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EXAMPLE 1.3.1

Stored in the memory of a digital signal processor is one cycle of the sinusoidal signal

$$x(n) = \sin\left(\frac{2\pi n}{N} + \theta\right)$$

where $\theta = 2\pi q / N$, where q and N are integers.

(a) Determine how this table of values can be used to obtain values of harmonically related sinusoids having the same phase.

(b) Determine how this table can be used to obtain sinusoids of the same frequency but different phase.

EXAMPLE 1.3.1 : **Solution**

(a) Let $x_k(n)$ denote the sinusoidal signal sequence

$$x_k(n) = sin\left(\frac{2\pi nk}{N} + \theta\right)$$

This is a sinusoid with frequency $f_k = k / N$, which is harmonically related *to* $x(n)$. But $x_k(n)$ may be expressed as

$$x_k(n) = sin\left[\frac{2\pi(kn)}{N} + \theta\right] = x(kn)$$

Thus we observe that $x_k(0) = x(0)$, $x_k(1) = x(k)$, $x_k(2) = x(2k)$, and so on.

EXAMPLE 1.3.1 : Solution

Hence the sinusoidal sequence $x_k(n)$ can be obtained from the table of values of $x(n)$ by taking ***every kth value of x(n), beginning with x(0)***.

In this manner we can generate the values of all harmonically related sinusoids with frequencies $f_k = k / N$ for $k = 0, 1, \dots, N - 1$.

EXAMPLE 1.3.1 : Solution

(b) We can control the phase θ of the sinusoid with frequency $f_k = k / N$ by taking the first value of the sequence from ***memory location*** $\boldsymbol{q = \theta N/ 2\pi}$, where q is an integer.

Thus the initial phase θ controls the starting location in the table and we wrap around the table each time the index (kn) exceeds N.

EXAMPLE 1.4.1

The implications of these frequency relations can be fully appreciated by considering the two analog sinusoidal signals.

$$x_1(t) = \cos 2\pi(10)t \qquad (1.4.12)$$

$$x_2(t) = \cos 2\pi(50)t$$

which are sampled at a rate $F_s = 40Hz$. The corresponding discrete-time signal or sequences are

$$x_1 = \cos 2\pi\left(\frac{10}{40}\right)n = \cos\frac{\pi}{2}n \qquad (1.4.13)$$

$$x_2 = \cos 2\pi\left(\frac{50}{40}\right)n = \cos\frac{5\pi}{2}n$$

EXAMPLE 1.4.1

However, $cos5\pi n/2 = cos(2\pi n + \pi n/2)$.

Hence $\boldsymbol{x_2(n) = x_1(n)}$.

Thus the sinusoidal signals are ***identical*** and, consequently, ***indistinguishable***.

If we are given the sampled values generated by $cos(\pi/2)n$, there is ***some ambiguity*** as to whether these sampled values correspond to $x_1(t)$ or $x_2(t)$.

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EXAMPLE 1.4.1

Since $x_2(t)$ yields exactly the same values as $x_2(t)$ when the two are sampled at $F_s = 40$ samples per second,

we say that the frequency $F_2 = 50Hz$ is an ***alias*** of the frequency $F_1 = 10Hz$ at the sampling rate of 40 samples per second.

It is important to note that F_2 is ***not the only alias*** of F_1.

EXAMPLE 1.4.1

In fact at the sampling rate of 40 samples per second, the frequency $\boldsymbol{F_3 = 90Hz}$ is also an ***alias of*** $\boldsymbol{F_1}$, as is the frequency $F_4 = 130\ Hz$, and so on.

All of the sinusoids $cos2\pi(F_1 + 40\ k)t$, $k = 1, 2, 3, 4, \ldots$, sampled at 40 samples per second, yield identical values. Consequently, they are ***all aliases*** of $F_1 = 10Hz.$

EXAMPLE 1.4.2

Consider the analog signal $x_a(t) = 3\cos 100\pi t$

(a) Determine the minimum sampling rate required to ***avoid aliasing***.

(b) Suppose that the signal is sampled at the rate $F_s = 200\ Hz$. What is the discrete-time signal obtained after sampling?

(c) Suppose that the signal is sampled at the rate $F_s = 75\ Hz$. What is the discrete-time signal obtained after sampling?

(d) What is the frequency $0 < F < F_s / 2$ of a sinusoid that yields samples identical to those obtained in part (c)?

EXAMPLE 1.4.2 **Solution**

(a) The frequency of the analog signal is *F = 50 Hz*. Hence the minimum sampling rate required to avoid aliasing is $F_s = 100$ *Hz*.

(b) If the signal is sampled at $F_s = 200$ *Hz*, the discrete-time signal is

$$x(n) = 3\cos\frac{100\pi}{200}n = 3\cos\frac{\pi}{2}n$$

EXAMPLE 1.4.2 **Solution**

(c) If the signal is sampled at F_s = *75 Hz*, the discrete-time signal is

$$x(n) = 3\cos\frac{100\pi}{75}n = 3\cos\frac{4\pi}{3}n$$

$$= 3\cos\left(2\pi - \frac{2\pi}{3}\right)n$$

$$= 3\cos\frac{2\pi}{3}n$$

EXAMPLE 1.4.2 **Solution**

(d) For the sampling rate F_s = *75 Hz*, we have

$$F = f\,F_s = 75\,f$$

The frequency of the sinusoid in part (c) is $f = ⅓$.
Hence F = 25 Hz.

Clearly, the sinusoidal signal

$$y_a\,(t) = 3\cos 2\pi\,Ft \;= 3\cos 50\pi t$$

Sampled at F_s = 75 samples/s yields *identical samples*.
Hence F = *50 Hz* is ***an alias*** of F = *25 Hz* for the sampling rate F_s = *75 Hz*.

EXAMPLE 1.4.3

Consider the analog signal

$$x_a(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t$$

What is the ***Nyquist rate*** for this signal ?

EXAMPLE 1.4.3 **Solution**

The frequencies present in the signal above are

$$F_1 = 25\ Hz \quad F_2 = 150\ Hz \quad F_3 = 50\ Hz$$

Thus $F_{max} = 150Hz$ and according to (1.4.19)

$F_s > 2F_{max} = 300\ Hz$

The ***Nyquist rate*** is $F_N > 2F_{max}$,

Hence $F_N = 300\ Hz$

EXAMPLE 1.4.4

Consider the analog signal

$x_a(t) = 3\cos 2000\pi t + 5\sin 6000\pi t + 10\cos 12.000\pi t$

(**a**) What is the ***Nyquist rate*** for this signal?

(**b**) Assume now that we sample this signal using a sampling rate $F_s = 5000$ samples/s. What is the ***discrete-time signal*** obtained after sampling?

(**c**) What is the analog signal $y_a(t)$ that we can ***reconstruct*** from the samples if we use ideal interpolation?

EXAMPLE 1.4.4 **Solution**

(a) The frequencies existing in the analog signal are

$$\boldsymbol{F_1 = 1\ kHz, \qquad F_2 = 3\ kHz, \qquad F_3 = 6\ kHz}$$

Thus $F_{max} = 6\ kHz$, and according to the sampling theorem,

$$F_s > 2\,F_{max} = 12\ kHz$$

The ***Nyquist rate*** is

$$F_N = 12\ kHz$$

EXAMPLE 1.4.4 **Solution**

(b) Since we have chosen $F_s = 5\ kHz$, the *folding frequency* is

$$\frac{F_s}{2} = 2.5\ kHz$$

and this is the maximum frequency that can be represented uniquely by the sampled signal. By making use of (1.4.2) we obtain

$$x(n) = x_a(nT) = x_a\left(\frac{n}{F_s}\right)$$

$$x(n) = 13cos2\pi\left(\frac{1}{5}\right)n - 5sin2\pi\left(\frac{2}{5}\right)n$$

EXAMPLE 1.4.4 **Solution**

Indeed, since $F_s = 5\ kHz$, the folding frequency is $F_s/2 = 2.5$ kHz.

This is the maximum frequency that can be represented uniquely by the sampled signal.

From(1.4.17) we have $F_0 = F_k - kF_s$.

Thus F_0 can be obtained by subtracting from F_k an integer multiple of F_s such that

$$-F_s/2 \le F_0 \le F_s/2.$$

The frequency F_1 is less than $F_s/2$ and thus it is ***not affected*** by aliasing.

EXAMPLE 1.4.4 **Solution**

However, the other two frequencies are above the folding frequency and they will be changed by the ***aliasing effect***

$$F_2' = F_2 - F_s = -2kHz$$

$$F_3' = F_3 - F_s = 1kHz$$

From (1.4.5) it follows that $f_1 = \frac{1}{5}, f_2 = -\frac{2}{5}$ and $f_3 = \frac{1}{5}$

which are in agreement with the result above.

EXAMPLE 1.4.4 **Solution**

(**c**) Since the frequency components at only *1 kHz* and *2 kHz* are present in the sampled signal,

the analog signal we can recover is

$y_a(t) = 13 \cos 2000\pi t - 5 \sin 4000\pi t$

which is obviously ***different*** from the original signal $x_a(t)$