Digital Signal Processing

405109

Chapter 1

Sampling and Reconstruction

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1. Introduction

❖ A typical signal processing system includes 3 stages:

❖ The analog signal is digitalized by an A/D converter

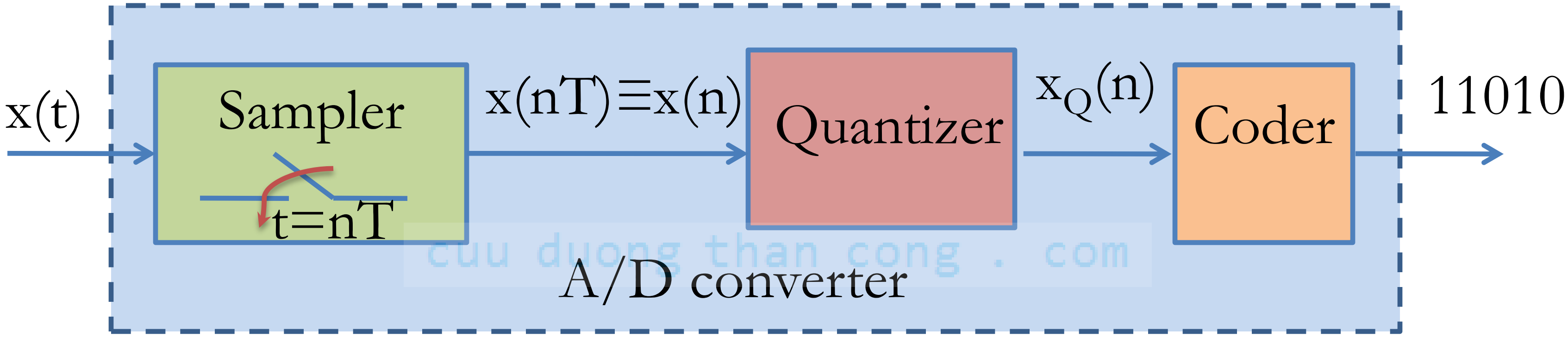
❖ The digitalized samples are processed by a digital signal processor.

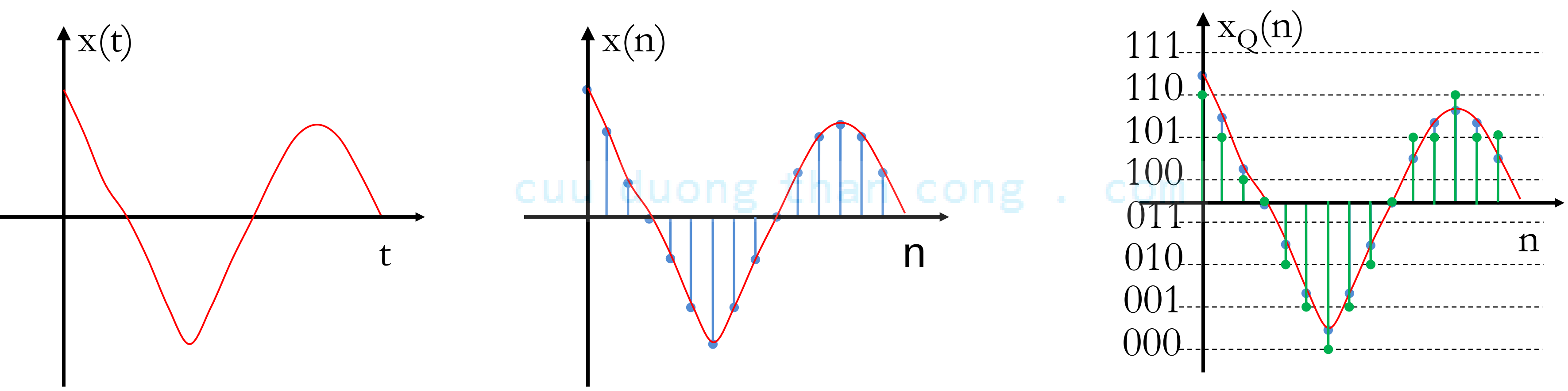
- The digital processor can be programmed to perform signal processing operations such as filtering, spectrum estimation. Digital signal processor can be a general purpose computer, DSP chip or other digital hardware.

❖ The resulting output samples are converted back into analog by a D/A converter.

2. Analog to digital conversion

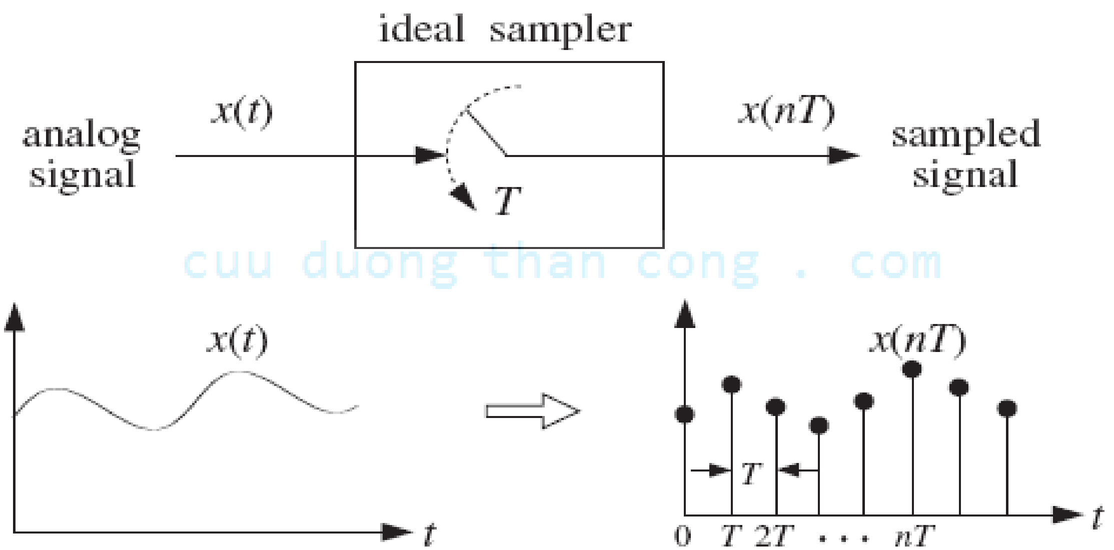
❖ Analog to digital (A/D) conversion is a three-step process.





3. Sampling

- Sampling is to convert a continuous time signal into a discrete time signal. The analog signal is periodically measured at every T seconds



- $x(n) \equiv x(nT) = x(t=nT)$, n=…-2, -1, 0, 1, 2, 3…
- T: sampling interval or sampling period (second);
- Fs=1/T: sampling rate or frequency (samples/second or Hz)

Example 1

- The analog signal $x(t)=2\cos(2\pi t)$ with t(s) is sampled at the rate $F_s=4$ Hz. Find the discrete-time signal x(n) ?

Solution:

- $x(n)\equiv x(nT)=x(n/Fs)=2\cos(2\pi n/Fs)=2\cos(2\pi n/4)=2\cos(\pi n/2)$

n	0	1	2	3	4
x(n)	2	0	-2	0	2

- Plot the signal

Example 2

❖ Consider the two analog sinusoidal signals

$$x_1(t) = 2\cos(2\pi\frac{7}{8}t), \quad x_2(t) = 2\cos(2\pi\frac{1}{8}t); \quad t(s)$$

These signals are sampled at the sampling frequency F_s=1 Hz.
Find the discrete-time signals ?

Solution:

$$x_1(n) \equiv x_1(nT) = x_1(n\frac{1}{F_s}) = 2\cos(2\pi\frac{7}{8}\frac{1}{1}n) = 2\cos(\frac{7}{4}\pi n)$$

$$= 2\cos((2-\frac{1}{4})\pi n) = 2\cos(\frac{\pi}{4}n)$$

$$x_2(n) \equiv x_2(nT) = x_2(n\frac{1}{F_s}) = 2\cos(2\pi\frac{1}{8}\frac{1}{1}n) = 2\cos(\frac{1}{4}\pi n)$$

❖ Observation: $x_1(n)=x_2(n)$ → based on the discrete-time signals, we cannot tell which of two signals are sampled ? These signals are called "alias"

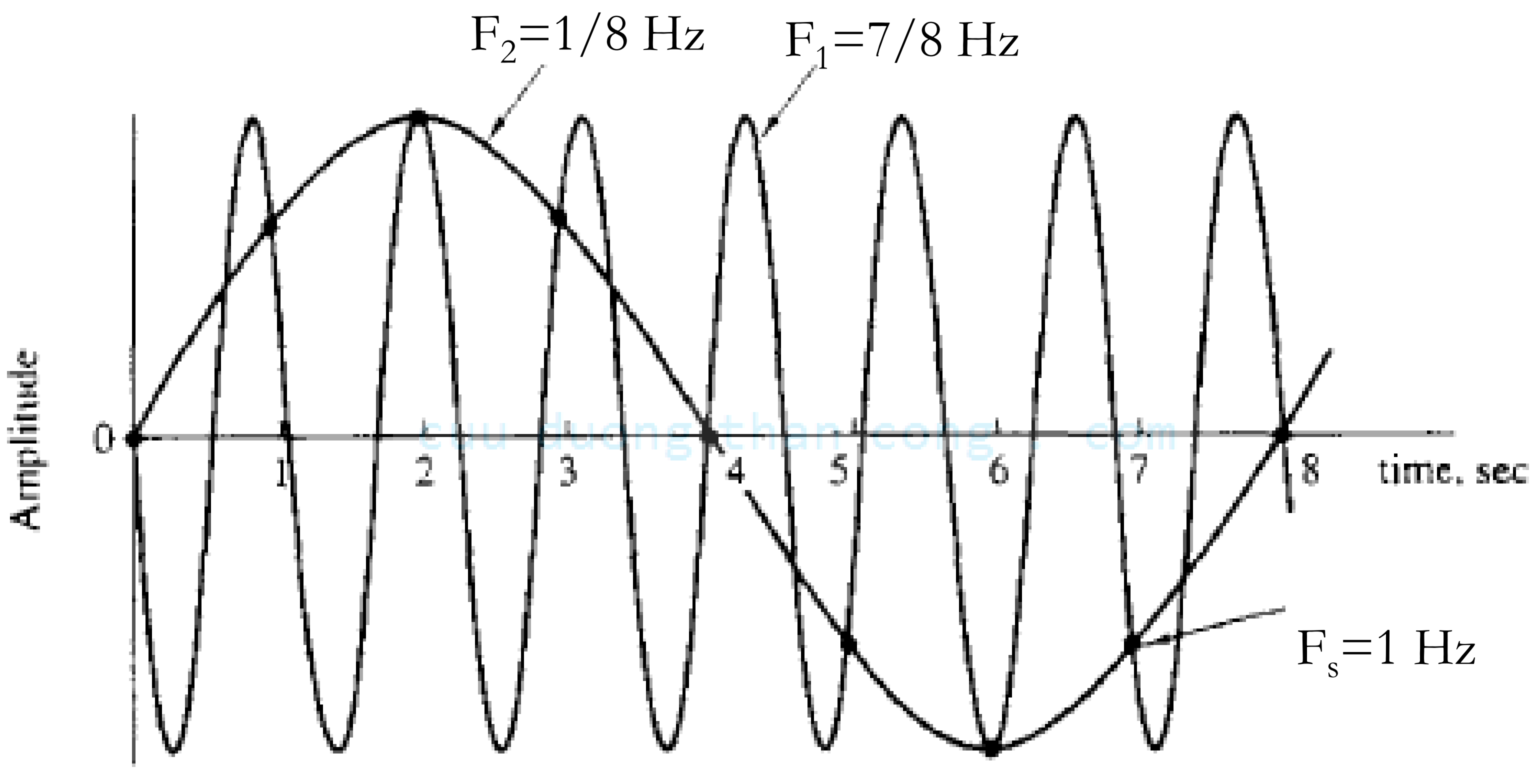


Fig: Illustration of aliasing

4. Aliasing of Sinusoids

- In general, the sampling of a continuous-time sinusoidal signal $x(t) = A\cos(2\pi F_0 t + \theta)$ at a sampling rate $F_s = 1/T$ results in a discrete-time signal x(n).

- The sinusoids $x_k(t) = A\cos(2\pi F_k t + \theta)$ is sampled at F_s , resulting in a discrete time signal $x_k(n)$.

- If $F_k = F_0 + kF_s$, $k = 0, \pm 1, \pm 2, \ldots$, then $x(n) = x_k(n)$.

Proof: (in class)

- Remarks: We can that the frequencies $F_k = F_0 + kF_s$ are indistinguishable from the frequency F_0 after sampling and hence they are aliases of F_0

5. Spectrum Replication

❖ Let $x(nT) = \hat{x}(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) = x(t)s(t)$ where $s(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT)$

❖ s(t) is periodic, thus, its Fourier series are given by

$$s(t) = \sum_{n=-\infty}^{\infty} S_n e^{j2\pi F_s nt} \quad \text{where } S_n = \frac{1}{T}\int_T \delta(t)e^{-j2\pi F_s nt}dt = \frac{1}{T}\int_T \delta(t)dt = \frac{1}{T}$$

Thus, $s(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{j2\pi F_s nt}$

which results in $\hat{x}(t) = x(t)s(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} x(t)e^{j2\pi n f_s t}$

❖ Taking the Fourier transform of $\hat{x}(t)$ yields $\hat{X}(F) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(F-nF_s)$

❖ Observation: The spectrum of discrete-time signal is a sum of the original spectrum of analog signal and its periodic replication at the interval $F_{s.}$

❖ $F_s/2 \geq F_{max}$

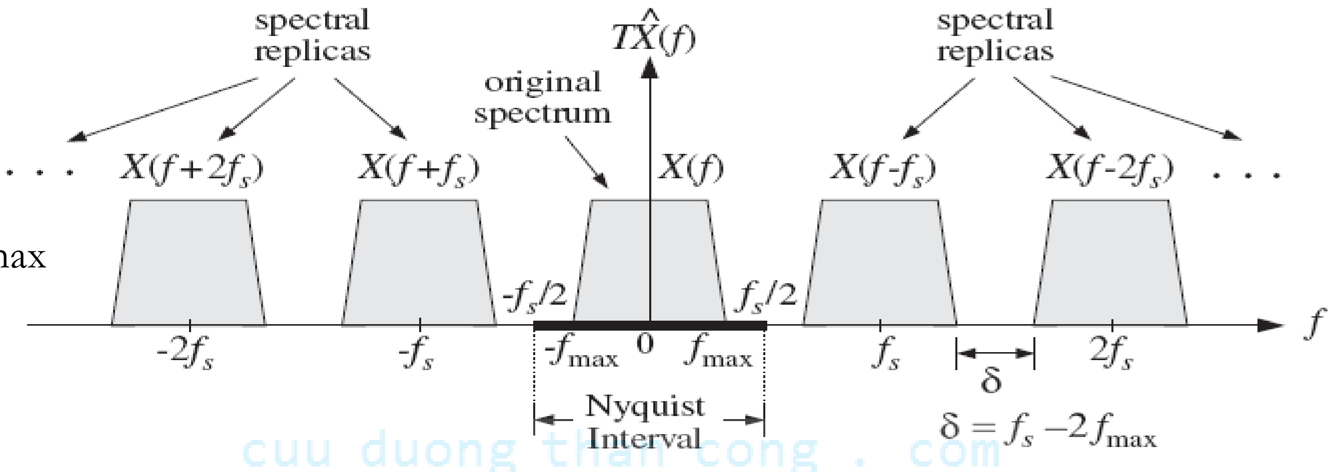


Fig: Spectrum replication caused by sampling

❖ $F_s/2 < F_{max}$

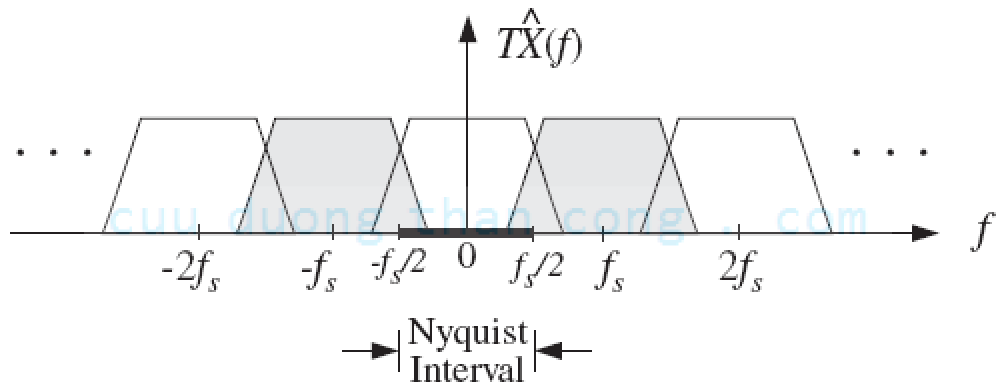


Fig: Aliasing caused by overlapping spectral replicas

6. Sampling Theorem

❖ For accurate representation of a signal x(t) by its time samples x(nT), two conditions must be met:

1) The signal x(t) must be band-limited, i.e., its frequency spectrum must be limited to F_{max}.

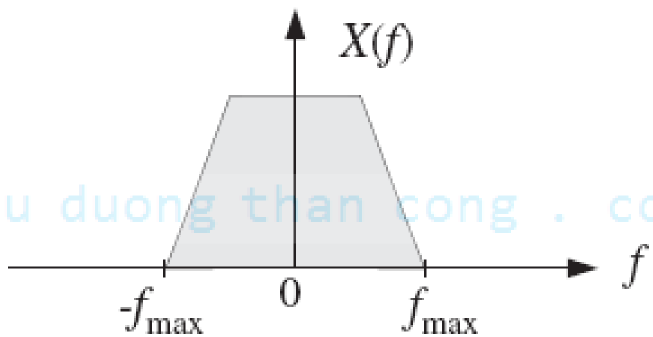


Fig: Typical band-limited spectrum

2) The sampling rate F_s must be chosen at least twice the maximum frequency F_{max}.

$$F_s \geq 2F_{max}$$

❖ $F_s=2F_{max}$ is called Nyquist rate; $F_s/2$ is called Nyquist frequency; $[-F_s/2, F_s/2]$ is Nyquist interval.

❖ The values of F_{max} and F_s depend on the application

Application	Fmax	Fs
Biomedical	1 KHz	2 KHz
Speech	4 KHz	8 KHz
Audio	20 KHz	40 KHz
Video	4 MHz	8 MHz

7. Ideal analog reconstruction

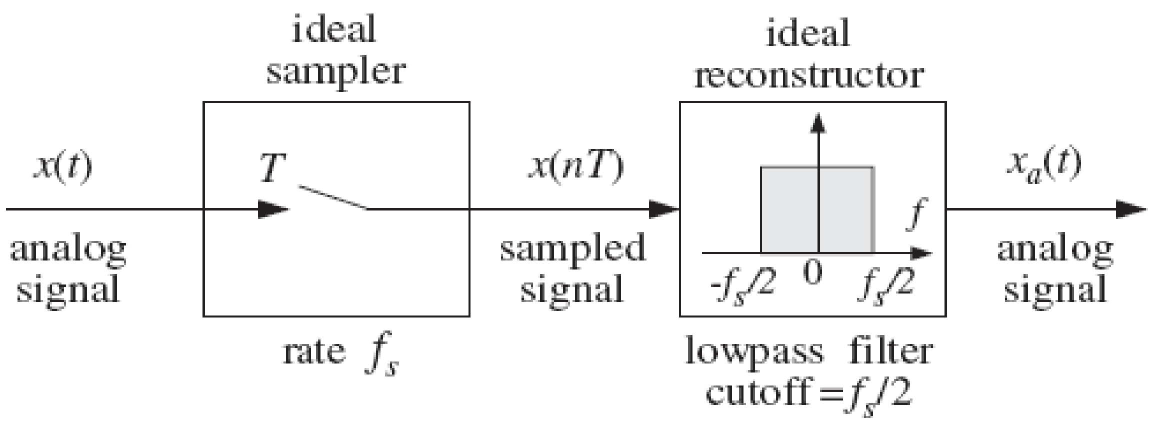


Fig: Ideal reconstructor as a lowpass filter

- An ideal reconstructor acts as a lowpass filter with cutoff frequency equal to the Nyquist frequency Fs/2.
- An ideal reconstructor (lowpass filter)

$$H(F) = \begin{cases} T & F \in [-F_s/2, F_s/2] \\ 0 & otherwise \end{cases}$$

Then

$$\hat{X}_a(F) = \hat{X}(F)H(F) = X(F)$$

Example 3

- The analog signal $x(t)=\cos(20\pi t)$ is sampled at the sampling frequency Fs=40 Hz.
  a) Plot the spectrum of signal x(t) ?
  b) Find the discrete time signal x(n) ?
  c) Plot the spectrum of signal x(n) ?
  d) The signal x(n) is an input of the ideal reconstructor, find the reconstructed signal $x_a(t)$?

Example 4

❖ The analog signal $x(t)=\cos(100\pi t)$ is sampled at the sampling frequency $F_s=40$ Hz.

a) Plot the spectrum of signal x(t) ?

b) Find the discrete time signal x(n) ?

c) Plot the spectrum of signal x(n) ?

d) The signal x(n) is an input of the ideal reconstructor, find the reconstructed signal $x_a(t)$?

- Remarks: $x_a(t)$ contains only the frequency components that lie in the Nyquist interval (NI) $[-F_s/2, F_s/2]$.

- $x(t)$, $F_0 \in NI$ --------------------> (sampling at F_s) $x(n)$ ------------------------> (ideal reconstructor) $x_a(t)$, $F_a=F_0$

- $x_k(t)$, $F_k=F_0+kF_s$ ------------------> (sampling at F_s) $x(n)$ ----------------------> (ideal reconstructor) $x_a(t)$, $F_a=F_0$

- The frequency F_a of reconstructed signal $x_a(t)$ is obtained by adding to or substracting from F_0 (F_k) enough multiples of Fs until it lies within the Nyquist interval $[-F_s/2, F_s/2]$. That is

$$F_a = F \bmod(F_s)$$

Example 5

- The analog signal $x(t)=10\sin(4\pi t)+6\sin(16\pi t)$ is sampled at the rate 20 Hz. Find the reconstructed signal $x_a(t)$?

Example 6

❖ Let x(t) be the sum of sinusoidal signals
$x(t)=4+3\cos(\pi t)+2\cos(2\pi t)+\cos(3\pi t)$ where t is in milliseconds.

a) Determine the minimum sampling rate that will not cause any aliasing effects ?
b) To observe aliasing effects, suppose this signal is sampled at half its Nyquist rate. Determine the signal $x_a(t)$ that would be aliased with x(t) ? Plot the spectrum of signal x(n) for this sampling rate?

Example 7

A wheel, rotating at 6 Hz, is seen in a dark room by means of a strobe light flashing at a rate of 8 Hz. Determine the apparent rotational speed and sense of rotation of the wheel. Repeat the question if the flashes occur at 12 Hz, 16 Hz, or 24 Hz.

8. Ideal antialiasing prefilter

❖ The signals in practice may not band-limited, thus they must be filtered by a lowpass filter

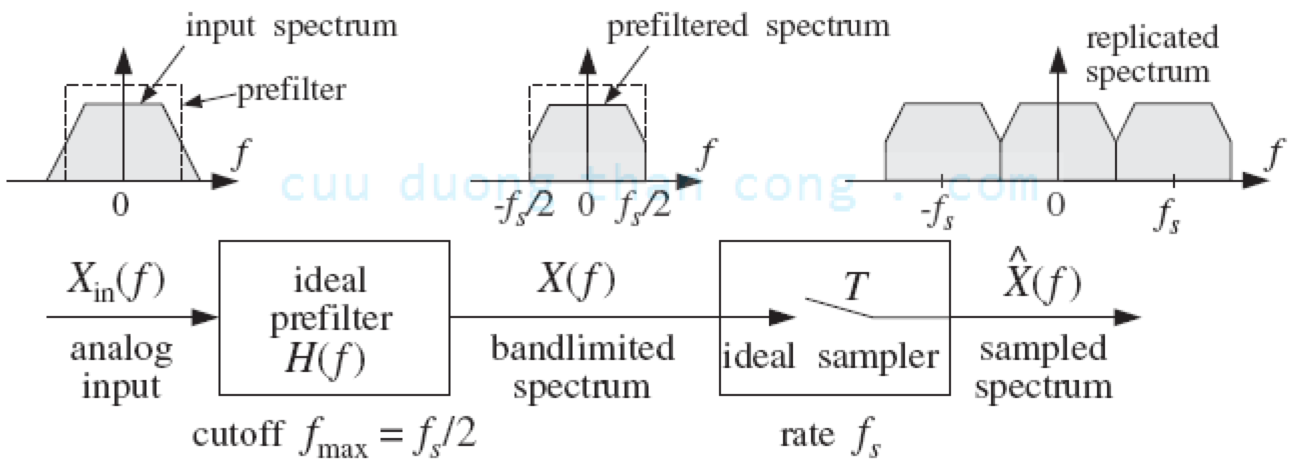


Fig: Ideal antialiasing prefilter

9. Practical antialiasing prefilter

- A lowpass filter: [$-F_{pass}$, F_{pass}] is the frequency range of interest for the application ($F_{max}=F_{pass}$)
- The Nyquist frequency Fs/2 is in the middle of transition region.
- The stopband frequency F_{stop} and the minimum stopband attenuation A_{stop} dB must be chosen appropriately to minimize the aliasing effects.

$$F_s = F_{pass} + F_{stop}$$

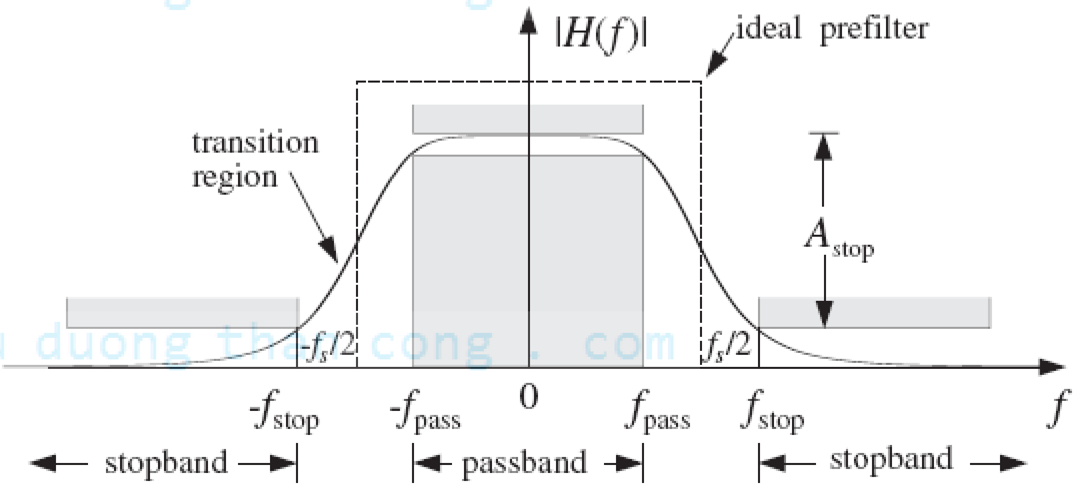


Fig: Practical antialiasing lowpass prefilter

❖ The attenuation of the filter in decibels is defined as

$$A(F) = -20\log_{10}\left|\frac{H(F)}{H(F_0)}\right| (dB)$$

where f_0 is a convenient reference frequency, typically taken to be at DC for a lowpass filter.

❖ α_{10} =A(10F)-A(F) (dB/decade): the increase in attenuation when F is changed by a factor of ten.

❖ α_2 =A(2F)-A(F) (dB/octave): the increase in attenuation when F is changed by a factor of two.

❖ Analog filter with order N, $|H(F)| \sim 1/F^N$ for large F, thus α_{10} =20N (dB/decade) and α_{10} =6N (dB/octave)

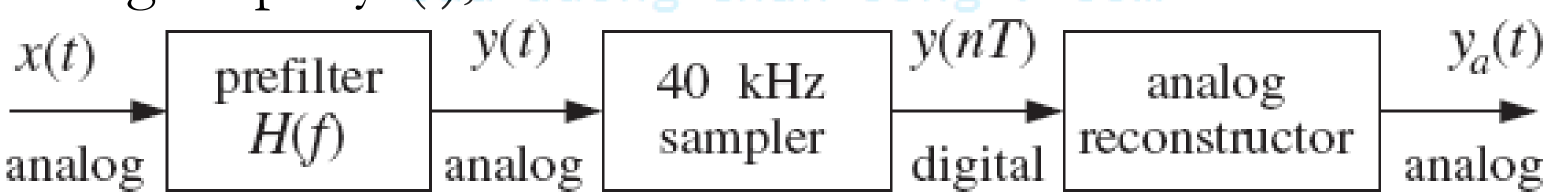
Example 6

❖ A sound wave has the form

$$x(t) = 2A\cos(10\pi t) + 2B\cos(30\pi t) + 2C\cos(50\pi t) \\ + 2D\cos(60\pi t) + 2E\cos(90\pi t) + 2F\cos(125\pi t)$$

where t is in milliseconds. What is the frequency content of this signal ? Which parts of it are audible and why ?

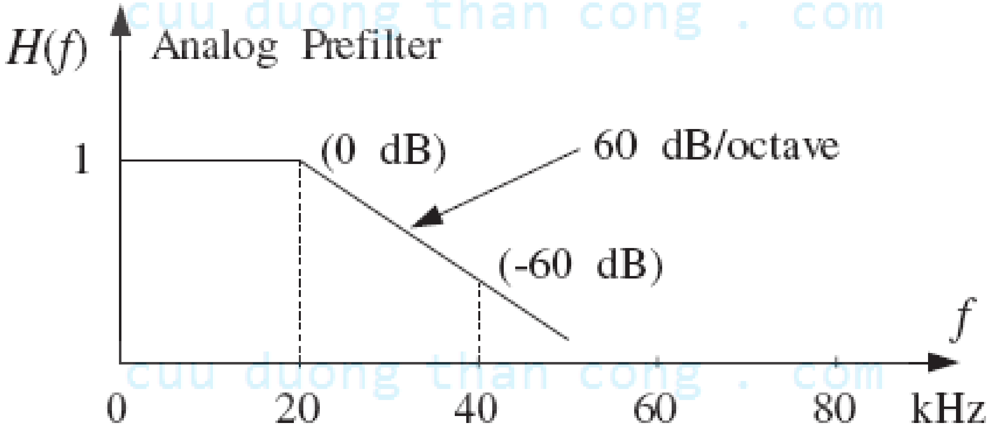
This signal is prefilter by an anlog prefilter H(f). Then, the output y(t) of the prefilter is sampled at a rate of 40KHz and immediately reconstructed by an ideal analog reconstructor, resulting into the final analog output ya(t), as shown below:



Example 7

Determine the output signal y(t) and ya(t) in the following cases:

a) When there is no prefilter, that is, H(F)=1 for all F.
b) When H(F) is the ideal prefilter with cutoff Fs/2=20 KHz.
c) When H(F) is a practical prefilter with specifications as shown below:

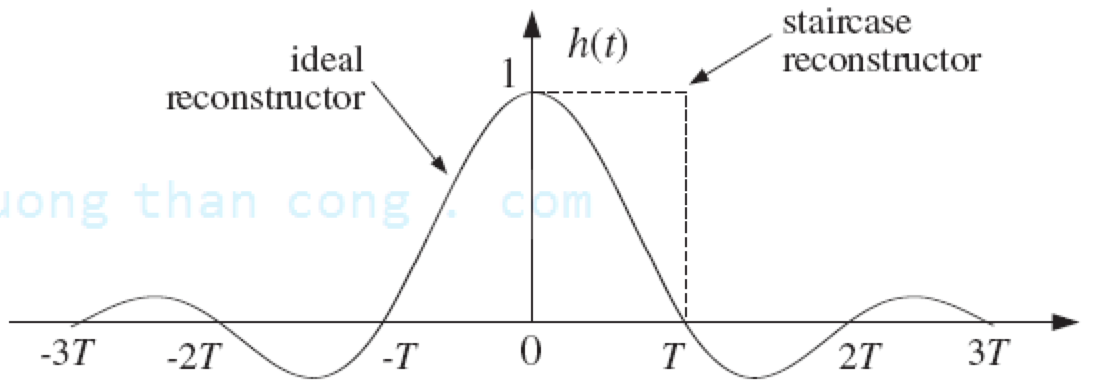


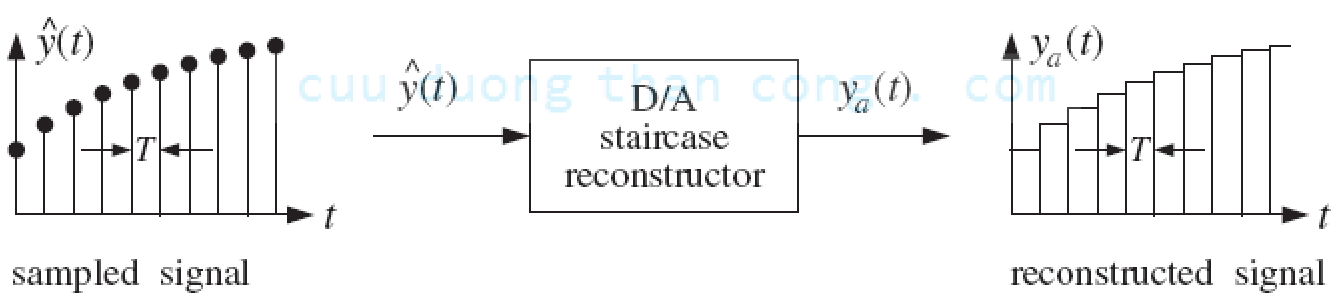
The filter's phase response is assumed to be ignored in this example.

10. Practical analog reconstructors

❖ The ideal reconstructor has the impulse response: $h(t) = \dfrac{\sin(\pi F_s t)}{\pi F_s t}$ which is not realizable since its impulse response is not casual

❖ It is practical to use a staircase reconstructor





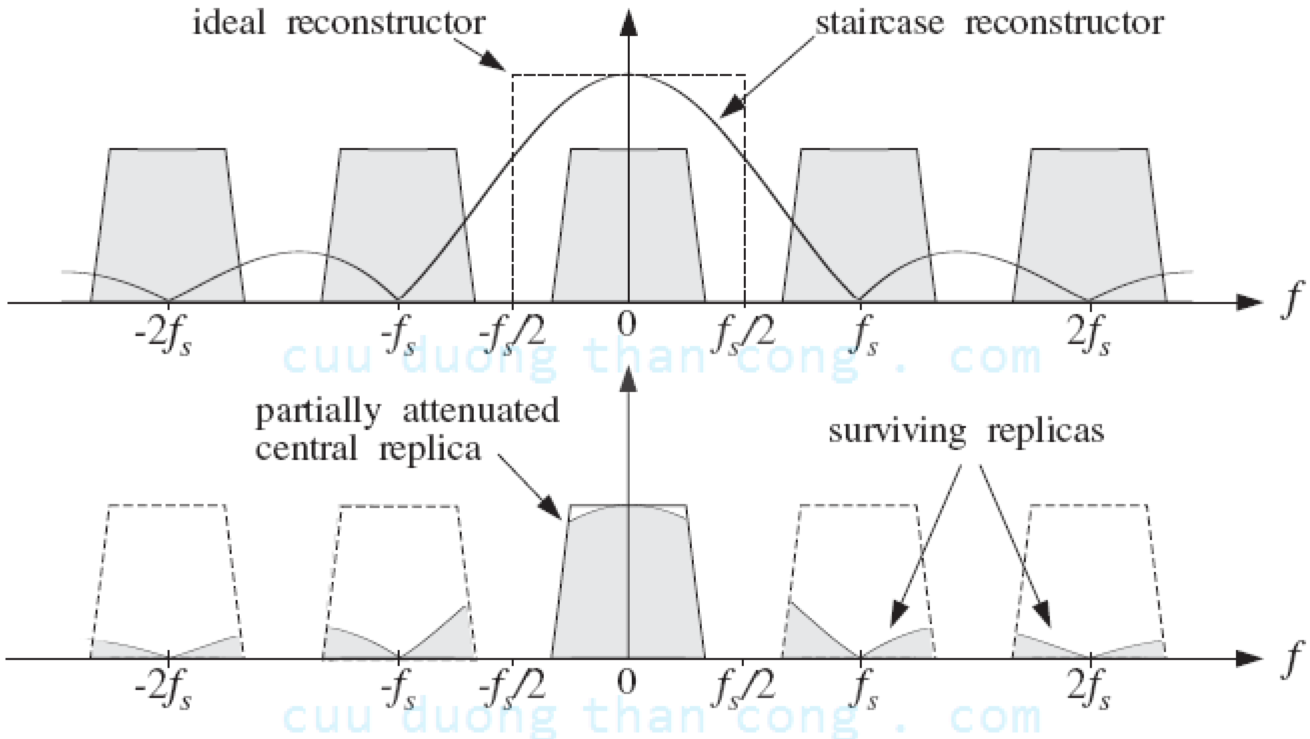


Fig: Frequency response of staircase reconstructor

11. Anti-image postfilter

❖ An analog lowpass postfilter whose cutoff is Nyquist frequency Fs/2 is used to remove the surviving spectral replicas.

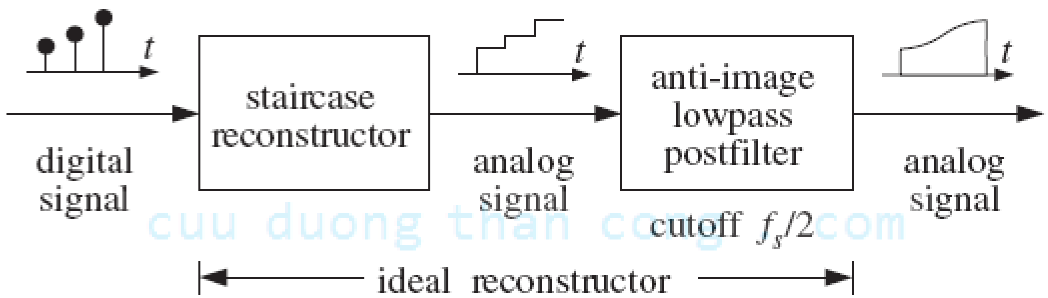


Fig: Analog anti-image postfilter

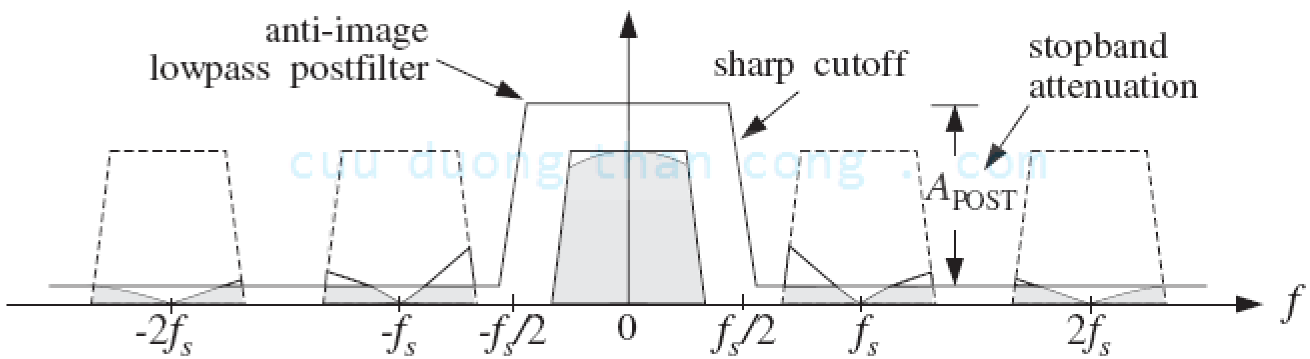


Fig: Spectrum after postfilter

Review

- ❖ Hoạt động của bộ lấy mẫu lý tưởng?
- ❖ Hiện tượng chồng lấn?
- ❖ Tính chất lặp phổ?
- ❖ Phát biểu định lý lấy mẫu?
- ❖ Hoạt động của bộ khôi phục lý tưởng?
- ❖ Tại sao phải dùng tiền lọc/hậu lọc?
- ❖ Hoạt động của bộ tiền lọc lý tưởng/thực tế?

Homework 1

Determine which of the following sinusoids are periodic and compute their fundamental period.

(a) $\cos 0.01\pi n$ **(b)** $\cos\left(\pi \frac{30n}{105}\right)$ **(c)** $\cos 3\pi n$ **(d)** $\sin 3n$ **(e)** $\sin\left(\pi \frac{62n}{10}\right)$

Homework 2

Determine whether or not each of the following signals is periodic. In case a signal is periodic, specify its fundamental period.

(a) $x_a(t) = 3\cos(5t + \pi/6)$

(b) $x(n) = 3\cos(5n + \pi/6)$

(c) $x(n) = 2\exp[j(n/6 - \pi)]$

(d) $x(n) = \cos(n/8)\cos(\pi n/8)$

(e) $x(n) = \cos(\pi n/2) - \sin(\pi n/8) + 3\cos(\pi n/4 + \pi/3)$

Homework 3

(a) Show that the fundamental period N_p of the signals

$$s_k(n) = e^{j2\pi kn/N}, \qquad k = 0, 1, 2, \ldots$$

is given by $N_p = N/\text{GCD}(k, N)$, where GCD is the greatest common divisor of k and N.

(b) What is the fundamental period of this set for $N = 7$?

(c) What is it for $N = 16$?

Homework 3

(a) Show that the fundamental period N_p of the signals

$$s_k(n) = e^{j2\pi kn/N}, \qquad k = 0, 1, 2, \ldots$$

is given by $N_p = N/\text{GCD}(k, N)$, where GCD is the greatest common divisor of k and N.

(b) What is the fundamental period of this set for $N = 7$?

(c) What is it for $N = 16$?

Homework 4

Consider the following analog sinusoidal signal:

$$x_a(t) = 3\sin(100\pi t)$$

(a) Sketch the signal $x_a(t)$ for $0 \le t \le 30$ ms.

(b) The signal $x_a(t)$ is sampled with a sampling rate $F_s = 300$ samples/s. Determine the frequency of the discrete-time signal $x(n) = x_a(nT)$, $T = 1/F_s$, and show that it is periodic.

(c) Compute the sample values in one period of $x(n)$. Sketch $x(n)$ on the same diagram with $x_a(t)$. What is the period of the discrete-time signal in milliseconds?

(d) Can you find a sampling rate F_s such that the signal $x(n)$ reaches its peak value of 3? What is the minimum F_s suitable for this task?

Homework 5

An analog signal $x_a(t) = \sin(480\pi t) + 3\sin(720\pi t)$ is sampled 600 times per second.

(a) Determine the Nyquist sampling rate for $x_a(t)$.

(b) Determine the folding frequency.

(c) What are the frequencies, in radians, in the resulting discrete time signal $x(n)$?

(d) If $x(n)$ is passed through an ideal D/A converter, what is the reconstructed signal $y_a(t)$?

Homework 6

The analog signal $x(t) = 10\sin(2\pi t) + 10\sin(8\pi t) + 5\sin(12\pi t)$, where t is in seconds, is sampled at a rate of $f_s = 5$ Hz. Determine the signal $x_a(t)$ aliased with $x(t)$. Show that the two signals have the *same* sample values, that is, show that $x(nT) = x_a(nT)$. Repeat the above questions if the sampling rate is $f_s = 10$ Hz.

Homework 7

The signal $x(t) = \cos(5\pi t) + 4\sin(2\pi t)\sin(3\pi t)$, where t is in milliseconds, is sampled at a rate of 3 kHz. Determine the signal $x_a(t)$ aliased with $x(t)$.

Determine two other signals $x_1(t)$ and $x_2(t)$ that are different from each other and from $x(t)$, yet they are aliased with the same $x_a(t)$ that you found.

Homework 8

Let $x(t) = \cos(8\pi t) + 2\cos(4\pi t)\cos(6\pi t)$, where t is in seconds. Determine the signal $x_a(t)$ aliased with $x(t)$, if the sampling rate is 5 Hz. Repeat for a sampling rate of 9 Hz.

Homework 9

The analog signal $x(t) = \sin(6\pi t)[1 + 2\cos(4\pi t)]$, where t is in milliseconds, is sampled at a rate of 4 kHz. The resulting samples are immediately reconstructed by an ideal reconstructor. Determine the analog signal $x_a(t)$ at the output of the reconstructor.

Homework 10

Consider the following sound wave, where t is in milliseconds:

$$x(t) = \sin(10\pi t) + \sin(20\pi t) + \sin(60\pi t) + \sin(90\pi t)$$

This signal is prefiltered by an analog antialiasing prefilter $H(f)$ and then sampled at an audio rate of 40 kHz. The resulting samples are immediately reconstructed using an ideal reconstructor. Determine the output $y_a(t)$ of the reconstructor in the following cases and compare it with the *audible part* of $x(t)$:

a. When there is no prefilter, that is, $H(f) \equiv 1$.

b. When $H(f)$ is an *ideal* prefilter with cutoff of 20 kHz.

c. When $H(f)$ is a *practical* prefilter that has a flat passband up to 20 kHz and attenuates at a rate of 48 dB/octave beyond 20 kHz. (You may ignore the effects of the phase response of the filter.)

Homework 11

❖ Cho tín hiệu ngõ vào tương tự $x(t) = 3\cos 10^3\pi t - 4\sin 10^4\pi t$ (t: s) đi qua hệ thống lấy mẫu và khôi phục lý tưởng với tần số lấy mẫu F_s = 8 KHz.

a) Viết biểu thức của tín hiệu sau lấy mẫu x[n]? Xác định giá trị mẫu x[n=2] của tín hiệu sau lấy mẫu.

b) Có hay không 1 tần số lấy mẫu khác ($F_{sb} \neq$ 8 KHz) cho cùng kết quả tín hiệu sau lấy mẫu x[n]? Nếu không, hãy chứng minh. Nếu có, hãy chỉ ra 1 tần số lấy mẫu khác đó.

c) Vẽ phổ biên độ của tín hiệu sau lấy mẫu trong phạm vi tần số từ 0 đến 10 KHz.

d) Xác định biểu thức của tín hiệu sau khôi phục.

e) Xác định biểu thức của tín hiệu sau khôi phục trong trường hợp dùng thêm bộ tiền lọc thông thấp thực tế có biên độ phẳng trong tầm [-4 4] KHz và suy giảm với tốc độ -1@0dB/decade bên ngoài dải thông.

Homework 12

❖ Cho tín hiệu ngõ vào tương tự $x(t) = 2 – 4\sin 6\pi t + 8\cos 10\pi t$ (t: s) đi qua hệ thống lấy mẫu và khôi phục lý tưởng với tần số lấy mẫu lựa chọn F_s = 7,@ KHz.

a) Vẽ phổ biên độ của tín hiệu ngõ vào x(t).

b) Vẽ phổ biên độ của một tín hiệu chồng lấn (aliased signal) với x(t).

c) Vẽ phổ biên độ của tín hiệu sau lấy mẫu trong phạm vi tần số từ 0 đến 10 KHz.

d) Tìm giá trị mẫu x[n=2] của tín hiệu sau lấy mẫu.

e) Xác định biểu thức (theo thời gian) của tín hiệu sau khôi phục.

f) Tìm điều kiện của tần số lấy mẫu để khôi phục đúng tín hiệu ngõ vào x(t).

Homework 13

❖ Cho tín hiệu ngõ vào tương tự $x(t) = 14\sin^2 3\pi t + 3\sin 14\pi t$ (t: ms) đi qua hệ thống lấy mẫu và khôi phục lý tưởng với tần số lấy mẫu $F_s = 8$ KHz.

a) Tìm giá trị mẫu x[n=4] của tín hiệu sau lấy mẫu?

b) Xác định biểu thức của 1 tín hiệu chồng lấn (aliased signal) với tín hiệu ban đầu x(t)?

c) Vẽ phổ biên độ của tín hiệu sau lấy mẫu trong phạm vi tần số từ 0 đến 8 KHz?

d) Xác định biểu thức của tín hiệu sau khôi phục?

e) Xác định biểu thức của tín hiệu sau khôi phục trong trường hợp dùng thêm bộ tiền lọc thông thấp thực tế có biên độ phẳng trong tầm 4 KHz và suy giảm với tốc độ -4@dB/decade bên ngoài dải thông?

f) Xác định 1 tập giá trị thích hợp ($A, B, F_A \neq F_B$) của tín hiệu ngõ vào $x(t) = A\sin F_A t + B\sin F_B t$ (t: ms) để tín hiệu sau khôi phục (khi không dùng thêm bộ tiền lọc) $y(t) = 2\sin 2\pi t$ (t: ms)?

Homework 14

❖ Cho tín hiệu ngõ vào tương tự $x(t) = 1 – 2\cos6\pi t + 3\sin14\pi t$ (t: ms) đi qua hệ thống lấy mẫu và khôi phục lý tưởng với tần số lấy mẫu $F_s = 8$ KHz.

a) Tìm giá trị mẫu x[n=2] của tín hiệu sau lấy mẫu?

b) Xác định biểu thức (theo thời gian) của 1 tín hiệu chồng lấn (aliased signal) với tín hiệu ban đầu x(t)?

c) Vẽ phổ biên độ của tín hiệu sau lấy mẫu trong phạm vi tần số từ 0 đến 8 KHz?

d) Xác định biểu thức (theo thời gian) của tín hiệu sau khôi phục?

e) Xác định biểu thức (theo thời gian) của tín hiệu sau khôi phục trong trường hợp dùng thêm bộ tiền lọc thông thấp thực tế có biên độ phẳng trong tầm 4 KHz và suy giảm với tốc độ -6@dB/decade bên ngoài dải thông?

f) Tìm điều kiện của chu kỳ lấy mẫu T_s sao cho tín hiệu sau khôi phục (khi không dùng thêm bộ tiền lọc) giống tín hiệu ban đầu x(t)?

g) Tìm tần số lấy mẫu F_s lớn nhất có thể sao cho tín hiệu sau khôi phục (khi không dùng thêm bộ tiền lọc) là tín hiệu một chiều không đổi. Xác định giá trị một chiều không đổi này?

Homework 15

❖ Cho tín hiệu $x(t) = 4cos(0.6\pi t) + 2\cos(6\pi t) + \cos(1@\pi t)$ với t (ms), được lấy mẫu ở tần số f_s=8KHz. Xác định tín hiệu sau khi qua bộ tiền lọc y(t) và tín hiệu khôi phục $y_a(t)$ trong các trường hợp sau (bỏ qua ảnh hưởng đáp ứng pha của bộ lọc):

a) Không có bộ tiền lọc, nghĩa là H(f)=1 cho tất cả giá trị của f.

b) H(f) là bộ lọc thông thấp lý tưởng có tần số cắt tại $f_s/2$=4KHz.

c) H(f) là bộ lọc có đáp ứng tần số như hình.