EXAMPLE 2.2.1

Determine the response of the following systems to the input signal.

$$x(n) = \begin{cases} |n|, & -3 \leq n \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

(a) *y(n) = x(n)* (identity system)

(b) *y(n) = x(n - 1)* (unit delay system)

(b) *y(n) = x(n + 1)* (unit advance system)

(d) *y(n) = ⅓ [x(n +1) + x(n) + x(n − 1)]* (moving average filter)

(c) *y(n) = median {x(n + 1), x(n), x(n − 1)}* (median filter)

(f)

$$y(n) = \sum_{k=-\infty}^{n} x(k) = x(n) + x(n-1) + x(n-2) + \cdots \text{ (accumulator)} \qquad (2.2.3)$$

EXAMPLE 2.2.1 Solution

First, we determine explicitly the sample values of the input signal

$$x(n) = \{ \ldots, 0, 3, 2, 1, 0, 1, 2, 3, 0, \ldots\}$$

$\uparrow$

Next, we determine the output of each system using its input-output relationship.

(a) In this case the output is exactly the same as the input signal. Such a system is known as the *identity* system.

EXAMPLE 2.2.1 Solution

(b) This system simply delays the input by one

$$y(n) = \{ \ldots, 0, 3, 2, 1, 0, 1, 2, 3, 0, \ldots\}$$

$$\uparrow$$

(c) In the case the system " advances" the input one sample into the future.

$$y(n) = \{ \ldots, 0, 3, 2, 1, 0, 1, 2, 3, 0, \ldots\}$$

$$\uparrow$$

d) The output of this system at any time is

$$y(n) = \left\{\ldots, 0, 1, \frac{5}{3}, 2, 1, \frac{2}{3}, 1, 2, \frac{5}{3}, 1, 0, \ldots\right\}$$

$$\uparrow$$

EXAMPLE 2.2.1 Solution

(e) This system selects as its output at time n the median value of the three input samples $x(n-1)$, $x(n)$, and $x(n+1)$

$$y(n) = \{\ 0,\ 3,\ 3,\ 3,\ 2,\ 1,\ 2,\ 3,\ 3,\ 3,\ 0,\ \dots\}$$

$\uparrow$

(f) This system is basically an *accumulator*

$$y(n) = \{\ \dots,\ 0,\ 3,\ 5,\ 6,\ 6,\ 7,\ 9,\ 12,\ 0,\ \dots\}$$

$\uparrow$

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EXAMPLE 2.2.4

Determine if the system shown in Fig. 2.2.8 are time invariant or time variant.

(a) This system is described by the input – output equations

$$y(n) = T[x(n)] = x(n) - x(n-1) \qquad (2.2.15)$$

Now if the input is delayed by k unit in time and applied to the system

$$y(n,k) = x(n-k) - x(n-k-1) \qquad (2.2.16)$$

If we delay $y(n)$ by k unit in time, we obtain

$$y(n-k) = x(n-k) - x(n-k-1) \qquad (2.2.17)$$

The system is *time invariant*

EXAMPLE 2.2.4

(b) The input-output equation for this system is

$$y(n) = T[x(n)] = nx(n) \qquad (2.2.18)$$

The response of this system to $x(n - k)$ is

$$y(n,k) = nx(n - k) \qquad (2.2.19)$$

Now if we delay $y(n)$ in (2.2.18) by k units in time, we obtain

$$y(n - k) = (n - k)x(n - k)$$

$$= nx(n - k) - kx(n - k) \qquad (2.2.20)$$

This system is ***time variant***, since $y(n,k) \neq y(n-k)$

EXAMPLE 2.2.4

(c) This system is described by the input-output relation

$$y(n) = T[x(n)] = x(-n) \qquad (2.2.21)$$

The response of this system to $x(n-k)$ is

$$y(n,k) = T[x(n-k)] = x(-n-k) \qquad (2.2.22)$$

If we delay the output $y(n)$, as given by (2.2.21), by k unit in time, the result will be

$$y(n-k) = x(-n+k) \qquad (2.2.23)$$

Since $y(n,k) \neq y(n-k)$, the system is *time variant*.

EXAMPLE 2.2.4

(d) The input-output equation for this system is

$$y(n) = x(n) \cos\omega_0 n \qquad (2.2.24)$$

The response of this system to $x(n - k)$ is

$$y(n,k) = x(n - k) \cos\omega_0 n \qquad (2.2.25)$$

If the expression in (2.2.24) is delayed by k unit and the result is compared to (2.2.25), it is evident that the system is *time variant*

EXAMPLE 2.2.5

Determine if the systems described by the following input-output equation are linear or nonlinear

(a) $y(n) = nx(n)$ **(b)** $y(n) = x(n^2)$

Solution.

(a) For two input sequence $x_1(n)$ and $x_2(n)$, the corresponding output are

$$y_1(n) = n\,x_1(n)$$
$$y_2(n) = n\,x_2(n) \qquad (2.2.31)$$

EXAMPLE 2.2.5

A linear combination of the two input sequences results in the output.

$$y_3(n) = T[a_1x_1(n) + a_2x_2(n)] = n[a_1x_1(n) + a_2x_2(n)]$$

$$= a_1 nx_1(n) + a_2 nx_2(n) \qquad (2.2.32)$$

A linear combination of the two outputs in (2.2.31) results in the output

$$a_1y_1(n) + a_2y_2(n) = a_1 nx_1(n) + a_2 nx_2(n) \quad (2.2.33)$$

Since the right-hand sides of (2.2.32) and 2.2.33) are identical, the system is *linear*.

EXAMPLE 2.2.5

(b) we find he response of the system to two separate input signals $x_1(n)$ and $x_2(n)$. The result is

$$y_1(n) = x_1(n^2) \qquad y_2(n) = x_2(n^2) \qquad (2.2.34)$$

The output of the system to a linear combination of $x_1(n)$ and $x_2(n)$ is

$$y_3(n) = T\,[a_1x_1(n) + a_2x_2(n)] = a_1x_1(n^2) + a_2x_2(n^2)]$$
$$(2.2.35)$$

A linear combination of the two outputs yields

$$a_1y_1(n) + a_2y_2(n) = a_1x_1(n^2) + a_2x_2(n^2) \qquad (2.2.36)$$

We conclude that the system is *linear*.

EXAMPLE 2.2.6

Determine if the systems described by the following input-output equations are causal or noncausal.

(a) $y(n) = x(n) - x(n-1)$ **(b)** $y(n) = \sum_{k=-\infty}^{n} x(k)$

(c) $y(n) = ax(n)$ **(d)** $y(n) = x(n) + 3x(n+4)$

(e) $y(n) = x(n^2)$ **(f)** $y(n) = x(2n)$ **(g)** $y(n) = x(-n)$

EXAMPLE 2.2.6

Solution. The systems described in parts **(a)**, **(b)**, and **(c)** are clearly *causal*, since the output depends only on the present and past inputs.

The system in parts **(d)**, **(e)**, and **(f)** are clearly *noncausal*, since the output depends on future values of the input. The system in **(g)** is also *noncausal*, as we note by selecting, for example, $n = -1$, which yields $y(-1) = x(1)$. Thus the output at $n = -1$ depends on the input at $n = 1$, which is two units of time into the future.

EXAMPLE 2.3.1

Consider the special case of a finite-duration sequence given as   *x(n) = { 2, 4, 0, 3 }*

↑

Resolve the sequence *x(n)* into a sum of weighted impulse sequences.

Solution.   Since the sequence *x(n)* is nonzero for the time instants *n = -1, 0, 2*, we need three impulses at delays *k = -1, 0, 2* Following (2.3.10) we find that

$$x(n) = 2\delta(n + 1) + 4\delta(n) + 3\delta(n - 2)$$

EXAMPLE 2.3.3

Determine the output *y(n)* of a relaxed linear time-invariant system with impulse response

$$h(n) = a^n u(n), \ |a| < 1$$

when the input is a unit step sequence, that is,

$$x(n) = u(n)$$

Solution

In this case both *h(n)* and *x(n)* are infinite-duration sequences. We use the form of the convolution formula given by (2.3.28) in which *x(k)* is folded.

EXAMPLE 2.3.3 Solution

The sequences *h(k)*, *x(k)*, and *x(- k)* are shown in Fig.2.24.

The product sequences $v_0(k)$, $v_1(k)$, and $v_2(k)$ corresponding to *x(- k)h(k)*, *x(1- k)h(k)*, and *x(2 - k)h(k)* are illustrated in Fig.2.24(c), (d), and (e), respectively.

Thus we obtain the outputs

$$y(0) = 1$$
$$y(1) = 1 + a$$
$$y(2) = 1 + a + a^2$$

EXAMPLE 2.3.3 Solution

Clearly, for *n > 0*, the output is
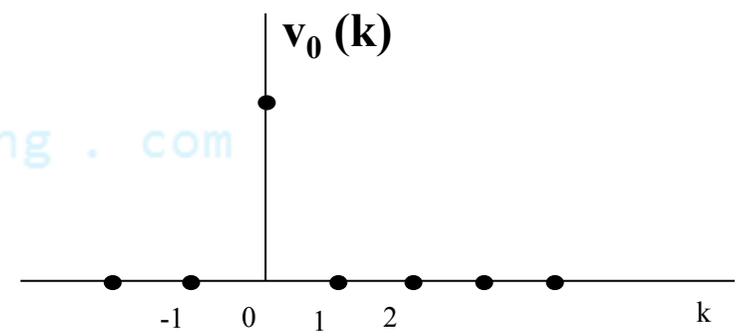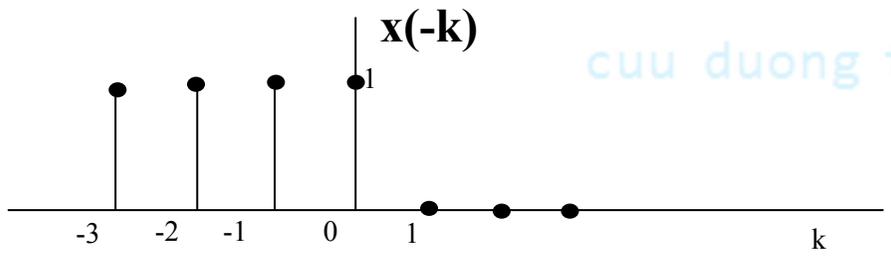
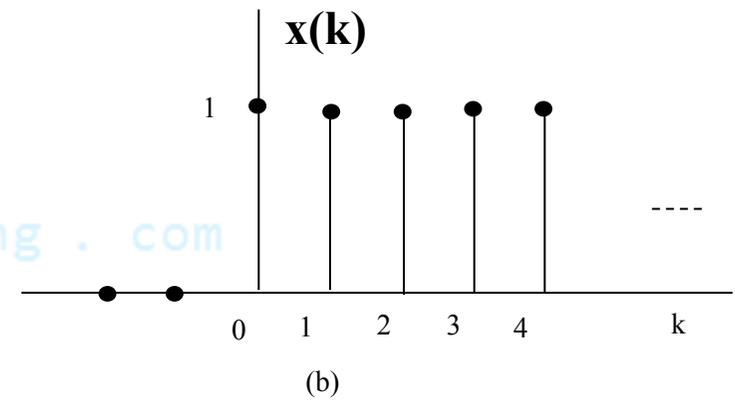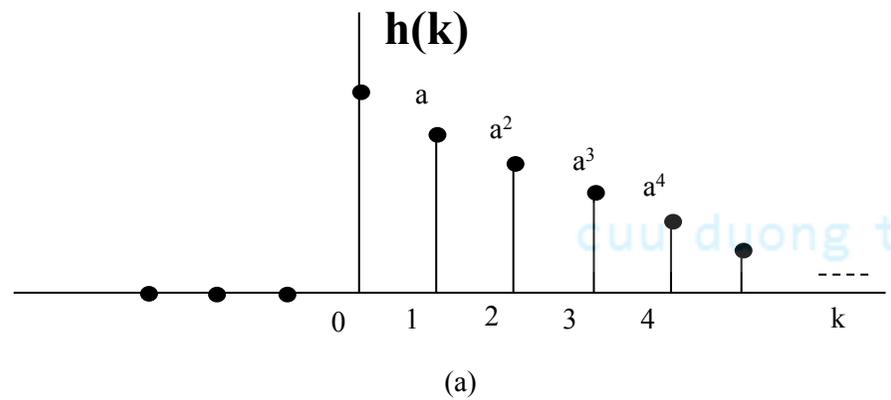$$y(n) = 1 + a + a^2 + \ldots + a^n = \frac{1 - a^{n+1}}{1 - a} \quad (2.3.29)$$

for *n < 0*, the product sequences consist of all zeros. Hence   *y(n) = 0,   n < 0*

A graph of the output *y(n)* is illustrated in Fig. 2.3.3(f) for the case *0 < a < 1*. Note the exponential rise in the output as a function of *n*. Since *|a| < 1*, the final value of the output as *n* approaches infinity is

$$y(\infty) = \lim_{n \to \infty} y(n) = \frac{1}{1 - a} \quad (2.3.30)$$
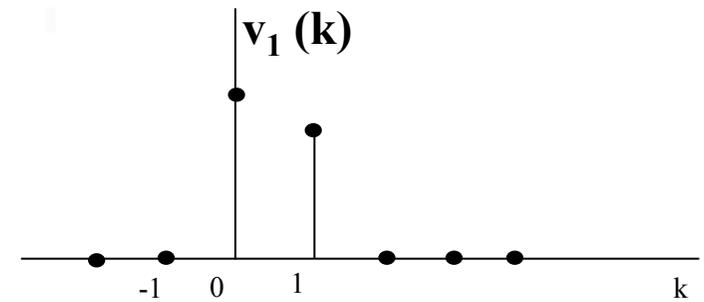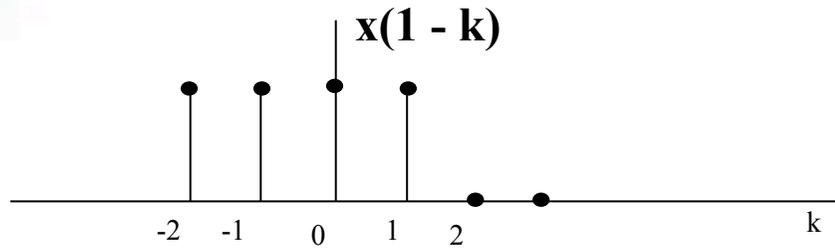
EXAMPLE 2.3.3 Solution

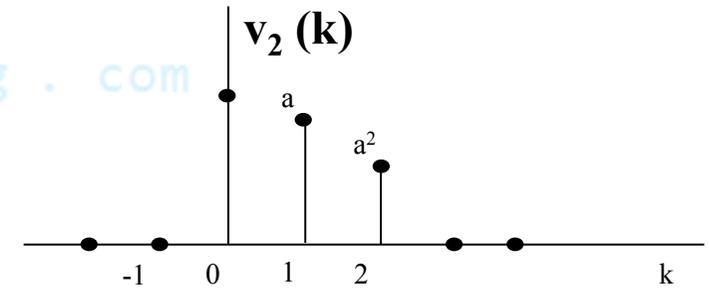Figure 2.24 Graphical computation of convolution in Example 2.3.3

(a)

(b)

(c)

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EXAMPLE 2.3.3 Solution

x(1 - k)

$v_1 (k)$

(d)

x(2 - k)

$v_2 (k)$

a

a^2

(e)

y(n)

asymptote

$a + 1 + a + a^2$
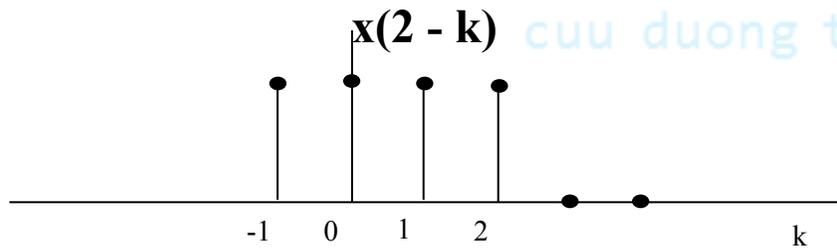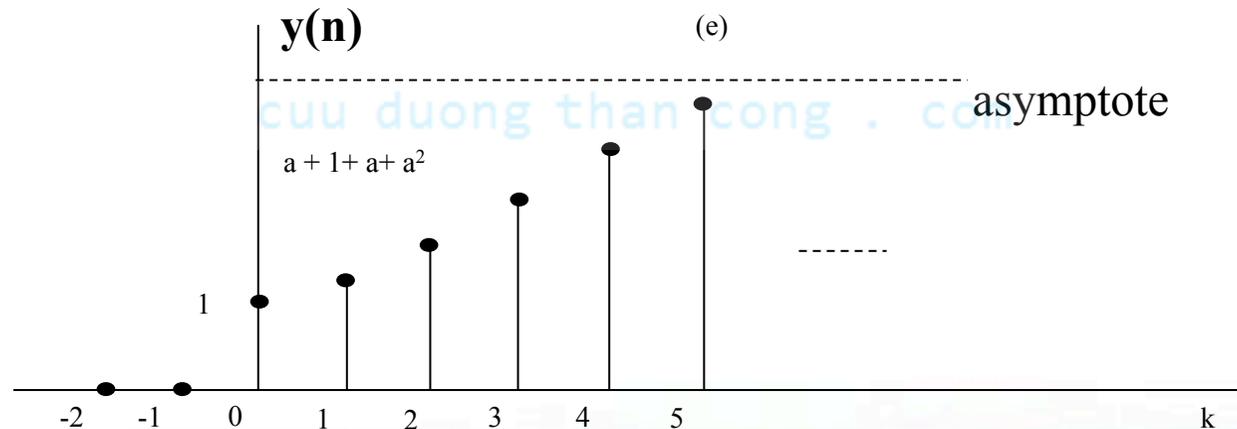
1

EXAMPLE 2.3.4

Determine the impulse response for the cascade of two linear time-invariant systems having impulse responses

$$h_1(n) = \left(\frac{1}{2}\right)^n u(n)$$

and

$$h_2(n) = \left(\frac{1}{4}\right)^n u(n)$$

Solution. To determine the overall impulse response of the two systems in cascade, we simply convolve $h_1(n)$ with $h_2(n)$.

EXAMPLE 2.3.4 Solution

Hence

$$h(n) = \sum_{k=-\infty}^{\infty} h_1(k)h_2(n-k)$$

where $h_2(n)$ is folded and shifted. We define the product sequence

$$v_n = h_1(k)h_2(n-k) = (\frac{1}{2})^k (\frac{1}{4})^{n-k}$$

which is nonzero for $k \geq 0$ and $n - k \geq 0$ or $n \geq k \geq 0$. On the other hand, for $n < 0$, we have $v_n(k) = 0$ for all k, and hence

$$h(n) = 0, \ n < 0$$

EXAMPLE 2.3.4 Solution

For $n \geq k \geq 0$, the sum of the values of the product sequence $v_n (k)$ over all k yields

$$h(n) = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \left(\frac{1}{4}\right)^{n-k} = \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} 2^k$$

$$h(n) = \left(\frac{1}{4}\right)^n (2^{n+1} - 1) = \left(\frac{1}{2}\right)^n \left[2 - \left(\frac{1}{2}\right)^n\right], \quad n \geq 0$$

EXAMPLE 2.4.4

Determine the homogeneous solution of the system described by the first-order difference equation

$$y(n) + a_1\, y(n-1) = x(n) \qquad (2.4.18)$$

Solution. The assumed solution obtained by setting $x(n) = 0$ is $y_h(n) = \lambda^n$

When we substitute this solution in (2.4.18), we obtain [with $x(n) = 0$]

$$\lambda^n + a_1\, \lambda^{n-1} = 0 \qquad \leftrightarrow \qquad \lambda = -a_1$$

The solution to the homogeneous difference equation is $y_h(n) = C\lambda^n = C(-a_1)^n \qquad (2.4.19)$

EXAMPLE 2.4.4 Solution

The zero-input response of the system can be determined from(2.4.18) and (2.4.19).

With $x(n) = 0$, (2.4.18) yields $y(0) = -a_1 y(-1)$

And from (2.4.19) we have $y_h(0) = C$

Hence the zero-input response of the system is

$$y_{zi}(n) = (-a_1)^{n+1} y(-1), n \geq 0 \qquad (2.4.20)$$

With $a = -a_1$, this result is consistent with (2.4.11) for the first-order system, which was obtained by iteration of the difference equation

EXAMPLE 2.4.5

Determine the zero-input response of the system described by the homogeneous second-order difference equation

$$y(n) - 3y(n - 1) - 4y(n - 2) = 0 \qquad (2.4.21)$$

Solution. First we determine the solution to the homogeneous equation. We assume the solution to be the exponential

$$y_h(n) = \lambda^n$$

Upon substitution of this solution into (2.4.21), we obtain the characteristic equation

$$\lambda^n - 3\lambda^{n-1} - 4\lambda^{n-2} = 0$$

EXAMPLE 2.4.5 Solution

$$\lambda^{n-2} - (\lambda^2 - 3\lambda - 4) = 0$$

Therefore, the roots are $\lambda = -1, 4$, and the general form of the solution to the homogeneous equation is

$$y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n$$

$$= C_1(-1)^n + C_2(4)^n \qquad (2.4.22)$$

The zero-input response of the system can be obtained from the homogenous solution by evaluating the constants in (2.4.22), given the initial conditions $y(-1)$ and $y(-2)$. From the difference equation in (2.4.21) we have

$$y(0) = 3y(-1) + 4y(-2)$$

EXAMPLE 2.4.5 Solution

$$y(1) = 3y(0) + 4y(-1)$$
$$= 3 [3y(-1) + 4y(-2)] + 4y(-1)$$
$$= 13y(-1) + 12y(-2)$$

On the other hand, from (2.4.22) we obtain

$$y(0) = C_1 + C_2 \qquad y(1) = -C_1 + 4C_2$$

By equating these two sets of relations is

$$C_1 + C_2 = 3y(-1) + 4y(-2)$$
$$-C_1 + 4C_2 = 13y(-1) + 12y(-2)$$

The solution of these two equations is

$$C_1 = -\frac{1}{5}y(-1) + \frac{4}{5}y(-2) \qquad C_2 = \frac{16}{5}y(-1) + \frac{15}{5}y(-2)$$

EXAMPLE 2.4.5 Solution

Therefore, the zero-input response of the system is

$$y_{zi}(n) = [-\frac{1}{5}y(-1) + \frac{4}{5}y(-2)](-1)^n \qquad (2.4.23)$$

$$+ [\frac{16}{5}y(-1) + \frac{16}{5}y(-2)](4)^n, \qquad n \geq 0$$

For example, if *y(- 2) = 0* and *y(- 1) = 5,* then $C_1 = 1, C_2 = 16,$ and hence

$$y_{zi}(n) = (-1)^{n+1} + (4)^{n+2}, \qquad n \geq 0$$

EXAMPLE 2.4.6

Determine the particular solution of the fist-order difference equation

$$y(n) + a_1\, y(n-1) = x(n), \qquad |a_1| < 1 \qquad (1.4.26)$$

when the input $x(n)$ is a unit step sequence, that is, $\qquad x(n) = u(n)$

Solution. Since the input sequence $x(n)$ is a constant for $n \geq 0$, hence the assumed solution of the difference equation to the forcing function $x(n)$, called the particular solution of the difference equation, is $y_p(n) = K\, u(n)$

EXAMPLE 2.4.6 Solution

where K is a scale factor determined so that (2.4.26) is satisfied. Upon substitution of this assumed solution into (2.4.26), we obtain

$$K\,u(n) + a_1\,K\,u(n-1) = u(n)$$

To determine K, we must evaluate this equation for any $n \geq 1$, where none of the terms vanish. Thus

$$K + a_1 K = 1$$

$$\leftrightarrow$$

$$K = \frac{1}{1 + a_1}$$

EXAMPLE 2.4.8

Determine the total solution $y(n)$, $n \geq 0$, to the difference equation

$$y(n) + a_1 y(n-1) = x(n) \qquad (2.4.28)$$

When $x(n)$ is a unit step sequence [i.e., $x(n)=u(n)$] and $y(-1)$ is the initial condition

Solution. From (2.4.19) of Example 2.4.4, the homogeneous solution is $y_h(n) = C(-a_1)^n$ and from (2.4.26) of Example 2.4.6, the particular solution is

$$y_p(n) = \frac{1}{1 + a_1}$$

EXAMPLE 2.4.8 Solution

Consequently, the total solution is

$$y(n) = C(-a_1)^n + \frac{1}{1+a_1} \qquad n \geq 0 \qquad (2.4.29)$$

Where the constant C is determined to satisfy the initial condition $y(-1)$.

We wish to obtain the zero-state response of the system described by the first-order difference equation in (2.4.8). Then we set $y(-1) = 0$. To evaluate C, we evaluate (2.4.28) at $n=0$ obtaining

$$y(0) + a_1\, y(-1) = 1 \quad \leftrightarrow \quad y(0) = 1$$

EXAMPLE 2.4.8 Solution

On the other hand, (2.4.29) evaluated at $n = 0$ yields

$$y(0) = c + \frac{1}{1 + a_1}$$

Consequently

$$c + \frac{1}{1 + a_1} = 1 \qquad \leftrightarrow \qquad c = \frac{a_1}{1 + a_1}$$

Substitution for C into (2.4.29) yields the zero-state response of the system

$$y_{zs}(n) = \frac{1 - (-a_1)^{n+1}}{1 + a_1} \qquad n \geq 0$$

EXAMPLE 2.4.8 Solution

If we evaluate the parameter C in (2.4.29) under the condition that *y(– 1) ≠ 0*, the total solution will include the zero-input response as well as the zero-state response of the system.

In this case (2.4.8) yields

y(0) + a₁ y(- 1) = 1   ↔   *y(0) = - a₁y(- 1) +1*

On the other hand, (2.4.29) yields

$$y(0) = C + \frac{1}{1 + a_1}$$

EXAMPLE 2.4.8 Solution

By equating these two relations, we obtain

$$C + \frac{1}{1+a_1} = -a_1 y(-1) + 1 \qquad \leftrightarrow \qquad C = -a_1 y(-1) + \frac{a_1}{1+a_1}$$

Finally, if we substitute this value of C into (2.4.29), we obtain

$$y(n) = (-a_1)^{n+1} y(-1) + \frac{1 - (-a_1)^{n+1}}{1+a_1} \qquad n \geq 0$$

$$= y_{zi}(n) + y_{zs}(n)$$

EXAMPLE 2.6.1

Determine the crosscorrelation sequence $r_{xy}(l)$ of the sequences

$$x(n) = \{\ldots, 0, 0, 2, -1, 3, 7, 1, 2, -3, 0, 0, \ldots\}$$

$$\uparrow$$

$$y(n) = \{\ldots, 0, 0, 1, -1, 2, -2, 4, 1, -2, 5, 0, 0, \ldots\}$$

$$\uparrow$$

Solution. Let us use the definition in (2.6.3) to compute $r_{xy}(l)$. For $l = 0$ we have

$$r_{xy}(0) = \sum_{n=-\infty}^{\infty} x(n)\, y(n)$$

EXAMPLE 2.6.1 Solution

The product sequence $v_0(n) = x(n)y(n)$ is

$v_0(n) = \{…, 0, 0, 2, 1, 6, - 14, 4, 2, 6, 0, 0, … \}$

$\uparrow$

and hence the sum over all values of n is $r_{xy}(0) = 7$
For $l > 0$, we simply shift $y(n)$ to the right relative to $x(n)$ by l units, compute the product sequence $v_1(n) = x(n) \, y(n - 1)$, and finally, sum over all values of the product sequence. Thus we obtain

$r_{xy}(1) = 13, \quad r_{xy}(2) = - 18, \, r_{xy}(3) = 16, \quad r_{xy}(4) = - 7,$
$r_{xy}(5) = 5, \quad r_{xy}(6) = - 3, \quad r_{xy}(l) = 0, \quad l \geq 7$

EXAMPLE 2.6.1 Solution

For $l < 0$, we shift $y(n)$ to the left relative to $x(n)$ by l units, compute the product sequence $v_1(n) = x(n) \, y(n - l)$, and sum over all values of the product sequence. Thus we obtain the values of the crosscorrelation sequence

$r_{xy}(-1) = 0,$ $r_{xy}(-2) = 33,$ $r_{xy}(-3) = -14,$

$r_{xy}(-4) = 36$ $r_{xy}(-5) = 19,$ $r_{xy}(-6) = -9,$

$r_{xy}(-7) = 10,$ $r_{xy}(l) = 0, \; l \leq -8$

Therefore, the crosscorrelation sequence of $x(n)$ and $y(n)$ is

$r_{xy}(l) = \{\, 10, -9, 19, 36, -14, 33, 0, 7, 13, -18, 16, -7, 5, -3 \,\}$