 \qquad 0 \le \omega \le \pi \qquad (4.5.42)$$

The simplest example : $H(z) = z^{-k}$ only for a delay of k samples.

A more interesting all-pass filter

$$H(z) = \frac{a_N + a_{N-1}z^{-1} + \cdots + a_1 z^{-N+1} + z^{-N}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} \qquad (4.5.43)$$

$$H(z) = \frac{\sum_{k=0}^{N} a_N z^{-N+k}}{\sum_{k=0}^{N} a_k z^{-k}}, \qquad a_0 = 1$$

{ ak } are real

$$H(z) = z^{-N}\frac{A(z^{-1})}{A(z)} \qquad (4.5.44)$$

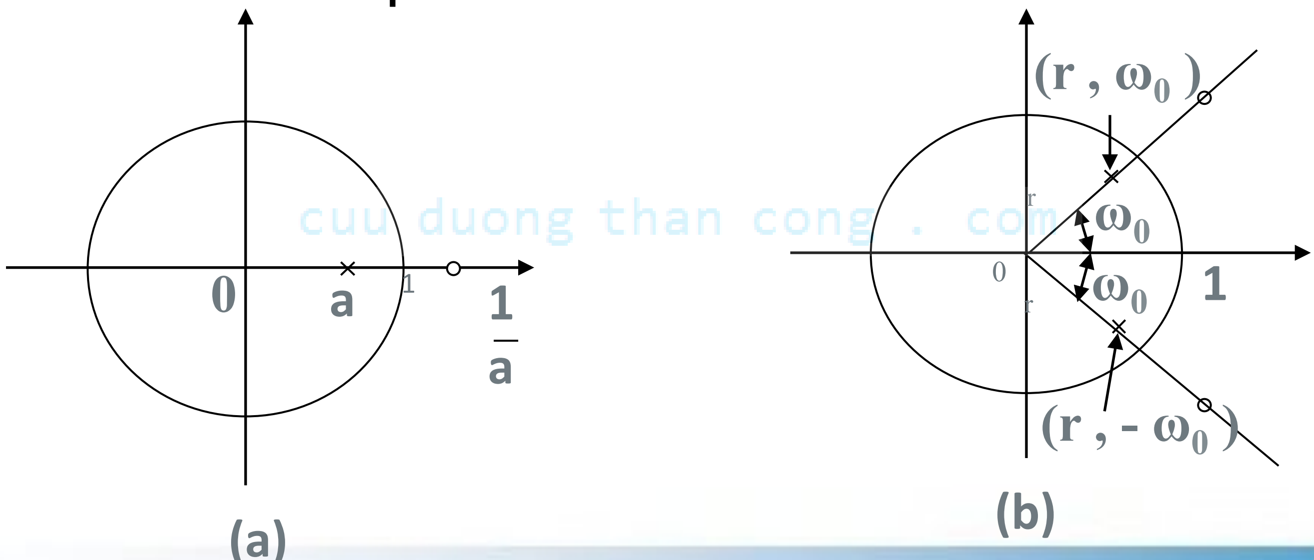
4.5.6 All-Pass Filters

with
$$A(z) = \sum_{k=0}^{N} a_k z^{-k}, \qquad a_0 = 1$$

Since
$$|H(\omega)|^2 = H(z)H(z^{-1})|_{z=e^{j\omega}} = 1$$

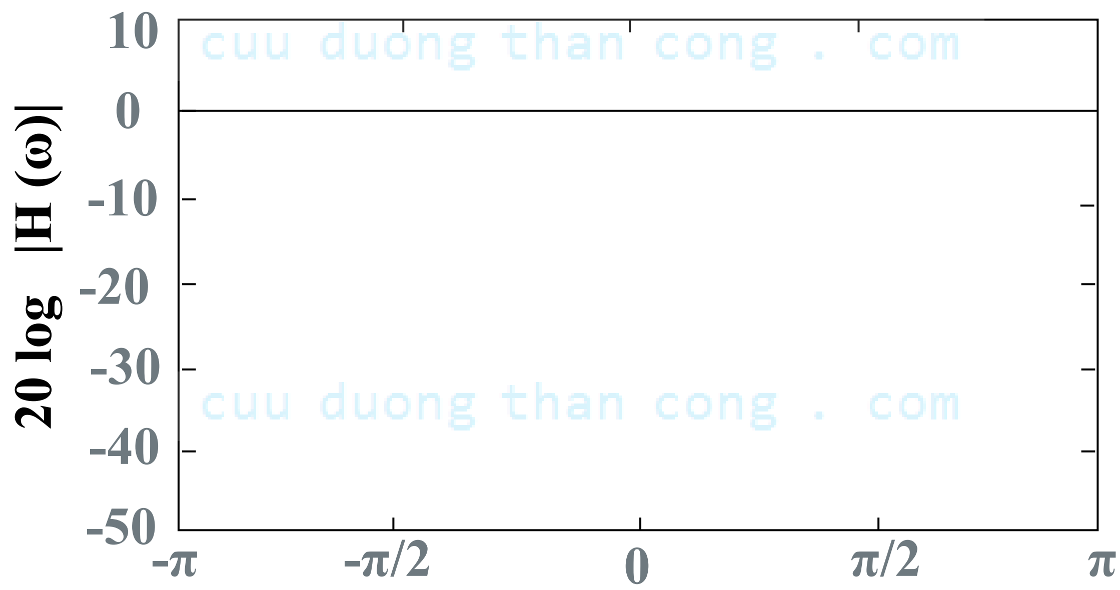
If z_0 is a pole of $H(z)$, then $1/z_0$ is a zero of $H(z)$.

Figure 4.58 Pole – zero patterns of (a) a first-order and (b) a second-order all-pass filter

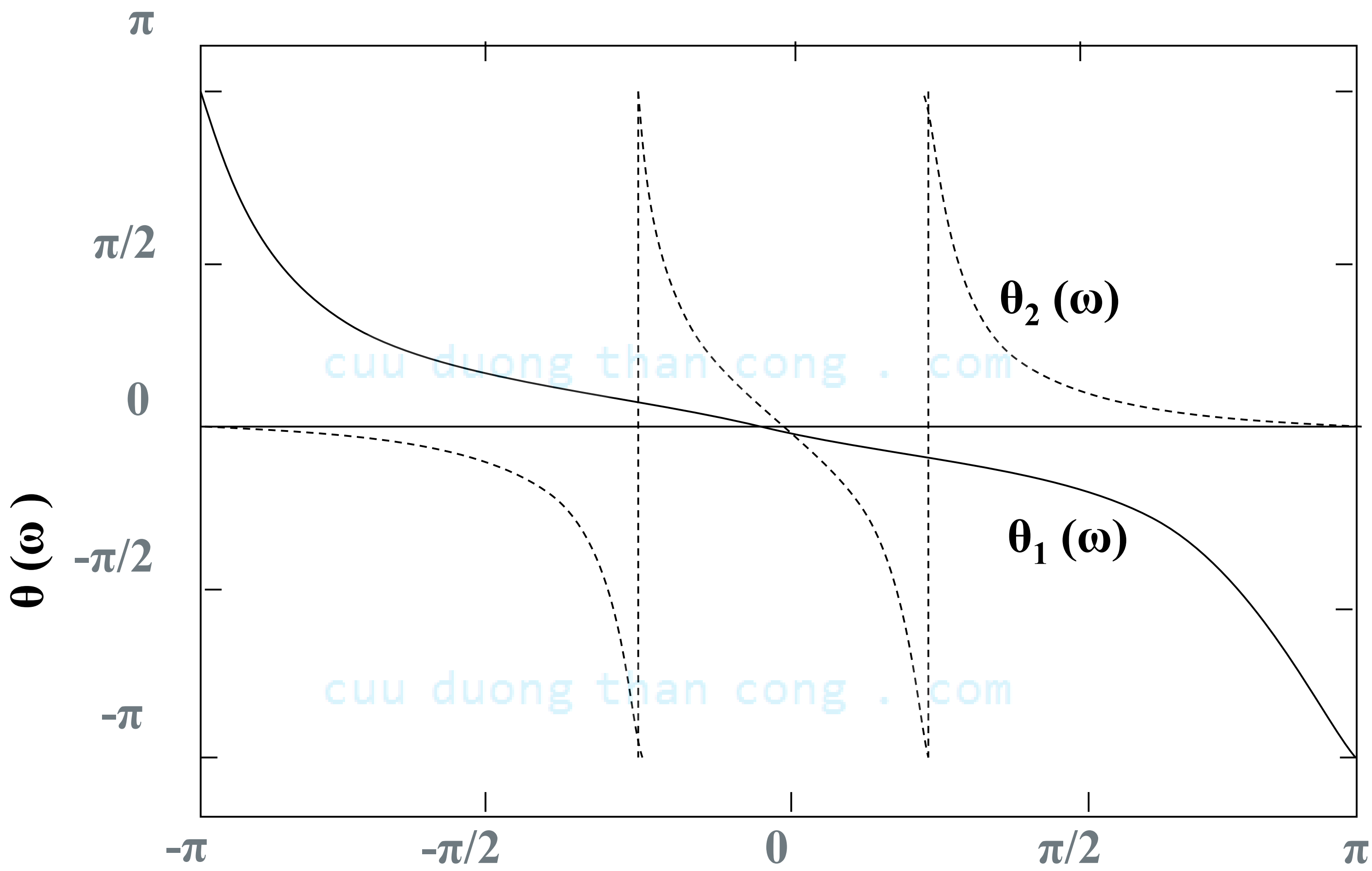


4.5.6 All-Pass Filters

Figure 4.59 Frequency response characteristics of an all-pass filter with system function (1) $H(z) = (0.6 + z^{-1}) / (1+0.6z^{-1})$, (2) $H(z) = (r^2 - 2r\cos\omega_0 z^{-1} + z^{-2}) / (1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2})$, $r = 0.9$, $\omega_0 = \pi/4$.



4.5.6 All-Pass Filters



4.5.6 All-Pass Filters

The *most general form* for the system function:

$$H_{ap}(z) = \prod_{k=1}^{N_R} \frac{z^{-1} - \alpha_k}{1 - \alpha_k z^{-1}} \prod_{k=1}^{N_c} \frac{(z^{-1} - \beta_k)(z^{-1} - \beta_k^*)}{(1 - \beta_k z^{-1})(1 - \beta_k^* z^{-1})} \qquad (4.5.45)$$

N_R – real poles and zeros

N_c – Complex- conjugate pairs of poles and zeros,

All- pass filters find application as ***phase equalizers***.

4.5.7 Digital Sinusoidal Oscillators

A digital sinusoidal oscillator can be viewed as a limiting form of *a two-pole resonator* for which the complex-conjugate poles lie on the unit circle.

The second-order system with system function

$$H(z) = \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-2}} \qquad (4.5.47)$$

With $a_1 = -2r\cos\omega_0$ and $a_2 = r^2$ (4.5.48)

has complex-conjugate poles at $p = re^{\pm j\omega 0}$

4.5.7 Digital Sinusoidal Oscillators

We have a unit sample response.

$$h(n) = \frac{b_0 r^n}{\sin \omega_0} \sin(n+1)\omega_0 u(n) \qquad (4.5.49)$$

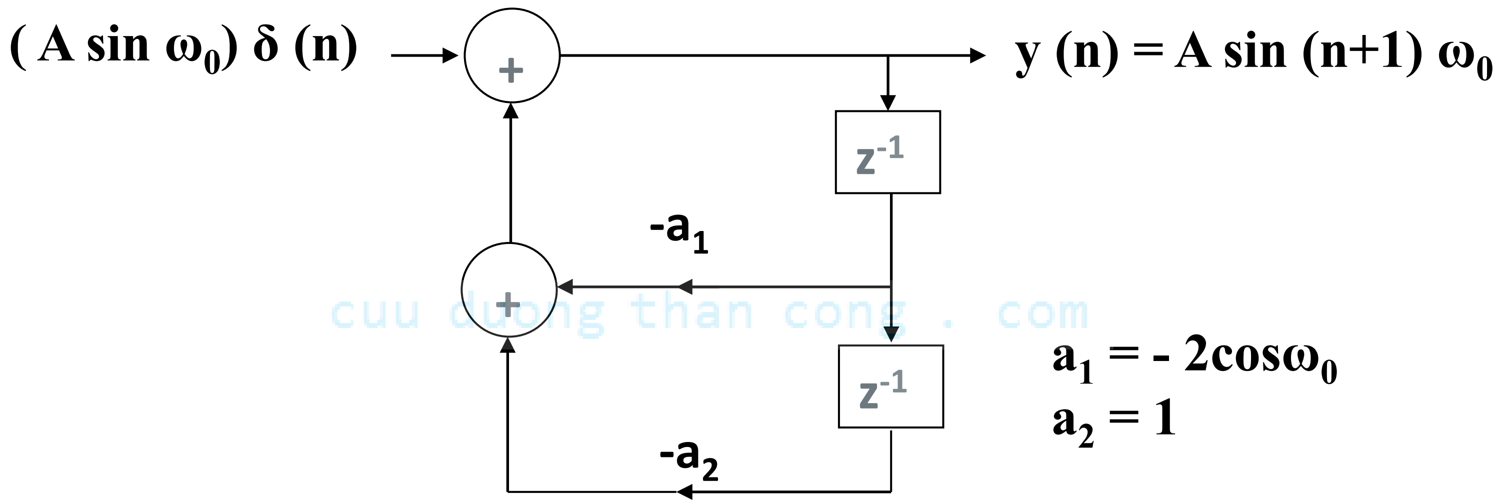
If the poles are placed on the unit circle and b_0 is set to $A \sin\omega_0$, then

$$h(n) = A \sin(n + 1)\omega_0 u(n) \qquad (4.5.50)$$

Thus system is called a ***digital sinusoidal oscillator*** or ***a digital sinusoidal generator***.

4.5.7 Digital Sinusoidal Oscillators

Figure 4.60 Digital sinusoidal generator



The corresponding difference equation for this system is

$$y(n) = -a_1 y(n-1) - y(n-2) + b_0 \delta(n) \qquad (4.5.51)$$

where $a_1 = -2\cos\omega_0$ and $b_0 = A\sin\omega_0$

4.6 Inverse System and Deconvolution

The intersymbol interference may causes errors when we attempt to recover the data.

In digital communication such *a corrective system* is called an ***equalizer*** or ***inverse system***.

The *inverse system*, operation that takes *y(n)* and produces *x(n)* is called ***deconvolution***.

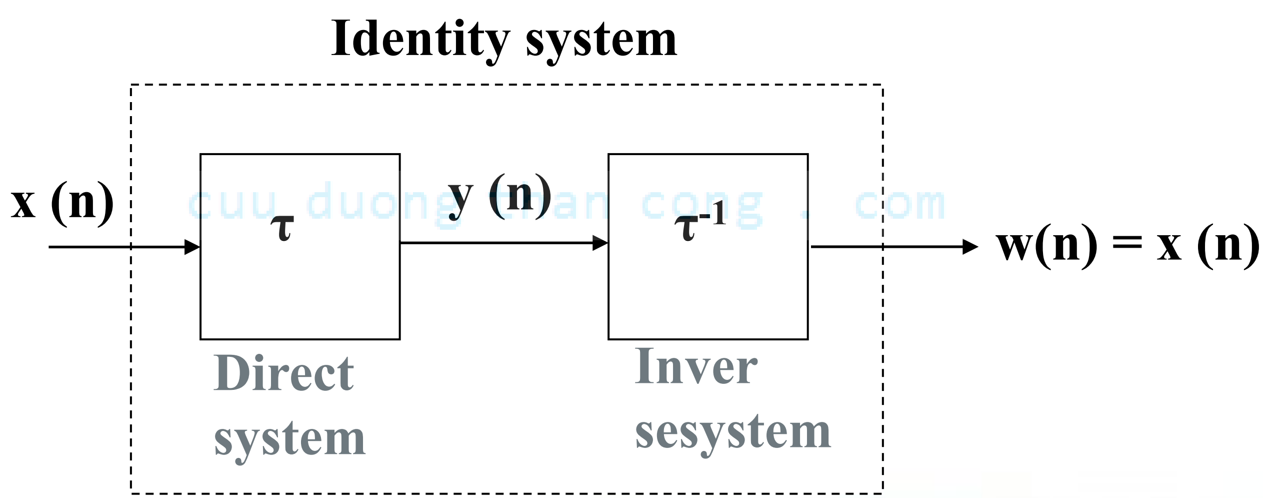
The process of determining the characteristics of the unknown system, either ***h(n)*** or ***H(ω)***, by a set of measurements performed on the system is called ***system identification***.

4.6.1 Invertibility of Linear Time-Invariant systems

A system is said to be ***invertible*** if there is a ***one – to – one correspondence*** between its input and output signals.

$$w(n) = \tau^{-1}[\, y(n)\,] = \tau^{-1}\{\, \tau[\, x(n)\,]\} = x(n) \qquad (4.6.1)$$

Figure 4.62 System τ in cascade with its inverse τ^{-1}



4.6.1 Invertibility of Linear Time-Invariant systems

For example:

$y(n) = ax(n)$ and $y = x(n - 5)$

are invertible

$y(n) = x^2(n)$ and $y(n) = 0$

represent noninvertible systems,

We limit *our discussion to the class of linear time-invariant discrete-time systems*.

Let $h_1(n)$ denote the impulse response of the inverse system T^{-1} , then

$$w(n) = h_1(n) * h(n) * x(n) = x(n) \qquad (4.6.2)$$

or $$h_1(n) * h(n) = \delta(n) \qquad (4.6.3)$$

4.6.1 Invertibility of Linear Time-Invariant systems

In the z- domain (4.6.3) be comes

$H(z)\, H_I(z) = 1$ or

$$H_I(z) = \frac{1}{H(z)} \qquad (4.6.4)$$

If

$$H(z) = \frac{B(z)}{A(z)} \qquad (4.6.5)$$

Then

$$H_I(z) = \frac{B(z)}{A(z)} \qquad (4.6.6)$$

The ***zero*** of *H(z)* become the ***poles*** of the *inverse system* and vice versa

4.6.2 Minimum-Phase, Maximum-Phase, and Mixed – Phase Systems

The ***invertibility*** of a linear time- invariant system is intimately related to the characteristics of the ***phase spectral function*** of the system.

For example

$$H_1(z) = 1 + \tfrac{1}{2} z^{-1} = z^{-1} (z + \tfrac{1}{2}) \qquad (4.6.10)$$

$$H_2(z) = \tfrac{1}{2} + z^{-1} = z^{-1} (\tfrac{1}{2} z + 1) \qquad (4.6.11)$$

$$|H_1(\omega)| = |H_2(\omega)| = \sqrt{\frac{5}{4} + \cos\omega} \qquad (4.6.12)$$

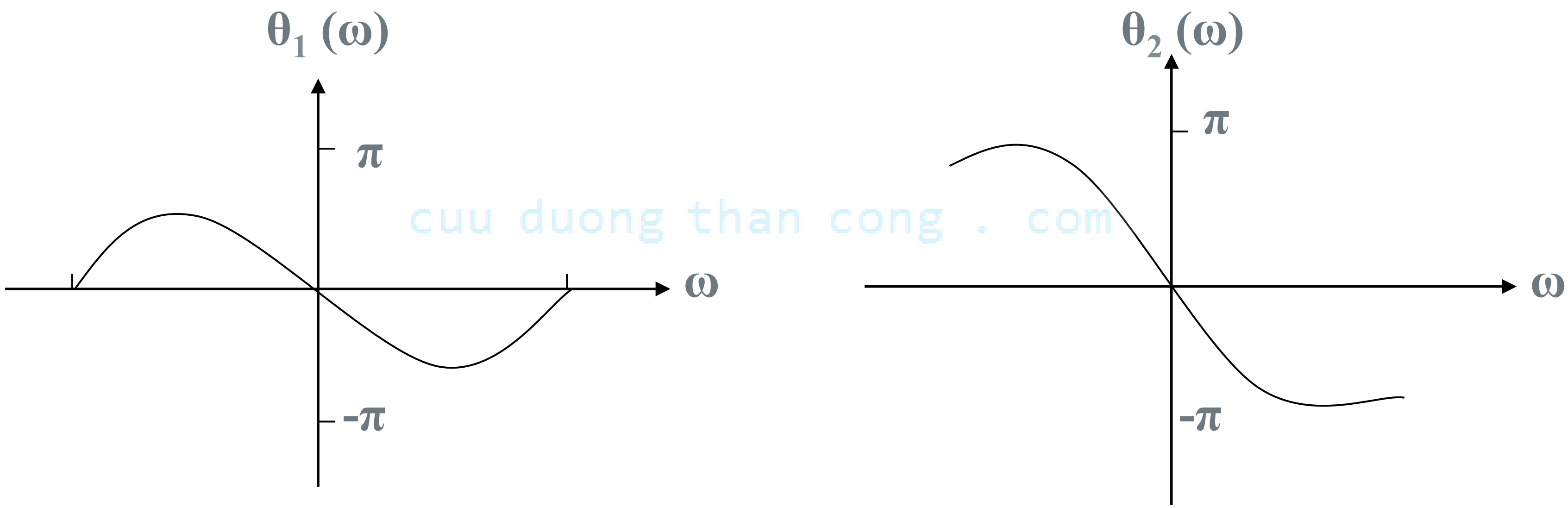
and

$$\Theta_1(\omega) = -\omega + tan^{-1}\frac{sin\omega}{\frac{1}{2} + cos\omega} \qquad (4.6.13)$$

$$\Theta_2(\omega) = -\omega + tan^{-1}\frac{sin\omega}{2 + cos\omega} \qquad (4.6.14)$$

4.6.2 Minimum-Phase, Maximum-Phase, and Mixed – Phase Systems

Figure 4.64 Phase response characteristics for the system in (4.6.10). and (4.6.11)



Net phase $\omega = 0 \rightarrow \pi$ then $\Theta_1(\pi) - \Theta_1(0) = 0$,

The first system is called a ***minimum-phase system***

4.6.2 Minimum-Phase, Maximum-Phase, and Mixed – Phase Systems

Net phase $\omega = 0 \rightarrow \pi$ then $\Theta_2(\pi) - \Theta_2(0) = \pi$,

The second system is called a ***maximum-phase system***

Extended to an FIR system of arbitrary length.

$$H(\omega) = b_0(1 - z_1 e^{-j\omega})(1 - z_2 e^{-j\omega}) \ldots (1 - z_M e^{-j\omega}) \quad (4.6.15)$$

where $\{z_i\}$ denote the zeros and b_0 constant.

4.6.2 Minimum-Phase, Maximum-Phase, and Mixed – Phase Systems

When ***all the zeros*** are *inside the unit circle*.

Therefore,

$$\angle H(\pi) - \angle H(0) = 0 \quad (4.6.16)$$

System is called a ***minimum-phase system***.

When ***all the zeros*** are *outside the unit circle*.

Therefore,

$$\angle H(\pi) - \angle H(0) = M\pi \quad (4.6.17)$$

System is called a ***maximum-phase system***.

4.6.2 Minimum-Phase, Maximum-Phase, and Mixed – Phase Systems

If the FIR system with *M* ***zero*** has *some of its zeros inside the unit circle* and remaining *zeros outside the unit circle*, is called a ***Mixed-phase system*** or a ***nonminimum-phase system*** .

A minimum-phase characteristic implies a minimum *delay function*.

A maximum-phase characteristic implies a maximum *delay function*.

4.6.2 Minimum-Phase, Maximum-Phase, and Mixed – Phase Systems

An *IIR system* with system function

$$H(z) = \frac{B(z)}{A(z)} \qquad (4.6.20)$$

is called ***minimum phase*** if all its poles and zeros are inside the unit circle.

For a *stable* and *causal system*, the system is called ***maximum phase*** if all the zeros are outside the unit circle,

and ***mixed phase*** if some, but not all, of the zeros are outside the unit circle.

4.6.2 Minimum-Phase, Maximum-Phase, and Mixed – Phase Systems

A *stable pole-zero system* that is *minimum phase* has a ***stable inverse*** which is also ***minimum phase***.

$$H^{-1}(z) = \frac{B(z)}{A(z)} \qquad (4.6.21)$$

Mixed-phase system and *maximum-phase system* result in ***unstable inverse systems***.

4.6.2 Minimum-Phase, Maximum-Phase, and Mixed – Phase Systems

Decomposition of nonminimum-phase pole-zero systems.

Any *nonminimum-phase pole-zero system* can be expressed as.

$$H(z) = H_{min}(z)H_{ap}(z) \qquad (4.6.22)$$

Where

$H_{min}(z)$ – minimum-phase system

$H_{ap}(z)$ – stable, all-pass, maximum-phase system.

Group delay of nonminimum-phase system

$$\tau_g(\omega) = \tau_g^{min}(\omega) + \tau_g^{ap}(\omega) \qquad (4.6.23)$$

Among all pole-zero systems having the *same magnitude response*, the ***minimum-phase system*** has the ***smallest group delay***.

4.6.2 Minimum-Phase, Maximum-Phase, and Mixed – Phase Systems

The ***partial energy*** of a causal system with impulse response *h(n)* is defined as.

$$E(n) = \sum_{k=0}^{n} |h(k)|^2 \qquad (4.6.24)$$

The ***minimum-phase system*** has the ***largest partial energy***.

4.6.3 System Identification and Deconvolution

We have an unknown linear time-invariant system with an input sequence *x(n)* and output sequence *y(n),*

we wish *to determine the impulse response* of the ***unknown system***.

This is a problem in ***system identification*** which can be solved by ***deconvolution***.

We have

$$y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \qquad (4.6.25)$$

z-transform $\qquad Y(z) = H(z)\,X(z)$

$$H(z) = \frac{Y(z)}{X(z)} \qquad (4.6.26)$$

4.6.4 Homomorphic Deconvolution

In LTI system

$$Y(z) = H(z)\,X(z) \qquad (4.6.30)$$

The logarithm of $Y(z)$ is

$$C_y(z) = \ln Y(z) = \ln X(z) + \ln H(z)$$

$$C_y(z) = C_x(z) + C_h(z) \qquad (4.6.31)$$

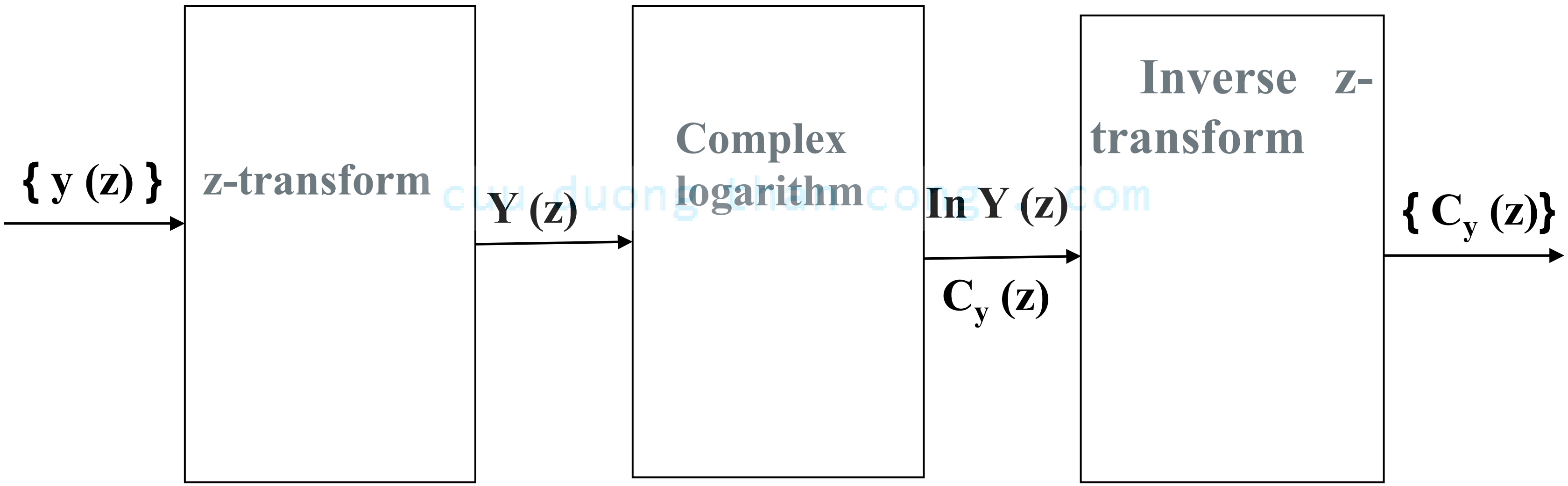
Complex cepstrum of the output sequence $\{ y(n) \}$ is expressed

$$C_y(n) = C_x(n) + C_h(n) \qquad (4.6.32)$$

The system for performing these transformations is called a ***homormorphic systems***, in Fig 4.65

4.6.4 Homomorphic Deconvolution

Figure 4.65 *Homomorphic system* for obtaining the cepstrum $\{ c_y (n) \}$ of the sequence $\{ y (n) \}$.



Problems: 4.4, 4.5, 4.9, 4.18, 4.23, 4.26, 4.27, 4.30, 4.33.