EXAMPLE 3.1.1

Determine the z-transform of the following finite-duration signals.

(a) $x_1(n) = \{ \underset{\uparrow}{1}, 2, 5, 7, 0, 1\}$ **(b)** $x_2(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

(c) $x_3(n) = \{ \underset{\uparrow}{0},0,1,2,5,7,0,1\}$ **(d)** $x_4(n) = \{ 2, 4, \underset{\uparrow}{5}, 7, 0,1\}$

(e) $x_5(n) = \delta(n)$ **(f)** $x_6(n) = \delta(n - k), \; k > 0$

(g) $x_7(n) = \delta(n + k), k > 0$

EXAMPLE 3.1.1 Solution

From definition (3.1.1), we have

(a) $X_1(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$,

ROC: entire *z*-plane except $z = 0$

(b) $X_2(z) = z^2 + 2z + 5 + 7z^{-1} + z^{-3}$,

ROC: entire z-plane except $z = 0$ and $z = \infty$

(c) $X_3(z) = z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}$,

ROC: entire *z*-plane except $z = 0$

(d) $X_4(z) = 2z^2 + 4z + 5 + 7z^{-1} + z^{-3}$,

ROC: entire *z*-plane except $z = 0$ and $z = \infty$

EXAMPLE 3.1.1 Solution

$$\text{(e) } X_5(z) = 1 \left[\text{i.e., } \delta(n) \overset{z}{\leftrightarrow} 1\right],$$

$$\text{ROC: entire } z - \text{plane}$$

$$\text{(f) } X_6(z) = z^{-k} \left[\text{i.e., } \delta(n-k) \overset{z}{\leftrightarrow} z^{-k}\right], k > 0,$$

$$\text{ROC: entire } z - \text{plane except } z = 0$$

$$\text{(g) } X_7(z) = z^{k} \left[\text{i.e., } \delta(n+k) \overset{z}{\leftrightarrow} z^{k}\right], k > 0,$$

$$\text{ROC: entire } z - \text{plane except } z = \infty$$

EXAMPLE 3.1.2

Determine the z-transform of the signal

$$x(n) = (\frac{1}{2})^n u(n)$$

Solution. The signal *x(n)* consists of an infinite number of nonzero values

$$x(n) = \{1, (\frac{1}{2}), (\frac{1}{2})^2, (\frac{1}{2})^3, \dots, (\frac{1}{2})^n, \dots\}$$

The z-transform of *x(n)* is the infinite power series

$$A(z) = 1 + \frac{1}{2} z^{-1} + (\frac{1}{2})^2 z^{-2} + (\frac{1}{2})^n z^{-n} + \cdots$$

EXAMPLE 3.1.2 Solution

$$= \sum_{n=0}^{\infty} (\frac{1}{2})^n z^{-n} = \sum_{n=0}^{\infty} (\frac{1}{2} z^{-1})^n$$

This is an infinite geometric series. We recall that

$$1 + A + A^2 + A^3 + \cdots = \frac{1}{1-A} \qquad \text{if } |A| < 1$$

Consequently, for $| \frac{1}{2} z^{-1}| < 1$, or equivalently, for $| z | > \frac{1}{2}$, $X(z)$ converges to

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad ROC: |z| > \frac{1}{2}$$

The *z*-transform provides a ***compact alternative representation*** of the signal *x(n)*.

EXAMPLE 3.1.4

Determine the z-transform of the signal

$$x(n) = -\alpha^n u(-n-1) = \begin{cases} 0, & n \geq 0 \\ -\alpha^n, & n \leq -1 \end{cases}$$

Solution. From the definition (3.1.1) we have

$$X(z) = \sum_{n=\infty}^{-1} (-\alpha^n) z^{-n} = -\sum_{l=1}^{\infty} (\alpha^{-1} z)^l$$

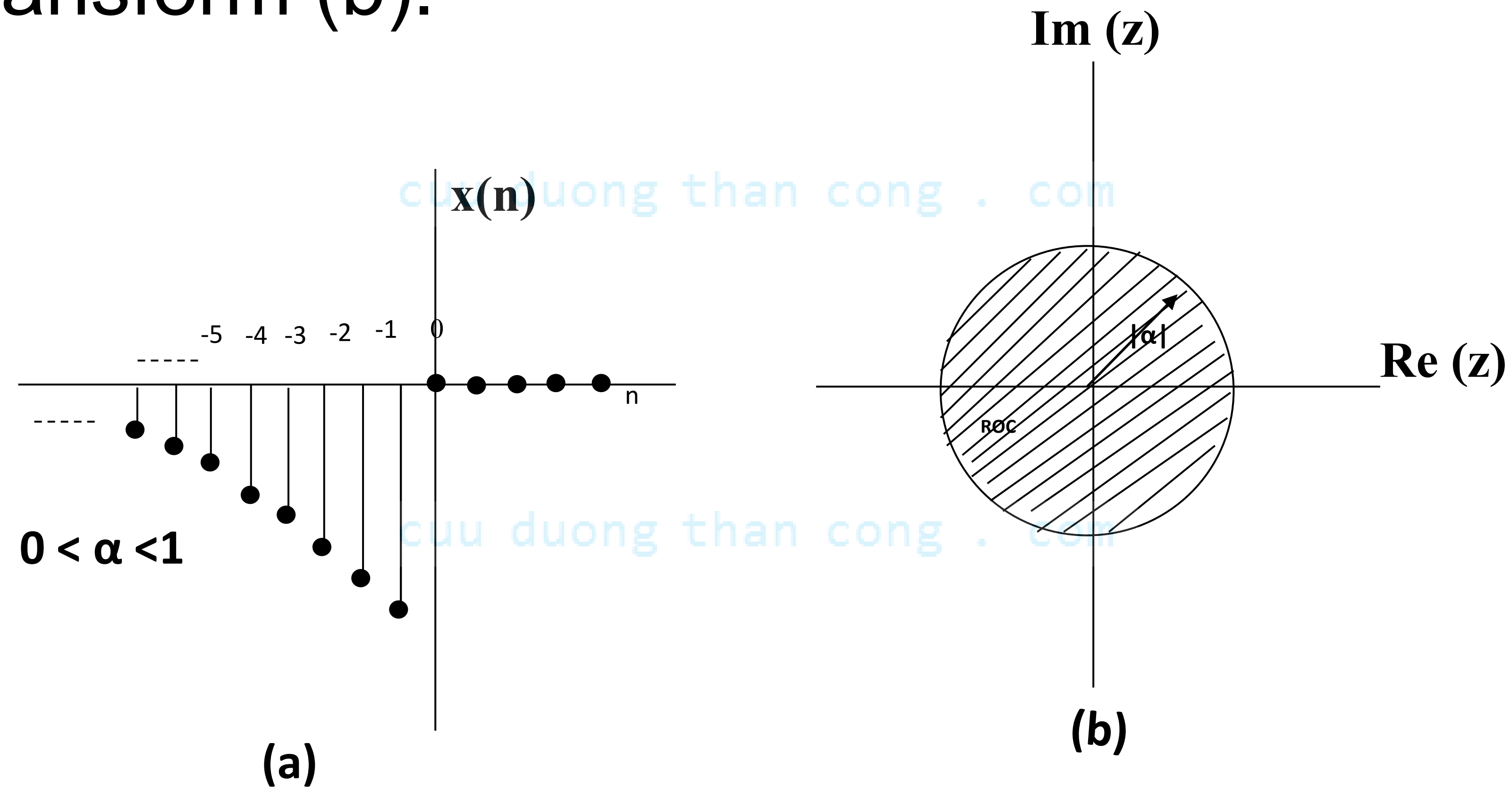
where *l* = - *n*. provided that $|\alpha^{-1} z| < 1$ or, $|z| < |\alpha|$

$$X(z) = \frac{\alpha^{-1} z}{1 - \alpha^{-1} z} = \frac{1}{1 - \alpha z^{-1}} \qquad \text{ROC: } |z| < |\alpha| \qquad (3.1.9)$$

This is shown in Fig. 3.3

EXAMPLE 3.1.4 Solution

Figure 3.3 Anti causal signal $x(n) = -\alpha^n u(-n-1)$ (a), and the ROC of its z-transform (b).



(a)

(b)

EXAMPLE 3.1.5

Determine the z-transform of the signal

$$x(n) = \alpha^n u(n) + b^n u(-n-1)$$

Solution. From definition (3.1.1) we have

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n + \sum_{l=1}^{\infty} (b^{-1} z)^l$$

The first power series converges if $|\alpha z^{-1}| < 1$ or $|z| > |\alpha|$. The second power series converges if $|b^{-1}z| < 1$ or $|z| < |b|$.

In determining the convergence of *X(z)*, we consider two different two different cases.

EXAMPLE 3.1.5 Solution

Case 1 $|b| < |\alpha|$: In this case the two ROC above do not overlap, as shown in Fig. 3.4(a).

Consequently, we cannot find values of z for which both power series converge simultaneously. Clearly, in this case, $X(z)$ does not exist.

Case 2 $|b| > |\alpha|$: In this case there is a ring in the z-plane where both power series converge simultaneously, as shown in Fig. 3.4(b).

EXAMPLE 3.1.5 Solution

Then we obtain

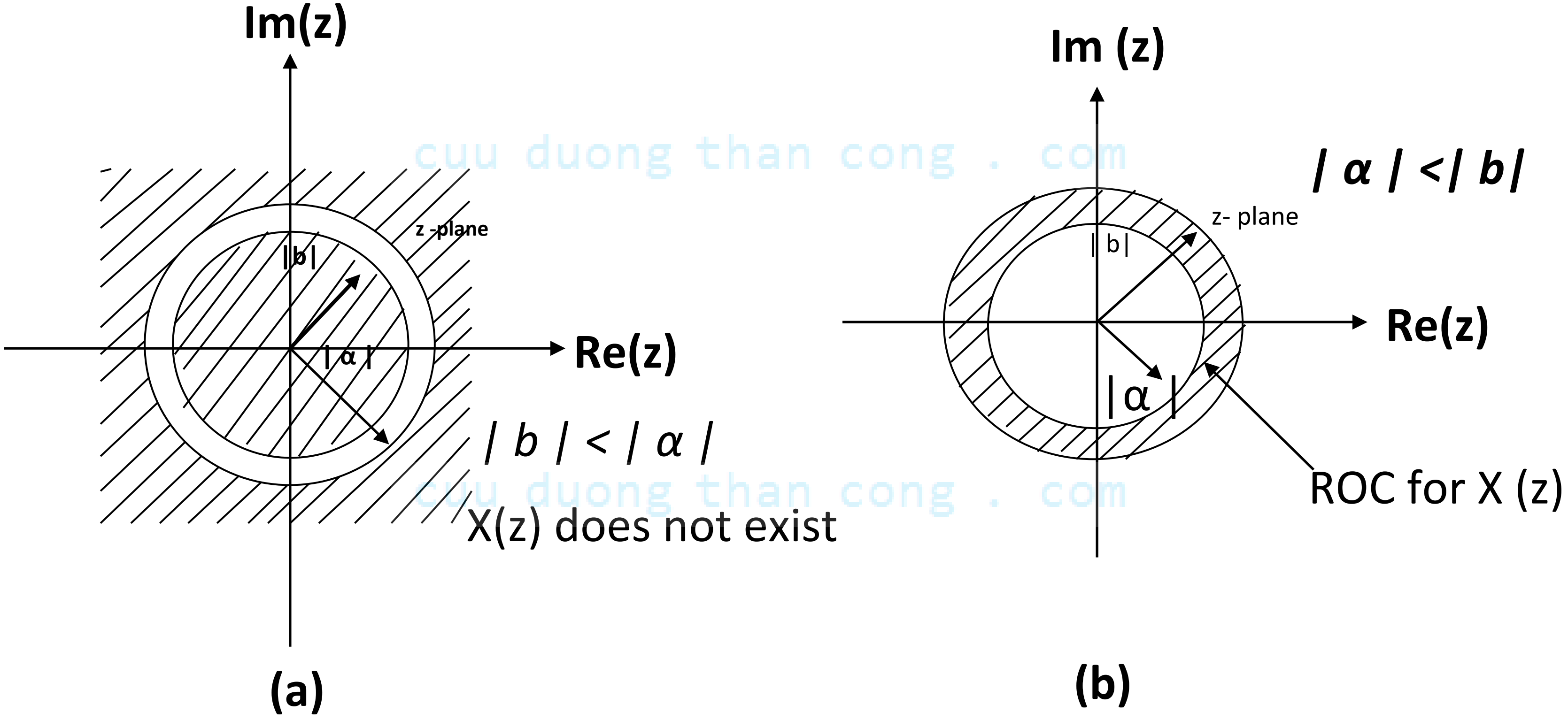
$$X(z) = \frac{1}{1-\alpha z^{-1}} - \frac{1}{1-bz^{-1}} \qquad (3.1.10)$$

$$= \frac{b-\alpha}{\alpha+b-z-\alpha bz^{-1}}$$

The ROC of $X(z)$ is $|\alpha| < |z| < |b|$.

EXAMPLE 3.1.5 Solution

Figure 3.4 ROC for *z*- transform in Example 3.1.5



(a)

(b)

EXAMPLE 3.3.1

Determine the pole-zero plot for the signal

$$x(n) = a^n u(n), \qquad a > 0$$

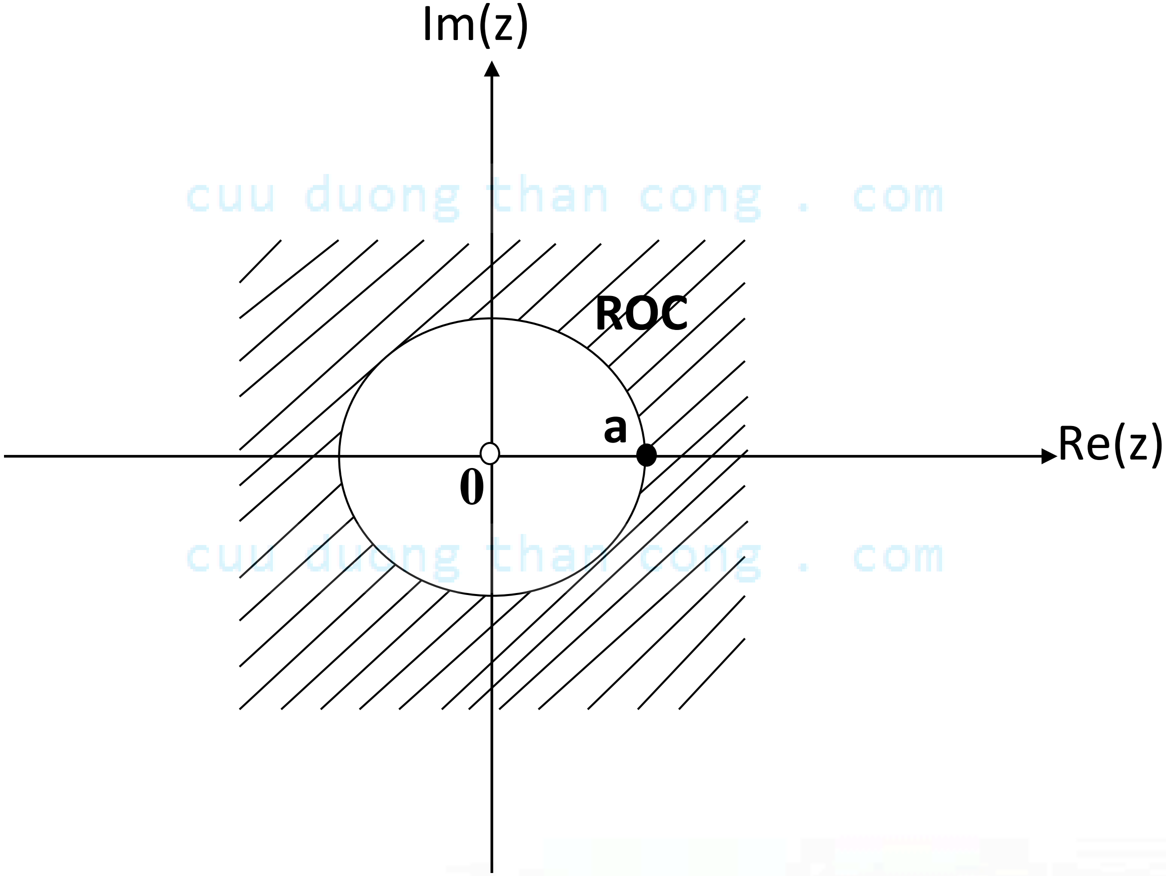
Solution. From Table 3.3 we find that

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad \text{ROC: } |z| > a$$

Thus $X(z)$ has one ***zero*** at $z_1 = 0$ and one ***pole*** at $p_1 = a$. The pole – zero plot is shown in Fig. 3.3.1. Note that the pole $p_1 = a$ is not included in the ROC since the z-transform does not converge at a pole.

EXAMPLE 3.3.1 Solution

Figure 3.3.1 Pole – zero plot for the causal exponential signal $x(n) = a^n u(n)$.



EXAMPLE 3.3.2

Determine the pole – zero plot for signal

$$x(n) = \begin{cases} a^n, & 0 \le n \le M-1 \\ 0, & \text{elsewhere} \end{cases}$$

where a > 0

Solution. From the definition (3.1.1) we obtain

$$X(z) = \sum_{n=0}^{M-1} (az^{-1})^n = \frac{1-(az^{-1})^M}{1-az^{-1}} = \frac{z^M - a^M}{z^{M-1}(z-a)}$$

Since $a > 0$, the equation $z^M = a^M$ has roots at

$z_k = ae^{j2\pi k/M}$ $k = 0, 1, \ldots, M - 1$

EXAMPLE 3.3.2 Solution

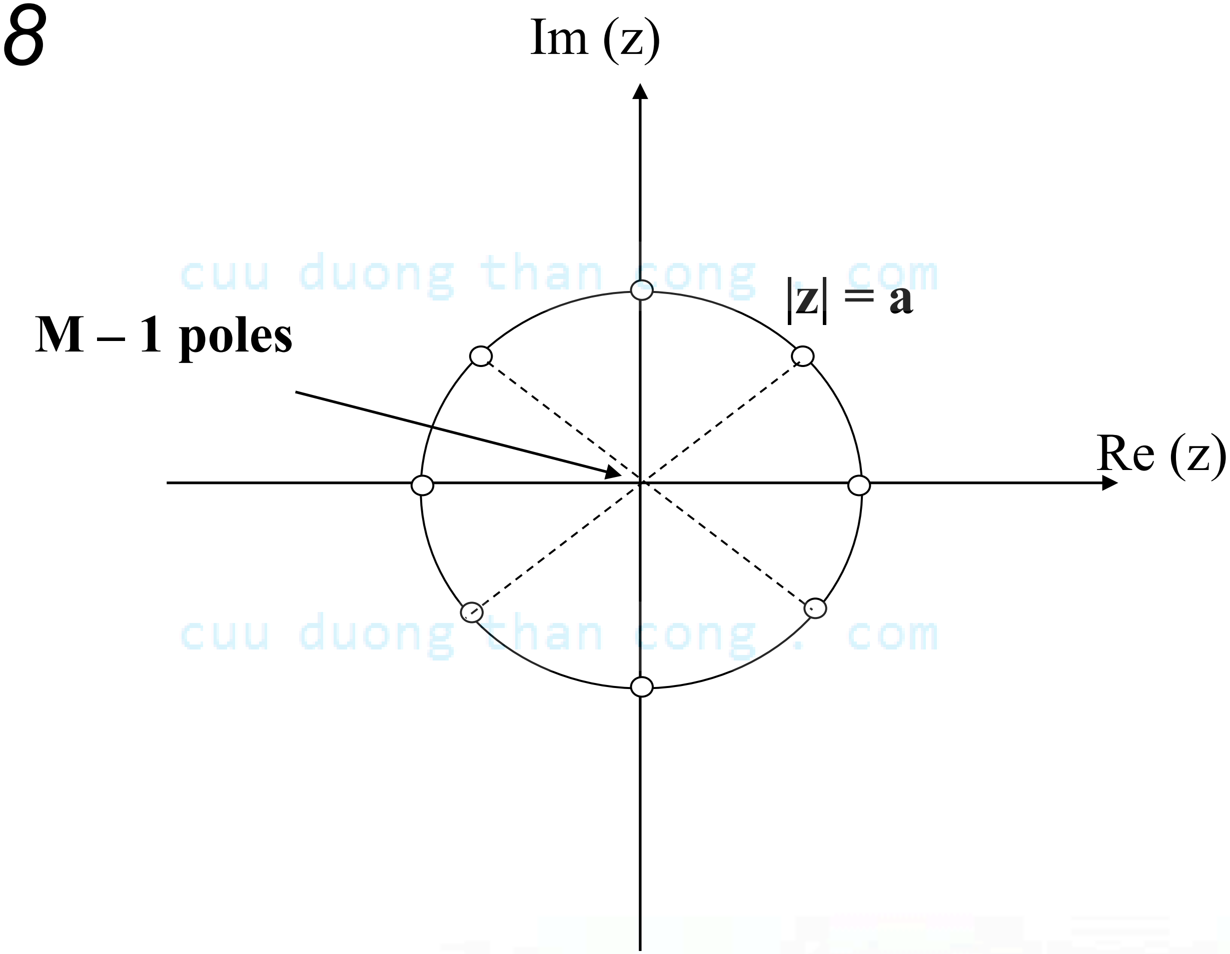
The zero $z_0 = a$ cancels the pole at $z = a$. Thus

$$X(z) = \frac{(z - z_1)(z - z_2)\ldots(z - z_{M-1})}{z^{M-1}}$$

which has $M - 1$ zeros and $M - 1$ poles, located as shown in Fig. 3.3.2 $M = 8$. Note that the ROC is the entire z-plane except $z = 0$ because of the $M - 1$ poles located at the origin.

EXAMPLE 3.3.2 Solution

Figure 3.3.2 Pole – zero pattern for the finite – duration signal $x(n) = a^n$, $0 \le n \le M - 1$ $(a > 0)$, for $M = 8$



EXAMPLE 3.3.3

Determine the *z*-transform and the signal that corresponds to the pole – zero pole of Fig. 3.3.3.

Solution. There are two zero ($M = 2$) at $z_1 = 0$, $z_2 = r\cos\omega_0$ and two poles ($N = 2$) at

$$p_1 = re^{j\omega_0} \quad \text{and} \quad p_2 = re^{j\omega_0}$$

By substitution of these relations into(3.3.2), we obtain

$$X(z) = G\frac{(z-z_1(z-z_2)}{(z-p_1)(z-p_2)} = G\frac{z(z-r\cos\omega 0)}{(z-re^{j\omega_0})(z-re^{-j\omega_0})}), \qquad ROC: |z| > r$$

EXAMPLE 3.3.3 Solution

After some simple algebraic manipulations, we obtain

$$X(z) = G\frac{1 - rz^{-1}cos\omega 0}{1 - 2rz^{-1}cos\omega_0 + r^2z^{-2}} \qquad ROC: |z| > r$$

From table 3.3 we find that $x(n) = G(r^n cos\omega_0 n)u(n)$

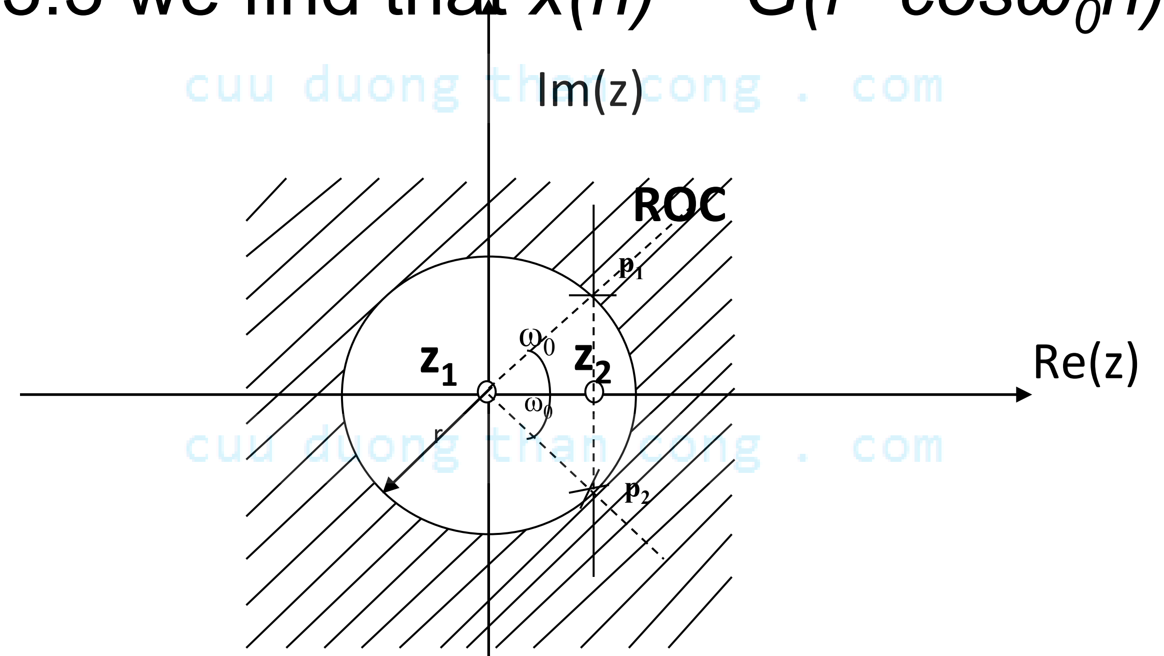


Figure 3.3.3 Pole-zero pattern for Example 3.3.3.

EXAMPLE 3.3.4

Determine the system function and the unit sample response of the system described by the difference equation $y(n) = \frac{1}{2} y(n-1) + 2x(n)$

Solutions. By computing the *z*-transform of the difference equation, we obtain

$$Y(z) = \frac{1}{2} z^{-1} Y(z) + 2X(z)$$

Hence the system function is $H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 - \frac{1}{2} z^{-1}}$

This system has a pole at $z = \frac{1}{2}$ and a zero at the origin. Using Table 3.3 we obtain the inverse transform

$$h(n) = 2(\frac{1}{2})^n u(n)$$

EXAMPLE 3.4.2

Determine the inverse z-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

when **(a)** ROC: $|z| > 1$ **(b)** ROC: $|z| < 0.5$

Solution.

(a) Since the ROC is the exterior of a circle, we expect *x(n)* to be a causal signal. Thus we seek a power series expansion in ***negative powers*** of *z*.

EXAMPLE 3.4.2 Solution

By dividing the numerator of *X(z)* by its denominator, we obtain the power series

$$X(z)=\frac{1}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}=1+\frac{3}{2}z^{-1}+\frac{7}{4}z^{-2}+\frac{15}{8}z^{-3}+\frac{31}{16}z^{-4}+\cdots$$

By comparing this relation with (3.1.1), we conclude that

$$x(n)=\{1,\frac{3}{2},\frac{7}{4},\frac{15}{8},\frac{31}{16},\cdots\}$$

Note that in each step of the long-division process, we eliminate the lowest-power term of z^{-1}

EXAMPLE 3.4.2 Solution

(b) In this case the ROC is the interior of a circle. Consequently, the signal *x(n)* is anticausal.

To obtain a power series expansion in ***positive*** of z, thus:

$$X(z) = \frac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \cdots$$

In this case *x(n) = 0* for *n ≥ 0*. By comparing this result to (3.1.1), we conclude that

$$x(n) = \{\ \ldots 62,\ 30,\ 14,\ 6,\ 2,\ 0,\ \underset{\uparrow}{0}\}$$

EXAMPLE 3.4.4

Express the improper rational transform

$$X(z)=\frac{1+3z^{-1}+\frac{11}{6}z^{-2}+\frac{1}{3}z^{-3}}{1+\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$$

in terms of a polynomial and a proper function

Solution.

First, we note that we should reduce the numerator so that the terms z^{-2} and z^{-3} are eliminated.

EXAMPLE 3.4.4 Solution

Thus we should carry out the long division with these two polynomials written in reverse order.

We stop the division when the order of the remainder becomes z^{-1}. Then we obtain

$$X(z) = 1 + 2z^{-1} + \frac{\frac{1}{6}z^{-1}}{1 + \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}$$

EXAMPLE 3.4.6

Determine the partial-faction expansion of

$$X(z) = \frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}} \qquad (3.4.22)$$

Solution. To eliminate negative powers of *z* in (3.4.22), we multiply both numerator and denominator by z^2. Thus

$$\frac{X(z)}{z} = \frac{z+1}{z^2 - z + 0.5}$$

The poles of *X(z)* are complex conjugates

$p_1 = \frac{1}{2} + j\frac{1}{2}$ and $p_2 = \frac{1}{2} - j\frac{1}{2}$

EXAMPLE 3.4.6 Solution

Since $p_1 \neq p_2$, we seek an expansion of the form (3.4.15). Thus

$$\frac{X(z)}{z} = \frac{z+1}{(z-p_1)(z-p_2)} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2}$$

To obtain A_1 and A_2, we use the formula (2.3.21). Thus we obtain

$$A_1 = \frac{(z-p_1)X(z)}{z}\Big|_{z=p_1} = \frac{z+1}{z-p_2}\Big|_{z=p_1} = \frac{\frac{1}{2}+j\frac{1}{2}+1}{\frac{1}{2}+j\frac{1}{2}-\frac{1}{2}+j\frac{1}{2}} = \frac{1}{2}-j\frac{3}{2}$$

$$A_2 = \frac{(z-p_2)X(z)}{z}\Big|_{z=p_2} = \frac{z+1}{z-p_1}\Big|_{z=p_2} = \frac{\frac{1}{2}-j\frac{1}{2}+1}{\frac{1}{2}-j\frac{1}{2}-\frac{1}{2}-j\frac{1}{2}} = \frac{1}{2}+j\frac{3}{2}$$

EXAMPLE 3.4.9

Determine the causal signal *x(n)* whose *z*-transform is given by

$$X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$$

Solution. In Example 3.4.6 we have obtained the partial-faction expansion as

$$X(z) = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}}$$

where $A_1 = A_2^* = \frac{1}{2} - j\frac{3}{2}$ and $p_1 = p_2^* = \frac{1}{2} + j\frac{3}{2}$

EXAMPLE 3.4.9 Solution

Since we have a pair of complex-conjugate poles, we should use (3.4.34). The polar forms of A_1 and p_1 are

$$A_1 = \frac{\sqrt{10}}{2} e^{-j71.565} \quad \text{and} \quad p_1 = \frac{1}{\sqrt{2}} e^{j\pi/4}$$

hence

$$x(n) = \sqrt{10}\left(\frac{1}{\sqrt{2}}\right)^n cos\left(\frac{\pi n}{4} - 71.565^\circ\right)u(n)$$

EXAMPLE 3.5.1

Determine the transient and steady-state response of the system characterized by the difference equation

$$y(n) = 0.5y(n-1) + x(n)$$

where the input signal is $x(n) = 10\cos(\pi n/4)u(n)$
The system is initially at rest (i.e., it is relaxed)

Solution. The system function for this system is

$$H(z) = \frac{1}{1 - 0.5z^{-1}}$$

and therefore the system has a pole at $z = 0.5$.

EXAMPLE 3.5.1 Solution

The *z*-transform of the input signal is (from Table 3.3)

$$X(z)=\frac{10(1-(1/\sqrt{2})z^{-1}}{1-\sqrt{2}z^{-1}+z^{-2}}$$

Consequently, *Y(z) = H(z)X(z)*

$$=\frac{10(1-(1/\sqrt{2})z^{-1}}{(1-0.5z^{-1})(1-e^{j\pi/4}z^{-1}(1-e^{-j\pi/4}z^{-1})}=\frac{6.3}{1-0.5z^{-1}}+\frac{6.78e^{-j28.7^{\circ}}}{1-e^{j\pi/4}z^{-1}}+\frac{6.78e^{j28.7^{\circ}}}{1-e^{-j\pi/4}z^{-1}}$$

The natural or transient response is

$$y_{nr}(n)=6.3(0.5)^{n}u(n)$$

EXAMPLE 3.5.1 Solution

and the forced or steady-state response is

$$y_{fr}(n) = \left[6.78e^{-j28.7}\left(e^{j\pi n/4}\right) + 6.78e^{j28.7}e^{-j\pi n/4}\right]u(n)$$

$$= 13.56\cos\left(\frac{\pi}{4}n - 28.7^{\circ}\right)u(n)$$

Thus we see that the steady – state response persists for all *n* ≥ *0*, just as the input signal persists for all *n* ≥ *0*.