# EXAMPLE 5.1.1

Consider the signal $x(n) = a^n u(n)$, $0 < a < 1$

The spectrum of this signal is sampled at frequencies $\omega_k = 2\pi k / N$, $k = 0, 1, \ldots, N - 1$.

Determine the reconstructed spectra for $a = 0.8$ when $N = 5$ and $N = 50$.



**Solution.** The Fourier transform of the sequence $x(n)$ is

$$X(\omega) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \frac{1}{1 - ae^{-j\omega}}$$



Suppose that we sample $X(\omega)$ at $N$ equidistant frequencies $\omega_k = 2\pi k / N$, $k = 0, 1, \ldots, N - 1$.

# EXAMPLE 5.1.1 Solution

Thus we obtain the spectral samples

$$X(\omega k) \equiv X\left(\frac{2\pi k}{N}\right) = \frac{1}{1-ae^{-j2\pi k/N}}, \qquad k = 0, 1, \ldots, N-1$$

The periodic sequence $x_p(n)$, corresponding to the frequency samples $X(2\pi k/N)$, $k = 0,1, \ldots,N-1$, can be obtained from either (5.1.4) or (5.1.8). Hence



$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN) = \sum_{l=-\infty}^{0} a^{n-lN}$$

# EXAMPLE 5.1.1 Solution

$$x_p(n) = a^n \sum_{l=0}^{\infty} a^{lN} = \frac{a^n}{1-a^N}, \qquad 0 \le n \le N-1$$

where the factor $1/(1-a^N)$ represents the effect of aliasing. Since $0 < a < 1$, the aliasing error tends toward zero as $N \to \infty$.

For $a = 0.8$, the sequence $x(n)$ and its spectrum $X(\omega)$ are shown in Fig.7.1.4(a) and 7.1.4(b), respectively. The aliased sequence $x_p(n)$ for $N = 5$ and $N = 50$ and the corresponding spectral samples are shown in Fig. 7.1.4 (c) and 7.1.4)d), respectively.

# EXAMPLE 5.1.1 Solution

If we define the aliased finite-duration sequence *x(n)* as

$$\hat{x}(n) = \begin{cases} x_p(n), & 0 \le n \le N-1 \\ 0, & otherwise \end{cases}$$

then its Fourier transform is

$$\hat{X}(\omega) = \sum_{n=0}^{N-1} \hat{x}(n) e^{-j\omega n} = \sum_{n=0}^{N-1} x_p(n)\, e^{-j\omega n} = \frac{1}{1-a^N} \cdot \frac{1-a^N e^{-j\omega N}}{1-ae^{-j\omega}}$$

Note that although $\hat{X}(\omega) \neq X(\omega)$, the sample values at $\omega_k = 2\pi k / N$ are identical. That is,

$$\hat{X}\left(\frac{2\pi}{N}k\right) = \frac{1}{1-a^N} \cdot \frac{1-a^N}{1-ae^{-j2\pi k/N}} = X\left(\frac{2\pi}{N}k\right)$$

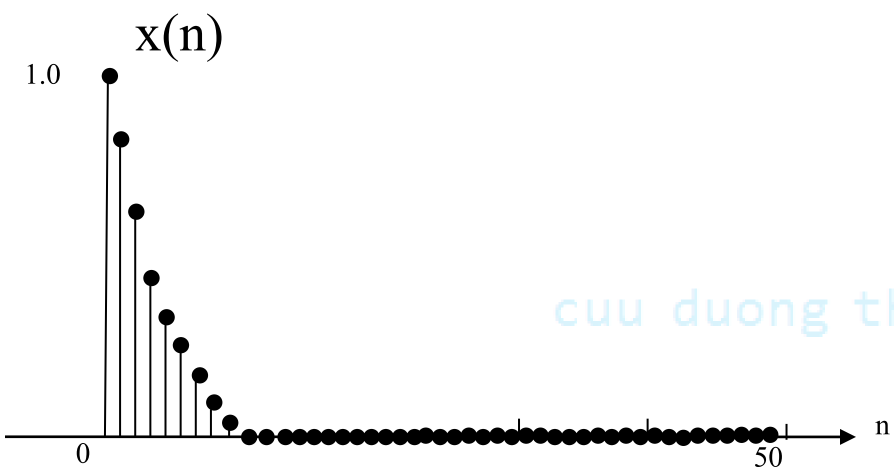
# EXAMPLE 5.1.1 Solution

Figure 5.4 (a) Plot of sequence $x(n) = (0.8)^n u(n)$;
(b) its Fourier transform (magnitude only);

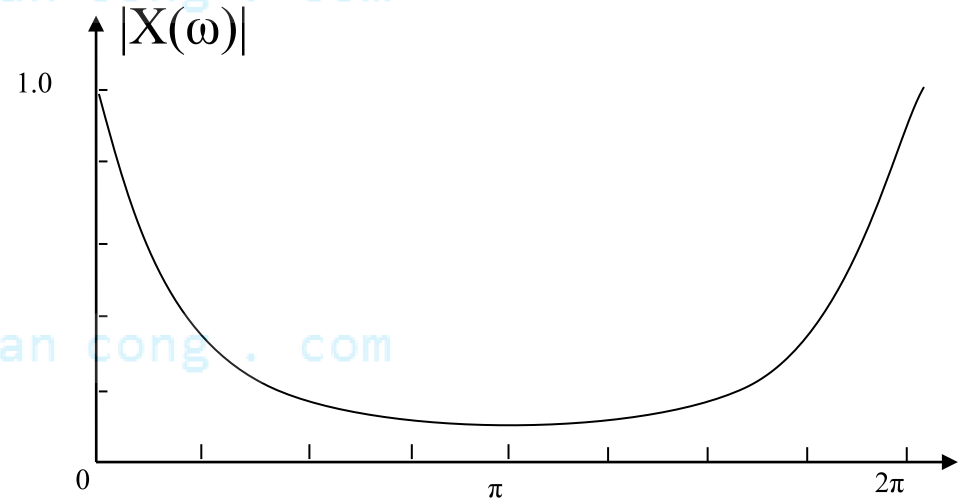
(c) effect of aliasing with N = 5;

(d) reduced effect of aliasing with N = 50



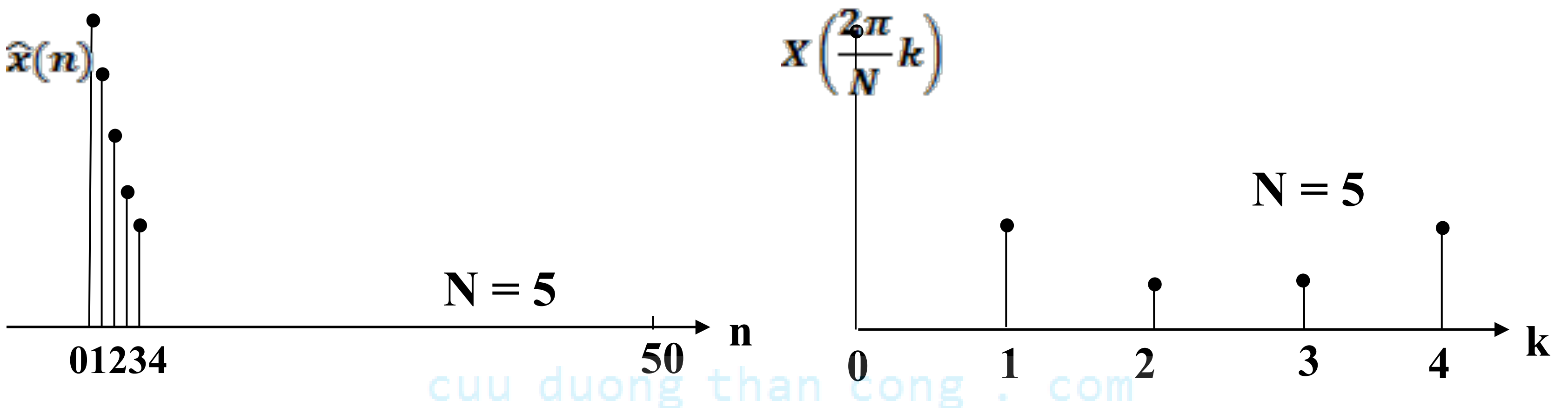



(a)

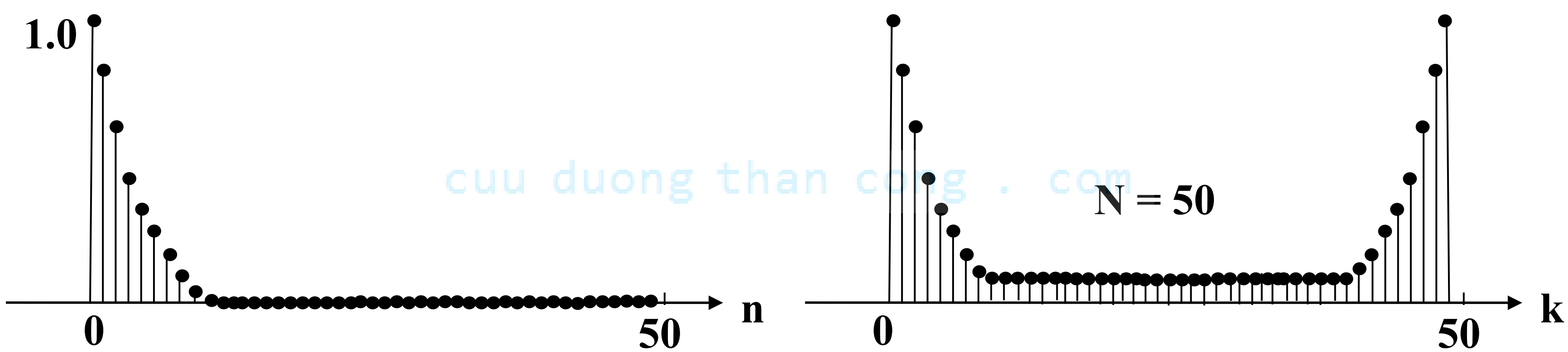



(b)

# EXAMPLE 5.1.1 Solution




(c)




(d)

# EXAMPLE 5.1.2

A finite-duration sequence of length $L$ is given as

$$x(n) = \begin{cases} 1, & 0 \le n \le L-1 \\ 0, & otherwise \end{cases}$$

Determine the $N$-point DFT of this sequence for $N \ge L$.



**Solution.** The Fourier transform of this sequence is



$$X(\omega) = \sum_{n=0}^{L-1} x(n)e^{-j\omega n} = \sum_{n=0}^{L-1} e^{-j\omega n} = \frac{1-e^{-j\omega L}}{1-e^{-j\omega}} = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2}$$

# EXAMPLE 5.1.2 Solution

The magnitude and phase of $X(\omega)$ are illustrated in Fig .7.1.5 for $L = 10$. The $N$-point DFT of $x(n)$ is simply $X(\omega)$ evaluated at the set of $N$ equally spaced frequencies $\omega_k = 2\pi k / N$, $k = 0, 1, \ldots N - 1$. Hence



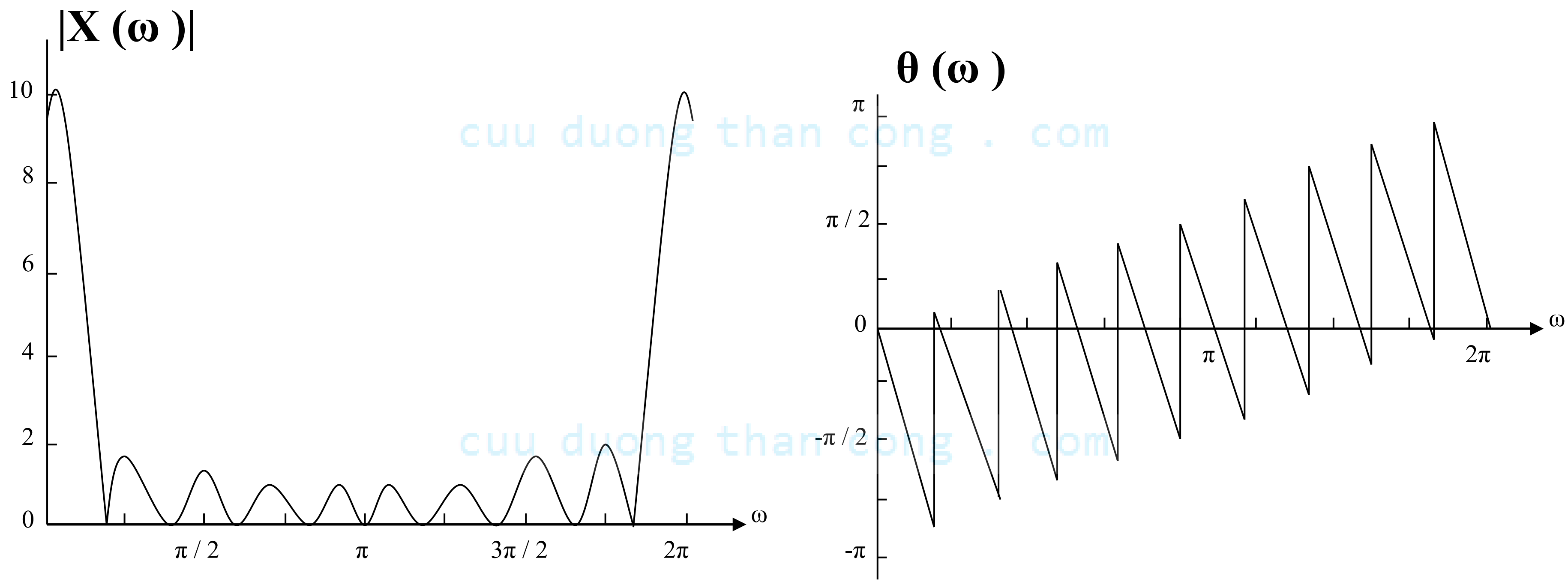
$$X(k) = \frac{1 - e^{-j2\pi kL/N}}{1 - e^{-j2\pi k/N}}, \quad k = 0, 1, \ldots N - 1$$



$$= \frac{\sin(\pi kL/N)}{\sin(\pi k/N)} e^{-j\pi k(L-1)/N}$$

# EXAMPLE 5.1.2 Solution

**Figure 5.5** Magnitude and phase characteristics of the Fourier transform for signal in Example 5.1.2.



# EXAMPLE 5.1.2 Solution

If *N* is selected such that *N*= *L*, then the DFT becomes

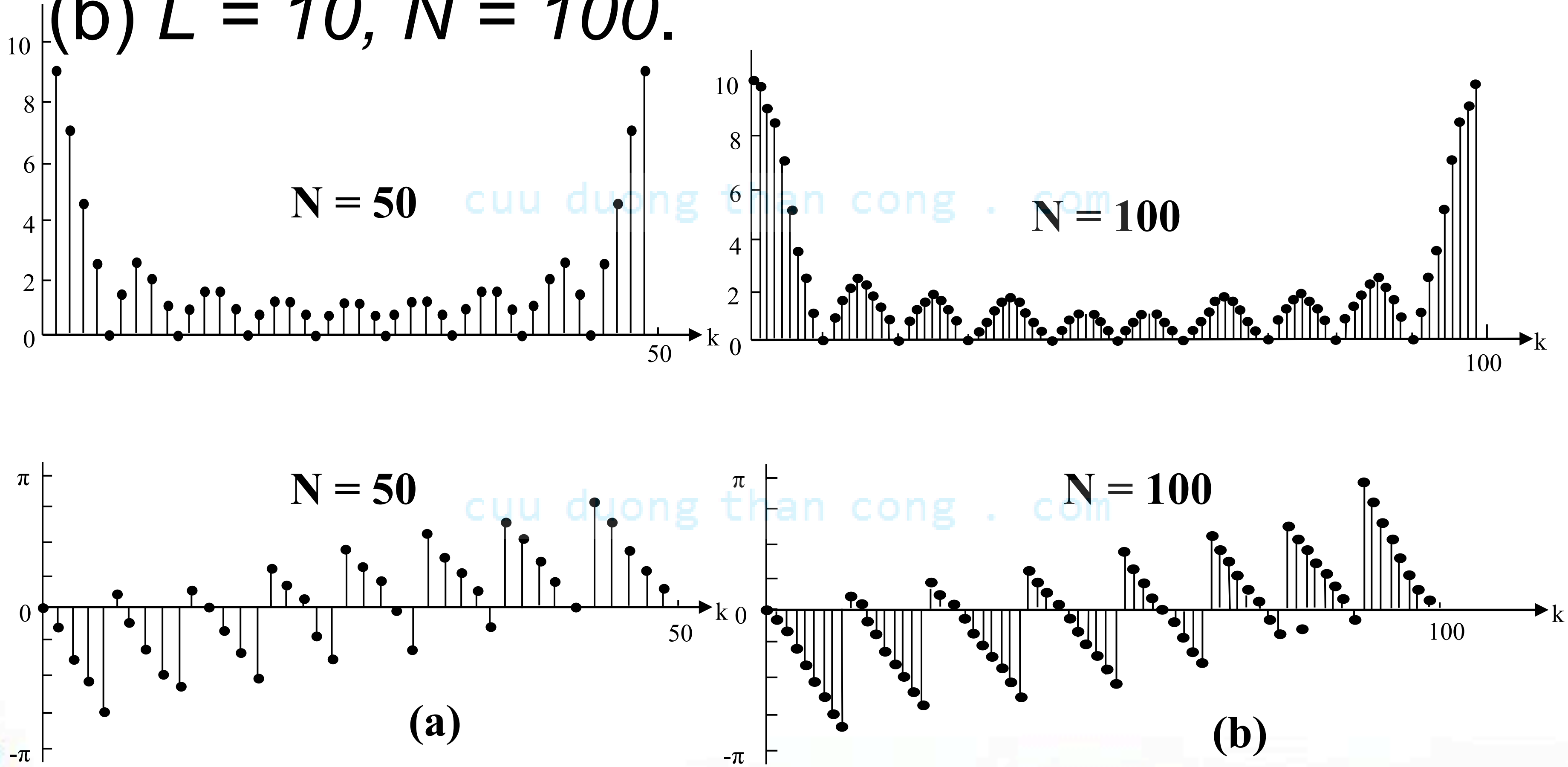
$$X(k) = \begin{cases} L, & k = 0 \\ 0, & k = 1, 2, \ldots, L-1 \end{cases}$$

Thus there only one nonzero value in the DFT . This is apparent from observation of $X(\omega)$, since $X(\omega) = 0$ at the frequencies $\omega_k = 2\pi k / L, k \neq 0$.

If we wish to have a better picture, we must evaluate $X(\omega)$ at more closely spaced frequencies, say $\omega_k = 2\pi k / N$, where $N > L$. Then the *N*-point DFT provides finer interpolation than the *L*-point DFT.

# EXAMPLE 5.1.2 Solution

Figure 5.6 Magnitude and phase of an N-point DFT in example 5.1.2; (a) $L = 10,\ N = 50;$ (b) $L = 10,\ N = 100$.



# EXAMPLE 5.1.3

Compute the DFT of the four-point sequence

$$x(n) = (0 \quad 1 \quad 2 \quad 3)$$

**Solution.** The first step is to determine the matrix $\boldsymbol{W}_4$. By exploiting the periodicity property of $\boldsymbol{W}_4$ and the symmetry property $W_N^{k+N/2} = -W_N^k$

The matrix $\boldsymbol{W}_4$ may be expressed as

$$W_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix}$$

# EXAMPLE 5.1.3 Solution

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Then

$$X_4 = W_4 x_4 = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$





The IDFT of $X_4$ may be determined by conjugating the elements in $W_4$ to obtain $W_4^*$ and then applying the formula (5.1.26).

# EXAMPLE 5.2.1

Perform the circular convolution of the following two sequences:

$x_1(n) = \{ \underset{\uparrow}{2}, 1, 2, 1 \}$  $x_2(n) = \{ \underset{\uparrow}{1}, 2, 3, 4 \}$

**Solution.** it is desirable to graph each sequence as points on a circle. Thus the sequences $x_1(n)$ and $x_2(n)$ are graphed as illustrated in fig.5.8(a) We note that the sequences are graphed in a counterclockwise direction on a circle.

Now, $x_3(m)$ is obtained by circularly convolving $x_1(n)$ with $x_2(n)$ as specified by (5.2.39).

# EXAMPLE 5.2.1 Solution

Beginning with $m = 0$ we have

$$x_3(0) = \sum_{n=0}^{3} x_1(n)x_2((-n))N$$

$x_2((-n))_4$ is simply the sequence $x_2(n)$ folded and graphed on a circle as illustrated in Fig. 5.8(b). In other words, the folded sequence is simply $x_2(n)$ graphed in a clockwise direction.

The product sequence is obtained by multiplying $x_1(n)$ with $x_2((-n))_4$, point by point. This sequence is a illustrated in Fig. 5.8(b). lso illustrated in Fig. 5.8(b).

# EXAMPLE 5.2.1 Solution

We sum the values in the product sequence to obtain $x_3(0) = 14$

For $m = 1$ we have

$$x_3(1) = \sum_{n=0}^{3} x_1(n)\, x_2((1-n))_4$$

$x_3(1) = 16$

For $m = 2$ we have

$$x_3(2) = \sum_{n=0}^{3} x_1(n)\, x_2((2-n))_4$$

$x_3(2) = 14$

For $m = 3$ we have

$$x_3(3) = \sum_{n=0}^{3} x_1(n)\, x_2((3-n))_4$$

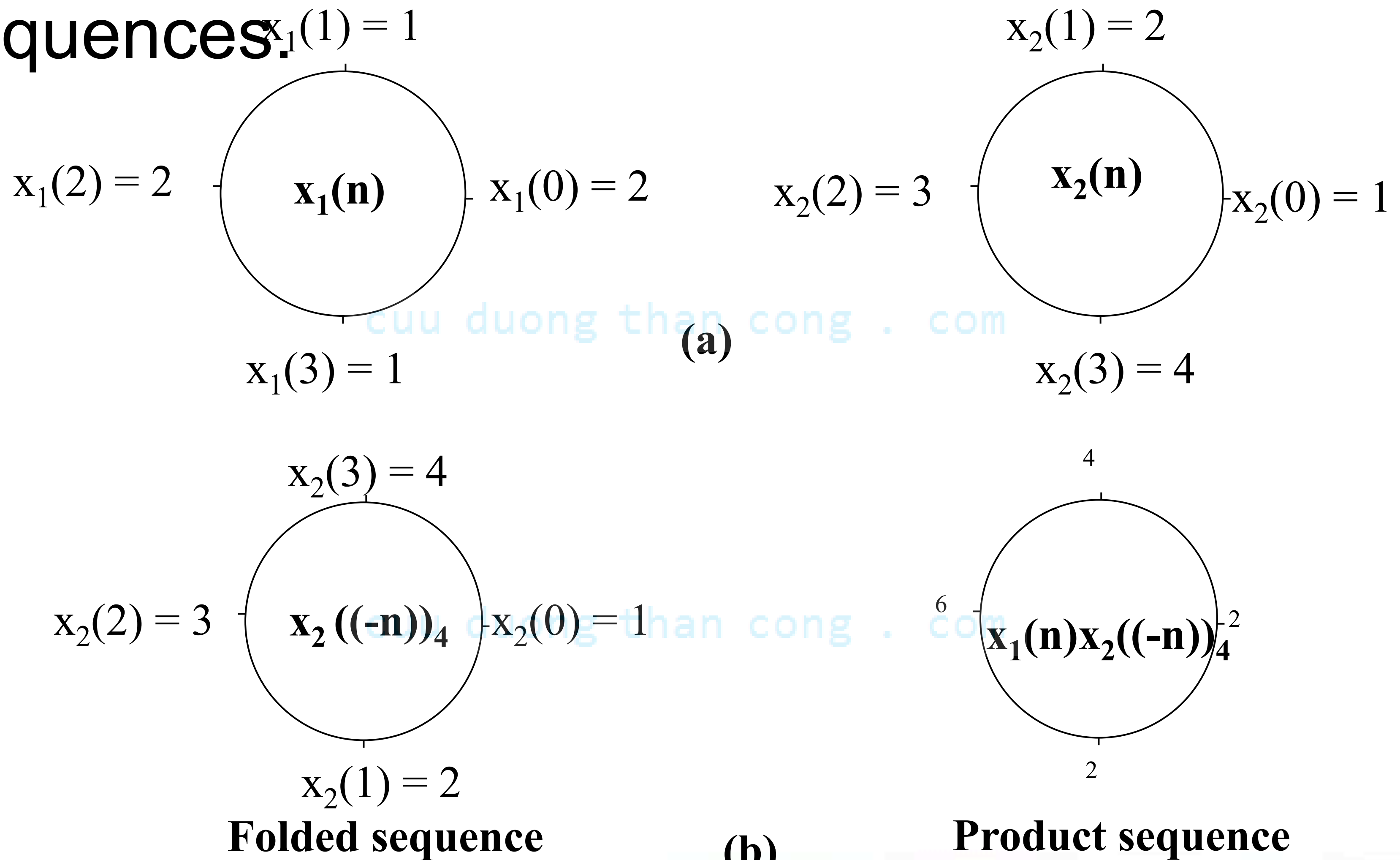
$x_3(3) = 16$

Finally $x_3(n) = \{ 14, 16, 14, 16 \}$

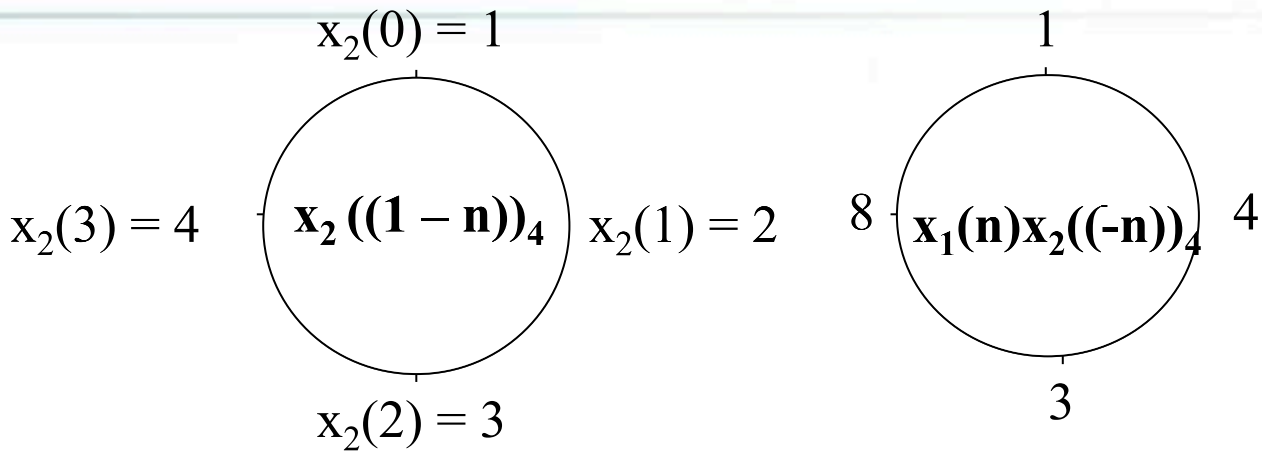
$\uparrow$

# EXAMPLE 5.2.1 Solution

**Figure 5.8** Circular convolution of two sequences.



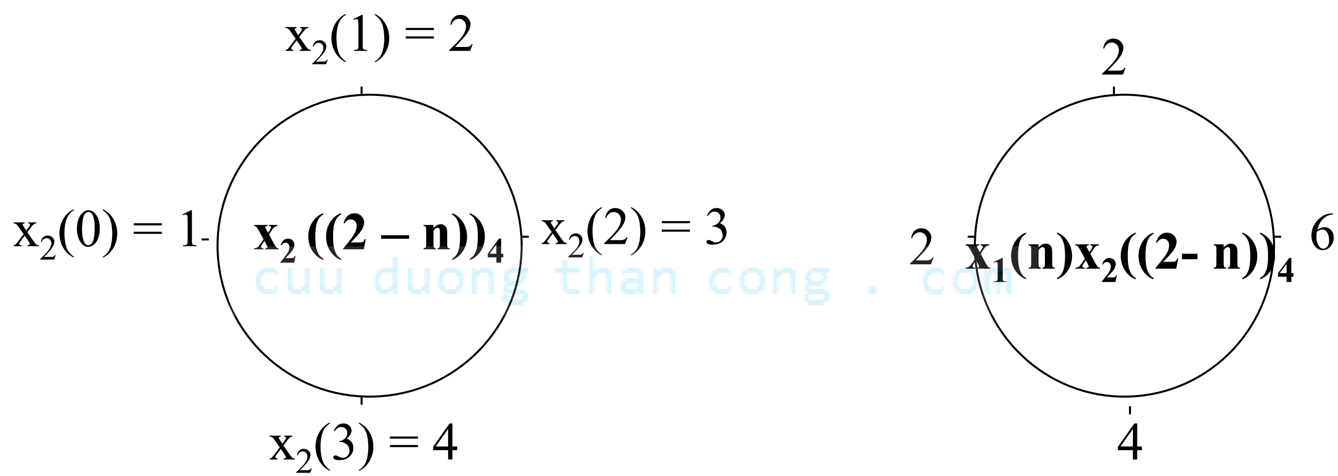
# EXAMPLE 5.2.1 Solution




**Folded sequence rotated by one unit in time**

**Product sequence**

(c)

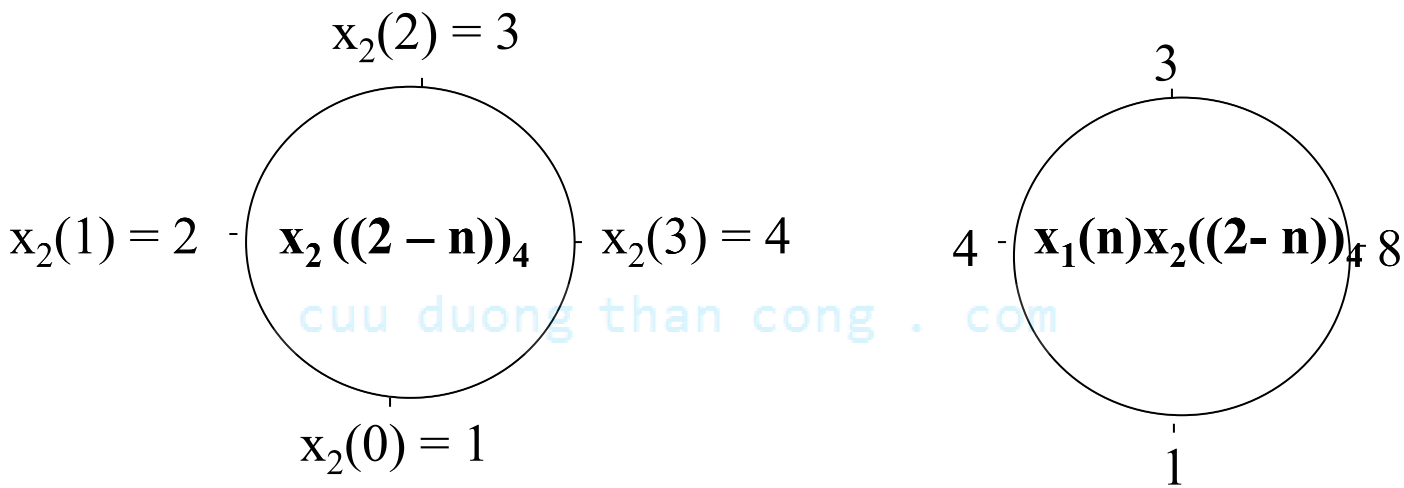



**Folded sequence rotated by one unit in time**

**Product sequence**

(d)

# EXAMPLE 5.2.1 Solution




**Folded sequence rotated by one unit in time**

**Product sequence**

(e)

# EXAMPLE 5.3.1

By means of the DFT and IDFT, determine the response of the FIR filter with impulse response

$$h(n) = \{ \underset{\uparrow}{1}, 2, 3 \}$$

to the input sequence

$$x(n) = \{ 1, \underset{\uparrow}{2}, 2, 1 \}$$



**Solution.** The input sequence has length $L = 4$ and the impulse response has length $M = 3$ Linear convolution of these two sequences produces a sequence of length $N = 6$.

# EXAMPLE 5.3.1 Solution

Consequently, the size of the DFTs must be at least six. For simplicity we compute eight-point DFTs. Hence the eight-point DFT of *x(n)* is

$$X(k)=\sum_{n=0}^{7} x(n)\, e^{-j2\pi kn/8} = 1+2e^{-j\pi k/4}+2e^{-j\pi k/2}+e^{-j3\pi k/4}, \quad k=0,1,\ldots,7$$



This computation yields

$$X(0)=6$$

$$X(1)=\frac{2+\sqrt{2}}{2}-j\left(\frac{4+3\sqrt{2}}{2}\right)$$

$$X(2)=-1-j,$$



$$X(3)=\frac{2-\sqrt{2}}{2}+j\left(\frac{4-3\sqrt{2}}{2}\right)$$

$$X(4)=0$$

$$X(5)=\frac{2-\sqrt{2}}{2}-j\left(\frac{4-3\sqrt{2}}{2}\right)$$

$$X(6)=-1+j,$$

$$X(7)=\frac{2+\sqrt{2}}{2}+j\left(\frac{4+3\sqrt{2}}{2}\right)$$

# EXAMPLE 5.3.1 Solution

The eight-point DFT of *h(n)* is

$$H(k)=\sum_{n=0}^{7} h(n)\,e^{-j2\pi kn/8} \quad = 1 + 2e^{-j\pi k/4} + 3e^{-j\pi k/2}$$

Hence

$$H(0)=6,\quad H(1)=1+\sqrt{2}-j\left(3+\sqrt{2}\right),\quad H(2)=-2-j2$$

$$H(3)=1-\sqrt{2}+j\left(3-\sqrt{2}\right),\quad H(4)=2$$

$$H(5)=1-\sqrt{2}-j\left(3-\sqrt{2}\right),\quad H(6)=-2+j2$$

$$H(7)=1+\sqrt{2}+j\left(3+\sqrt{2}\right),$$

# EXAMPLE 5.3.1 Solution

The product of these two DFTs yields *Y(k)*, which is

$Y(0) = 36,$ $\quad Y(1) = -14.07 - j17.48$

$Y(2) = j4$ $\quad Y(3) = 0.07 + j0.515$

$Y(4) = 0,$ $\quad Y(5) = 0.07 - j0.515$

$Y(6) = -j4$ $\quad Y(7) = -14.07 + j17.48$



Finally, the eight-point IDFT is



$$y(k) = \sum_{k=0}^{7} Y(k)\, e^{j2\pi kn/8}, \qquad n = 0, 1, \ldots, 7$$

# EXAMPLE 5.3.1 Solution

This computation yields the result

$$y(n) = \{\, 1, 4, 9, 11, 8, 3, 0, 0 \,\}$$

↑

We observe that the first six values of *y(n)* constitute the set of desired output values.

The last two values are zero because we used an eight-point DFT and IDFT, when, in fact, the minimum number of points required is six.

# EXAMPLE 5.3.2

Determine the sequence *y(n)* that results from the use of four-point DFTs in Example 5.3.1.

**Solution.** The four-point DFT of *h(n)* is

$$H(k)=\sum_{n=0}^{3} h(n)\,e^{-j2\pi kn/4} = 1+2e^{-j\pi k/2}+3e^{-jk\pi}, \qquad k=0,1,2,3$$



Hence $H(0)=6,\quad H(1)=-2-j2,\quad H(2)=0,\quad H(3)=-2+j2$

The four-point DFT of *x(n)* is



$$X(k)=1+2e^{-j\pi k/2}+2e^{-jk\pi}+1e^{-j3\pi k/2}, \qquad k=0,1,2,3$$

Hence

$X(0)=6,\quad X(1)=-1-j,\quad X(2)=0,\quad H(3)=-1+j$

# EXAMPLE 5.3.2 Solution

The product of these two four-point DFTs is

$$\tilde{Y}(0) = 36, \qquad \tilde{Y}(1) = j4, \qquad \tilde{Y}(2) = 0, \qquad \tilde{Y}(3) = -j4$$

The four-point IDFT yields

$$\hat{y}(n) = \frac{1}{4}\sum_{k=0}^{3} \hat{Y}(k) e^{j2\pi kn/4}, \qquad n = 0, 1, 2, 3$$



$$= \frac{1}{4}\left(36 + j4e^{j\pi n/2} - j4e^{j3\pi n/2}\right)$$



Therefore, $\hat{y}(n) = \{ 9, 7, 9, 11 \}$

$\uparrow$

# EXAMPLE 5.4.1

The exponential signal

$$x_a(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

is sampled at the rate $F_s = 20$ samples per second, and a block of *100* samples used to estimate its spectrum. Determine the spectral characteristics of the signal $x_a(t)$ by computing the DFT of the finite-duration sequence. Compare the spectrum of the truncated discrete-time signal to the spectrum of the analog signal.

# EXAMPLE 5.4.1 Solution

The spectrum of the signal is

$$X_a(F) = \frac{1}{1 + j2\pi F}$$

The exponential analog signal sampled at the rate of *20* samples per second yields the sequence



$$x(n) = e^{-nT} = e^{-n/20}, \qquad n \geq 0$$

$$= \left(e^{-1/20}\right)^n = (0.95)^n, \qquad n \geq 0$$



Now, let

$$x(n) = \begin{cases} (0.95)^n, & 0 \leq n \leq 99 \\ 0, & otherwise \end{cases}$$

# EXAMPLE 5.4.1 Solution

The *N*-point DFT of the *L = 100* point sequence is

$$\hat{X}(k) = \frac{1}{4}\sum_{k=0}^{99} \hat{x}(n)e^{-j2\pi k/N}, \quad k = 0, 1, \dots, N-1$$

To obtain sufficient detail in the spectrum we choose *N = 200*. This is equivalent to padding the sequence *x(n)* with 100 zeros.

# EXAMPLE 5.4.1 Solution

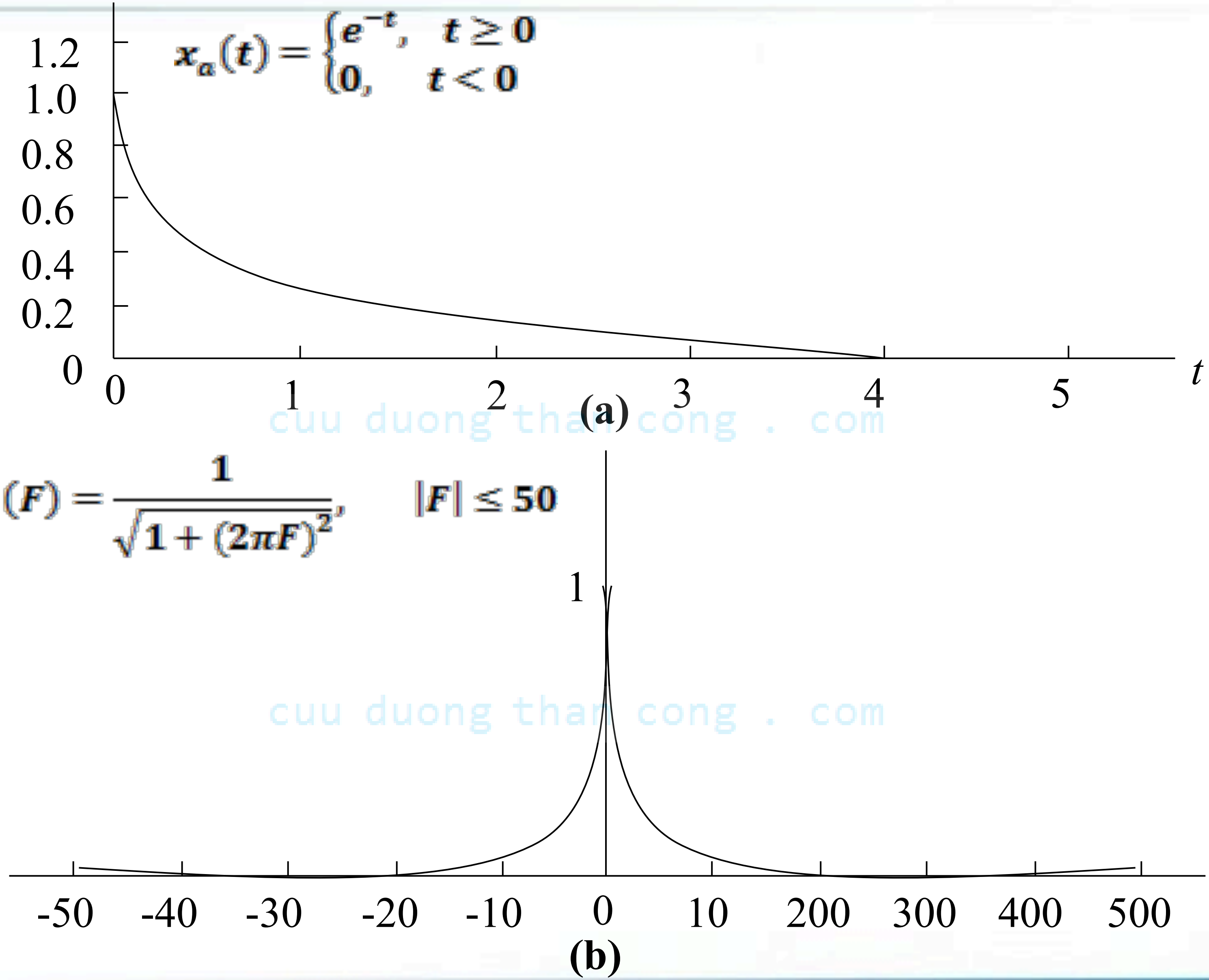
The graph of the analog signal $x_a(t)$ and its magnitude spectrum $|Xa(F)|$ are illustrated in Fig 5.16(a) and (b), respectively. The truncated sequence $x(n)$ and its $N = 200$ point DFT (magnitude) are illustrated in Fig 5.16(c) and (d), respectively.

**Figure 5.16** Effect of windowing (truncating) the sampled version of the analog signal in Example 5.4.1

# EXAMPLE 5.4.1 Solution

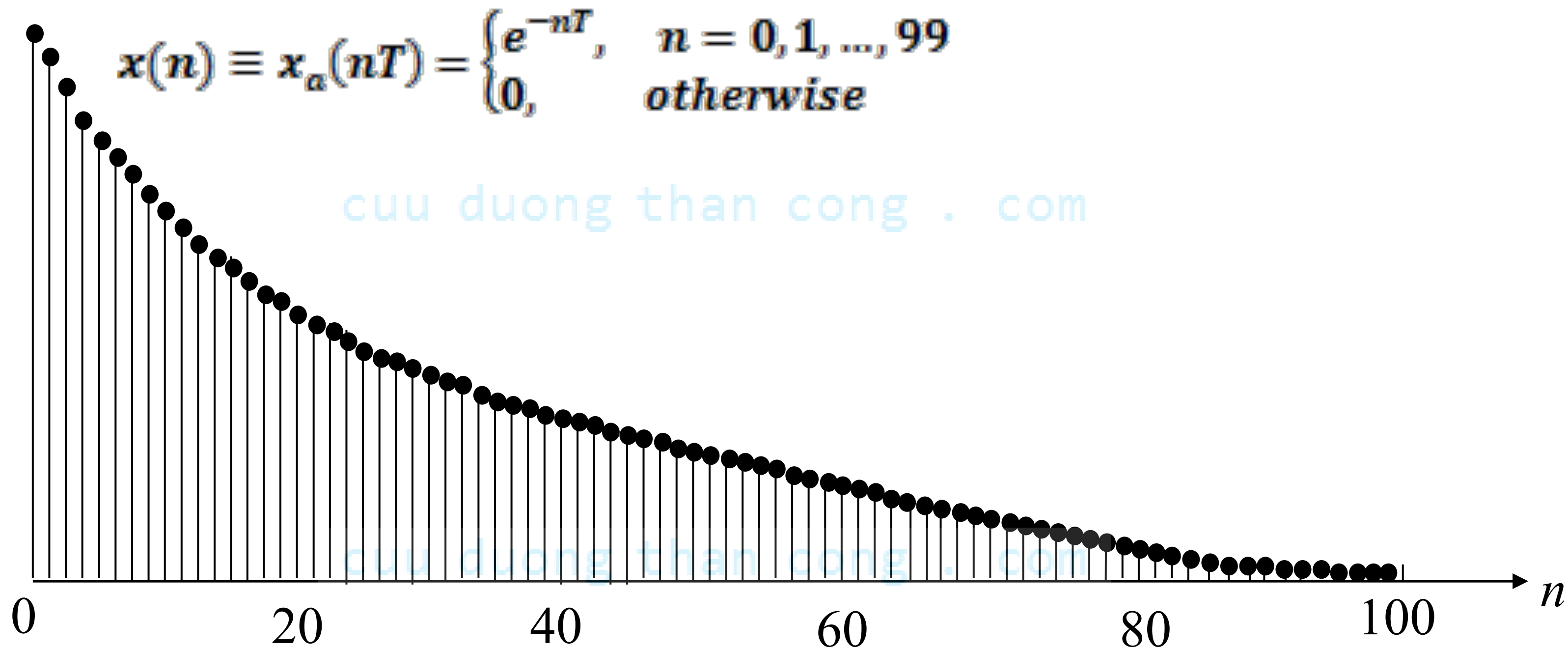
$$x_a(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$|X_a(F)| = \frac{1}{\sqrt{1 + (2\pi F)^2}}, \qquad |F| \leq 50$$



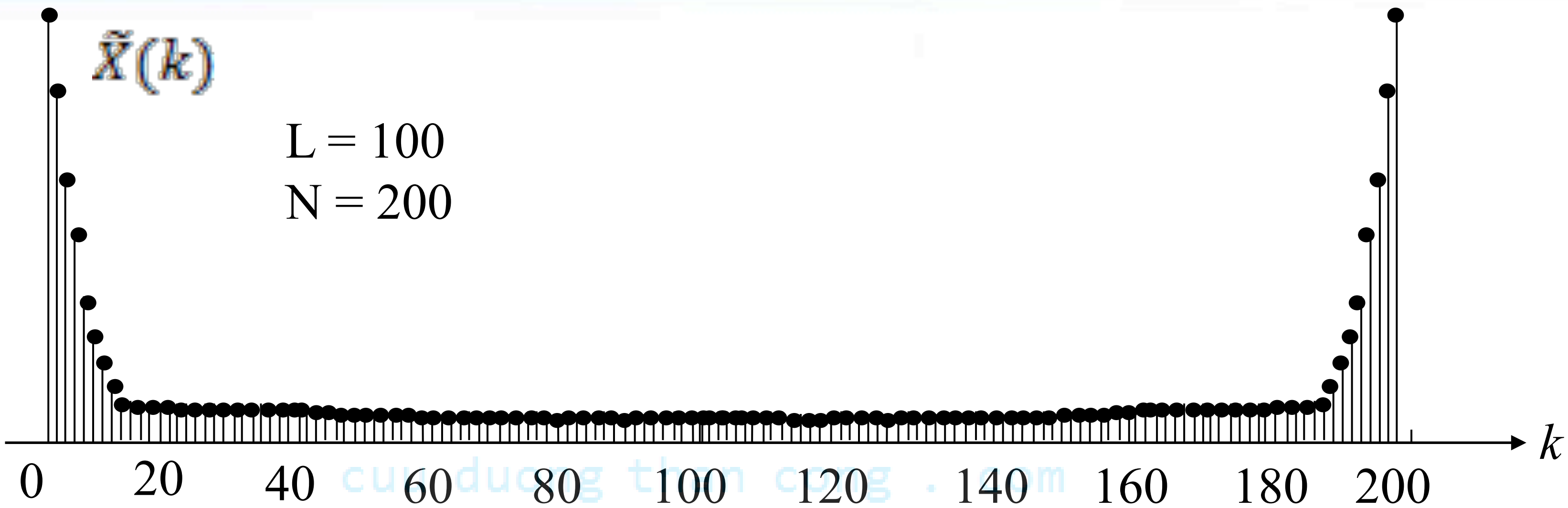
# EXAMPLE 5.4.1 Solution

## Figure 5.16 *Continued*

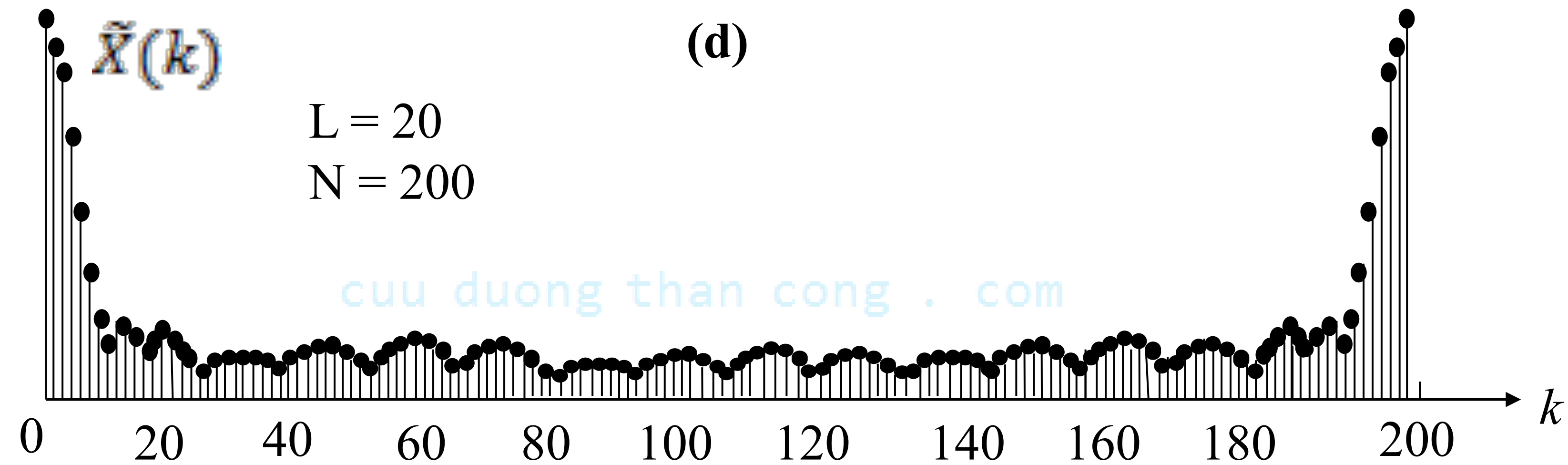



(c)

# EXAMPLE 5.4.1 Solution




(d)




(e)