CHAPTER 5 The Discrete Fourier Transform: Its Properties and Applications

X(ω) is a ***continuous function of frequency*** and therefore, it is ***not computationally convenient representation*** of the sequence *{ x(n) }*.

A frequency-domain representation leads to the discrete Fourier transform (***DFT***), which is a *powerful computational tool* for performing frequency analysis discrete-time signals.

5.1 Frequency – Domain Sampling: The Discrete Fourier Transform

Let us consider

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \qquad (5.1.1)$$

Suppose that we sample *X(ω)* periodically in frequency at a spacing of *δω* radians between successive samples. At *ω = 2πk/N*, we obtain

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi kn/N}, \qquad (5.1.2)$$

$k = 0, 1, \ldots, N - 1$

5.1 Frequency – Domain Sampling: The Discrete Fourier Transform

If we change the index in the inner summation from *n* to *n – lN*, we obtain the result.

$$X\left(\frac{2\pi}{N}k\right)=\sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{\infty}x(n-lN)\right]e^{-j2\pi kn/N}, \qquad (5.1.3)$$

For $k = 0, 1, 2, \ldots, N-1$

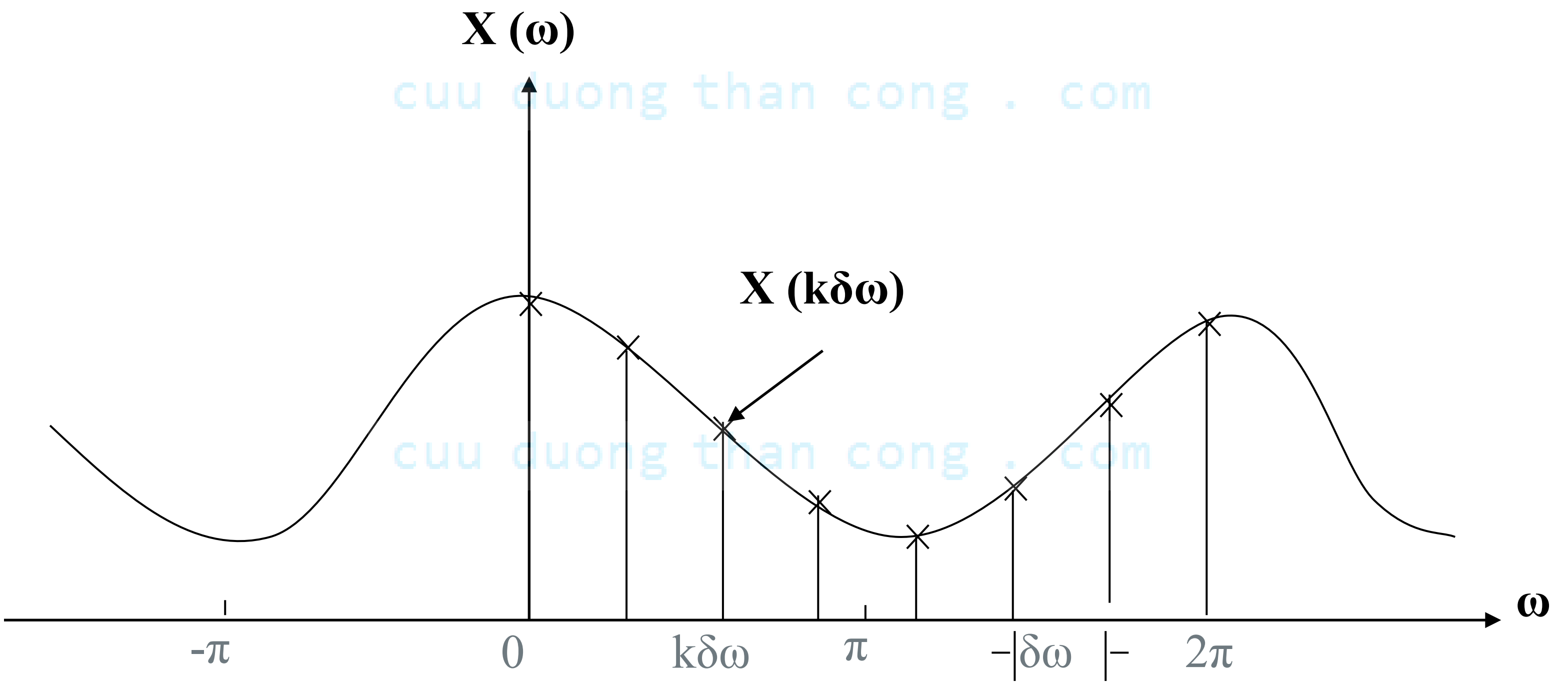
The signal

$$x_p(n)=\sum_{l=-\infty}^{\infty}x(n-lN) \qquad (5.1.4)$$

obtained by the ***periodic repetition*** of *x(n)* every *N* samples, is clearly periodic with fundamental period *N*.

5.1 Frequency – Domain Sampling: The Discrete Fourier Transform

Figure 5.1.1: Frequency – domain sampling of the Fourier transform



5.1 Frequency – Domain Sampling: The Discrete Fourier Transform

and

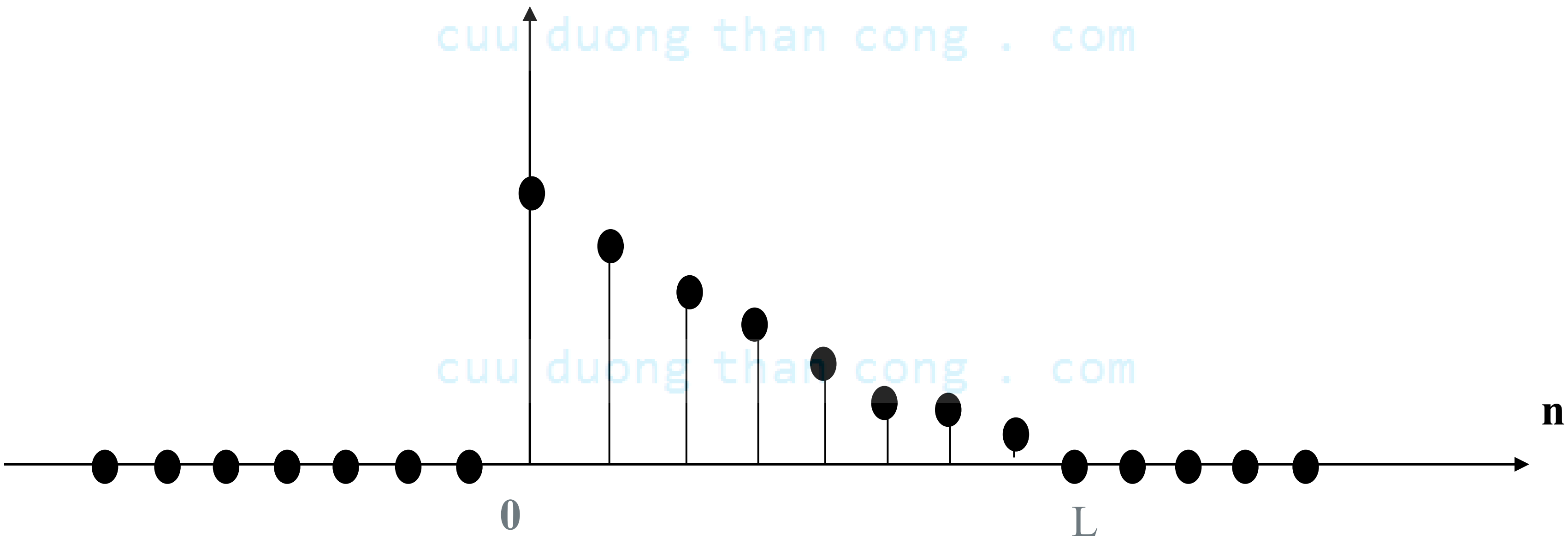
$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N}$$

$n = 0, 1, \ldots, N - 1$ (5.1.8)

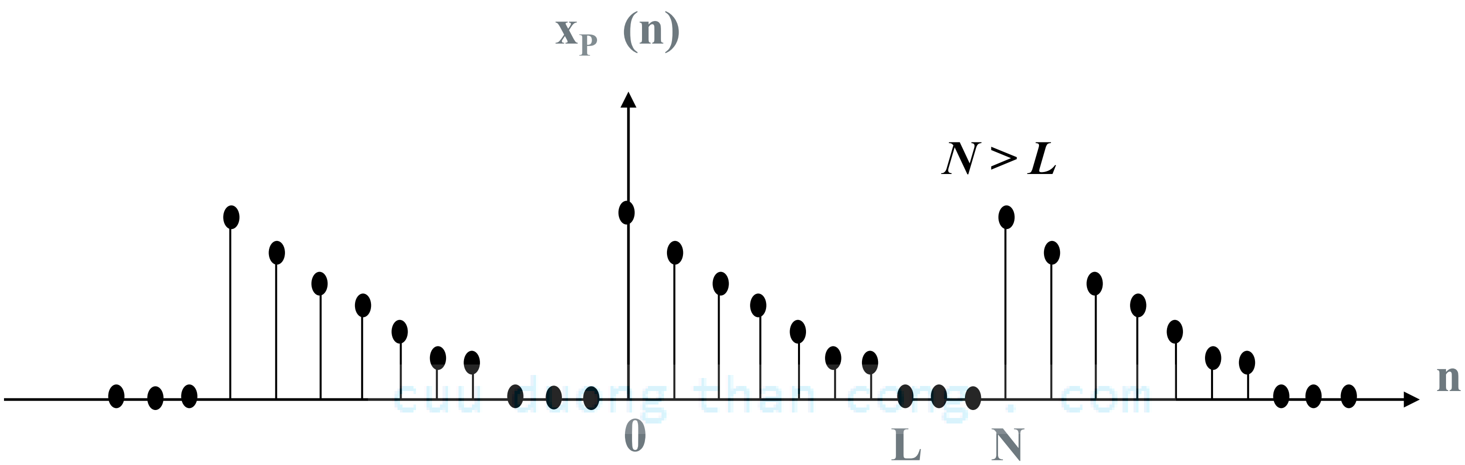
We need to consider the ***relation between*** $x_p(n)$ and $x(n)$

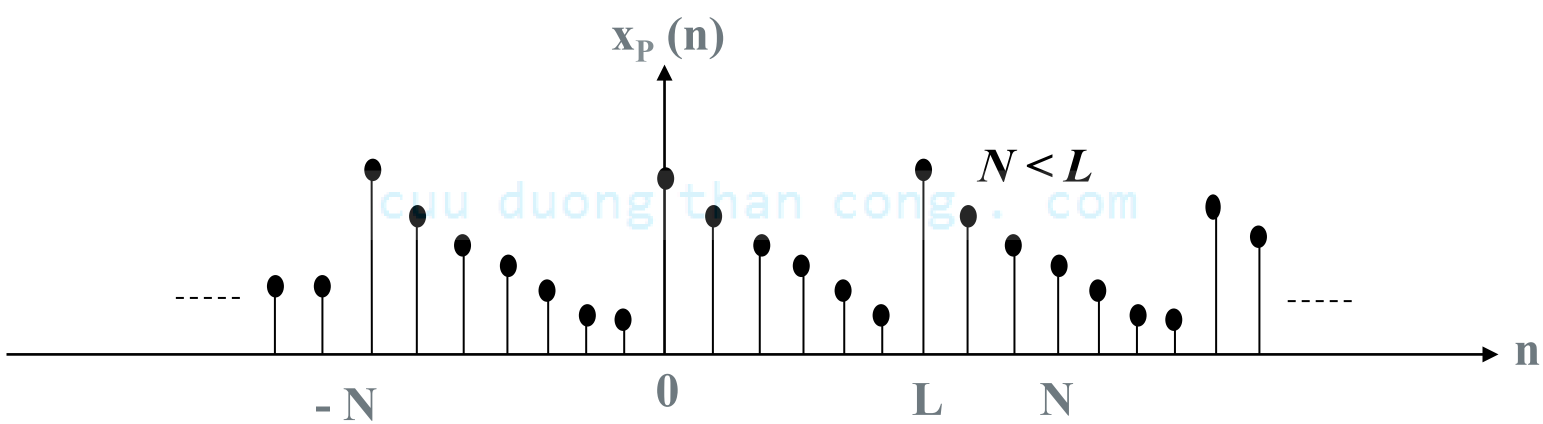
5.1: Frequency – Domain Sampling: The Discrete Fourier Transform

Figure 5.1.2 Apriodic sequence ***x(n)*** of length ***L*** and its periodic extension for ***N ≥ L*** (***no aliasing***) and ***N < L*** (***aliasing***)



5.1 Frequency – Domain Sampling: The Discrete Fourier Transform





5.1 Frequency – Domain Sampling: The Discrete Fourier Transform

we consider a finite-duration sequence $x(n)$, which is nonzero in the interval $0 \le n \le L-1$.

When $\boldsymbol{N \ge L}$

$$x(n) = x_p(n) \qquad\qquad 0 \le n \le N-1$$

If $\boldsymbol{N < L}$,

it is ***not possible to recover*** $x(n)$ from its periodic extension due to ***time – domain aliasing***.

5.1 Frequency – Domain Sampling: The Discrete Fourier Transform

The ***spectrum of as a periodic discrete – time signal*** with finite duration *L*, can be ***exactly recovered*** from its samples at frequency as

$$\omega_k = 2\pi k/N \qquad \text{if} \qquad N \geq L$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right)e^{j2\pi kn/N} \qquad 0 \leq n \leq N-1 \qquad (5.1.10)$$

$$x(\omega) = \sum_{n=0}^{N-1} X\left(\frac{2\pi}{N}k\right)\left[\frac{1}{N}\sum_{n=0}^{N-1} e^{-j\left(\omega-\frac{2\pi k}{N}\right)n}\right] \qquad (5.1.11)$$

5.1 Frequency – Domain Sampling: The Discrete Fourier Transform

If we define

$$P(\omega) = \frac{\sin(\omega N/2)}{N\sin(\omega/2)} e^{-j\omega(N-1)/2} \qquad (7.1.12)$$

Then

$$X(\omega) = \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) P\left(\omega - \frac{2\pi}{N}k\right) \quad N \geq L \qquad (5.1.13)$$

The interpolation formula in (5.1.13) gives ***exactly the sample values*** *X(2πk/N)* for *ω = 2πk/N*

5.1.2 The Discrete Fourier Transform (DFT)

When the sequence $x(n)$ has a finite duration of length $L \le N$, then $x_p(n)$ is simply a periodic repetition of $x(n)$.

$$x_p(n) = \begin{cases} x(n), & 0 \le n \le L-1 \\ 0, & L \le n \le N-1 \end{cases} \qquad (5.1.16)$$

The frequency samples $X(2\pi k/N)$, $k = 0, 1, 2, \ldots, N-1$ ***uniquely represent*** the finite – duration sequence $x(n)$.

Zero padding does not provide any additional information about the spectrum $X(\omega)$ of the sequence $\{ x(n) \}$.

5.1.2 The Discrete Fourier Transform (DFT)

When we sample *X(ω)* at equally spaced frequencies *ωk = 2πk/N, k = 0,1,….N-1*

$$X(k) \equiv X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{L-1} x(n)\, e^{-j2\pi kn/N} \qquad (5.1.18)$$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}, \qquad k = 0,1,2,\ldots N-1$$

The frequency samples are obtained by evaluating the Fourier transform *X(ω)* at a set of ***N***(equally spaced) *discrete frequencies.* (5.1.18) is called the ***discrete Fourier transform*** (***DFT***) of *x(n)*.

5.1.2 The Discrete Fourier Transform (DFT)

To ***recover*** the sequence *x(n)* from the frequency samples.

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N} \qquad (5.1.19)$$

n = 0, 1, ...N - 1

is called the ***inverse DFT***. (***IDFT***)

5.1.3 The DFT as a Linear Transformation

The formulas for the ***DFT*** and ***IDFT*** may be expressed as.

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \qquad k = 0, 1, \ldots, N-1 \qquad (5.1.20)$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}, \qquad n = 0, 1, \ldots, N-1 \qquad (5.1.21)$$

where, by definition $W_N = e^{-j2\pi/N}$

Which is an *Nth* root of unity .

5.1.3 The DFT as a Linear Transformation

The N-point DFT values can be computed in a total of N^2 complex multiplications and $N(N-1)$ complex additions. Let us define

an N-point vector $\boldsymbol{x}_N$ of the signal sequence $x(n)$, $n = 0, 1, \ldots, N-1$.

an N-point vector $\boldsymbol{X}_N$ of frequency samples

$$\boldsymbol{x}_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \qquad \boldsymbol{X}_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

5.1.3 The DFT as a Linear Transformation

and an $N \times N$ matrix $\mathbf{W}_N$ as.

$$W_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \quad (5.1.23)$$

With these definitions,

$$\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N \quad (5.1.24)$$

$$\mathbf{x}_N = \mathbf{W}_N^{-1} \mathbf{X}_N \quad (5.1.25)$$

$$\mathbf{W}_N^{-1} = \frac{1}{N} \mathbf{W}_N^{*} \quad (5.1.27)$$

5.1.4 Relationship of the DFT to Other Transforms.

Relationship to the *Fourier series coefficients* of a periodic sequence.

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi nk/N}, \qquad -\infty < n < \infty \qquad (5.1.29)$$

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi nk/N}, \qquad k = 0, 1, \ldots, N-1 \qquad (5.1.30)$$

If $x(n) = x_p(n)$ with $0 \le n \le N - 1$, the DFT of this Simply

$$X(k) = N c_k \qquad (5.1.31)$$

5.1.4 Relationship of the DFT to Other Transforms

Relationship to the *Fourier transform* of an aperiodic sequence.

$$X(k) = X(\omega)\Big|_{\omega=\frac{2\pi k}{N}} = \sum_{l=-\infty}^{\infty} x(n)\, e^{-j2\pi nk/N}, \qquad (5.1.32)$$

k = 0, 1, …, N – 1

The finite – duration sequence

$$\hat{x}(n) = \begin{cases} x_p(n), & 0 \le n \le N-1 \\ 0, & Otherwise \end{cases} \qquad (5.1.34)$$

In which case: $x(n) = \hat{x}(n) \quad 0 \le n \le N-1 \qquad (5.1.35)$

Only will the IDFT of *{ X(k) }* yield the original sequence *{ x(n) }*.

5.1.4 Relationship of the DFT to Other Transforms

Relationship to the *z – transform*

k = 0, 1, …, N – 1

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad (5.1.36)$$

with ROC that includes the unit circle

$$X(k) \equiv X(z)\big|_{z=e^{j2\pi nk/N}}$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi nk/N} \qquad (5.1.37)$$

k = 0, 1, …, N – 1

5.1.4 Relationship of the DFT to Other Transforms

If the sequence *x(n)* has a finite duration of length *N* or less, the sequence can be ***recovered*** from its *N*-point DFT.

When evaluated on the unit circle

$$X(\omega) = \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{-j(\omega - 2\pi k/N)}} \qquad (5.1.39)$$

This expression for the Fourier transform is a ***polynomial (Lagrange) interpolation formula*** for *X(ω)*.

5.1.4: Relationship of the DFT to Other Transforms

Relationship to the *Fourier series coefficients* of a continuous – time signal.

$$x(n) \equiv x_a(nT) = \sum_{k=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} c_k - lN \right] e^{j2\pi kn/N} \qquad (5.1.41)$$

$$X(k) = N \sum_{l=-\infty}^{\infty} C_{k-lN} \equiv N\tilde{c}_k \qquad (5.1.42)$$

and

$$\tilde{c}_k = \sum_{l=-\infty}^{\infty} c_k - lN \qquad (5.1.43)$$

5.2 Properties of the DFT

Some important differences exist, one of which is the ***circular convolution property*** derived in the following section.

The notation used below to denote the *N*-point DFT pair *x(n)* and *X(k)* is.

$$x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)$$

5.2.1 Periodicity, Linearity, and symmetry properties

Periodicity. If $x(n)$ and $X(k)$ are an N-point DFT pair, then

$$x(n + N) = x(n) \quad \text{for all } n \qquad (5.2.4)$$

$$X(k + N) = X(k) \quad \text{for all } k \qquad (5.2.5)$$

Linearity if

$$x_1(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X_1(k) \quad \text{and} \quad x_2(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X_2(k)$$

then for any real valued or complex valued constants a_1, and a_2,

$$a_1x_1(n) + a_2x_2(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} a_1X_1(k) + a_2X_2(k) \qquad (5.2.6)$$

5.2.1 Periodicity, Linearity, and symmetry properties.

Circular symmetries of a sequence

The circular shift of the sequence can be represented as the ***index modulo***. This we can write

$$x'(n) = x(\,n - k, \text{ modulo } N)$$

$$\equiv x((\,n - k\,))_N$$

For example, if $k = 2$ and $N = 4$, we have

$$x'(n) = x((\,n - 2\,))_4$$

$$x'(0) = x((-2\,))_4 = x(2)$$

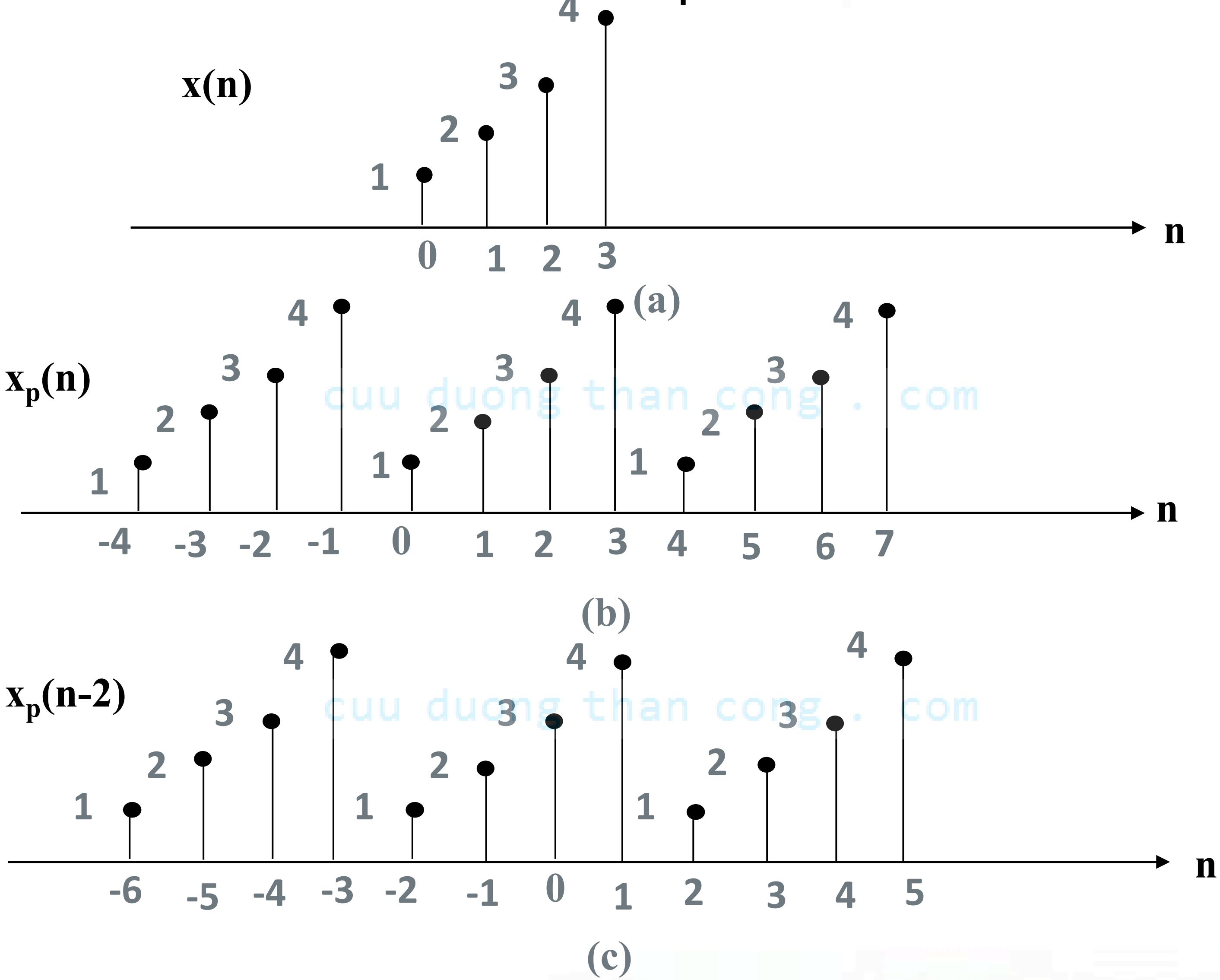
$$x'(1) = x((-1\,))_4 = x(3)$$

$$x'(2) = x(\,0\,))_4 = x(0)$$

$$x'(3) = x((\,1\,))_4 = x(1)$$

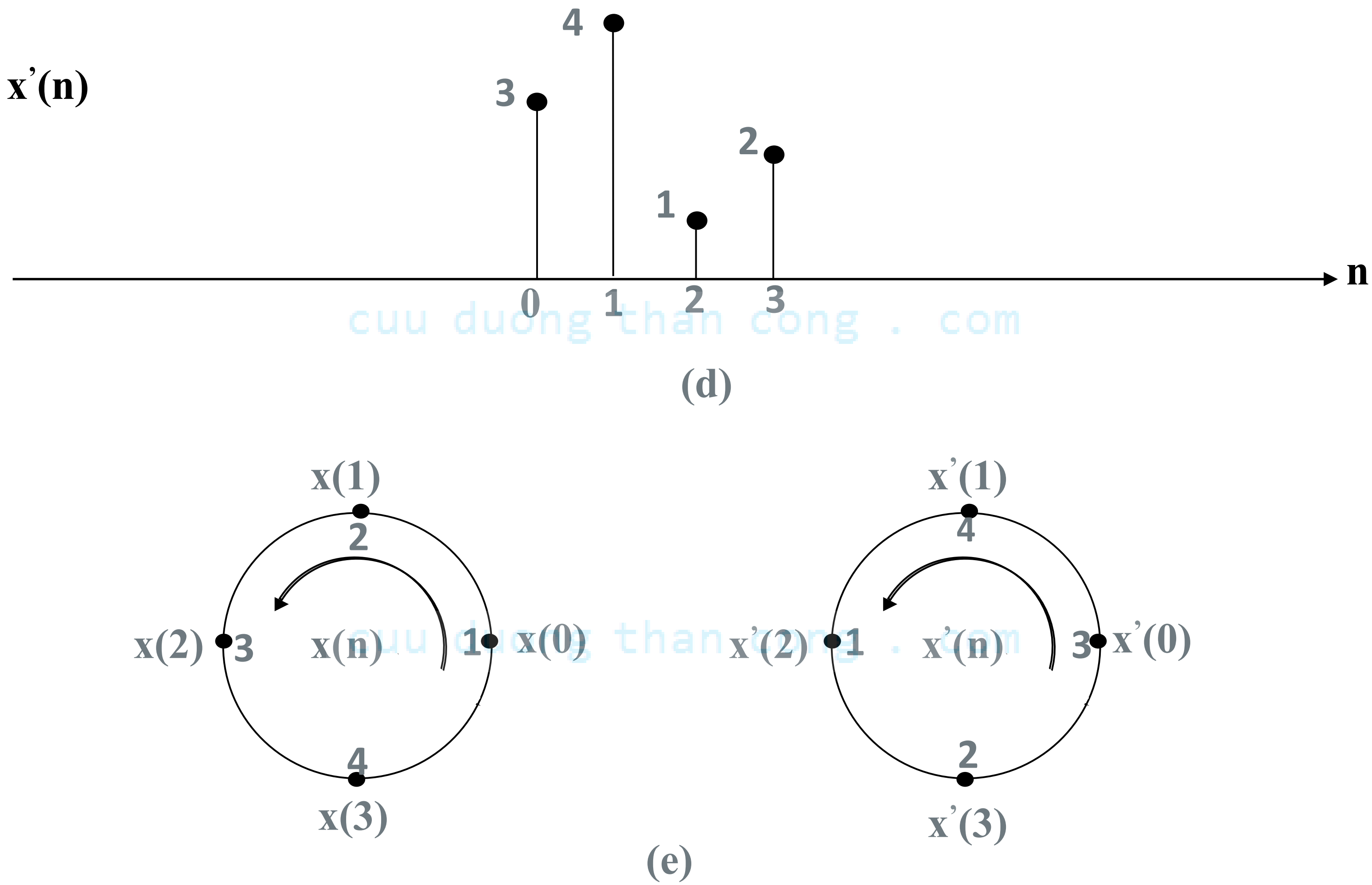
5.2.1 Periodicity, Linearity, and symmetry properties

Figure 5.7 Circular shift of a sequence.



5.2.1 Periodicity, Linearity, and symmetry properties

Figure 5.7 Circular shift of a sequence.



5.2.1 Periodicity, Linearity, and symmetry properties

An N-point sequence is called ***circularly even*** if it is symmetric about the point zero on the circle

$$x(N-n) = x(n) \quad 1 \le n \le N-1 \qquad (5.2.11)$$

An N – point sequence is called ***circularly odd*** if it is antisymmetric about the point zero on the circle

$$x(N-n) = -x(n) \quad 1 \le n \le N-1 \qquad (5.2.12)$$

The ***time reversal*** of an N-point sequence is attained by *reversing its samples* about the point zero on the circle.

$$x((-n))_N = x(N-n) \quad 0 \le n \le N-1 \qquad (5.2.13)$$

5.2.1 Periodicity, Linearity, and symmetry properties

An equivalent definition :

even: $x_p(n) = x_p(-n) = x_p(N-n)$ (5.2.14)

odd: $x_p(n) = -x_p(-n) = -x_p(N-n)$

conjugate even : $x_p(n) = x_p^*(N-n)$ (5.2.15)

conjugate odd : $x_p(n) = -x_p^*(N-n)$

$$x_p(n) = x_{pe}(n) + x_{po}(n) \quad (5.2.16)$$

Where

$$x_{pe}(n) = \frac{1}{2}[x_p(n) + x_p^*(N-n)] \quad (5.2.17)$$

$$x_{po}(n) = \frac{1}{2}[x_p(n) - x_p^*(N-n)]$$

5.2.1 Periodicity, Linearity, and symmetry properties

Symmetric properties of the DFT

if $x(n) = x_R(n) + jx_I(n), \quad 0 \le n \le N-1$ (5.2.18)

$X(k) = X_R(k) + jX_I(k), \quad 0 \le k \le N-1$ (5.2.19)

then

$$X_R(k) = \sum_{n=0}^{N-1}\left[x_R(n)\cos\frac{2\pi kn}{N} + x_I(n)\sin\frac{2\pi kn}{N}\right] \quad (5.2.20)$$

$$X_I(k) = -\sum_{n=0}^{N-1}\left[x_R(n)\sin\frac{2\pi kn}{N} - x_I(n)\cos\frac{2\pi kn}{N}\right] \quad (5.2.21)$$

$$x_R(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\cos\frac{2\pi kn}{N} - X_I(k)\sin\frac{2\pi kn}{N}\right] \quad (5.2.22)$$

$$x_I(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\sin\frac{2\pi kn}{N} + X_I(k)\cos\frac{2\pi kn}{N}\right] \quad (5.2.23)$$

5.2.1 Periodicity, Linearity, and symmetry properties

Real – valued sequences

If $x(n)$ is real then:

$$X(N-k) = X^{*}(k) = X(-k) \qquad (5.2.24)$$

Consequently: $|X(N-k)| = |X(k)|$ and

$$\angle X(N-k) = -\angle X(k)$$

$x(n)$ can be determined from (5.2.22), which is ***another form*** for the ***IDFT***.

5.2.1: Periodicity, Linearity, and symmetry properties

Real and even sequences:

If *x(n)* is *real* and *even*, that is

$$x(n) = x(N-n) \qquad 0 \le n \le N-1$$

then

$$X(k) = \sum_{n=0}^{N-1} x(n) \cos\frac{2\pi kn}{N}, \qquad 0 \le k \le N-1 \qquad (5.2.25)$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) \cos\frac{2\pi kn}{N}, \qquad 0 \le n \le N-1 \qquad (5.2.26)$$

5.2.1 Periodicity, Linearity, and symmetry properties

Real and odd sequence

If *x(n)* is *real* and *odd*, that is

$$x(n) = -x(N-n) \qquad 0 \le n \le N-1$$

then

$$X(k) = -j\sum_{n=0}^{N-1} x(n)\sin\frac{2\pi kn}{N}, \quad 0 \le k \le N-1 \qquad (5.2.27)$$

$$x(n) = j\frac{1}{N}\sum_{k=0}^{N-1} X(k)\sin\frac{2\pi kn}{N}, \quad 0 \le n \le N-1 \qquad (5.2.28)$$

5.2.1 Periodicity, Linearity, and symmetry properties

Purely imaginary sequences.

if $x(n) = j\, x_I(n)$

then

$$X_R(k) = \sum_{n=0}^{N-1} x_I(n) \sin \frac{2\pi k n}{N} \qquad (5.2.29)$$

$$X_I(k) = \sum_{n=0}^{N-1} x_I(n) \cos \frac{2\pi k n}{N} \qquad (5.2.30)$$

$X_R(k)$ is *old* and $X_L(k)$ is *even*

5.2.1 Periodicity, Linearity, and symmetry properties

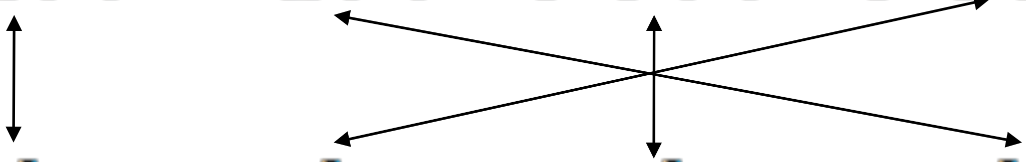
Table 5.1 Symmetry Properties of the DFT

N- Point Sequence $0 \leq n \leq N-1$	*N* – Point DFT
$x(n)$	$X(k)$
$x^*(n)$	$X^*(N-k)$
$x^*(N-n)$	$X^*(k)$
$x_R(n)$	$X_{ce}(k) = ½[X(k) + X^*(N-k)]$
$j x_I(n)$	$X_{c0}(k) = ½[X(k) - X^*(N-k)]$
$x_{ce}(n) = ½[x(n) + x^*(N-n)]$	$x_R(k)$
$x_{c0}(n) = ½[x(n) - x^*(N-n)]$	$j x_I(k)$
Real Signals	
Any real signal $x(n)$	$X(k) = X^*(N-k)$ $X_R(k) = X_R(N-k)$ $X_I(k) = -X_I(N-k)$ $\lvert X(k) \rvert = \lvert X(N-k) \rvert$ $\angle X(k) = -\angle X(N-k)$
$x_{ce}(n) = ½[x(n) + x(N-n)]$	$X_R(k)$
$x_{c0}(n) = ½[x(n) - x(N-n)]$	$j X_I(k)$

5.2.1 Periodicity, Linearity, and symmetry properties

The ***symmetry properties*** of the **DFT** are summarized as follows

$$x(n) = x_R^e(n) + x_R^o(n) + jx_I^e(n) + jx_I^o(n)$$



$$X(k) = X_R^e(k) + X_R^o(k) + jX_I^e(k) + jX_I^o(k)$$

(5.2.31)

5.2.2 Multiplication of Two DFTs and Circular Convolution

If we have two finite-duration sequences of length *N*, $x_1(n)$, $x_2(n)$. Their respective *N*-point DFTs are.

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi nk/N}, \qquad k = 0, 1, \ldots, N-1 \qquad (5.2.32)$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n) e^{-j2\pi nk/N}, \qquad k = 0, 1, \ldots, N-1 \qquad (5.2.33)$$

We have

$$X_3(k) = X_1(k) X_2(k), \qquad k = 0, 1, \ldots, N-1 \qquad (5.2.34)$$

5.2.2 Multiplication of Two DFTs and Circular Convolution

The IDFT of $\{X_3(k)\}$ is

$$x_3(m) = \frac{1}{N}\sum_{n=0}^{N-1} X_1(k)X_2(k)e^{j2\pi km/N} \qquad (5.2.35)$$

$$x_3(m) = \frac{1}{N}\sum_{n=0}^{N-1} x_1(n)\sum_{l=0}^{N-1} x_2(l)\left[\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N}\right] \qquad (5.2.36)$$

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & a = 1 \\ \dfrac{1-a^N}{1-a}, & a \neq 1 \end{cases} \qquad (5.2.37)$$

Where $a = e^{j2\pi(m-n-l)/N}$

5.2.2 Multiplication of Two DFTs and Circular Convolution

If $a = 1$ then

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N, & l = m - n + pN = ((m-n))_N, \ p \text{ an integer} \\ & \text{otherwise} \end{cases} \qquad (5.2.38)$$

$$x_3(m) = \frac{1}{N}\sum_{n=0}^{N-1} x_1(n)x_2((m-n))_N, \quad m = 0,1,\ldots N-1 \qquad (5.2.39)$$

The convolution sum in (5.2.39) involves the index $((m-n))_N$ and is called ***circular convolution.***

5.2.2 Multiplication of Two DFTs and Circular Convolution

Circular convolution.

if

$$x_1(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X_1(k)$$

and

$$x_2(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X_2(k)$$

then

$$x_1(n) \circledN x_2(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X_1(k)X_2(k) \qquad (5.2.41)$$

Where $x_1(n)$ Ⓝ $x_2(n)$ denotes the ***circular convolution*** of the sequence $x_1(n)$ and $x_2(n)$

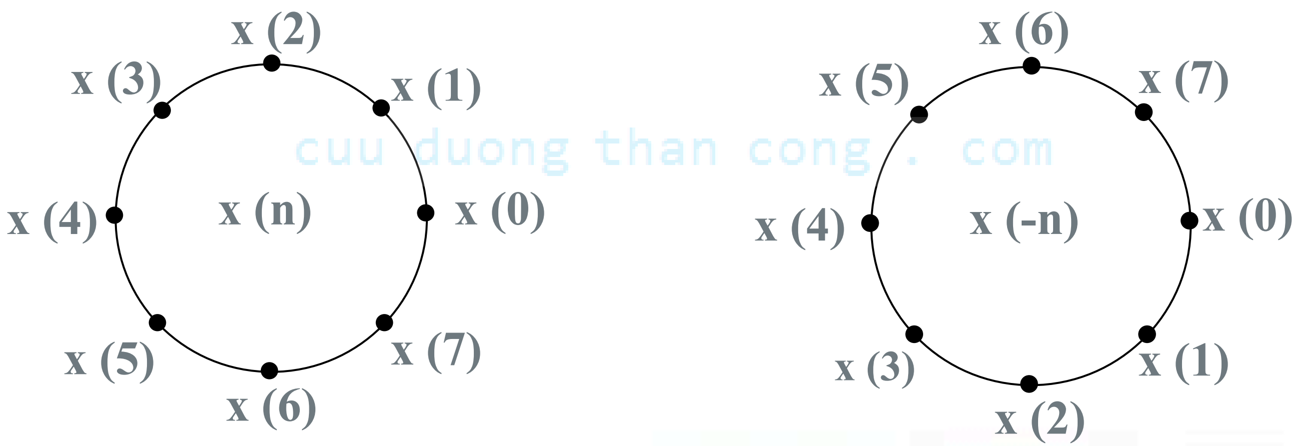
5.2.3 Additional DFT Properties.

Time reversal of a sequence

if $$x(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X(k)$$

then $$x((-n))_N = x(N-n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X((-k))_N = X(N-k) \qquad (5.2.42)$$

Figure 5.9 Time reversal of a sequence.



5.2.3 Additional DFT Properties.

Circular time shift of a sequence.

if

$$x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)$$

then

$$x((n-l))_N \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)e^{-j2\pi kl/N} \qquad (5.2.43)$$

Circular frequency shift

if

$$x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)$$

then

$$x(n)e^{j2\pi ln/N} \overset{DFT}{\underset{N}{\longleftrightarrow}} X((k-l))_N \qquad (5.2.44)$$

5.2.3 Additional DFT Properties.

Complex-conjugate properties.

If

$$x(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X(k)$$

then

$$x^*(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X^*((-k))_N = X^*(N-k) \qquad (5.2.45)$$

Circular correlation

If

$$x(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X(k) \quad and \quad y(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} Y(k)$$

then

$$\tilde{r}_{xy}(l) \underset{N}{\overset{DFT}{\longleftrightarrow}} X^*((-k))_N = X^*(N-k) \qquad (5.2.47)$$

5.2.3 Additional DFT Properties.

Where $\tilde{r}_{xy}(l)$ is the *circular crosscorrelation* sequence, defined as.

$$\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*((n-l))_N$$

Multiplication of two sequences.

if $x_1(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X_2(k) \quad and \quad x_2(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X_2(k)$

then

$$x_1(n)x_2(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} \frac{1}{N} X_1(k) \circledN X_2(k) \qquad (5.2.49)$$

5.2.3 Additional DFT Properties.

Parseval's Theorem

if $x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)$ and $y(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} Y(k)$

then

$$\sum_{n=0}^{N-1} x(n)y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k) \qquad (5.2.50)$$

5.2.3 Additional DFT Properties.

Table 5.2 Properties of the DFT

Property	Time Domain	Frequency Domain
Notation	$x(n), y(n)$	$X(k), Y(k)$
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(k) + a_2 X_2(k)$
Time revers	$x(N-n)$	$X(N-k)$
Circular time shift	$x((n-l))_N$	$X(k)e^{-j\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l))_N$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular convolution	$x_1(n)$ N $x_2(n)$	$X_1(k)X_2(k)$
Circular correlatio	$x(n)$ N $y^*(-n)$	$X(k)Y^*(k)$
Multiplication of two sequence	$x_1(n)\,x_2(n)$	$\frac{1}{N} X_1(k)$ N $X_2(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

5.3 Linear Filtering Methods Based on the DFT

Suppose that *x(n)*-a finite-duration sequence of length *L*, *h(n)* – the impulse response of the FIR filter of length *M*

$x(n) = 0 \qquad n < 0 \text{ and } n \geq L$

$h(n) = 0 \qquad n < 0 \text{ and } n \geq M$

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k) \qquad (5.3.1)$$

The duration of *y(n)* is *L + M – 1*. In the frequency domain

$$y(\omega) = X(\omega)\, H(\omega) \qquad (5.3.2)$$

5.3 Linear Filtering Methods Based on the DFT

A DFT of size $N \geq L + M - 1$ is

$$Y(k) = Y(\omega)|_{\omega = 2\pi k/N}, \qquad k = 0, 1, \ldots, N-1$$
$$= X(\omega) H(\omega)|_{\omega = 2\pi k/N}$$

Then

$$Y(k) = X(k) H(k) \qquad k = 0, 1, \ldots, N-1$$

Where $\{X(k)\}$ and $\{H(k)\}$ are the ***N-point DFTs*** of the corresponding sequences $x(n)$ and $h(n)$.

By the computation of the ***N-point IDFT***, must yield the sequences $\{y(n)\}$.

The ***aliasing*** that results in the time domain when the size of the DFTs is ***smaller*** than $L + M - 1$.

5.3.2 Filtering of Long Data Sequences

In practical applications, the input sequences *x(n)* is often a ***very long sequence***.

A long input signal sequence must be segmented to ***fixed-size blocks*** prior to processing.

Since the filtering is linear, *successive blocks* can be processed ***one at a time*** via the ***DFT*** and the output blocks are ***fitted together*** to form the ***overall output signal sequence.***

5.3.2 Filtering of Long Data Sequences

The two methods are called the ***overlap – save method*** and the ***overlap – add method***.

We assume that the *FIR filter* has ***duration M*** the input data sequence is segmented into ***blocks of L points,***

where, by assumption, ***L > > M*** without loss of generality.

5.3.2 Filtering of Long Data Sequences

Overlap – add method.

The size of the *input data blocks* is $N = L + M - 1$

The size of the *DFTs* and *IDFT* are of length N.

Each data block consists of the last $M - 1$ data point of the previous data block, followed by L new data points to form a data sequence of length

$$N = L + M - 1.$$

An N-point DFT is computed for each data block.

The impulse response of the FIR filter is increased in length by appending $L - 1$ zeros and N-points DFTs $\{H(k)\}$ and $\{X_m(k)\}$

5.3.2 Filtering of Long Data Sequences

for the ***mth*** block of data yields

$$\tilde{Y}_m(k) = H(k)X_m(k), \qquad k = 0, 1, \ldots, N-1 \qquad (5.3.7)$$

Then

$$\tilde{Y}_m(n) = \{\hat{y}_m(0)\hat{y}_m(1)\ldots\hat{y}_m(M-1)\hat{y}_m(M)\ldots\hat{y}_m(N-1)\} \qquad (5.3.8)$$

Since the data record is of length *N*, the first ***M – 1 points*** of $y_m(n)$ are corrupted by ***aliasing*** and must discarded. The ***last L points*** of $y_m(n)$ are ***exactly the same*** as the result form linear convolution, and as

$$\tilde{Y}_m(n) = y_m(n), \qquad n = M, M+1, \ldots, N-1 \qquad (5.3.9)$$

5.3.2 Filtering of Long Data Sequences

To ***avoid loss of data*** due to ***aliasing***, the ***last M – 1 points*** of each record are save and these points become the ***first M – 1 data points*** of the subsequent record.

To begin processing, the ***first M – 1 points*** of the ***first record*** are set to ***zero***.

5.3.2 Filtering of Long Data Sequences

$$x_1(n) = \{\underbrace{0, 0, \ldots, 0,}_{M-1\ point}\ x(0), x(1), \ldots, x(L-1)\} \qquad (5.3.10)$$

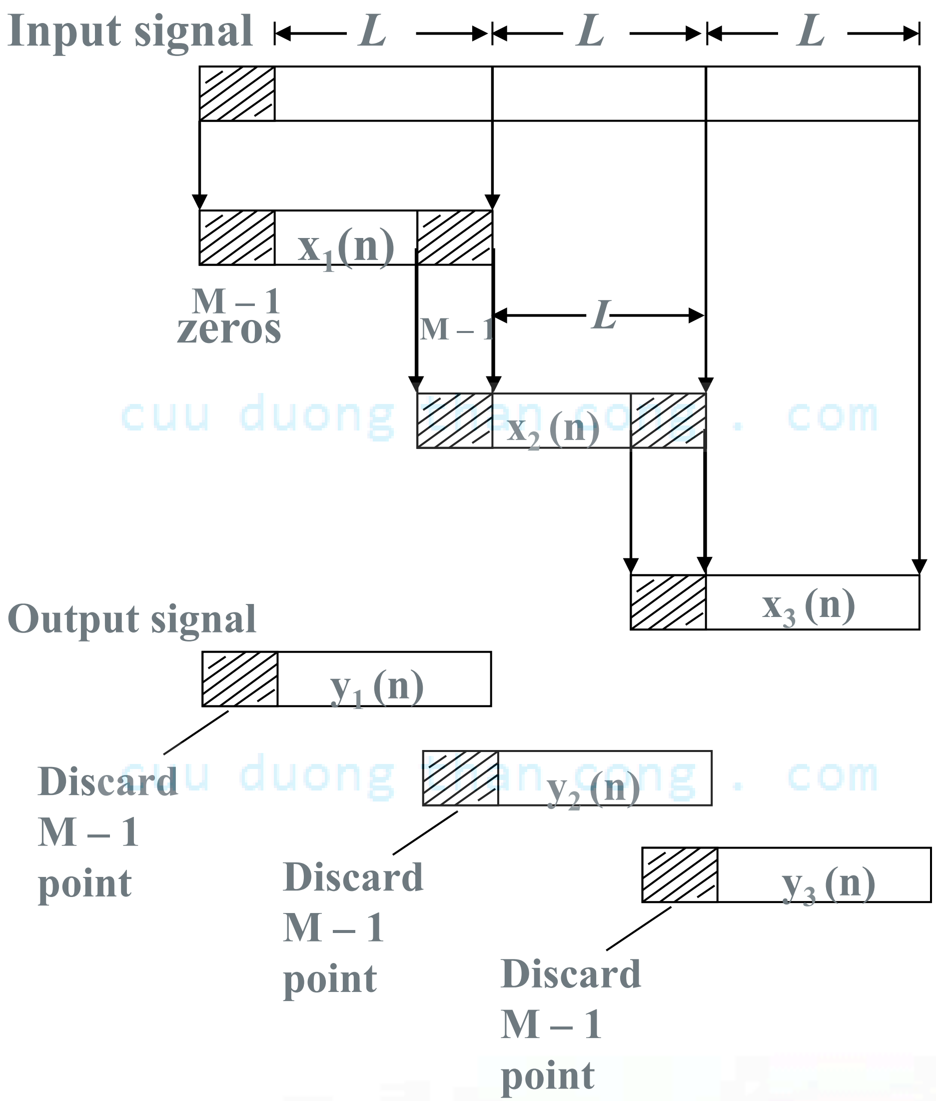
$$x_2(n) = \{\underbrace{x(L-M+1), \ldots, x(L-1),}_{M-1 data\ point\ form\ x_1(n)}\ \underbrace{x(L), \ldots, x(2L-1)}_{L\ new\ data\ points}\} \qquad (5.3.11)$$

$$x_3(n) = \{\underbrace{x(2L-M+1), \ldots, x(2L-1),}_{M-1 data\ point\ form\ x_2(n)}\ \underbrace{x(2L), \ldots, x(3L-1)}_{L\ new\ data\ points}\} \qquad (5.3.12)$$

and so forth.

5.3.2 Filtering of Long Data Sequences

Figure 5.10: Linear FIR filtering by the overlap- save method



5.3.2 Filtering of Long Data Sequences

Overlap-add method.

The size of the input data block is *L* points

The size of the DFTs and IDFT is $N = L + M - 1$

To each data block we append $M - 1$ zero and compute the N – point DFT.

$$x_1(n) = \{x(0), x(1), \dots, x(L-1), \underbrace{0, 0, \dots, 0}_{M-1\ zeros}\} \qquad (5.3.13)$$

$$x_2(n) = \{x(L), x(L+1), \dots, x(2L-1), \underbrace{0, 0, \dots, 0}_{M-1\ zeros}\} \qquad (5.3.14)$$

$$x_3(n) = \{x(2L), \dots, x(3L-1), \underbrace{0, 0, \dots, 0}_{M-1\ zeros}\} \qquad (5.3.15)$$

and so on

5.3.2 Filtering of Long Data Sequences

The two ***N*-point DFTs** are multiplied together to form

$$Y_m(k) = H(k)\,X_m(k), \quad k = 0, 1, \dots, N-1 \qquad (5.3.16)$$

The IDFT yields data blocks of length *N* that are ***free of aliasing***.

Since each data block is terminated with *M – 1* zeros, the ***last M – 1*** points form each output block must be ***overlapped and added*** to the first ***M – 1* points** of the succeeding.

5.3.2 Filtering of Long Data Sequences

The output sequence

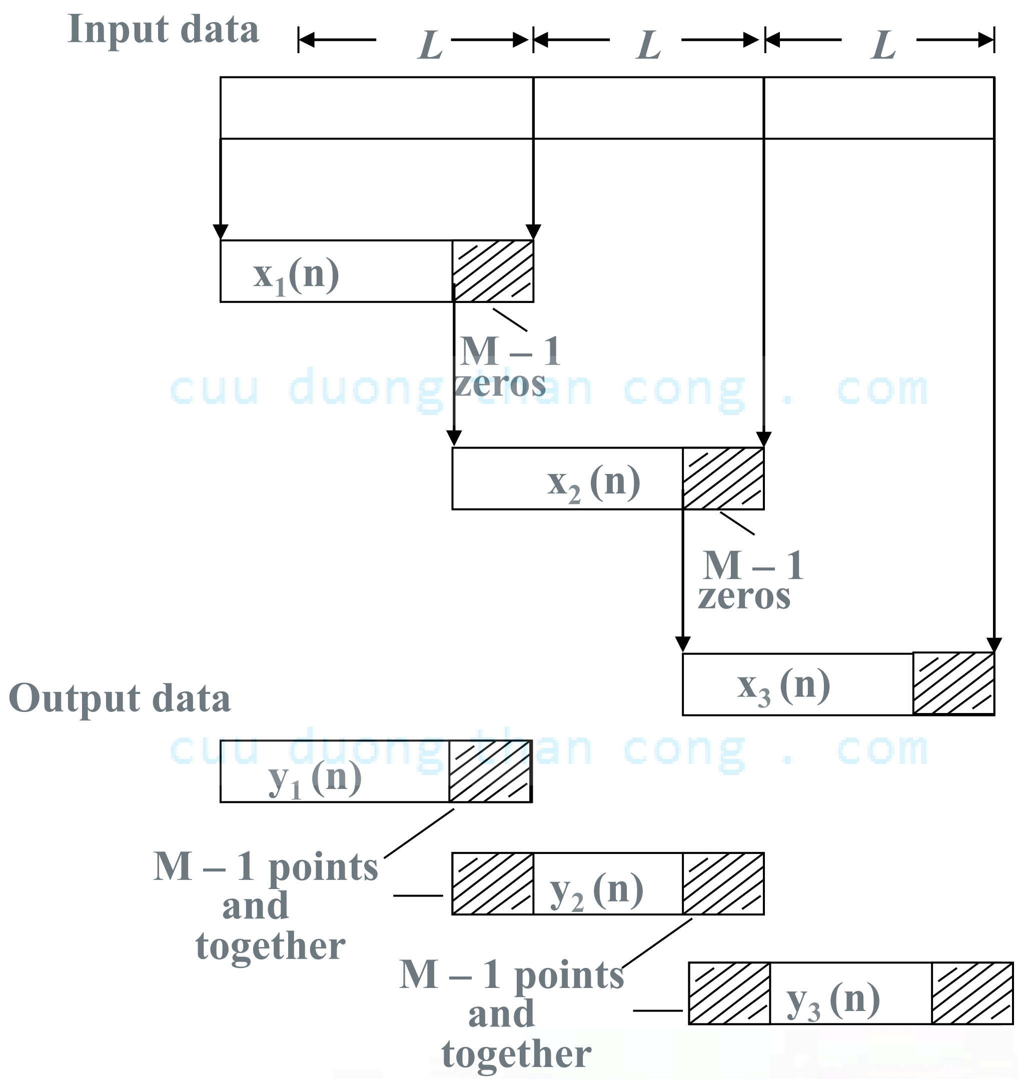
$$y(n) = \{\, y_1(0), y_1(1), \;\ldots\;, y_1(L\text{-}1), y_1(L) + y_2(0), y_1(L+1) + y_2(1), \ldots, y_1(N-1) + y_2(M-1), y_2(M), \ldots \} \quad (5.3.17)$$

The **segmentation** of the input data into ***blocks*** and the ***fitting*** of the output data blocks to form the ***output sequence*** are graphically illustrated in Fig. 5.11

It is an ***indirect method*** of computing the output of an FIR filter. It is ***more expensive*** computationally.

5.3.2 Filtering of Long Data Sequences.

Figure 5.11: Linear FIR filtering by the overlap-add method



5.4 Frequency Analysis of signals using the DFT.

If the signal to be analyzed is an ***analog signal***, we would first pass it through an *antialiasing filter*,

and then ***sample it*** at rate $F_S \geq 2B$, where B is the bandwidth of the filtered signal.

Finally, to limit the duration of the signal to the time interval $\boldsymbol{T_0 = LT}$, where $\boldsymbol{L}$ is the number of samples and $\boldsymbol{T}$ is the sample interval.

5.4 Frequency Analysis of signals using the DFT.

It limit our ability to ***distinguish*** two frequency components that are separated by less than $1/T_0 = 1/LT$ in frequency.

Now suppose that, $\{x(n)\}$ is limit the duration of the sequence to L samples :

$$x(n) = \cos\omega_0 n \qquad (5.4.3)$$

$$\hat{x}(n) = x(n)w(n) \qquad (5.4.1)$$

where

$$w(n) = \begin{cases} 1, & 0 \le n \le L-1 \\ 0, & \text{otherwise} \end{cases} \qquad (5.4.2)$$

$w(n)$ - a rectangular window.

5.4 Frequency Analysis of signals using the DFT.

then

$$\tilde{X}(\omega) = \frac{1}{2}\left[W(\omega - \omega_0) + W(\omega + \omega_0)\right] \qquad (5.4.4)$$

Where

$$W(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2} \qquad (5.4.5)$$

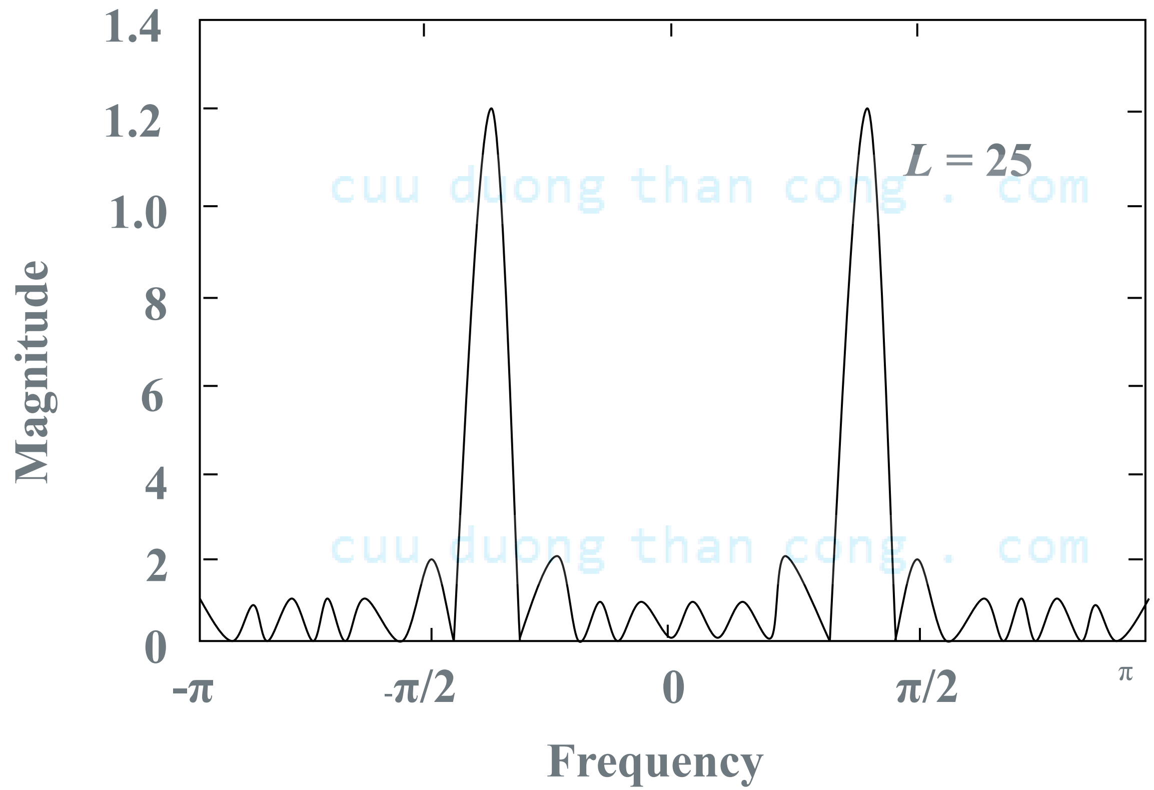
The magnitude spectrum

$$|\tilde{X}(k)| = |\tilde{W}(\omega_k)| \quad \text{for } \omega_k = 2\pi k/N$$

k = 0, 1,…N, is illustrated in Fig 5.12

5.4 Frequency Analysis of signals using the DFT.

Figure 5.12 Magnitude spectrum for L = 25 and N = 2048, illustrating the occurrence of leakage



5.4 Frequency Analysis of signals using the DFT.

Windowing not only ***distorts the spectral estimate*** due to the leakage effects, it also ***reduces spectral resolution***

For example : $x(n) = \cos\omega_1 n + \cos\omega_2 n$ (5.4.6)

when truncated to L samples :

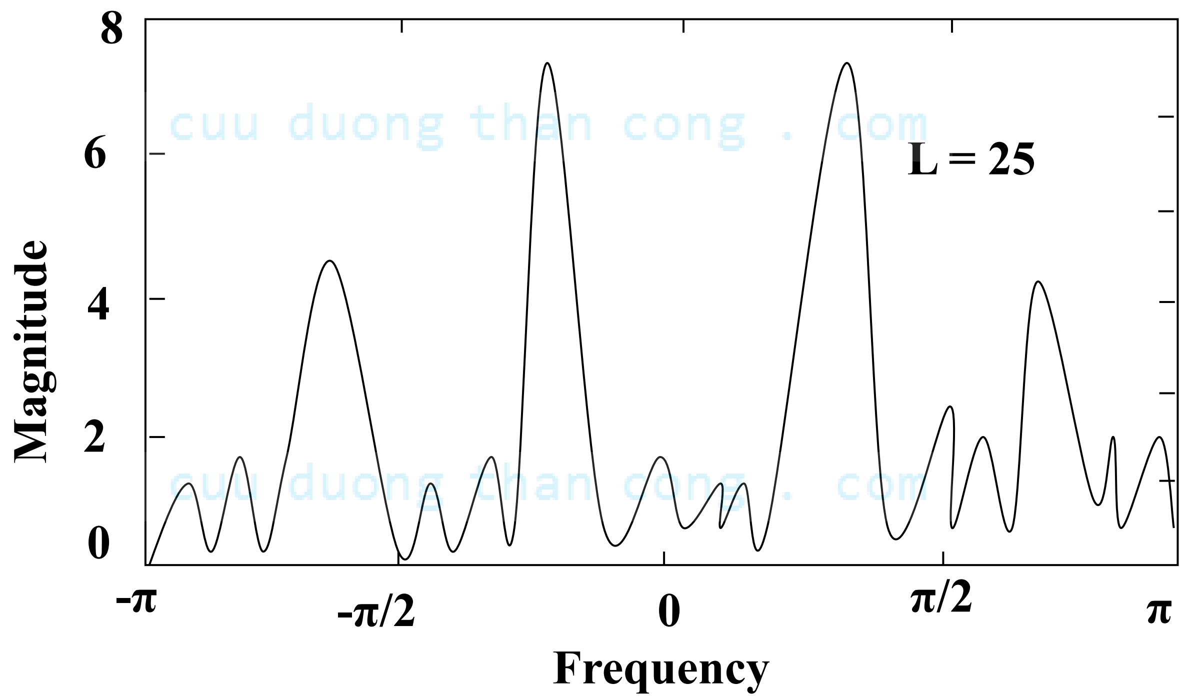
$$\tilde{X}(\omega) = \frac{1}{2}[W(\omega-\omega_1)+W(\omega-\omega_2)+W(\omega+\omega_1)+W(\omega+\omega_2)] \quad (5.4.7)$$

If $|\omega_1 - \omega_2| < 2\pi/L$, the two window function, $W(\omega - \omega_1)$ and $W(\omega - \omega_2)$ **overlap**, thus the two spectral lines in $x(n)$ are not distinguishable.

Suppose that $x(n) = \cos\omega_0 n + \cos\omega_1 n + \cos\omega_2 n$ (5.4.8)

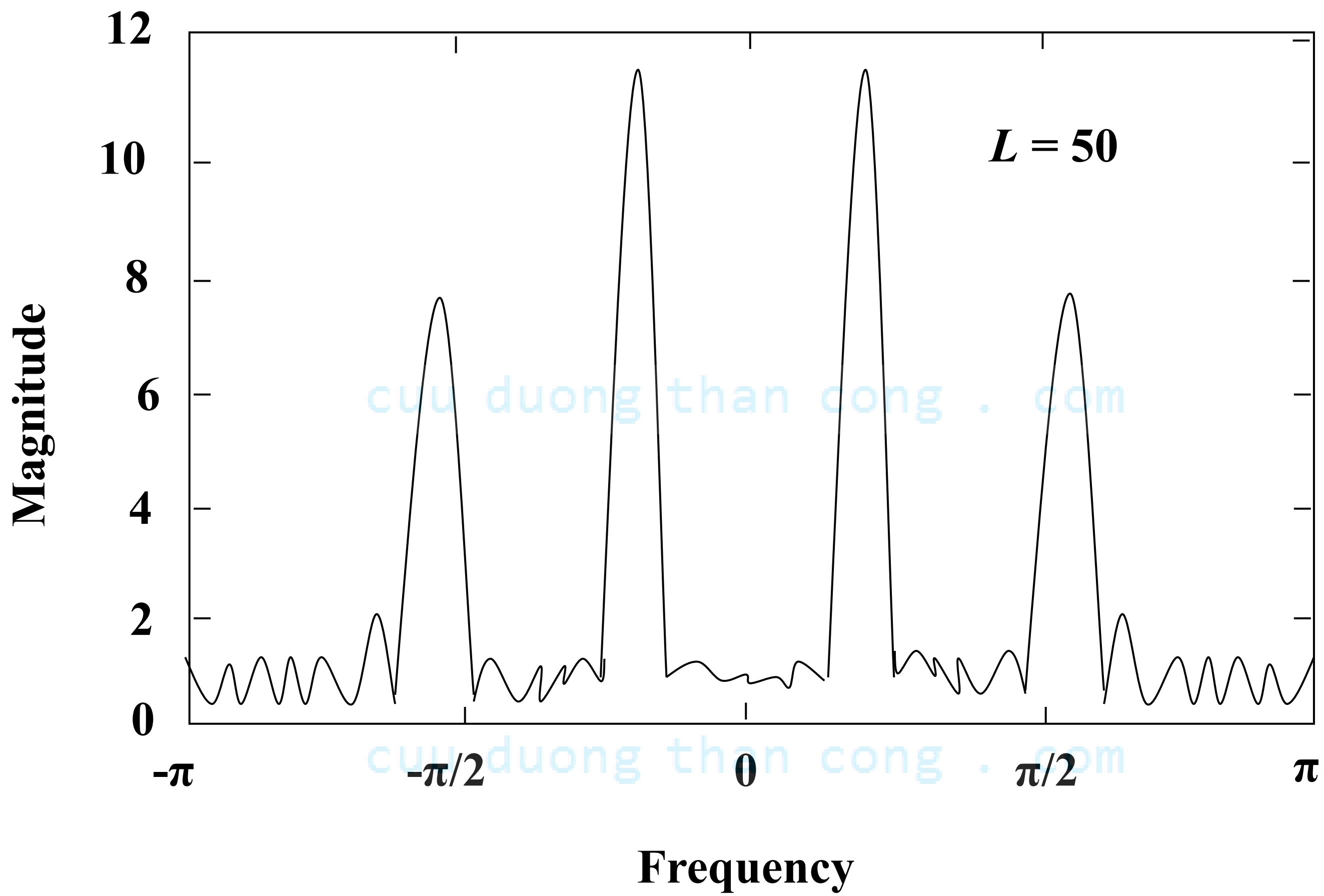
5.4 Frequency Analysis of signals using the DFT.

Fig, 5.13 illustrates the magnitude spectrum $|X(\omega)|$, given by (5.4.8) as observed through a ***rectangular window***, where $\omega_0 = 0.2\pi$, $\omega_1 = 0.22\pi$, $\omega_2 = 0.6\pi$



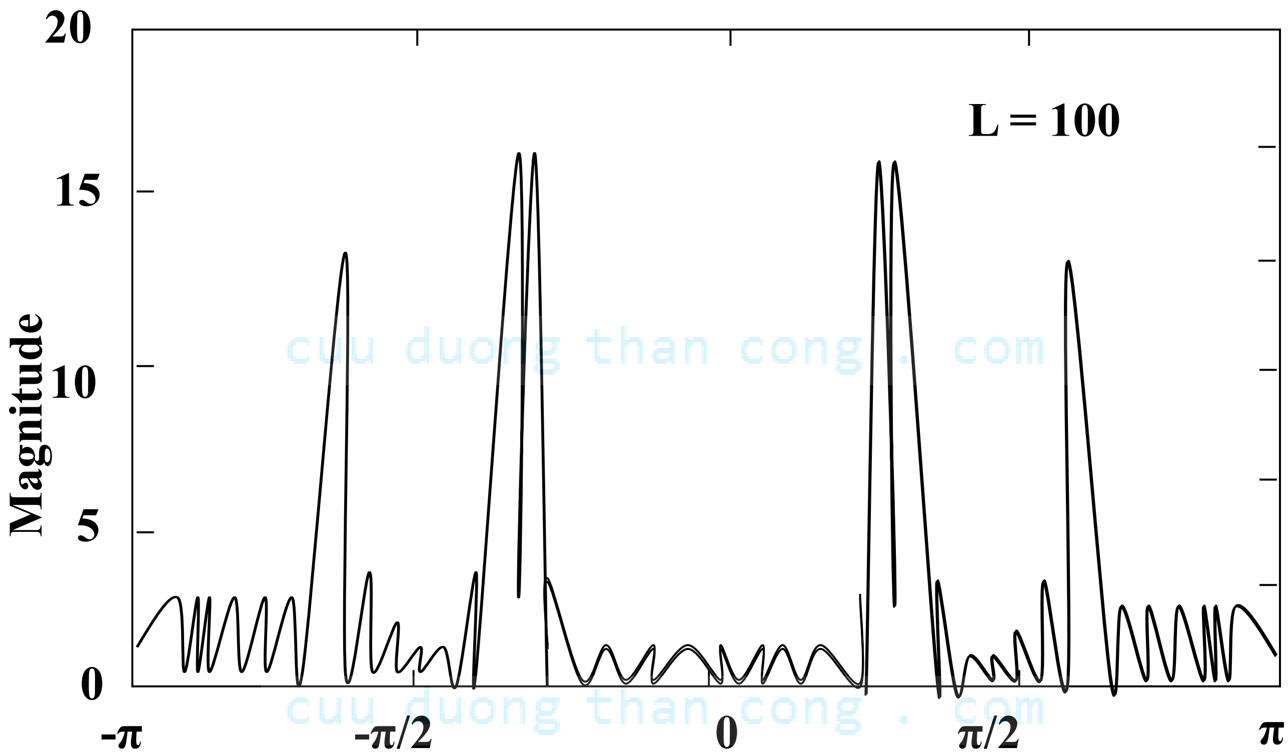
(a)

5.4 Frequency Analysis of signals using the DFT.



(b)

5.4 Frequency Analysis of signals using the DFT.



(c)

5.4 Frequency Analysis of signals using the DFT

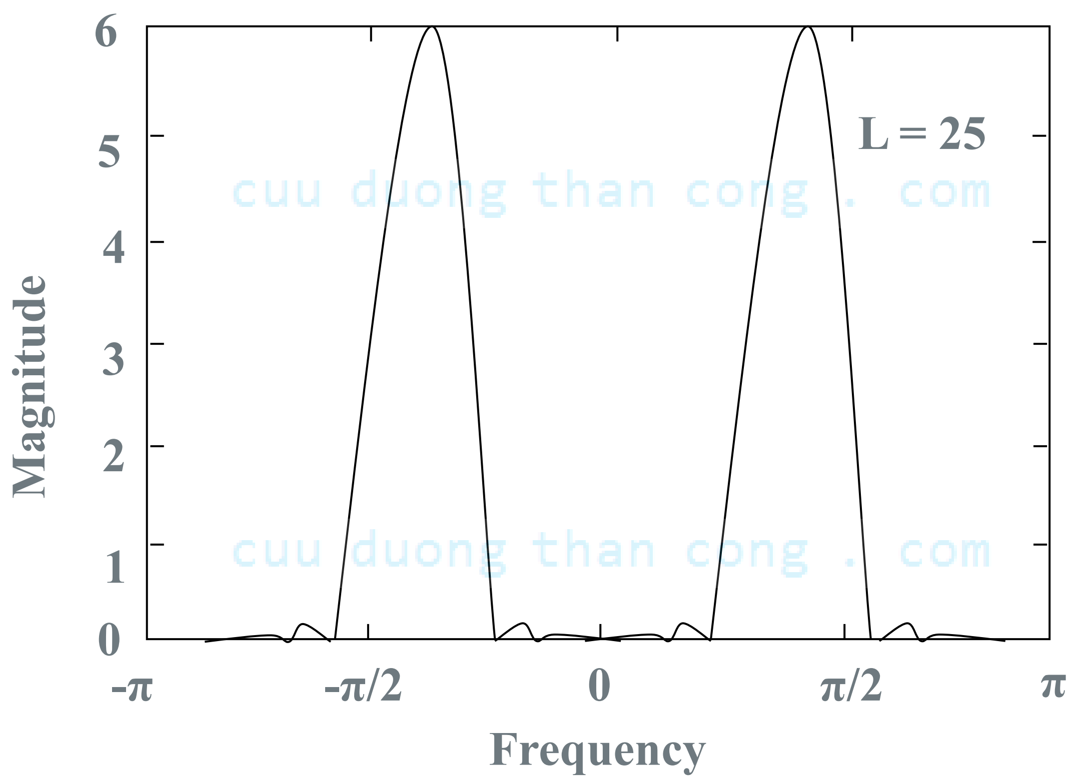
To ***reduce leakage***, we can select a data window *w(n)* that has ***lower sidelobes*** in the frequency domain compared with the rectangular window.

For example the *Hanning* window

$$w(n)=\begin{cases}\frac{1}{2}\left(1-\cos\frac{2\pi}{L-1}n\right), & 0\le n\le L-1 \\ 0, & \text{otherwise}\end{cases} \qquad (5.4.9)$$

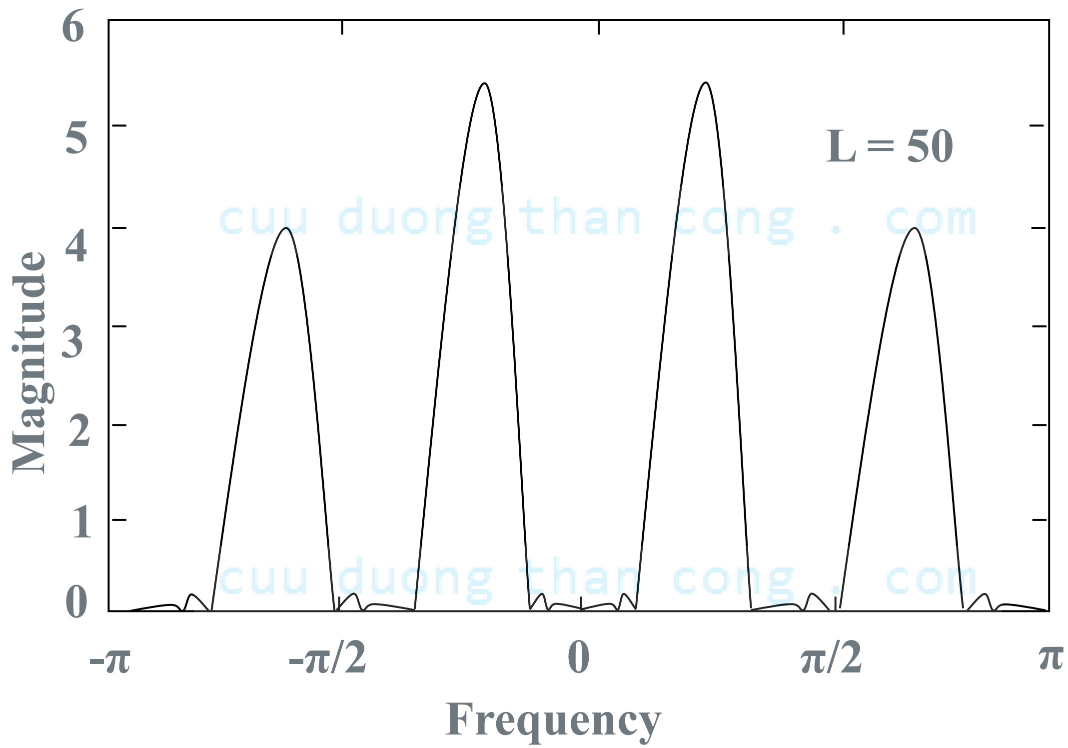
Fig. 5.14 shows $|\tilde{X}(\omega)|$ for the window of (5.4.9)

Figure 5.14 Magnitude spectrum of the Hanning window



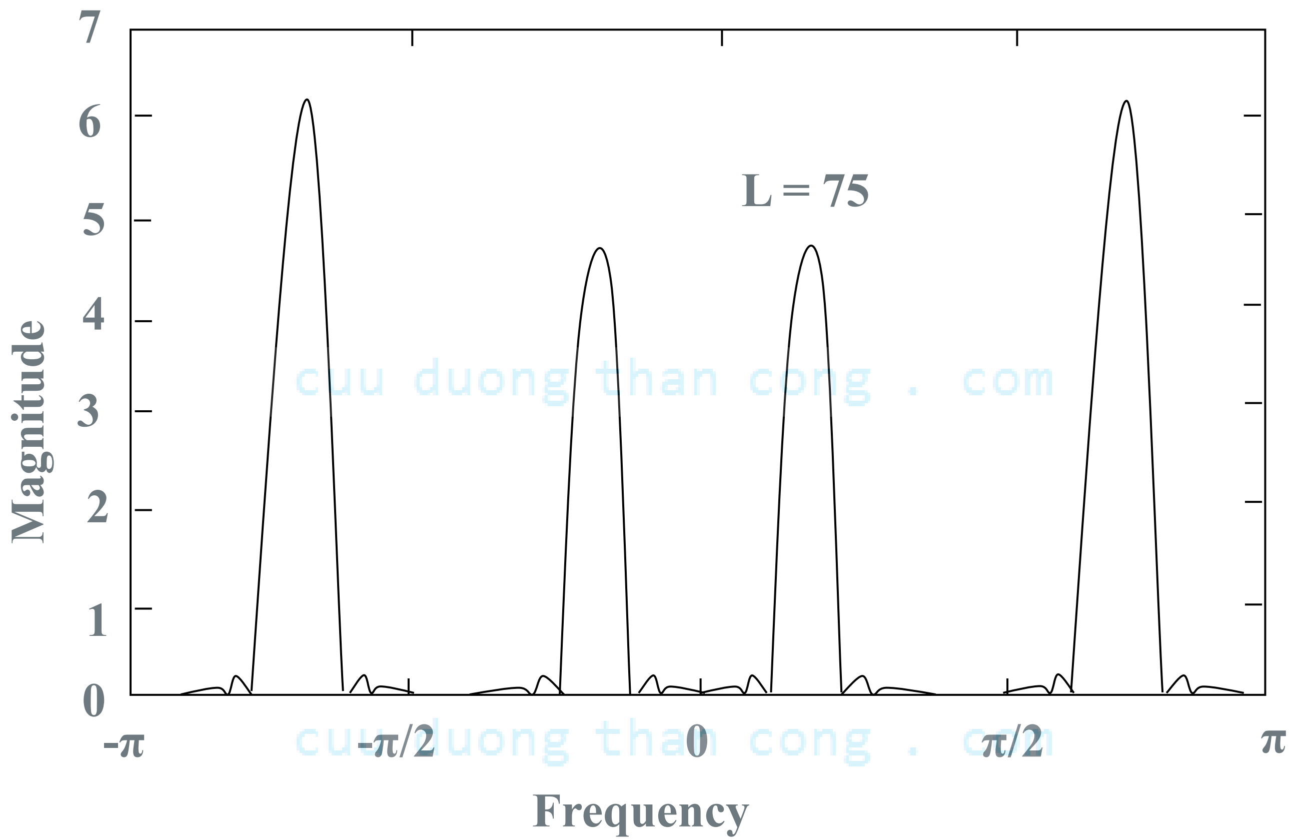
5.4 Frequency Analysis of signals using the DFT

Figure 5.15 Magnitude spectrum of the signal in (5.4.8) as observed through a Hanning window.



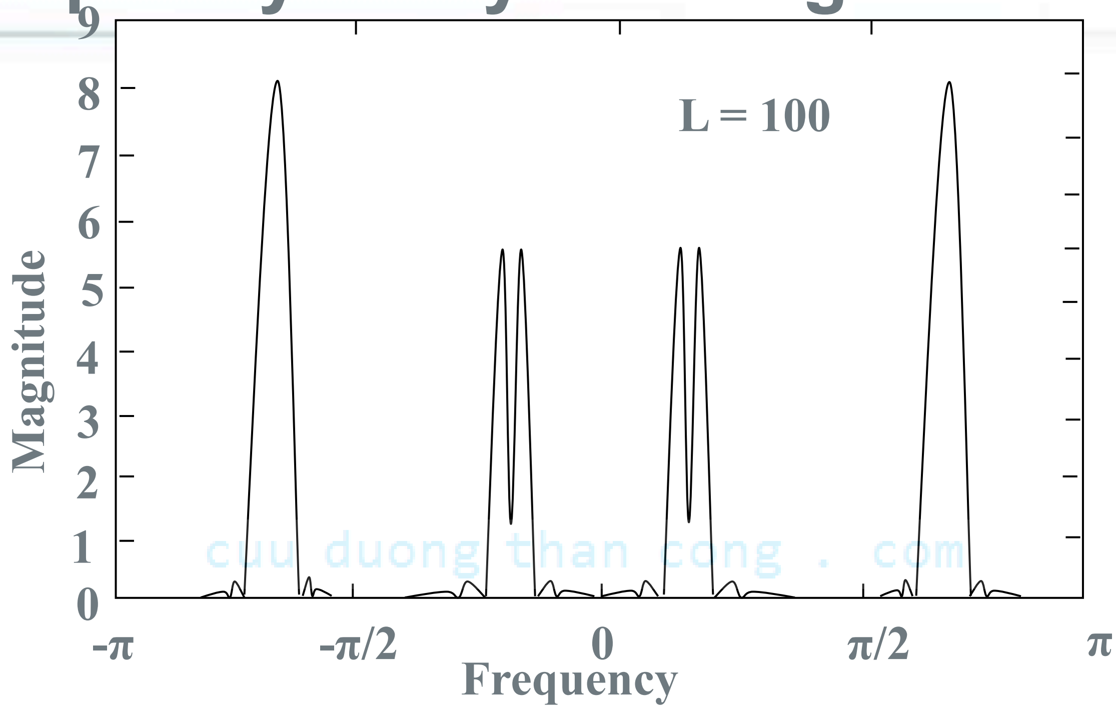
(a)

5.4 Frequency Analysis of signals using the DFT



(b)

5.4 Frequency Analysis of signals using the DFT



(c)

However, a ***reduction*** of the *sidelobes* in a window *w(ω)* is obtained at the expense of an ***increase in the width*** of the main lobe of *W(ω)* and hence a ***loss in resolution.***

Problems: 5.2, 5.4, 5.6, 5.7, 5.20, 5.23, 5.25.