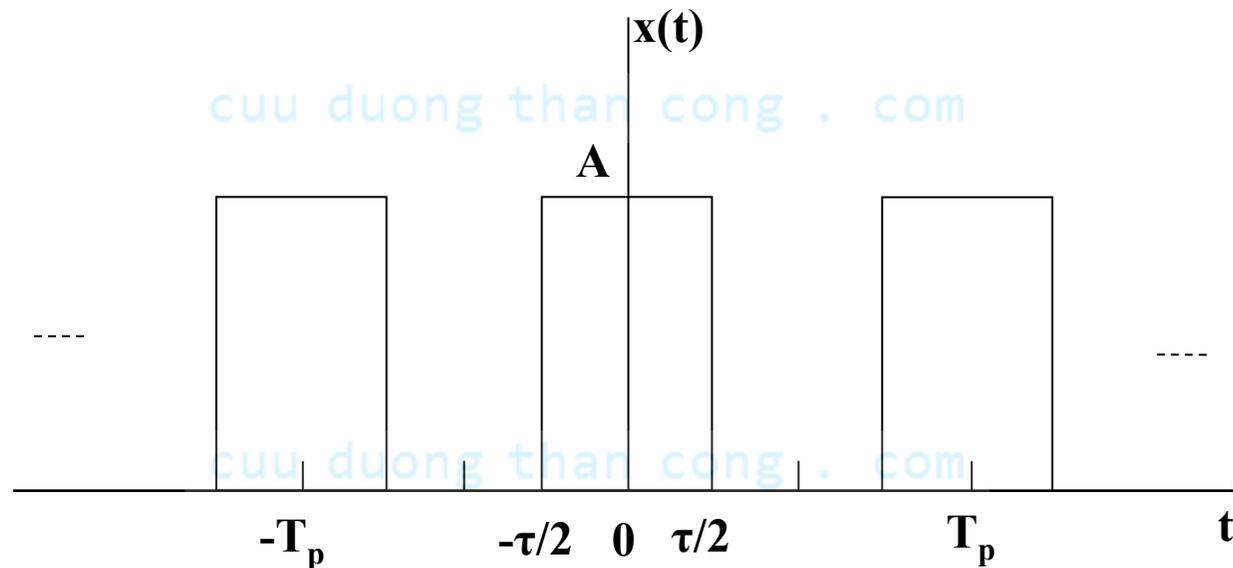# EXAMPLE 4.1.1

Determine the Fourier series and the power density spectrum of the rectangular pulse train signal illustrated in Fig 4.1.3

EXAMPLE 4.1.1 Solution

The signal is periodic with fundamental period T_p and, clearly, satisfies the Dirichlet conditions. Since *x(t)* is an even signal [i.e., *x(t) = x(- t)*], it is convenient to select the integration interval from $-P_T/2$ to $P_T/2$. Thus (4.1.9) evaluated for *k = 0* yields

$$c_0 = \frac{1}{T_p}\int_{-\tau/2}^{\tau/2} x(t)\,dt = \frac{1}{T_p}\int_{-\tau/2}^{\tau/2} A\,dt = \frac{A\tau}{T_p} \qquad (4.1.17)$$

EXAMPLE 4.1.1 Solution

The term c_0 represents the average value (dc component) of the signal $x(t)$. For $k \neq 0$ we have

$$c_k = \frac{1}{T_p} \int_{-\tau/2}^{\tau/2} A e^{-j2\pi k F_0 t} dt = \frac{A}{T_p} \left[\frac{e^{-j2\pi F_0 kt}}{-j2\pi k F_0} \right]_{-\tau/2}^{\tau/2}$$
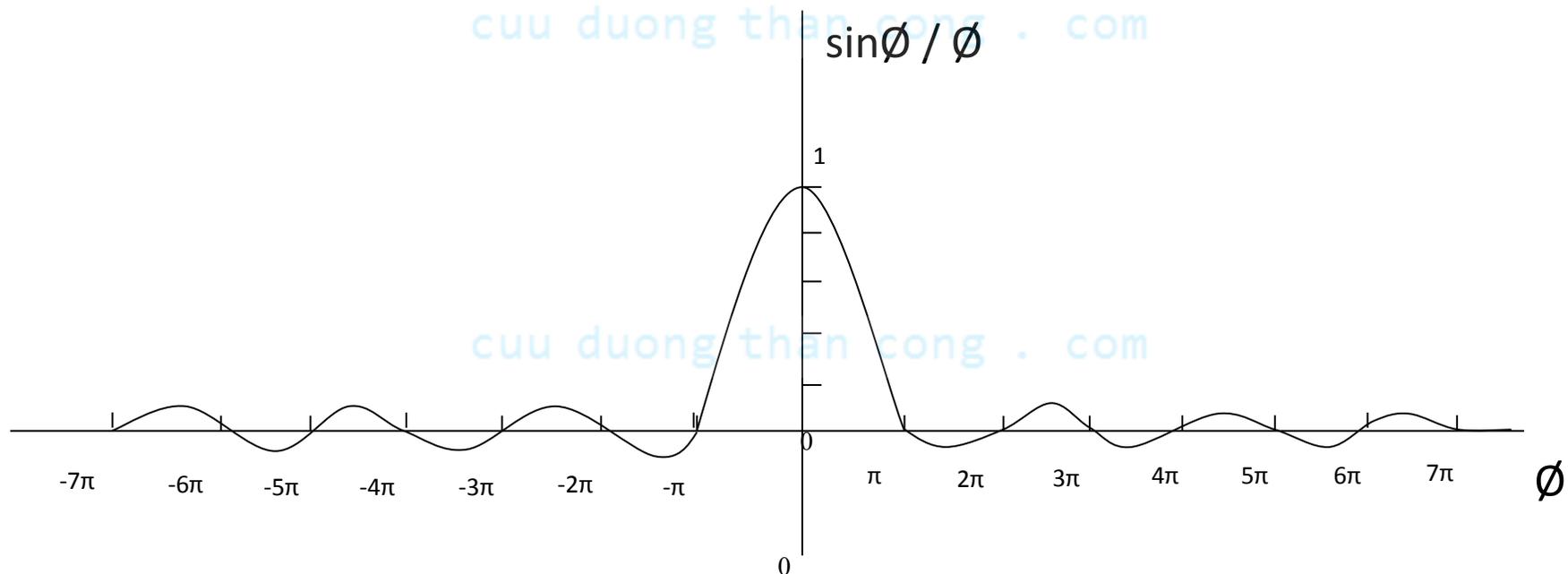
$$= \frac{A}{\pi F_0 k T_p} \frac{e^{j\pi k F_0 \tau} - e^{-j\pi k F_0 \tau}}{j2} \qquad (4.1.18)$$

$$= \frac{A\tau}{T_p} \frac{\sin \pi k F_0 \tau}{\pi k F_0 \tau}, \qquad k = \pm 1, \pm 2, \ldots$$
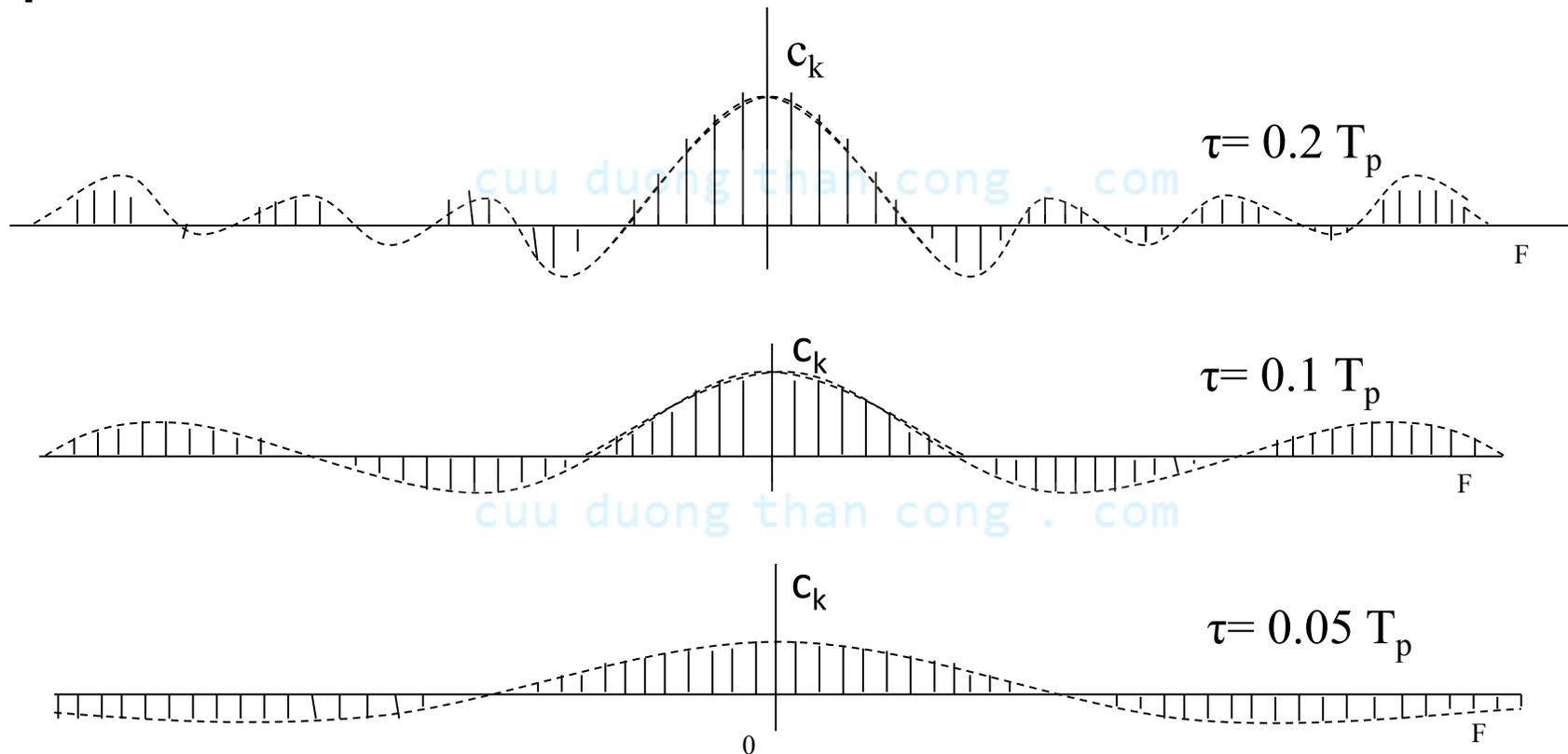
EXAMPLE 4.1.1 Solution

If we plot *(sin Ø) / Ø* with *Ø* as a continuous parameter over the range - ∞ < *Ø* < ∞, we obtain the graph show in Fig 4.1.4

Figure 4.1.4 The function *(sin Ø) / Ø*.

EXAMPLE 4.1.1 Solution

Figure 4.1.5 Fourier coefficients of the rectangular pulse train when T_p is fixed and the pulse width τ varies.

c_k

$\tau = 0.2\ T_p$

F

c_k

$\tau = 0.1\ T_p$

F

c_k

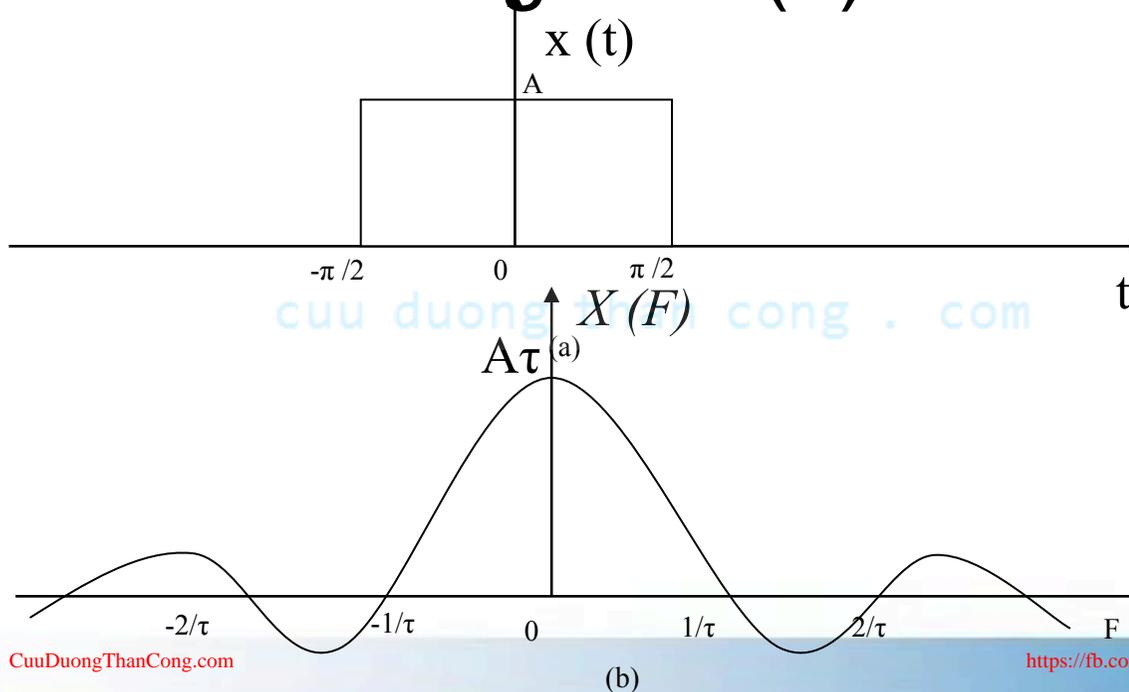$\tau = 0.05\ T_p$

0

F

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EXAMPLE 4.1.2

Determine the Fourier transform and the energy density spectrum of a rectangular pulse signal defined as

$$x(t) = \begin{cases} A, & |t| \leq \tau/2 \\ 0, & |t| \leq \tau/2 \end{cases} \qquad (4.1.43)$$

and illustrated in Fig 4.1.8(a).

EXAMPLE 4.1.2 Solution

Clearly, this signal is aperiodic and satisfies the Dirichlet conditions. Hence its Fourier transform exists. By applying (4.1.30), we find that

$$X(F) = \int_{-\tau/2}^{\tau/2} A e^{-j2\pi Ft} \, dt = A\tau \frac{\sin\pi F\tau}{\pi F\tau} \qquad (4.1.44)$$

The Fourier coefficients c_k in the corresponding periodic signal $x_p(t)$ are simply samples of $X(F)$ at frequencies $kF_0 = k / T_p$. Specifically,

$$c_k = \frac{1}{T_p} X(kF_0) = \frac{1}{T_p} X\left(\frac{k}{T_p}\right) \qquad (4.1.45)$$

EXAMPLE 4.2.1

Determine the spectra of the signals

(a) $x(n) = \cos\sqrt{2}\pi n$ **(b)** $x(n) = \cos(\pi n/3)$

(c) $x(n)$ is periodic with period $N = 4$ and

$$x(n) = \{\ 1,\ 1,\ 0,\ 0\ \}$$

Solution.

(a) For $\omega_0 = \sqrt{2}\pi$, we have $f_0 = 1/\sqrt{2}$. Since f_0 is not a rational number, the signal is not periodic. This signal cannot be expanded in a Fourier series. Its spectral content consists of the single frequency component at $\omega = \omega_0 = \sqrt{2}\pi$.

EXAMPLE 4.2.1 Solution

(b) In this case and hence x(n) is periodic with fundamental period *n = 6*. From (4.2.8) we have

$$c_k = \frac{1}{6}\sum_{n=0}^{5} x(n)\, e^{-j2\pi kn/6}, \qquad k = 0, 1, ..., 5$$

However, *x(n)* can be expressed as

$$x(n) = \cos\frac{2\pi n}{6} = \frac{1}{2}e^{j2\pi n/6} + \frac{1}{2}e^{-j2\pi n/6}$$

which is already in the form of the exponential Fourier series in (4.2.7).

EXAMPLE 4.2.1 Solution

In comparing the two exponential terms in *x(n)* with (4.2.7), $c_1 = \frac{1}{2}$.

The second exponential in *x(n)* corresponds to the term *k = - 1* in (4.2.7). This term can also be written as $e^{-j2\pi n/6} = e^{j2\pi(5-6)n/6} = e^{j2\pi(5n)/6}$

which means that $c_{-1} = c_5$

Consequently, we conclude that

$$c_0 = c_2 = c_3 = c_4 = 0 \qquad c_1 = \frac{1}{2} \qquad c_5 = \frac{1}{2}$$

EXAMPLE 4.2.1 Solution

(c) From (4.2.8), we have

$$c_k = \frac{1}{4} \sum_{n=0}^{3} x(n)\, e^{-j2\pi n/4}, \qquad k = 0, 1, 2, 3$$

or

$$c_k = \frac{1}{4} \sum_{n=0}^{3} (1 + e^{-j2\pi k/2}), \qquad k = 0, 1, 2, 3$$

For *k = 0, 1, 2, 3* we obtain

$$c_0 = \frac{1}{2} \quad c_1 = \frac{1}{4}(1-j), \quad c_2 = 0, \qquad c_3 = \frac{1}{4}(1+j)$$
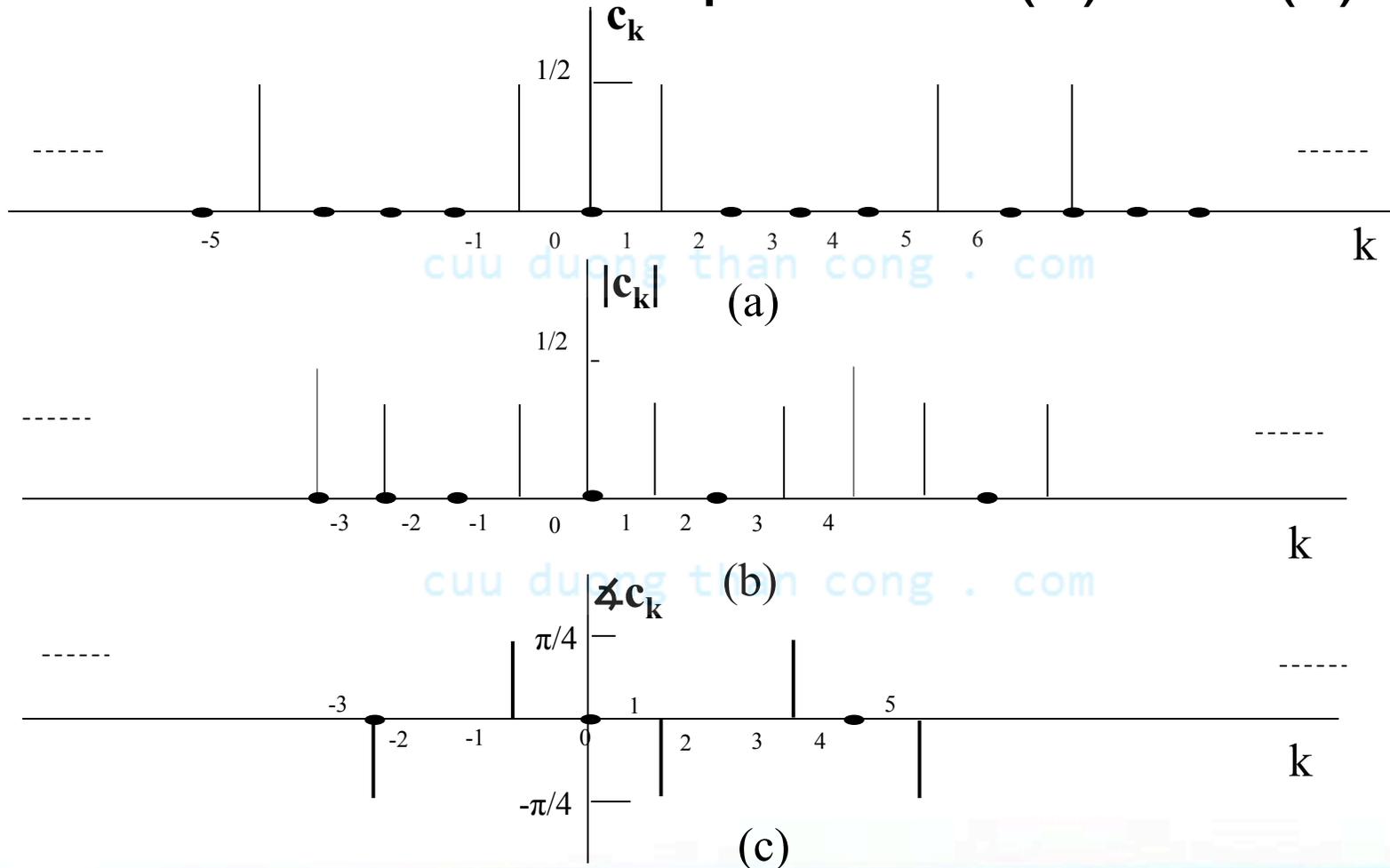
The magnitude and phase spectra are

$$|c_0| = \frac{1}{2}, \quad |c_1| = \frac{\sqrt{2}}{4\pi}, \qquad |c_2| = 0, \qquad |c_3| = \frac{\sqrt{2}}{4},$$

$$\angle c_0 = 0, \qquad \angle c_1 = -\frac{\pi}{4}, \qquad \angle c_2 = undefined, \qquad \angle c_3 = \frac{\pi}{4}$$
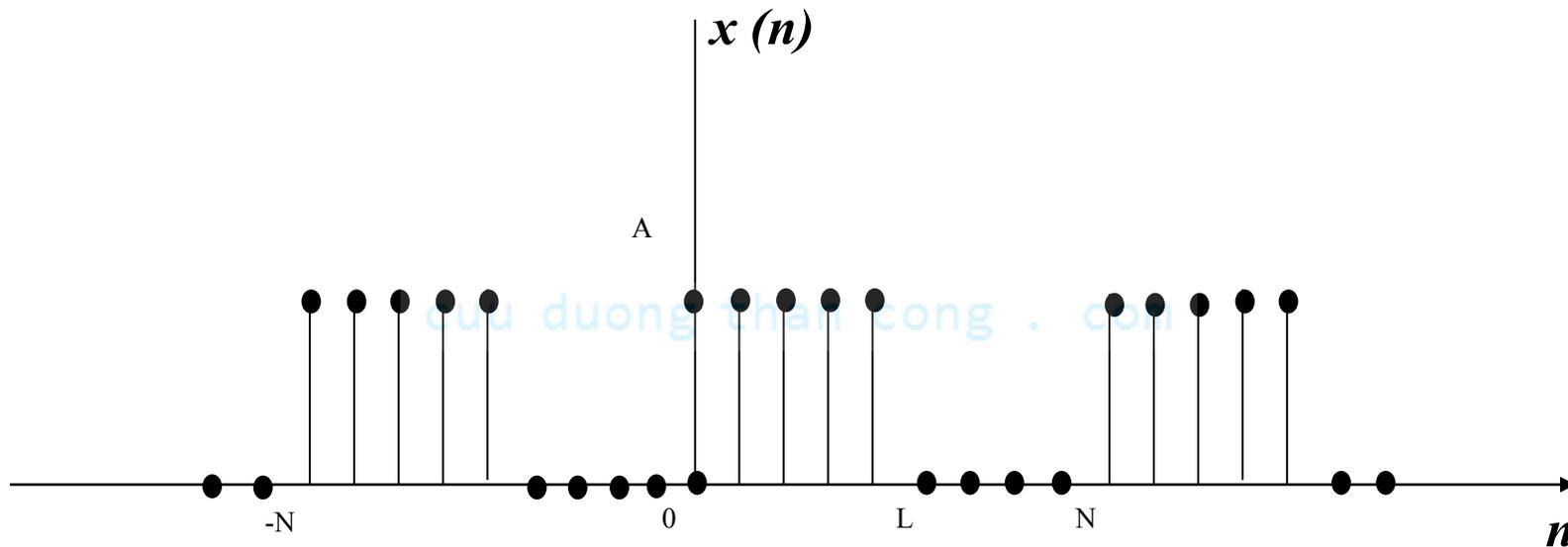
EXAMPLE 4.2.1 Solution

Figure 4.2.1 Spectra of the periodic signals discussed in Example 4.2.1 (b) and (c)

(a)

(b)

(c)

EXAMPLE 4.2.2 Periodic "Square – Wave" Signal

Determine the Fourier series coefficients and the power density spectrum of the periodic signal shown in Fig 4.2.2.

Figure 4.2.2 Discrete-time periodic square-wave signal.

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By applying the analysis equation (4.2.8) to the signal shown in Fig 4.2.2, we obtain

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} = \frac{1}{N} \sum_{n=0}^{L-1} A e^{-j2\pi kn/N}, \qquad k = 0, 1, \dots, N-1$$

which is a geometric summation. Now we can use (4.2.3) to simplify the summation above. Thus we obtain

$$c_k = \frac{A}{N} \sum_{n=0}^{L-1} (e^{-j2\pi k/N})^n = \begin{cases} \dfrac{AL}{N}, & k = 0 \\[2ex] \dfrac{A}{N} \cdot \dfrac{1 - e^{-j2\pi kL/N}}{1 - e^{-j2\pi k/N}}, & k = 1, 2, \dots, N-1 \end{cases}$$

EXAMPLE 4.2.2   Solution

The last expression can be simplified further if we note that

$$\frac{1 - e^{-j2\pi kL/N}}{1 - e^{-j2\pi k/N}} = \frac{e^{-j\pi kL/N}}{e^{-j\pi k/N}} \frac{e^{j\pi kL/N} - e^{-j\pi kL/N}}{e^{j\pi k/N} - e^{-j\pi k/N}} = e^{-j\pi k(L-1)/N} \frac{\sin(\pi kL/N)}{\sin(\pi k/N)}$$
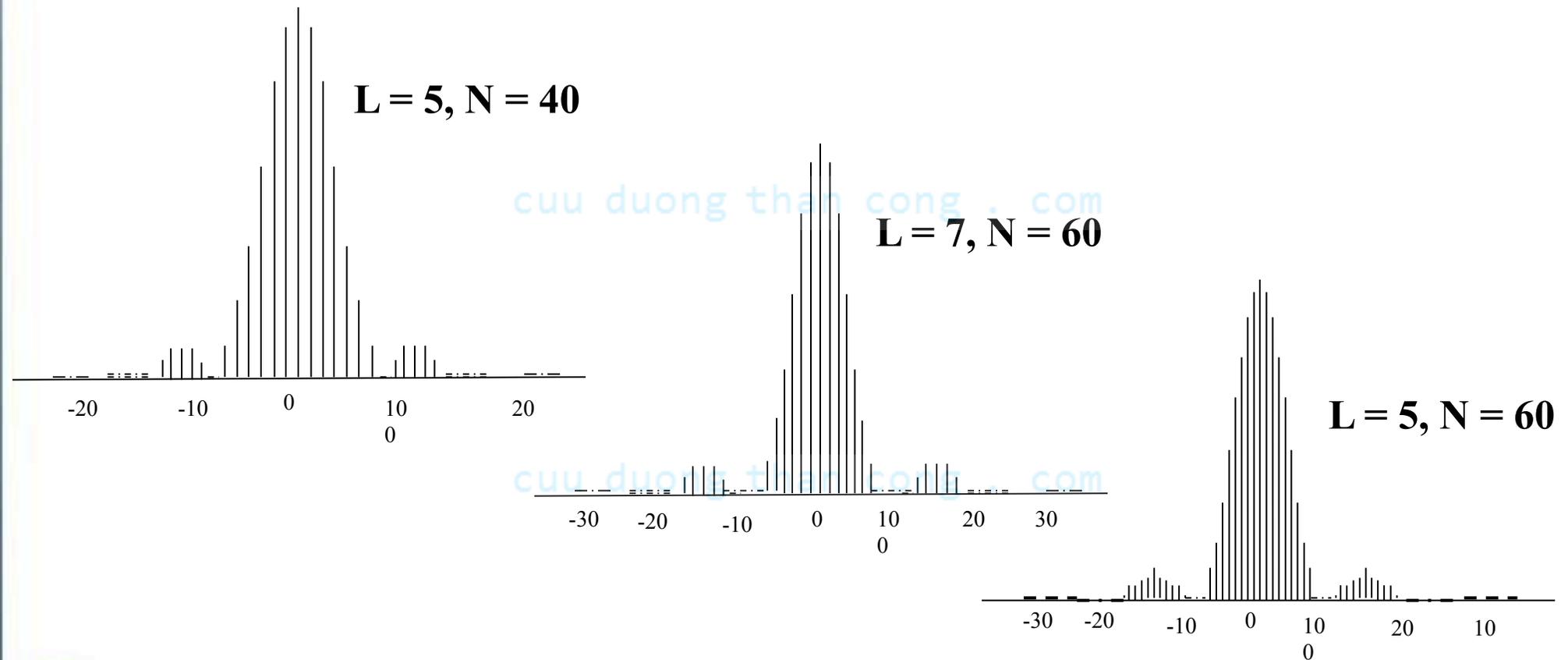
Therefore,

$$c_k = \begin{cases} \dfrac{AL}{N}, & k = 0, +N, \pm 2N, \dots \\ \dfrac{A}{N} e^{-j\pi k(L-1)/N} \dfrac{\sin(\pi kL/N)}{\sin(\pi k/N)}, & \text{otherwise} \end{cases} \qquad (4.2.21)$$

The power density spectrum of this periodic signal is

$$|c_k|^2 = \begin{cases} \left(\dfrac{AL}{N}\right)^2, & k = 0, +N, \pm 2N, \dots \\ \left(\dfrac{A}{N}\right)^2 \left(\dfrac{\sin \pi kL/N}{\sin \pi k/N}\right)^2, & \text{otherwise} \end{cases} \qquad (4.2.22)$$

EXAMPLE 4.2.2 Solution

Figure 4.2.3 illustrates the plots of $|c_k|$ ^{2} of $L = 2, N = 10$ and 40, and $A = 1$.

L = 5, N = 40

L = 7, N = 60

L = 5, N = 60

EXAMPLE 4.2.3

Determine and sketch the energy density spectrum $S_{xx}(\omega)$ of the signal

$$x(n) = a^n \, u(n), \qquad -1 < a < 1$$

Solution. Since $|a| < 1$, as can be verified by applying the geometric summation formula,

$$\sum_{n=-\infty}^{\infty} \|x(n)\| = \sum_{n=0}^{\infty} |a|^n = \frac{1}{1 - |a|} < \infty$$

Hence the Fourier transform $x(n)$ exists and is obtained by applying (4.2.29). Thus

$$X(\omega) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

EXAMPLE 4.2.3 Solution

Since $|ae^{-j\omega}| = |a| < 1$, use of the geometric summation formula again yields

$$X(\omega) = \frac{1}{1 - ae^{-j\omega}}$$

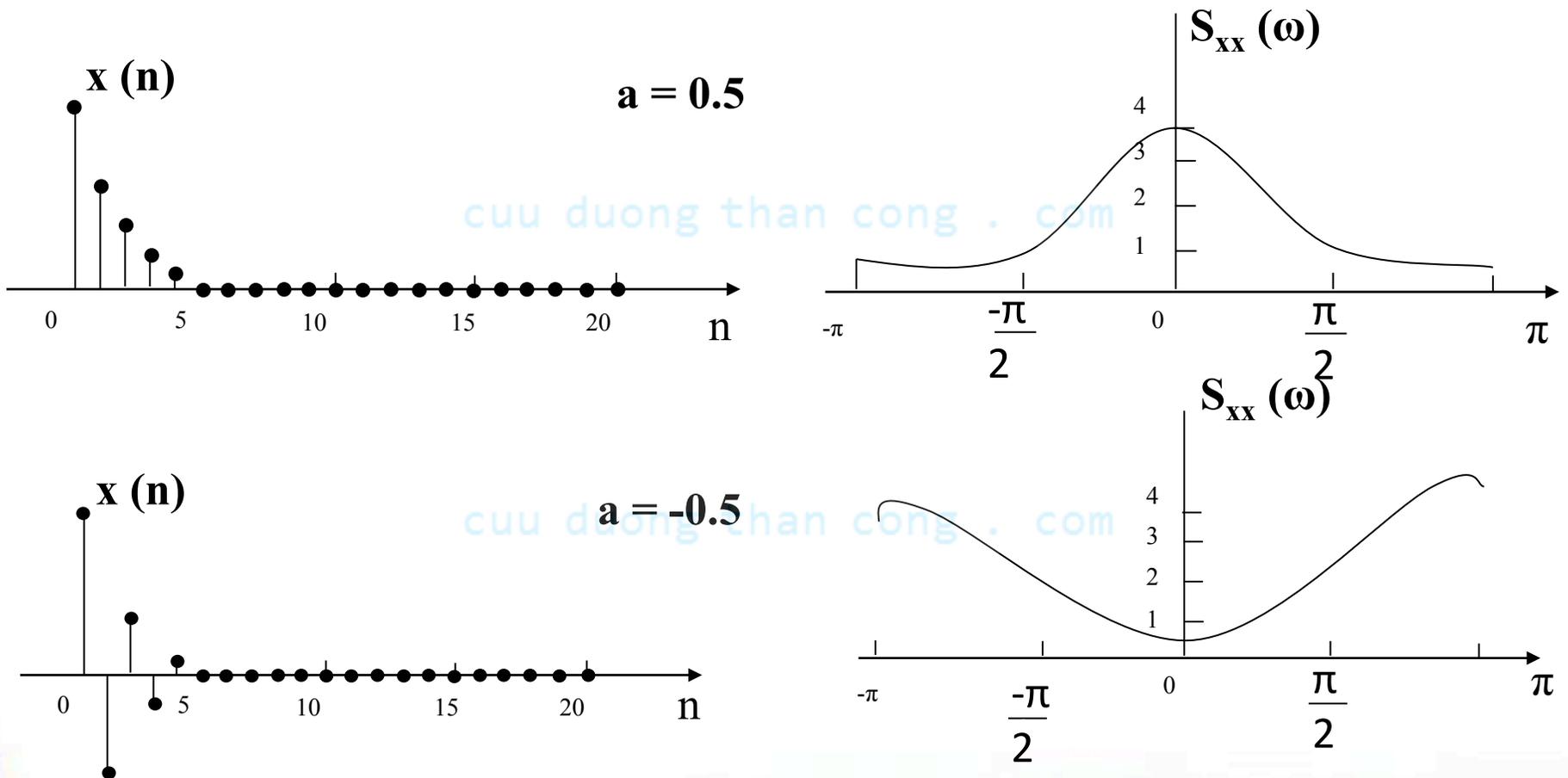The energy density spectrum is given by

$$S_{xx}(\omega) = |X(\omega)|^2 = X(\omega)X^*(\omega) = \frac{1}{(1 - ae^{-j\omega})(1 - ae^{j\omega})}$$

or, equivalently, as

$$S_{xx}(\omega) = \frac{1}{1 - 2a\cos\omega + a^2}$$

Note that $S_{xx}(-\omega) = S_{xx}(\omega)$ in accordance with (4.2.47).

EXAMPLE 4.2.3 Solution

Figure 4.15 **(a)** Sequence $x(n) = (1/2)^n u(n)$ and $x(n) = (-1/2)^n u(n);$ **(b)** their energy density spectra
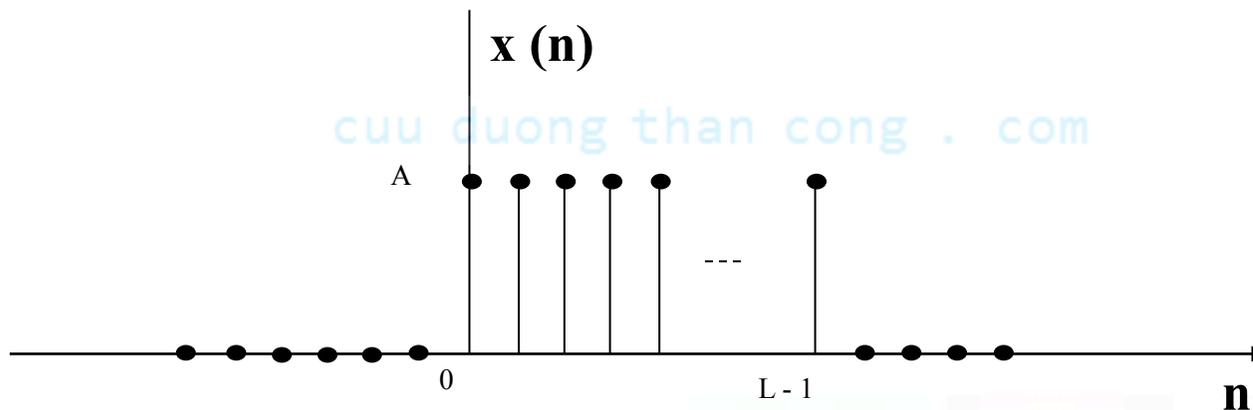
EXAMPLE 4.2.4

Determine the Fourier transform and the energy density spectrum of the sequence

$$x(n) = \begin{cases} A, & 0 \le n \le L - 1 \\ 0, & \text{otherwise} \end{cases} \qquad (4.2.48)$$

which is illustrated in Fig 4.16.

Figure 4.16 Discrete – time rectangular pulse.

EXAMPLE 4.2.4 Solution

Before computing the Fourier transform, we observe that

$$\sum_{n=-\infty}^{\infty} |x(n)| = \sum_{n=0}^{L-1} |A| = L|A| < \infty$$

Hence *x(n)* is absolutely summable and its Fourier transform exists. We note that *x(n)* is a finite – energy signal with $E_x = |A|^2 L$.

The Fourier transform of this signal is

$$X(\omega) = \sum_{n=0}^{L-1} A e^{-j\omega n} \qquad (4.2.29)$$

$$= A \frac{1 - e^{-j\omega L}}{1 - e^{-j\omega}} = A e^{-j(\omega/2)(L-1)} \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

EXAMPLE 4.2.4 Solution

The magnitude and phase spectra of *x(n)* are

$$|X(\omega)| = \begin{cases} |A|L, & \omega = 0 \\ |A| \left| \dfrac{\sin(\omega L/2)}{\sin(\omega/2)} \right|, & otherwise \end{cases} \qquad (4.2.50)$$
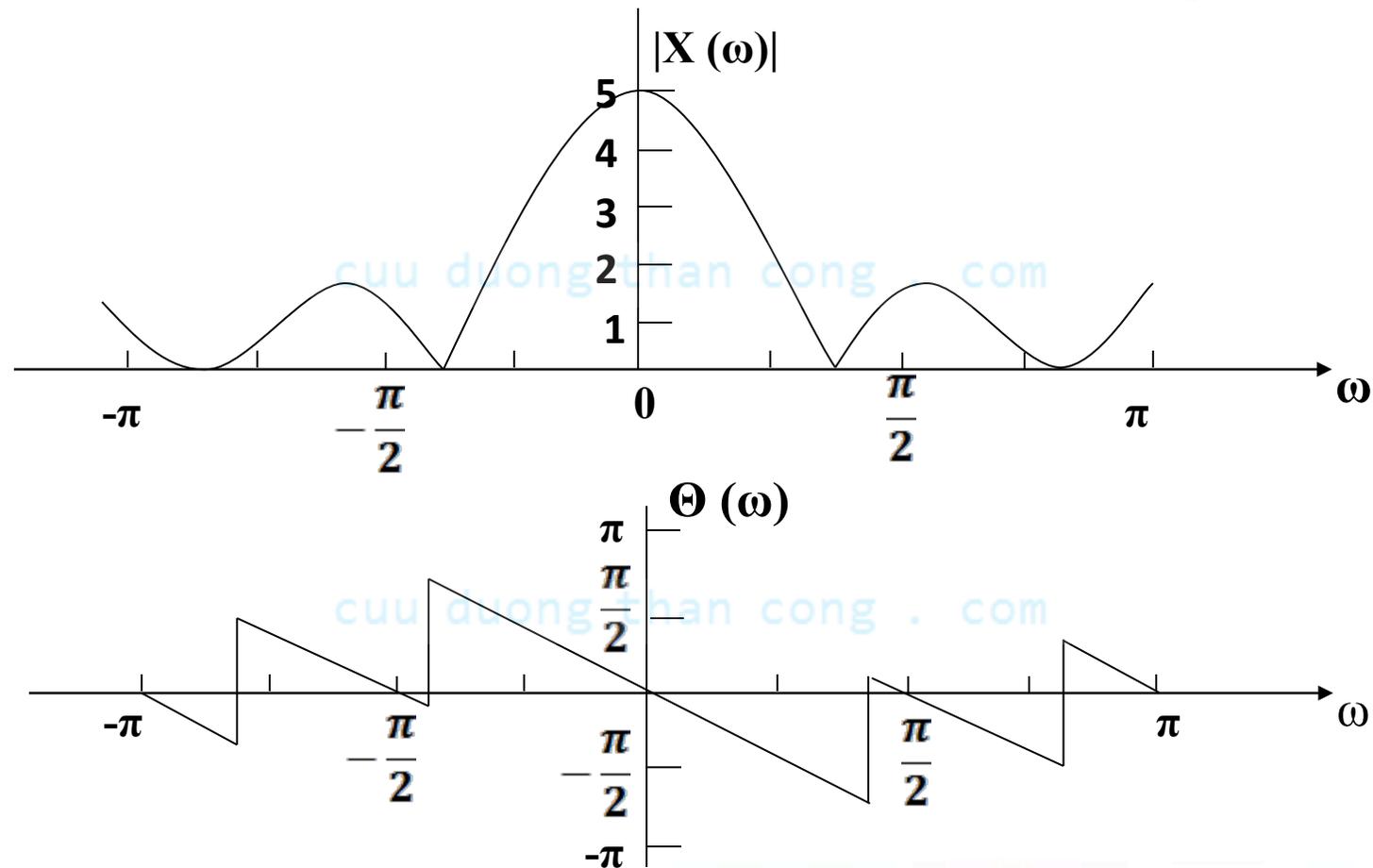
and

$$\angle X(\omega) = \angle A - \frac{\omega}{2}(L-1) + \angle \frac{\sin(\omega L/2)}{\sin(\omega/2)} \qquad (4.2.51)$$

The spectra $|X(\omega)|$ and ▣$X(\omega)$ are shown in Fig 4.17 for the case *A= 1* and *L = 5*. The energy density spectrum is simply the square of the expression given in (4.2.50).

EXAMPLE 4.2.4 Solution

Figure 4.17 Magnitude and phase of Fourier transform of the discrete- time rectangular pulse.

EXAMPLE 4.2.5

Determine the Fourier transform of the following signals.

(a) $x_1(n) = u(n)$ **(b)** $x_2(n) = (-1)^n u(n)$

(c) $x_3(n) = (\cos\omega_0 n) u(n)$

by evaluating their z-transform on the unit circle.

Solution. **(a)** From Table 3.3 we find that

$$X_1(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1} \quad ROC: |z| > 1$$

$X_1(z)$ has a pole, $p_1 = 1$, on the unit circle, but converges for $|z| > 1$.

EXAMPLE 4.2.5 Solution

If we evaluate $X_1(z)$ on the unit circle, except at $z = 1$, we obtain

$$X_1(\omega) = \frac{e^{j\omega/2}}{2j\sin(\omega/2)} = \frac{1}{2\sin(\omega/2)}e^{j(\omega-\pi/2)}, \qquad \omega \neq 2\pi k, \qquad k = 0, 1, \dots$$

At $\omega = 0$ and multiples of 2π, $X_1(\omega)$ contains impulses of area π.

Hence the presence of a pole at z = 1 (i.e., at $\omega = 0$) creates a problem only when we want to compute $|X_1(\omega)|$ at $\omega = 0$, because $|X_1(\omega)| \to \infty$ as $\omega \to 0$.

EXAMPLE 4.2.5 Solution

(b) From Table 3.3 we find that the *z*-transform of a^n *u(n)* with *a = -1* reduces to

$$X_2(z) = \frac{1}{1 + z^{-1}} = \frac{z}{z + 1}, \qquad ROC: |z| > 1$$

which has a pole at *z = -1 = $e^{j\pi}$*. The Fourier transform evaluated at frequencies other than *ω = π* and multiples of 2π is

$$X_2(\omega) = \frac{e^{j\omega/2}}{2\cos(\omega/2)}, \qquad \omega \neq 2\pi\left(k + \frac{1}{2}\right), \qquad k = 0, 1, \dots$$

In this case the impulse occurs at *ω = π + 2πk*.

EXAMPLE 4.2.5 Solution

Hence the magnitude is

$$|X_2(\omega)| = \frac{1}{2|\cos(\omega/2)|}, \qquad \omega \neq 2\pi k + \pi, \qquad k = 0, 1, \ldots$$

and the phase is

$$\angle X_2(\omega) = \begin{cases} \dfrac{\omega}{2}, & \text{if } \cos\dfrac{\omega}{2} \geq 0 \\[2mm] \dfrac{\omega}{2} + \pi & \text{if } \cos\dfrac{\omega}{2} < 0 \end{cases}$$

Due to the presence of the pole at *a = -1* (i.e., at frequency ω = π) , the magnitude of the Fourier transform becomes infinite. Now *|X(ω)| → ∞ as ω → π* . We observe that *(-1)ⁿ u(n)= (cosπn)u(n)* which is the fastest possible oscillating signal in discrete time.

EXAMPLE 4.2.5 Solution

(c) $X_3(\omega)$ is infinite at the frequency component $\omega=\omega_0$, $\omega=-\omega_0$. Indeed, from Table 3.3, we find that

$$x_3(n) = (\cos\omega_0 n)u(n) \overset{z}{\longleftrightarrow} X_3(z) = \frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}, \quad ROC: |z| > 1$$

The Fourier transform is

$$X_3(\omega) = \frac{1-e^{-j\omega}\cos\omega_0}{\left(1-e^{-j(\omega-\omega_0)}\right)\left(1-e^{-j(\omega+\omega_0)}\right)}, \quad \omega \neq \pm\omega_0+2\pi k, \quad k=0,1,\ldots$$

The magnitude of $X_3(\omega)$ is given by

$$|X_3(\omega)| = \frac{\left|1-e^{-j\omega}\cos\omega_0\right|}{\left|1-e^{-j(\omega-\omega_0)}\right|\left|1-e^{-j(\omega+\omega_0)}\right|}, \quad \omega \neq \pm\omega_0+2\pi k, \quad k=0,1,\ldots$$

EXAMPLE 4.3.1

Determine and sketch $X_R(\omega)$, $X_I(\omega)$, $|X(\omega)|$, and ☑$X(\omega)$ for the Fourier transform

$$X(\omega) = \frac{1}{1 - ae^{-j\omega}}, \quad -1 < a < 1 \quad (4.3.38)$$

Solution. By multiplying both the numerator and denominator of (4.4.38), we obtain

$$X(\omega) = \frac{1 - ae^{j\omega}}{(1 - ae^{-j\omega})(1 - ae^{j\omega})} = \frac{1 - a\cos\omega - ja\sin\omega}{1 - 2a\cos\omega + a^2}$$

This expression can be subdivided into real imaginary parts.

EXAMPLE 4.3.1 Solution

Thus we obtain

$$X_R(\omega) = \frac{1 - a\cos\omega}{1 - 2a\cos\omega + a^2} \qquad X_I(\omega) = \frac{a\sin\omega}{1 - 2a\cos\omega + a^2}$$

Substitution of the last two equations into (4.3.15) and (4.3.16) yields the magnitude and phase spectra as

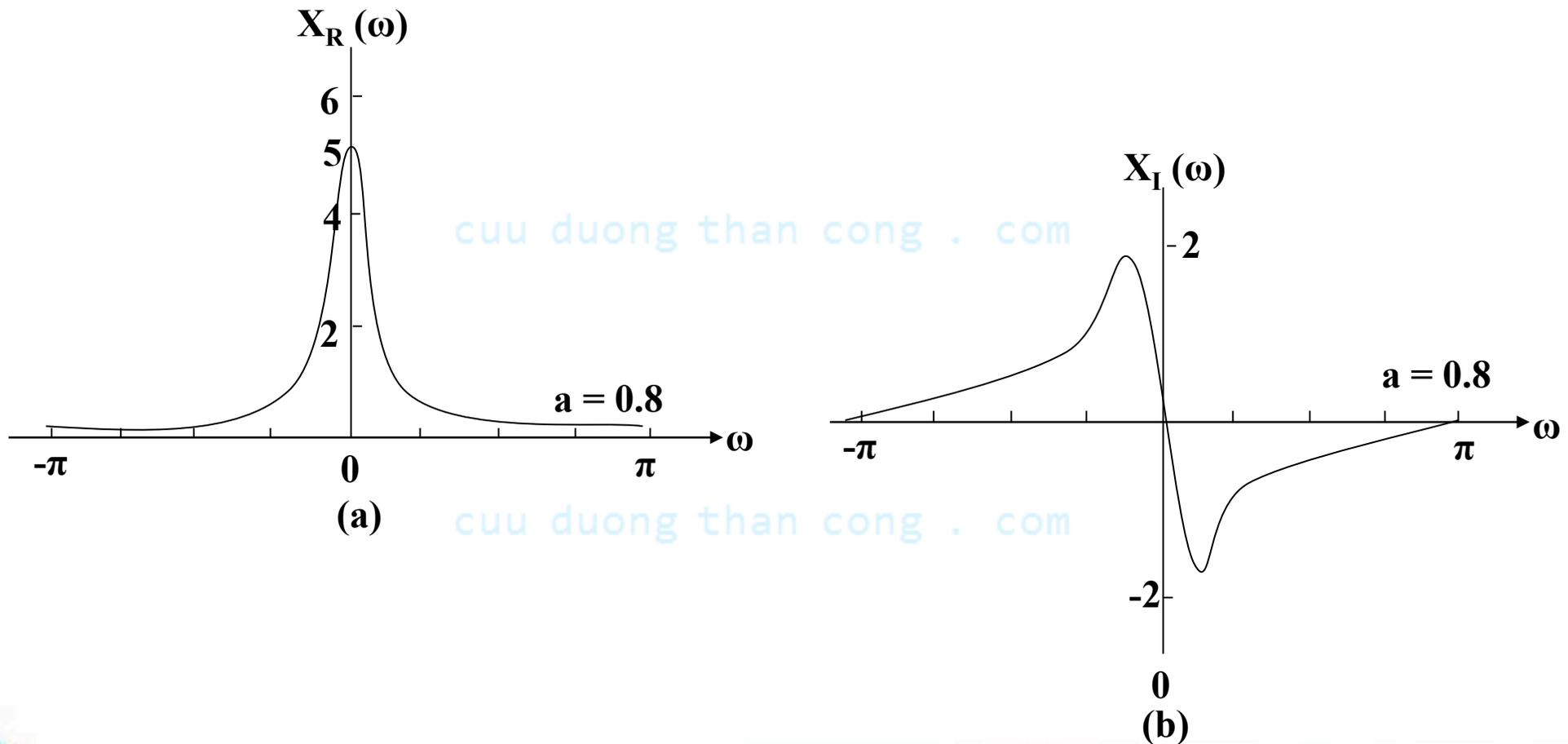$$|X(\omega)| = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}} \qquad (4.3.39)$$

and

$$\angle X(\omega) = -\tan^{-1}\frac{a\sin\omega}{1 - a\cos\omega} \qquad (4.3.40)$$

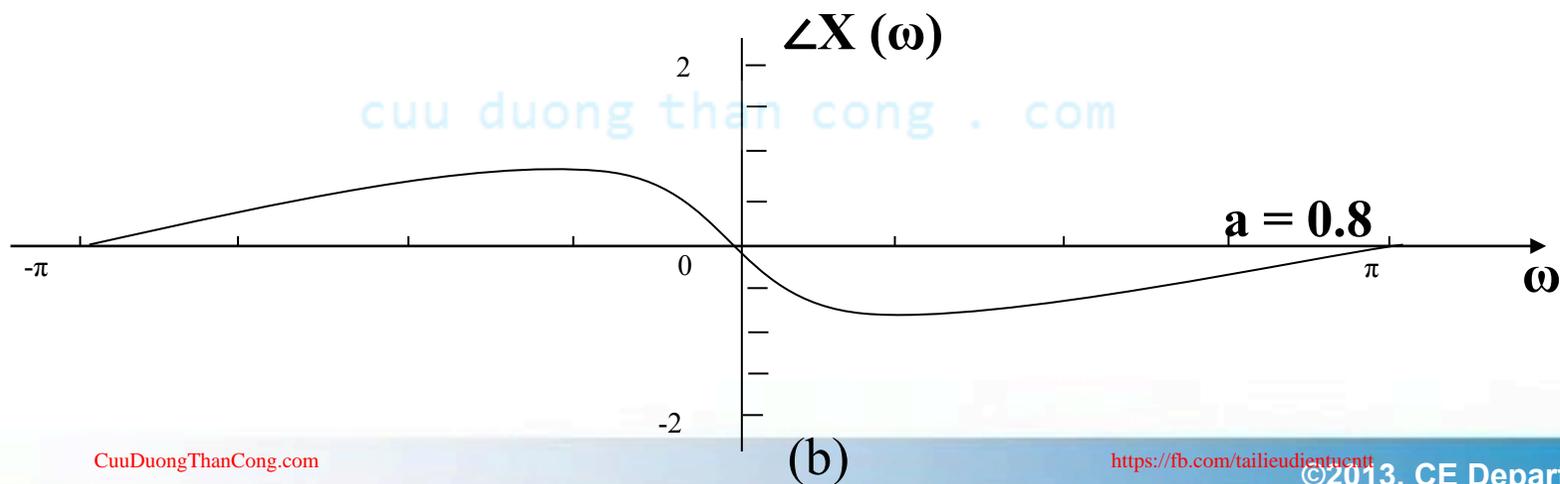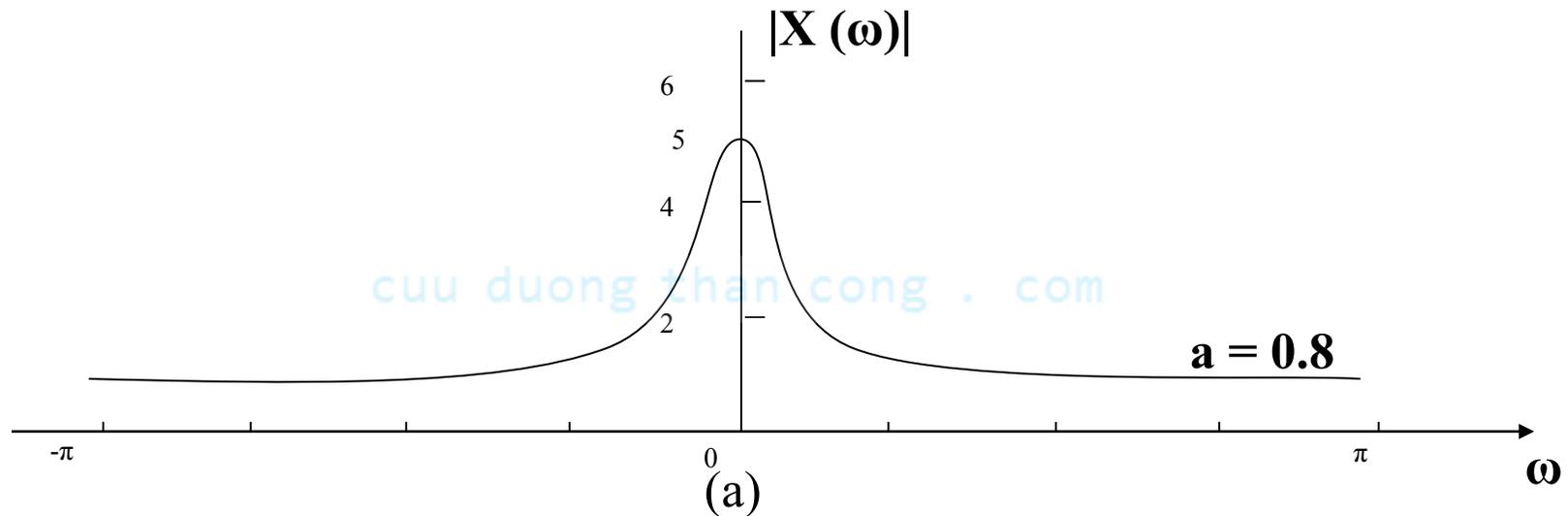The representation of these spectra for *a = 0.8*.

EXAMPLE 4.3.1 Solution

Figure 4.30 Graph of $X_R(\omega)$ and $X_I(\omega)$ for the transform in Example 4.3.1.

EXAMPLE 4.3.1 Solution

Figure 4.31 Magnitude and phase spectra of the transform in Example 4.3.1

(a)

(b)

EXAMPLE 4.4.1

Determine the output sequence of the system with impulse response

$$h(n) = (\frac{1}{2})^n u(n) \qquad\qquad (4.4.6)$$

when the input is the complex exponential sequence $x(n) = Ae^{j\pi n/2}$, $-\infty < n < \infty$

Solution. First we evaluate the Fourier transform of the impulse response $h(n)$, and then we use (4.4.5) to determine $y(n)$. From Example 4.2.3 we recall that

$$H(\omega) = \sum_{n-\infty}^{\infty} h(n)\, e^{-j\omega n} = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} \qquad\qquad (4.4.7)$$

EXAMPLE 4.4.1 Solution

At $\omega = \pi /2$, (4.4.7) yields

$$H\left(\frac{\pi}{2}\right) = \frac{1}{1 + j\frac{1}{2}}\frac{2}{\sqrt{5}}e^{-j26.6^\circ}$$

and therefore the output is

$$y(n) = A\left(\frac{2}{\sqrt{5}}e^{-j26.6^\circ}\right)e^{j\pi n/2} \qquad (4.4.8)$$

$$y(n) = \frac{2}{\sqrt{5}}Ae^{j(\pi n/2 - 26.6^\circ)}, \qquad -\infty < n < \infty$$

Determine the magnitude and phase of *H(ω)* for the three-point moving average (MA) system

$$y(n) = \frac{1}{3}[x(n+1) + x(n) + x(n-1)]$$

and plot these two functions for *0 ≤ ω ≤ π*.

Solution. Since $h(n) = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$

it follows that

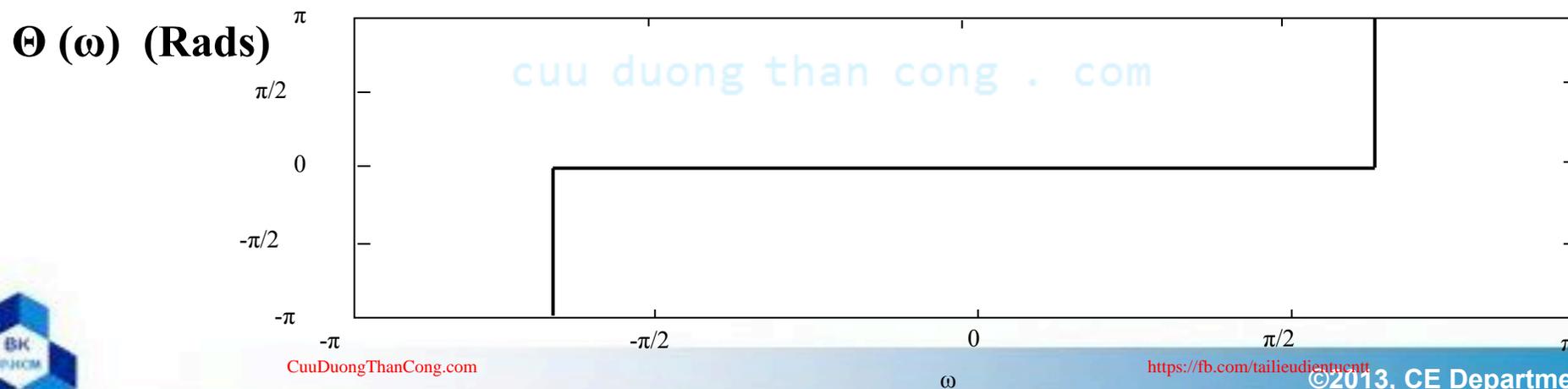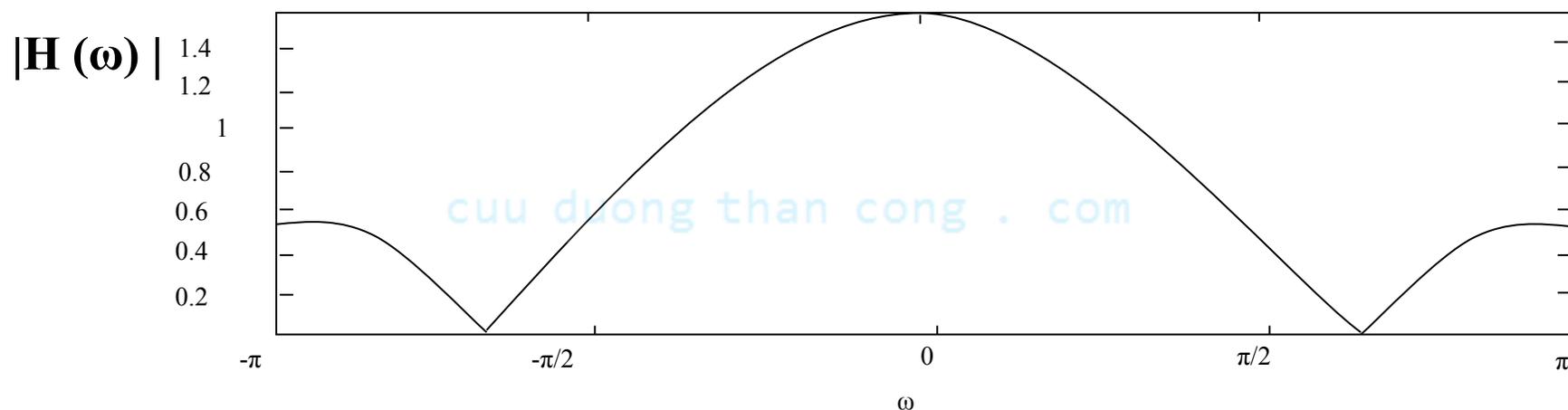$$H(\omega) = \frac{1}{3}(e^{j\omega} + 1 + e^{-j\omega}) = \frac{1}{3}(1 + 2cos\omega)^{\uparrow}$$

Hence

$$|H(\omega)| = \frac{1}{3}|1 + 2cos\omega| \qquad (4.4.16)$$

$$\Theta(\omega) = \begin{cases} 0, & 0 \le \omega \le 2\pi/3 \\ \pi, & 2\pi/3 \le \omega \le \pi \end{cases}$$

EXAMPLE 4.4.2 Moving Average Filter

Figure 4.37 Magnitude and phase response for the MA system in Example 4.4.2.

EXAMPLE 4.4.3

Determine the response of the system in Example 4.4.1 to the input signal

$$x(n) = 10 - 5\sin\frac{\pi}{2}n + 20\cos\pi n, \qquad -\infty < n < \infty$$

Solution. The frequency response of the system is given in (4.4.7) as

$$H(\omega) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

The first term in the input signal is a fixed signal component corresponding to $\omega = 0$. Thus

$$H(0) = \frac{1}{1 - \frac{1}{2}} = 2$$

EXAMPLE 4.4.3 Solution

The second term in *x(n)* has a frequency *π/2*. At this frequency the frequency response of the system is

$$H\left(\frac{\pi}{2}\right) = \frac{2}{\sqrt{5}} e^{-j26.6^\circ}$$

Finally, the third term in *x(n)* has a frequency *ω = π*. At this frequency

$$H(\pi) = \frac{2}{3}$$

Hence the response system to *x(n)* is

$$y(n) = 20 - \frac{10}{\sqrt{5}} \sin\left(\frac{\pi}{2}n - 26.6^\circ\right) + \frac{40}{3} \cos\pi n, \quad -\infty < n < \infty$$

EXAMPLE 4.4.5

A linear time-invariant system is characterized by its impulse response

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

Determine the spectrum and the energy density spectrum of the output signal when the system is excited by the signal

$$x(n) = \left(\frac{1}{4}\right)^n u(n)$$

Solution. The frequency response function of the system

$$H(\omega) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\omega n} = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

EXAMPLE 4.4.5 Solution

Hence the spectrum of the signal at the output of the system is

$$Y(\omega) = H(\omega)X(\omega) = \frac{1}{(1-\frac{1}{2}e^{-j\omega})(\frac{1}{4}e^{-j\omega})}$$

The corresponding energy density spectrum is

$$S_{yy}(\omega) = |Y(\omega)|^2 = |H(\omega)|^2|X(\omega)|^2 = \frac{1}{(\frac{5}{4}-\cos\omega)(\frac{17}{16}-\frac{1}{2}\cos\omega)}$$

EXAMPLE 4.4.7

Evaluate the frequency response of the system described by the system function

$$H(z) = \frac{1}{1 - 0.8z^{-1}} = \frac{z}{z - 0.8}$$

Solution. Clearly, $H(z)$ has a zero at $z = 0$ and a pole at $p = 0.8$. Hence the frequency response of the system is

$$H(\omega) = \frac{e^{j\omega}}{e^{j\omega} - 0.8}$$

The magnitude response is

$$|H(\omega)| = \frac{|e^{j\omega}|}{|e^{j\omega} - 0.8|} = \frac{1}{\sqrt{1.64 - 1.6\cos\omega}}$$

EXAMPLE 4.4.7 Solution

and the phase response is

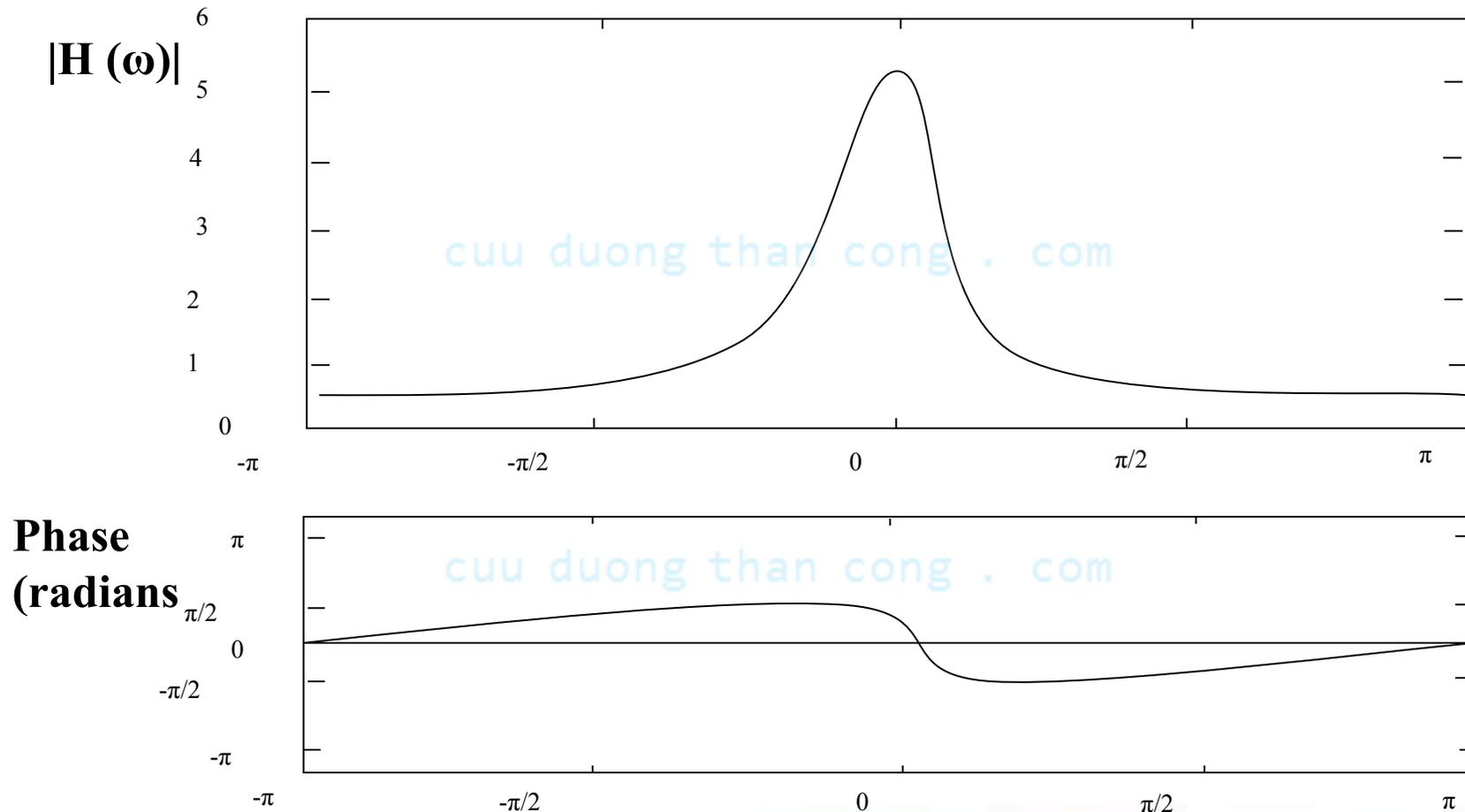$$\theta(\omega) = \omega - \tan^{-1} \frac{\sin\omega}{\cos\omega - 0.8}$$

The magnitude and phase response are illustrated in Fig.4.42. Note that the peak of the magnitude response occurs at $\omega = 0$, the point on the unit circle closest to the pole located at *0.8*.

EXAMPLE 4.4.7 Solution

Figure 4.42 Magnitude and phase of system with $H(z) = 1 / (1 - 0.8z^{-1})$.

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EXAMPLE 4.5.1

A two-pole lowpass filter has the system function

$$H(\omega) = \frac{b_0}{(1 - pz^{-1})^2}$$

Determine the values of b_0 and p such that the frequency response $H(\omega)$ satisfies the conditions

$$H(0) = 1 \quad \text{and} \quad \left| H\left(\frac{\pi}{4}\right) \right|^2 = \frac{1}{2}$$

Solution.

At $\omega = 0$ we have

$$H(0) = \frac{b_0}{(1 - p)^2} = 1$$

Hence $\quad b_0 = (1 - p)^2$

At $\omega = \pi/4$,

$$H\left(\frac{\pi}{4}\right) = \frac{(1 - p)^2}{(1 - pe^{-j\pi/4})^2}$$

EXAMPLE 4.5.1 Solution

$$H\left(\frac{\pi}{4}\right) = \frac{(1-p)^2}{(1-p\cos(\pi/4)+jp\sin(\pi/4))^2} = \frac{(1-p)^2}{(1-p/\sqrt{2}+jP/\sqrt{2})^2}$$

Hence
$$\frac{(1-p)^4}{[(1-p/\sqrt{2})^2+p^2/2]^2} = \frac{1}{2}$$

or, equivalently,
$$\sqrt{2}(1-p)^2 = 1+p^2-\sqrt{2}p$$

The value of *p = 0.32* satisfies this equation. Consequently, the system function for the desired filter is

$$H(z) = \frac{0.46}{(1-0.32z^{-1})^2}$$

EXAMPLE 4.5.2

Design a two-pole bandpass filter that has the center of its passband at $\omega = \pi/2$, zero in its frequency response characteristic at $\omega = 0$ and $\omega = \pi$, and a magnitude response of at $\omega = 4\pi/9$.

Solution.

Cleary, the filter must have poles at $p_{1,2} = re^{\pm j\pi/2}$ and zeros at $z = 1$ and $z = -1$. Consequently, the system function is

$$H(z) = G \frac{(z-1)(z+1)}{(z-jr)(z+jr)} = G \frac{z^2 - 1}{z^2 + r^2}$$

EXAMPLE 4.5.2 Solution

The gain factor is determined by evaluating the frequency response *H(ω)* of the filter at *ω = π/2*. Thus we have

$$H\left(\frac{\pi}{2}\right) = G\frac{1}{1-r^2} = 1 \qquad G = \frac{1-r^2}{2}$$

 The value of *r* is determine by evaluating *H(ω)* at *ω = 4π/9*. Thus we have

$$\left|H\left(\frac{4\pi}{9}\right)\right|^2 = \frac{(1-r^2)^2}{4}\frac{2-2\cos(8\pi/9)}{1+r^4+2r^2\cos(8\pi/9)} = \frac{1}{2}$$

EXAMPLE 4.5.2 Solution

or, equivalently,

$$1.94(1 - r^2)^2 = 1 - 1.88r^2 + r^4$$

The value of $r^2 = 0.7$ satisfies this equation. Therefore, the system function for the desired filter is
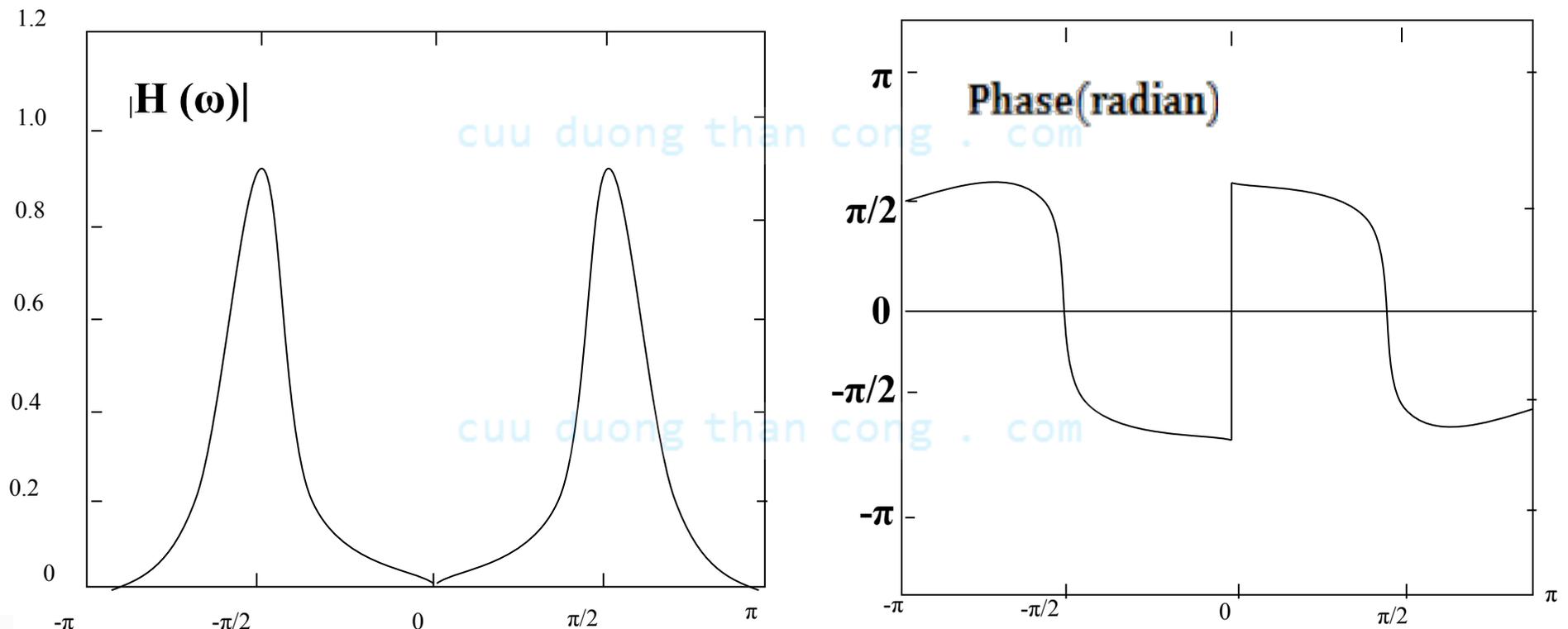
$$H(z) = 0.15 \frac{1 - z^{-2}}{1 + 0.7z^{-2}}$$

Its frequency response is illustrated in Fig 4.47.

EXAMPLE 4.5.2 Solution

Figure 4.47 Magnitude and phase response of a simple bandpass filter:

$H(z) = 0.15 [(1 – z^{-2}) / (1 + 0.7z^{-2})]$

EXAMPLE 4.6.1

Determine the inverse of the system with impulse response $h(n) = (\frac{1}{2})^n u(n)$

Solution. The system function corresponding to *h(n)* is

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} \qquad ROC: |z| > \frac{1}{2}$$

This system is both causal and stable. Since *H(z)* is an all-pole system, is inverse is FIR and is given by the system function $H_I(z) = 1 - \frac{1}{2}z^{-1}$

Hence its impulse response is

$$h_I(n) = \delta(n) - \frac{1}{2}\delta(n-1)$$

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EXAMPLE 4.6.2

Determine the inverse of the system with impulse response

$$h(n) = \delta(n) - \frac{1}{2}\delta(n-1)$$

Solution. This is an FIR system and its system function is

$$H(z) = 1 - \frac{1}{2}z^{-1}, \qquad ROC: |z| > 0$$

The inverse system has the system function

$$H_I(z) = \frac{1}{H(z)} = \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{z}{z - \frac{1}{2}}$$

Thus $H_I(z)$ has a zero at the origin and a pole at z=½ .

EXAMPLE 4.6.2 Solution

In this case there are two possible regions of convergence and hence two possible inverse system, as illustrated in Fig 4.63. If we take the ROC of $H_I(z)$ as $|z| > \frac{1}{2}$, the inverse system has an impulse response:(a)

$$h_I(n) = \left(\frac{1}{2}\right)^n u(n)$$

which is the impulse response of a causal and stable system. On the other hand, if the ROC is assumed to be $|z| < \frac{1}{2}$, the inverse system has an impulse response:(b)
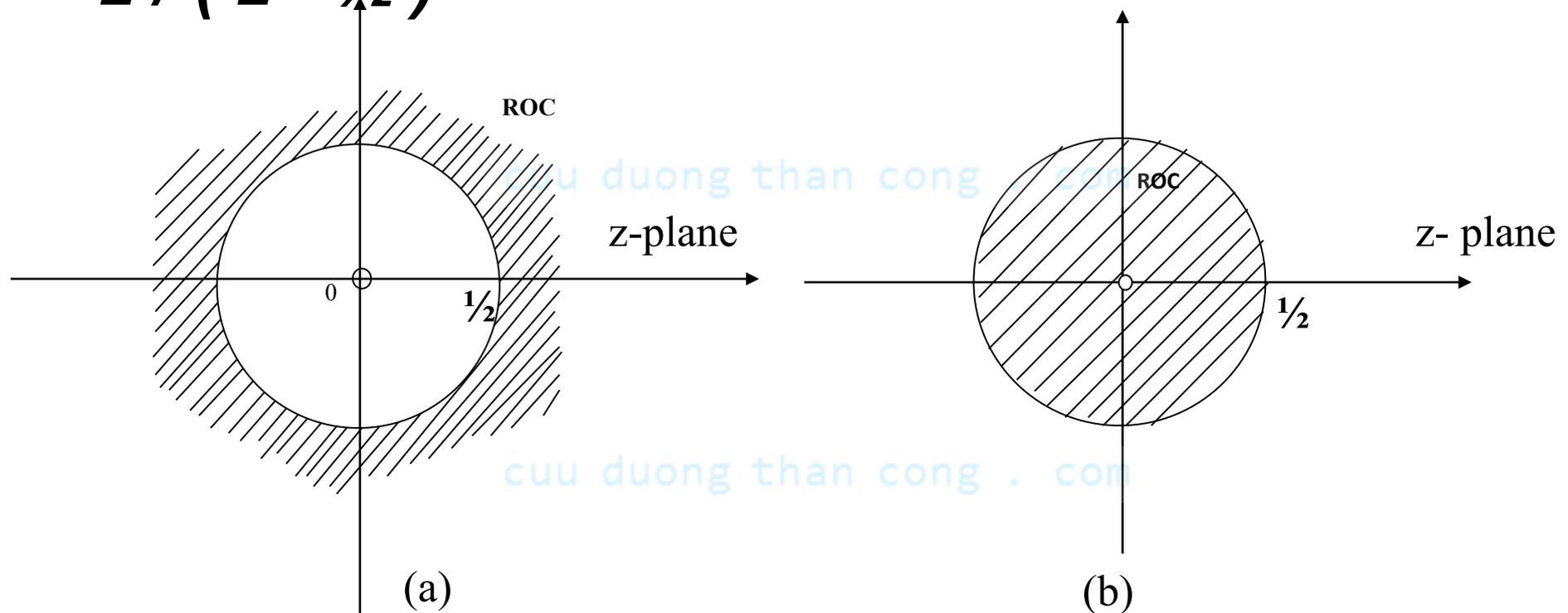
$$h_I(n) = \left(\frac{1}{2}\right)^n u(-n-1)$$

The inverse system is anticausal and unstable.

EXAMPLE 4.6.2 Solution

Figure 4.63 Two possible regions of convergence *H(z) =* *z / (z - ½)*

(a)

(b)

EXAMPLE 4.6.3

Determine the causal inverse of the FIR system with impulse response

$$h(n) = \delta(n) - \alpha \, \delta(n - 1)$$

Solution. Since $h(0) = 1$, $h(1) = -\alpha$, and $h(n) = 0$ for $n \geq \alpha$, we have :

$h_I(0) = 1 / h(0) = 1$ and $h_I(n) = \alpha h_I(n - 1)$, $n \geq 1$

Consequently, $h_I(1) = \alpha$, $h_I(2) = \alpha^2, \ldots, h_I(n) = \alpha^n$ which corresponds to a causal IIR system as expected.

EXAMPLE 4.6.5

A causal system produces the output sequence

$$y(n) = \begin{cases} 1, & n = 0 \\ \dfrac{7}{10}, & n = 1 \\ 0, & otherwise \end{cases}$$

when excited by the input sequence

$$x(n) = \begin{cases} 1, & n = 0 \\ -\dfrac{7}{10}, & n = 1 \\ \dfrac{1}{10}, & n = 2 \\ 0, & otherwise \end{cases}$$

Determine its impulse response and its input-output equation.

EXAMPLE 4.6.5 Solution

The system function is easily determine by taking the *z*-transforms of *x(n)* and *y(n).*

Thus we have

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{7}{10}z^{-1}}{1 - \frac{7}{10}z^{-1} + \frac{1}{10}z^{-2}} = \frac{1 + \frac{7}{10}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{5}z^{-1}\right)}$$

Since the system is causal, its ROC is |z| > ½ . The system is also stable since its poles lie inside the unit circle.

EXAMPLE 4.6.5 Solution

The input – output difference equation for the system is

$$y(n) = \frac{7}{10}y(n-1) - \frac{1}{10}y(n-2) + x(n) + \frac{7}{10}x)n-1)$$

Its impulse response is determine by performing a partial-faction expansion of *H*(z) and inverse transforming the result. This computation yields

$$h(n) = [4(\frac{1}{2})^n - 3(\frac{1}{5})^n]u(n)$$