Digital Signal Processing

405109

Chapter 3
Discrete-Time Systems

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Content

❖ Input/output relationship of the systems

❖ Linear time-invariant (LTI) systems
  ❑ convolution

❖ FIR and IIR filters

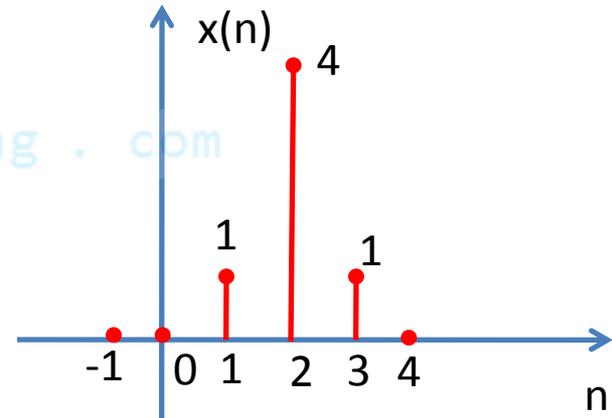❖ Causality and stability of the systems

1. Discrete-time signal

❖ The discrete-time signal x(n) is obtained from sampling an analog signal x(t), i.e., x(n)=x(nT) where T is the sampling period.

❖ There are some representations of the discrete-time signal x(n):

✓ Graphical representation:

✓ Function:

$$x(n) = \begin{cases} 1 & for \ n = 1,3 \\ 4 & for \ n = 2 \\ 0 & elsewhere \end{cases}$$

✓ Table:

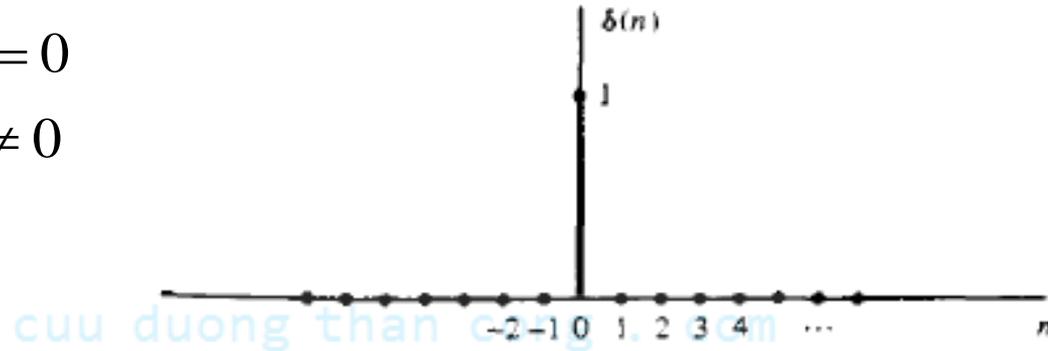| n	…	-2	-1	0	1	2	3	4	5	…
x(n)	…	0	0	0	1	4	1	0	0	…

✓ Sequence: x(n)=[… 0, 0, 1, 4, 1, 0, …]=[0, 1, 4, 1]

Some elementary discrete-time signals
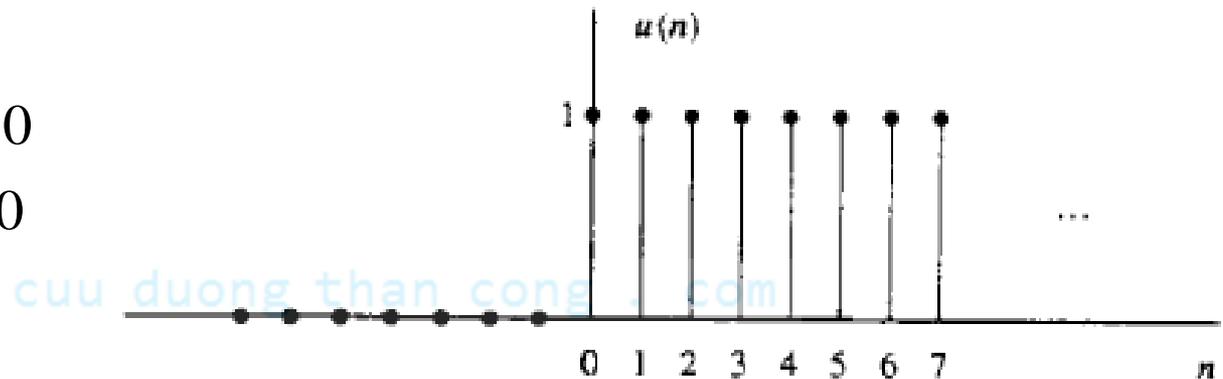
❖ Unit sample sequence (unit impulse):

$$\delta(n) = \begin{cases} 1 & for\ n = 0 \\ 0 & for\ n \neq 0 \end{cases}$$

❖ Unit step signal

$$u(n) = \begin{cases} 1 & for\ n \geq 0 \\ 0 & for\ n < 0 \end{cases}$$

2. Input/output rules

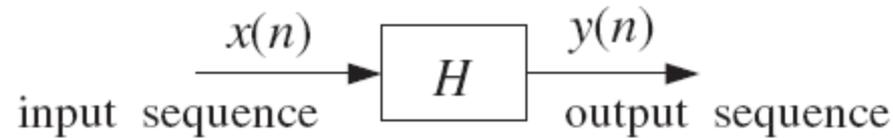❖ A discrete-time system is a processor that transform an input sequence x(n) into an output sequence y(n).

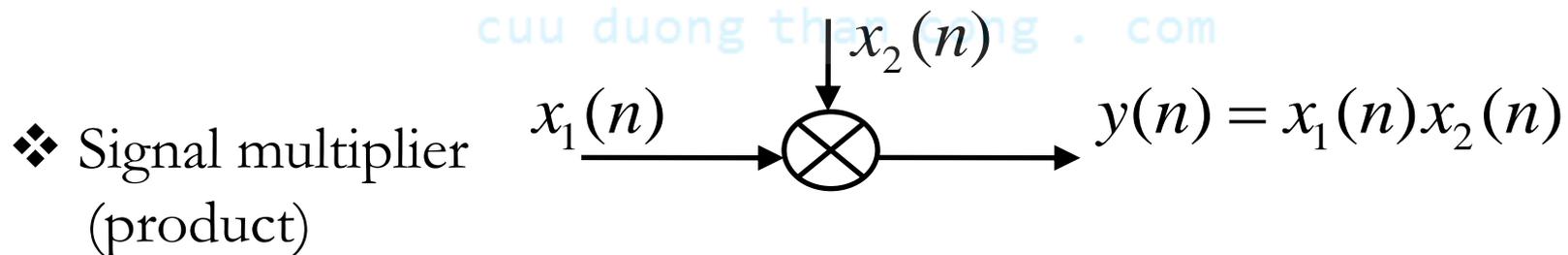$$\xrightarrow{\ x(n)\ } \boxed{H} \xrightarrow{\ y(n)\ }$$
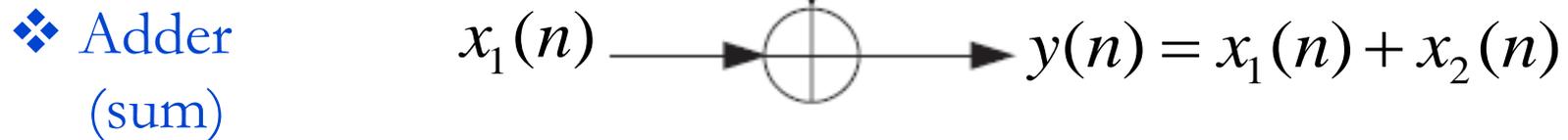input sequence ⟶ H ⟶ output sequence

Fig: Discrete-time system

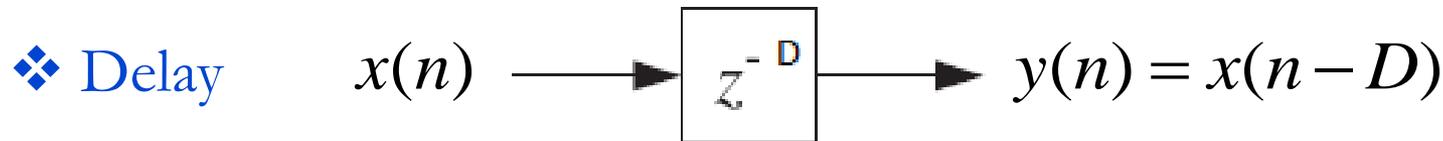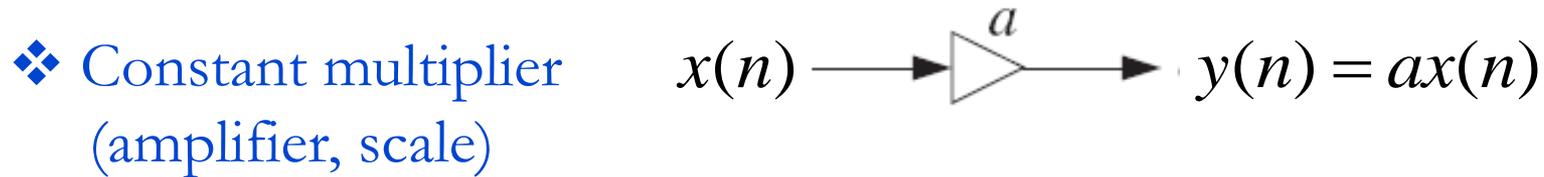❖ Sample-by-sample processing:

$$\{x_0, x_1, x_2, \ldots, x_n, \ldots\} \xrightarrow{H} \{y_0, y_1, y_2, \ldots, y_n, \ldots\}$$

that is, $x_0 \xrightarrow{H} y_0, x_1 \xrightarrow{H} y_1, x_2 \xrightarrow{H} y_2$ and so on.

❖ Block processing: $\mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} \xrightarrow{H} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix} = \mathbf{y}$

Basic building blocks of DSP systems

❖ Constant multiplier (amplifier, scale)

$$x(n) \xrightarrow{\quad} \overset{a}{\triangleright} \xrightarrow{\quad} y(n) = ax(n)$$

❖ Delay

$$x(n) \xrightarrow{\quad} \boxed{z^{-D}} \xrightarrow{\quad} y(n) = x(n-D)$$

❖ Adder (sum)

$$x_2(n) \downarrow$$
$$x_1(n) \xrightarrow{\quad} \oplus \xrightarrow{\quad} y(n) = x_1(n) + x_2(n)$$

❖ Signal multiplier (product)

$$x_2(n) \downarrow$$
$$x_1(n) \xrightarrow{\quad} \otimes \xrightarrow{\quad} y(n) = x_1(n)x_2(n)$$

Example 1

❖ Let x(n)={1, 3, 2, 5}. Find the output and plot the graph for the systems with input/out rules as follows:

 a) y(n)=2x(n)

 b) y(n)=x(n-4)

 c) y(n)=x(n+4)

 d) y(n)=x(n)+x(n-1)

Example 2

❖ A weighted average system y(n)=2x(n)+4x(n-1)+5x(n-2). Given the input signal x(n)=[x_0, x_1, x_2, x_3]

a) Find the output y(n) by sample-sample processing method?

b) Find the output y(n) by block processing method.

c) Plot the block diagram to implement this system from basic building blocks ?

3. Linearity and time invariance

❖ A **linear system** has the property that the output signal due to a linear combination of two input signals can be obtained by forming the same linear combination of the individual outputs.
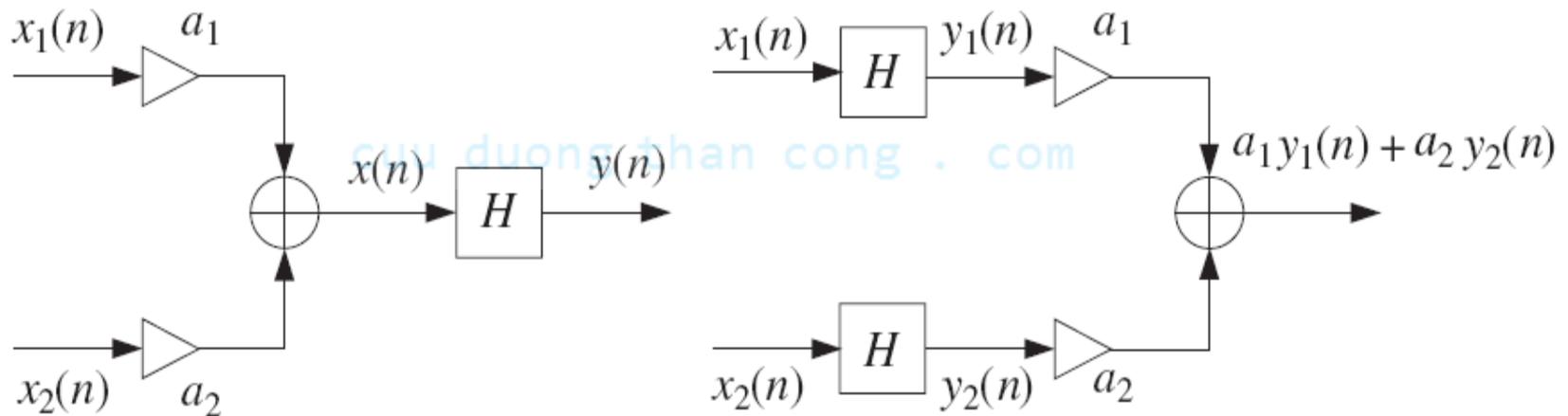

Fig: Testing linearity

❖ If $y(n)=a_1y_1(n)+a_2y_2(n)$ $\forall$ a_1, a_2 → linear system. Otherwise, the system is nonlinear.

Example 3

❖ Test the linearity of the following discrete-time systems:

a) $y(n)=nx(n)$

b) $y(n)=x(n^2)$

c) $y(n)=x^2(n)$

d) $y(n)=Ax(n)+B$

3. Linearity and time invariance

❖ A **time-invariant system** is a system that its input-output characteristics do not change with time.
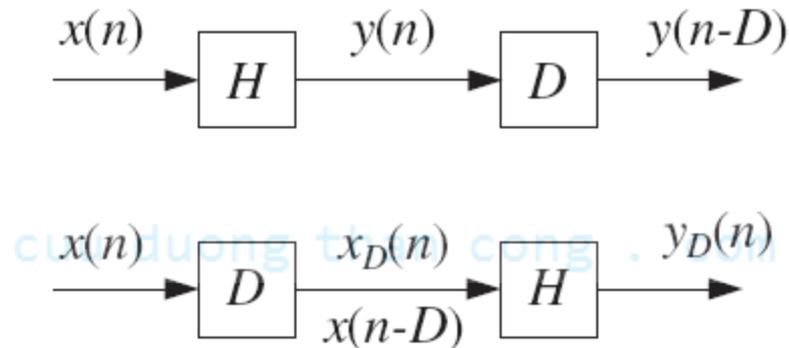

Fig: Testing time invariance

❖ If $y_D(n) = y(n-D)$ $\forall D \rightarrow$ time-invariant system. Otherwise, the system is time-variant.

Example 4

❖ Test the time-invariance of the following discrete-time systems:

a) y(n)=x(n)-x(n-1)

b) y(n)=nx(n)

c) y(n)=x(-n)

d) y(n)=x(2n)

4. Impulse response

❖ **Linear time-invariant (LTI) systems** are characterized uniquely by their impulse response sequence h(n), which is defined as the response of the systems to a unit impulse δ(n).
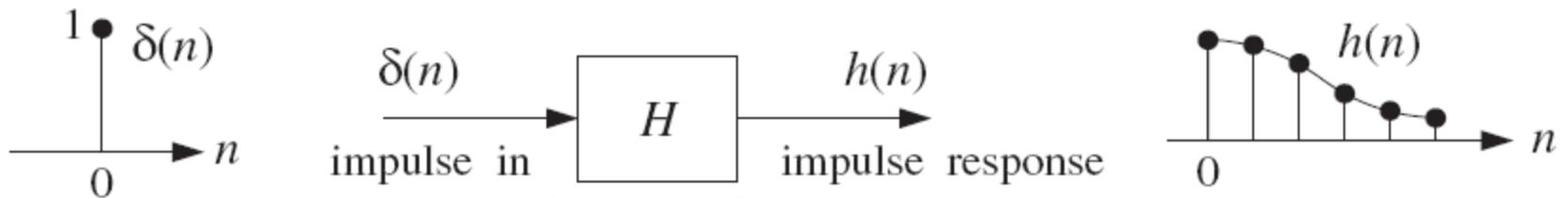
Fig: Impulse response of an LTI system

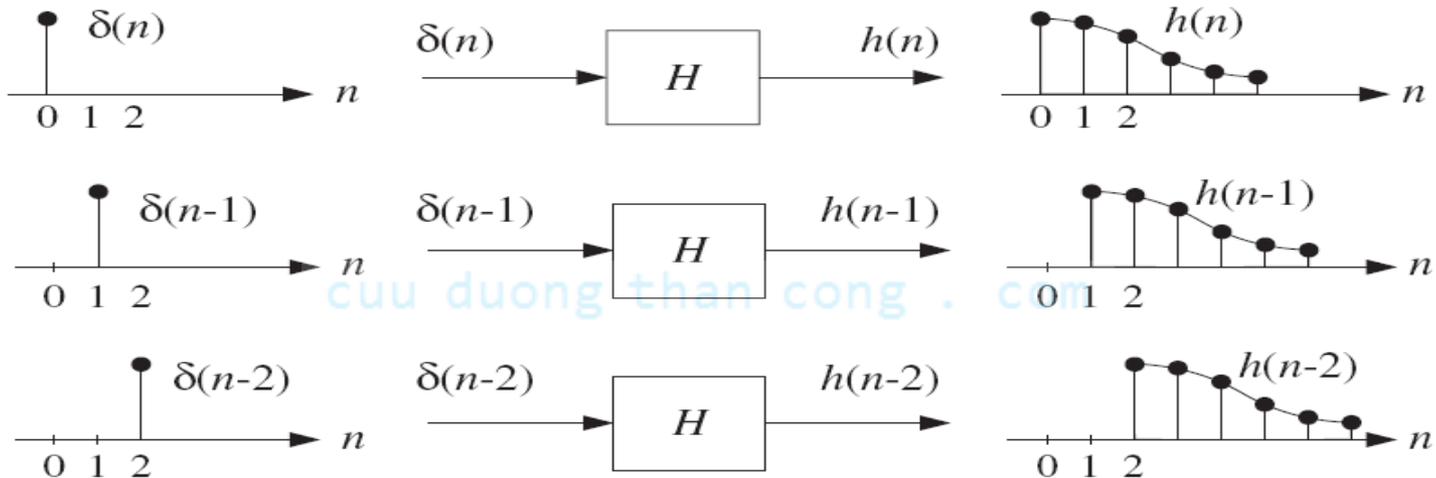Fig: Delayed impulse responses of an LTI system
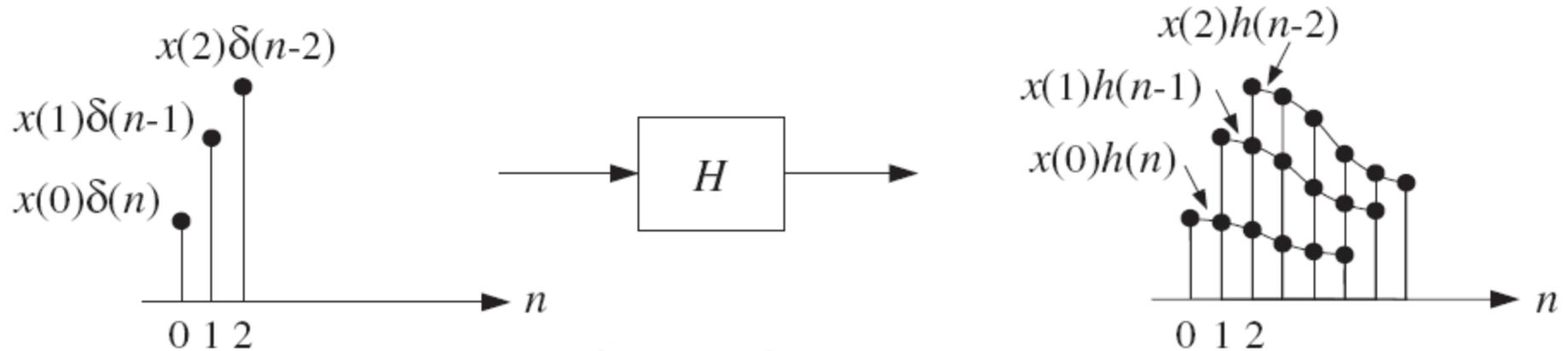
5. Convolution of LTI systems

Fig: Response to linear combination of inputs

❖ **Convolution:**

$$y(n) = \sum_m x(m)h(n-m) = x(n) * h(n) \quad \text{(LTI form)}$$

$$y(n) = \sum_m h(m)x(n-m) = h(n) * x(n) \quad \text{(direct form)}$$

6. FIR versus IIR filters

❖ **A finite impulse response (FIR) filter** has impulse response h(n) that extend only over a finite time interval, say $0 \leq n \leq M$.
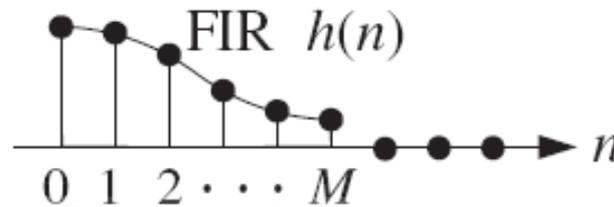

Fig: FIR impulse response

❖ M: filter order; L_h=M+1: the length of impulse response

❖ h={h_0, h_1, …, h_M} is referred by various name such as filter coefficients, filter weights, or filter taps.

❖ FIR filtering equation:

$$y(n) = h(n) * x(n) = \sum_{m=0}^{M} h(m) x(n-m)$$

Example 5

❖ The third-order FIR filter has the impulse response h=[1, 2, 1, -1]

a) Find the I/O equation, i.e., the relationship of the input x(n) and the output y(n) ?

b) Given x=[1, 2, 3, 1], find the output y(n) ?

6. FIR versus IIR filters

❖ **A infinite impulse response (IIR) filter** has impulse response h(n) of infinite duration, say $0 \le n \le \infty$.
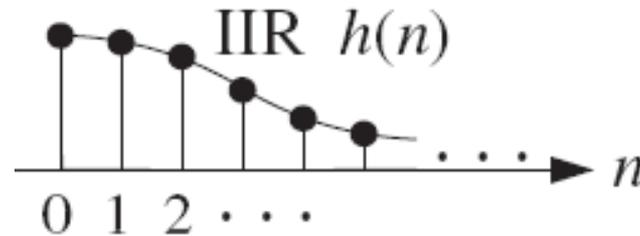

Fig: IIR impulse response

❖ IIR filtering equation: $$y(n) = h(n) * x(n) = \sum_{m=0}^{\infty} h(m)x(n-m)$$

❖ The I/O equation of IIR filters are expressed as the recursive difference equation.

Example 6

❖ Determine the output of the LTI system which has the impulse response $h(n)=a^n u(n)$, $|a| \leq 1$ when the input is the unit step signal $x(n)=u(n)$?

❖ Remark:

$$\sum_{k=m}^{n} r^k = \frac{r^m - r^{n+1}}{1-r}$$

❖ When $n = \infty$ and $|r| \leq 1$

$$\sum_{k=m}^{\infty} r^k = \frac{r^m}{1-r}$$

Example 7

❖ Assume the IIR filter has a **casual** h(n) defined by

$$h(n) = \begin{cases} 2 & \text{for } n = 0 \\ 4(0.5)^{n-1} & \text{for } n \geq 1 \end{cases}$$

a) Find the I/O difference equation ?

b) Find the difference equation for h(n)?
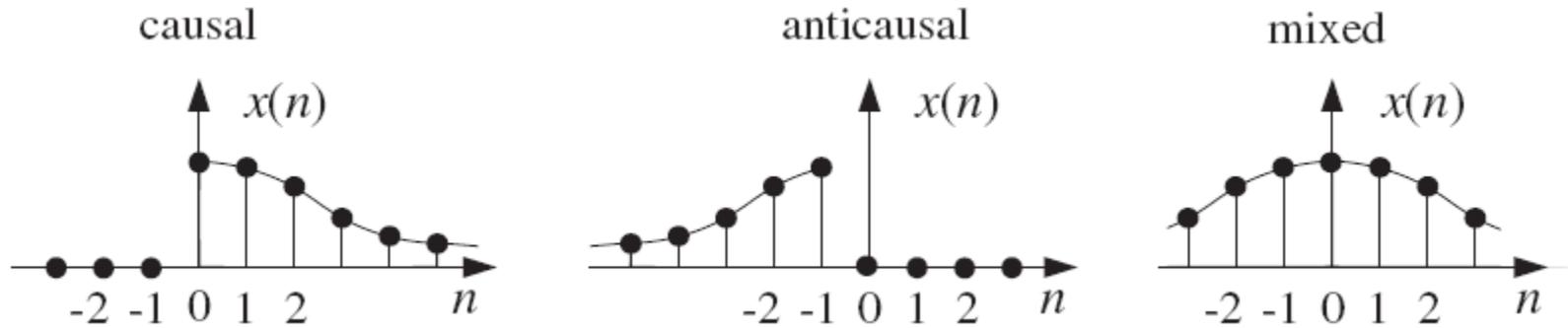
Fig: Causal, anticausal, and mixed signals

❖ LTI systems can also classified in terms of causality depending on whether h(n) is casual, anticausal or mixed.

❖ A system is stable (BIBO) if **bounded inputs** ($|x(n)| \leq A$) always generate **bounded outputs** ($|y(n)| \leq B$).

❖ A LTI system is stable $\Longleftrightarrow \displaystyle\sum_{n=-\infty}^{\infty} |h(n)| < \infty$

Example 8

❖ Consider the causality and stability of the following systems:

a) $h(n)=(0.5)^n u(n)$

b) $h(n)=(-0.5)^n u(-n-1)$

8. Static versus Dynamic systems

❖ Static (memoryless): output at any instant depends at most on the input sample at the same time, but not on past or future samples of the inputs.

$$y(n) = \mathcal{T}\big[x(n), n\big]$$

❖ Otherwise, the system is dynamic.
  - ✓ Finite memory
  - ✓ Infinite memory

9. Interconnection of discrete time systems

❖ Cascade (series):
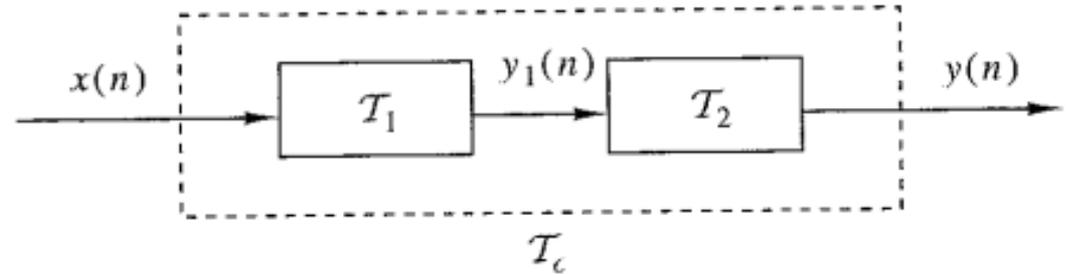
➢ LTI systems: $\mathcal{T}_2\mathcal{T}_1 = \mathcal{T}_1\mathcal{T}_2$

❖ Parallel: $\mathcal{T}_p = \mathcal{T}_1 + \mathcal{T}_2$
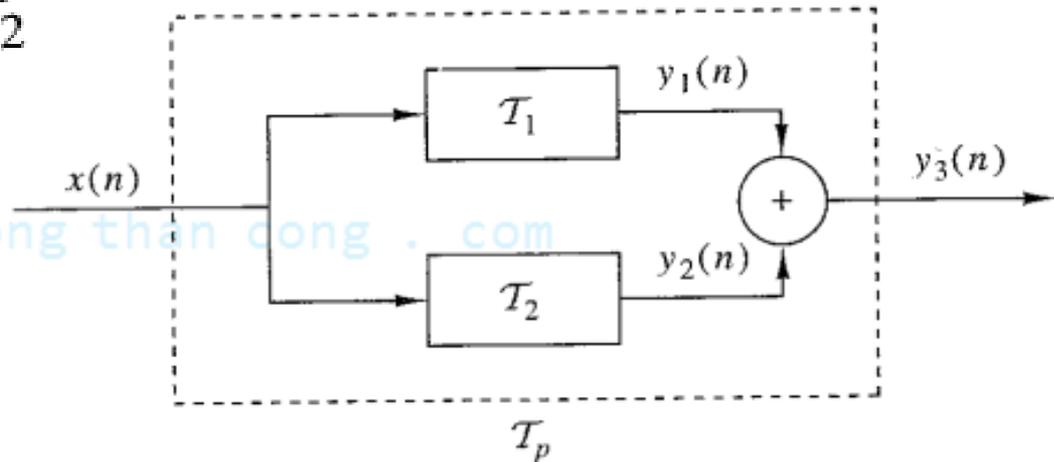
10. Energy versus Power signals

❖ Energy: $E \equiv \displaystyle\sum_{n=-\infty}^{\infty} |x(n)|^2$

❖ Power: $P = \displaystyle\lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$

❖ Periodic: $x(n + N) = x(n)$ for all n

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

❖ Otherwise, the signal is nonperiodic or aperiodic.

❖ Symmetric (even): $x(-n) = x(n)$

$$x(n) = x_e(n) + x_o(n)$$

❖ Antisymmetric (odd): $x(-n) = -x(n)$
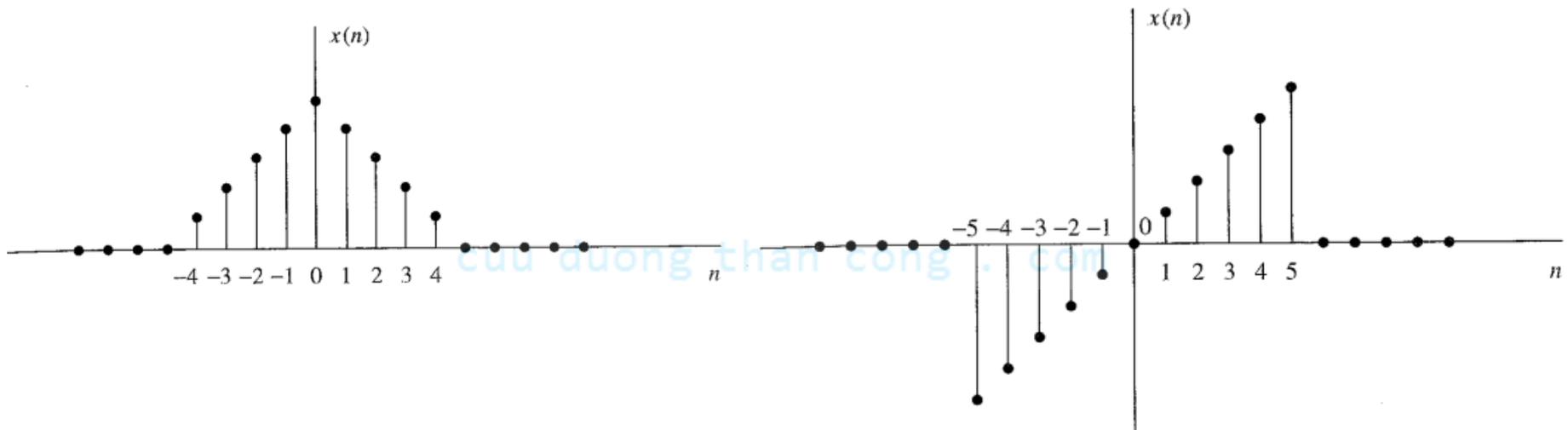
13. Crosscorrelation and Autocorrelation

❖ Crosscorrelation:

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\,y(n-l)$$

$$r_{xy}(l) = r_{yx}(-l)$$

$$r_{xy}(l) = x(l) * y(-l)$$

$$|r_{xy}(l)| \leq \sqrt{r_{xx}(0)\,r_{yy}(0)} = \sqrt{E_x E_y}$$

❖ Autocorrelation:

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\,x(n-l)$$

$$|r_{xx}(l)| \leq r_{xx}(0) = E_x$$

Example 9

Suppose that a signal sequence $x(n) = \sin(\pi/5)n$, for $0 \leq n \leq 99$ is corrupted by an additive noise sequence $w(n)$, where the values of the additive noise are selected independently from sample to sample, from a uniform distribution over the range $(-\Delta/2, \Delta/2)$, where Δ is a parameter of the distribution. The observed sequence is $y(n) = x(n) + w(n)$. Determine the autocorrelation sequence $r_{yy}(l)$ and thus determine the period of the signal $x(n)$.
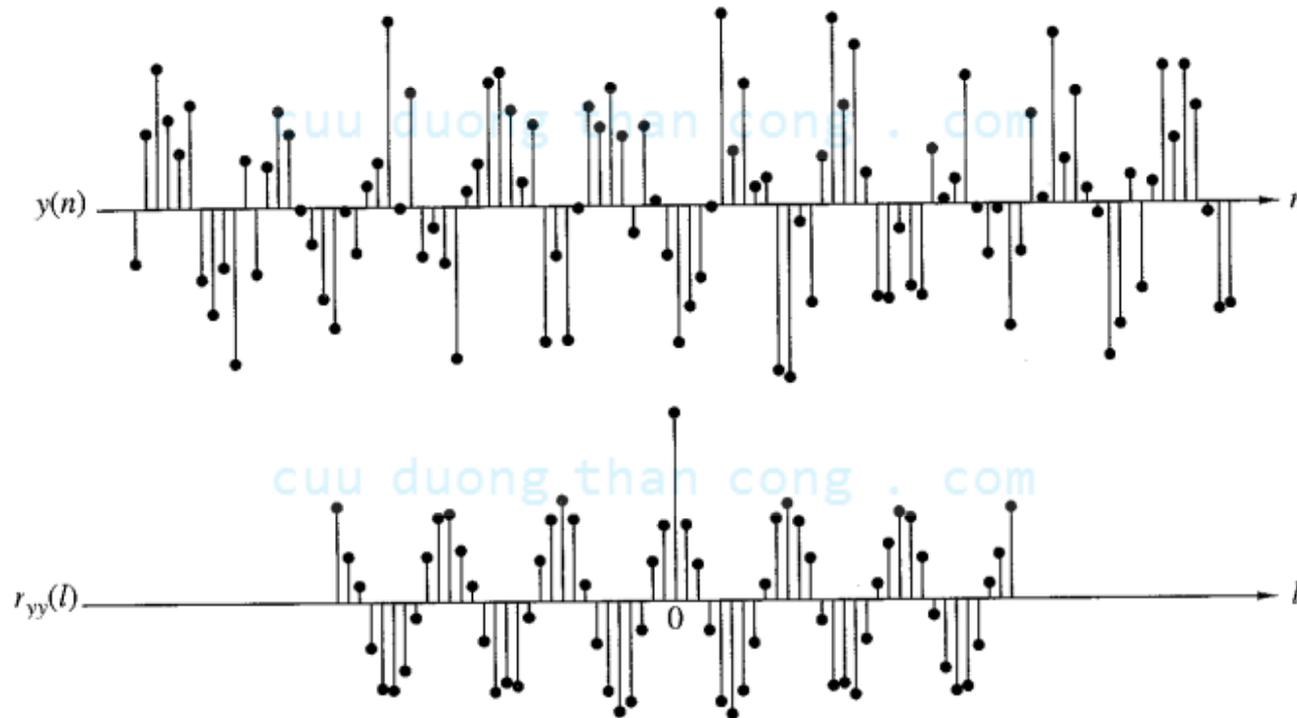
Consider the interconnection of LTI systems as shown in Fig. P2.35.
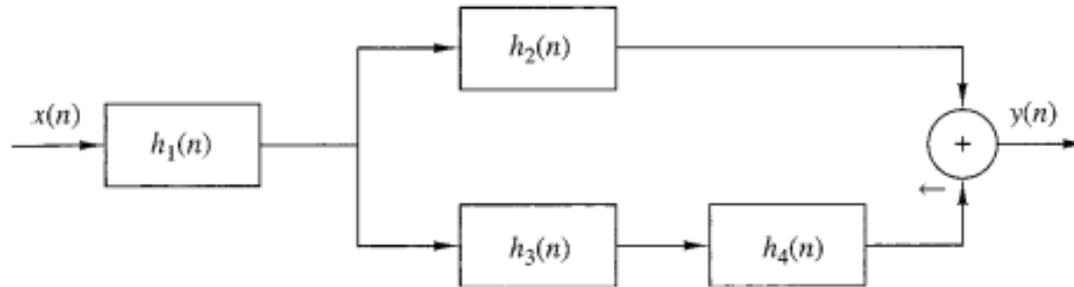
Figure P2.35

(a) Express the overall impulse response in terms of $h_1(n)$, $h_2(n)$, $h_3(n)$, and $h_4(n)$.

(b) Determine $h(n)$ when

$$h_1(n) = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{2}\}$$

$$h_2(n) = h_3(n) = (n+1)u(n)$$

$$h_4(n) = \delta(n-2)$$

(c) Determine the response of the system in part (b) if

$$x(n) = \delta(n+2) + 3\delta(n-1) - 4\delta(n-3)$$

Consider the system in Fig. P2.36 with $h(n) = a^n u(n)$, $-1 < a < 1$. Determine the response $y(n)$ of the system to the excitation

$$x(n) = u(n+5) - u(n-10)$$
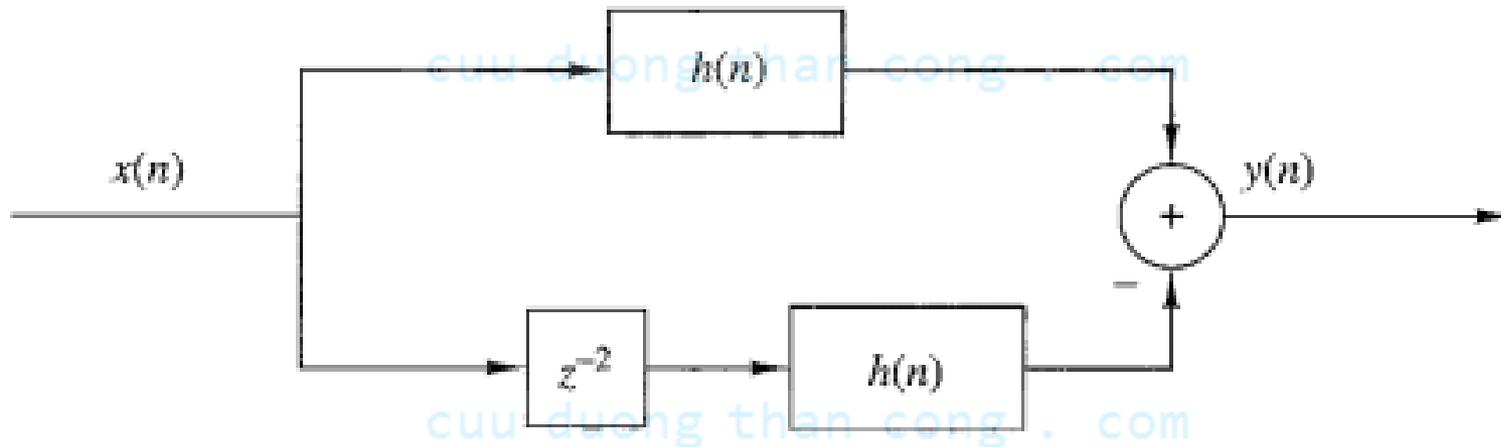
Figure P2.36

Homework 3

Consider the discrete-time system shown in Fig. P2.47.
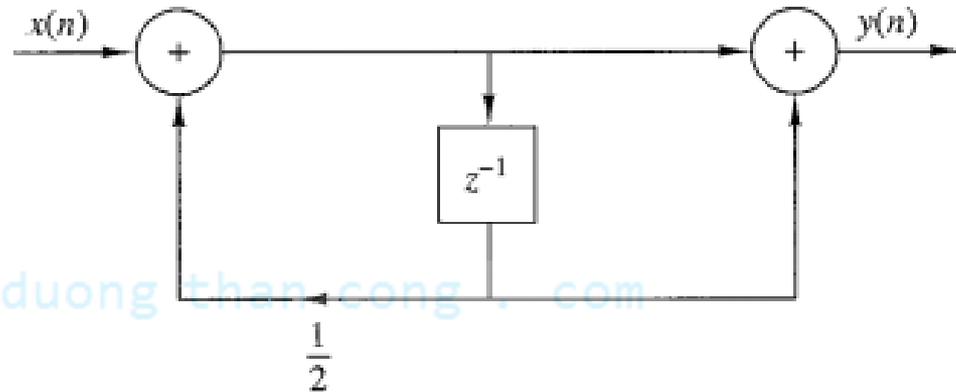
Figure P2.47

(a) Compute the 10 first samples of its impulse response.

(b) Find the input–output relation.

(c) Apply the input $x(n) = \{1, 1, 1, \ldots\}$ and compute the first 10 samples of the output.

(d) Compute the first 10 samples of the output for the input given in part (c) by using convolution.

(e) Is the system causal? Is it stable?

A discrete-time system is realized by the structure shown in Fig. P2.49.

(a) Determine the impulse response.

(b) Determine a realization for its inverse system, that is, the system which produces $x(n)$ as an output when $y(n)$ is used as an input.
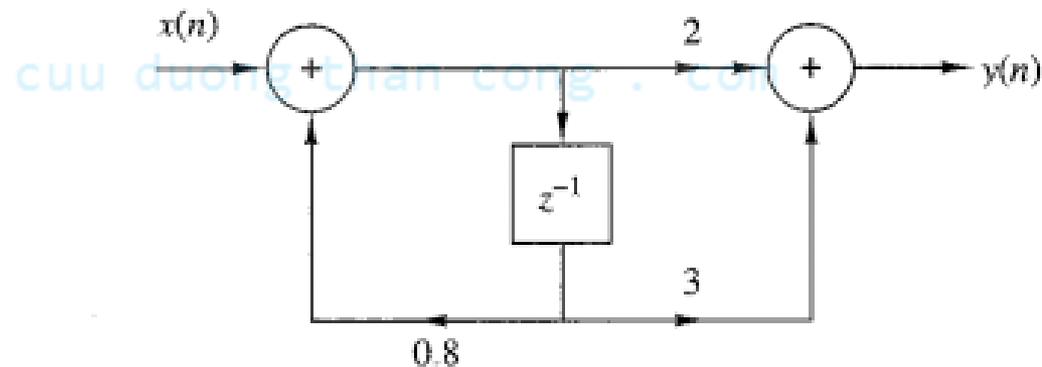
Figure P2.49

Consider the systems shown in Fig. P2.52.
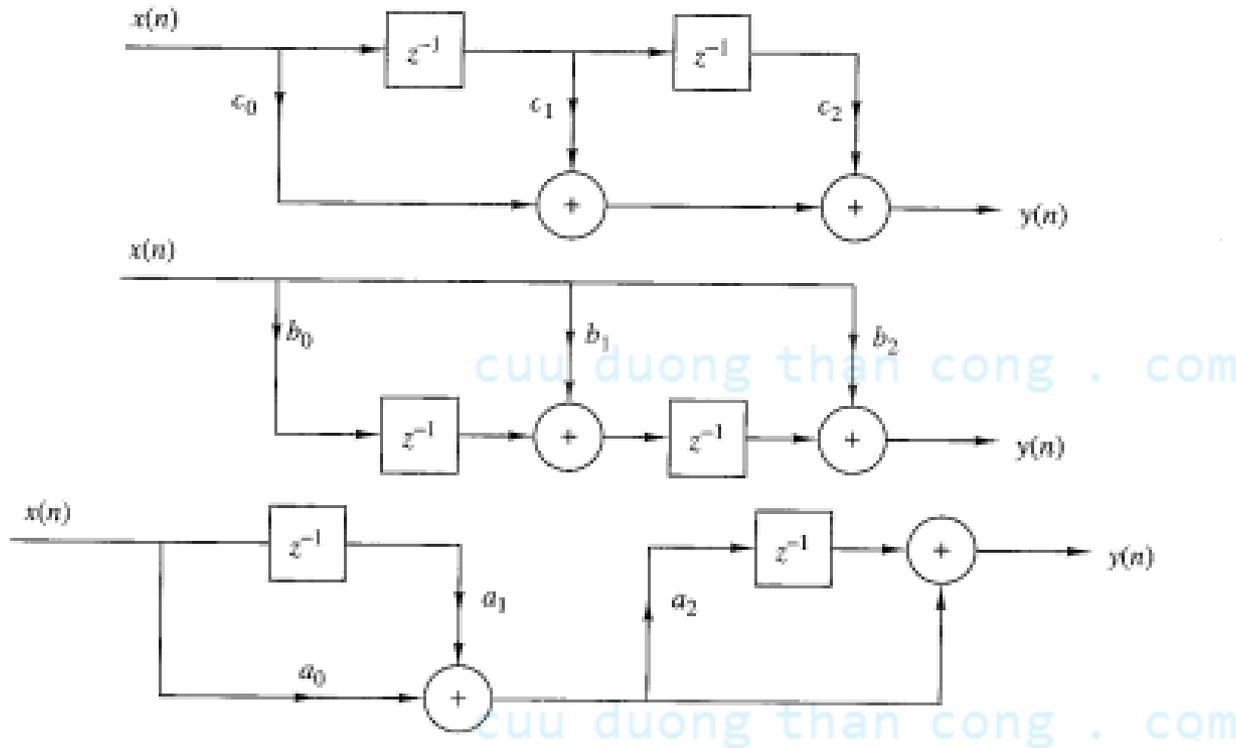
Figure P2.52

(a) Determine and sketch their impulse responses $h_1(n)$, $h_2(n)$, and $h_3(n)$.

(b) Is it possible to choose the coefficients of these systems in such a way that

$$h_1(n) = h_2(n) = h_3(n)$$

Consider the system shown in Fig. P2.53.
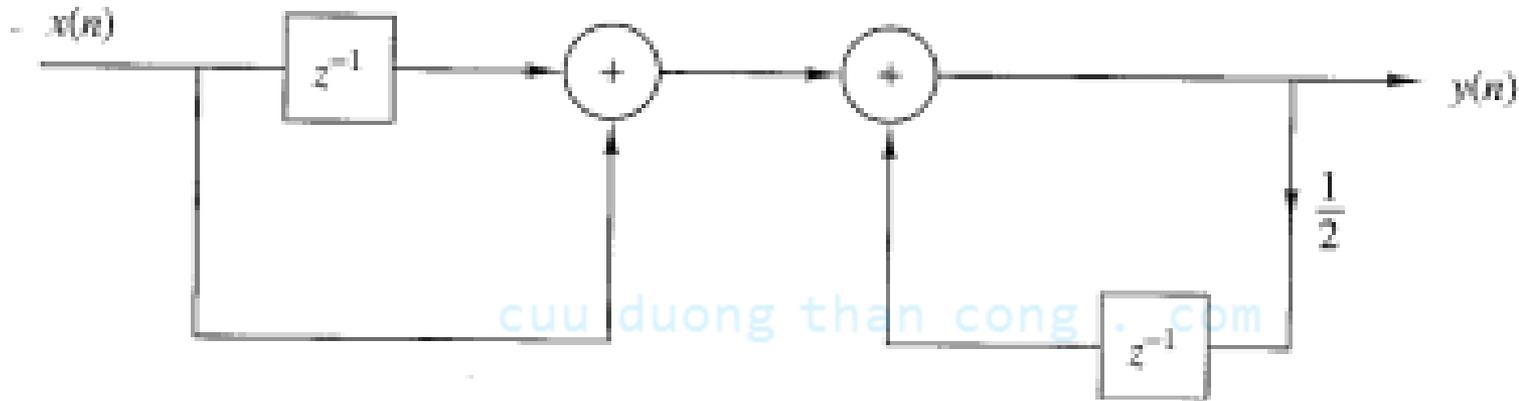
Figure P2.53

(a) Determine its impulse response $h(n)$.

(b) Show that $h(n)$ is equal to the convolution of the following signals:

$$h_1(n) = \delta(n) + \delta(n-1)$$

$$h_2(n) = (\frac{1}{2})^n u(n)$$

Homework 7

An audio signal $s(t)$ generated by a loudspeaker is reflected at two different walls with reflection coefficients r_1 and r_2. The signal $x(t)$ recorded by a microphone close to the loudspeaker, after sampling, is

$$x(n) = s(n) + r_1 s(n - k_1) + r_2 s(n - k_2)$$

where k_1 and k_2 are the delays of the two echoes.

(a) Determine the autocorrelation $r_{xx}(l)$ of the signal $x(n)$.

(b) Can we obtain r_1, r_2, k_1, and k_2 by observing $r_{xx}(l)$?

(c) What happens if $r_2 = 0$?

A causal linear time-invariant filter has impulse response:

$$h_n = C_1 p_1^n + C_2 p_2^n, \qquad n \geq 0$$

Working in the time domain, show that the difference equation satisfied by h_n for all $n \geq 0$ and the difference equation relating the input and output signals are of the form:

$$h_n + a_1 h_{n-1} + a_2 h_{n-2} = b_0 \delta(n) + b_1 \delta(n-1)$$

$$y_n + a_1 y_{n-1} + a_2 y_{n-2} = b_0 x_n + b_1 x_{n-1}$$

Determine $\{a_1, a_2, b_0, b_1\}$ in terms of $\{C_1, C_2, p_1, p_2\}$.

A causal linear time-invariant filter has impulse response:

$$h_n = C_0\delta(n) + C_1 p_1^n + C_2 p_2^n, \qquad n \geq 0$$

Show that the difference equation satisfied by h_n for all $n \geq 0$ and the difference equation relating the input and output signals are of the form:

$$h_n + a_1 h_{n-1} + a_2 h_{n-2} = b_0\delta(n) + b_1\delta(n-1) + b_2\delta(n-2)$$

$$y_n + a_1 y_{n-1} + a_2 y_{n-2} = b_0 x_n + b_1 x_{n-1} + b_2 x_{n-2}$$

Determine $\{a_1, a_2, b_0, b_1, b_2\}$ in terms of $\{C_0, C_1, C_2, p_1, p_2\}$.

A causal linear time-invariant filter has impulse response:

$$h_n = C_1 p_1^n + C_2 p_2^n + C_3 p_3^n, \qquad n \geq 0$$

Working in the time domain, show that the difference equation satisfied by h_n for all $n \geq 0$ and the difference equation relating the input and output signals are of the form:

$$h_n + a_1 h_{n-1} + a_2 h_{n-2} + a_3 h_{n-3} = b_0 \delta(n) + b_1 \delta(n-1) + b_2 \delta(n-2)$$

$$y_n + a_1 y_{n-1} + a_2 y_{n-2} + a_3 y_{n-3} = b_0 x_n + b_1 x_{n-1} + b_2 x_{n-2}$$

Determine $\{a_1, a_2, a_3, b_0, b_1\}$ in terms of $\{C_1, C_2, C_3, p_1, p_2, p_3\}$.

Cho hệ thống rời rạc tuyến tính bất biến có đáp ứng xung h(n)={0↑, @, -1}.

a) Xác định phương trình sai phân vào-ra của hệ thống trên.

b) Vẽ 1 sơ đồ khối thực hiện hệ thống trên.

c) Tìm giá trị của mẫu tín hiệu ngõ ra y(n = 1) khi tín hiệu ngõ vào x(n) = {1, 0↑, -1}.

d) Tìm giá trị của mẫu tín hiệu ngõ ra y(n = 2) khi tín hiệu ngõ vào x(n) = δ(n) − δ(n–2).

e) Tìm giá trị của mẫu tín hiệu ngõ ra y(n = 3) khi tín hiệu ngõ vào x(n) = u(n) − u(n–3).

f) Tìm giá trị của mẫu tín hiệu ngõ ra y(n = 4) khi tín hiệu ngõ vào x(n) = u(n+4) − u(n–4).

g) Tìm giá trị của mẫu tín hiệu ngõ ra y(n = 5) khi tín hiệu ngõ vào x(n) = u(–n) − u(–n–5).

Cho hệ thống rời rạc có phương trình sai phân vào-ra y(n) = 2x(n) – 3x(n–3).

a) Tìm đáp ứng xung của hệ thống trên.

b) Tìm các giá trị của tín hiệu ngõ ra khi tín hiệu ngõ vào x(n) = δ(n+@) + 2δ(n – 2).

c) Tìm 5 giá trị (n=0,1,2,3,4) của tín hiệu ngõ ra khi tín hiệu ngõ vào x(n) = u(n).

d) Tìm 5 giá trị (n=0,1,2,3,4) của tín hiệu ngõ ra khi tín hiệu ngõ vào x(n) = u(– n).

e) Tìm 5 giá trị (n=0,1,2,3,4) của tín hiệu ngõ ra khi tín hiệu ngõ vào x(n) = u(2 – n).

f) Tìm 5 giá trị (n=0,1,2,3,4) của tín hiệu ngõ ra khi tín hiệu ngõ vào x(n) = u(n – 2).

Cho hệ thống rời rạc tuyến tính bất biến nhân quả có phương trình sai phân vào-ra y(n) = 2x(n–2) – y(n–1).

a) Vẽ 1 sơ đồ khối thực hiện hệ thống trên với số bộ trễ là ít nhất có thể.

b) Tìm giá trị của đáp ứng xung h(n = @).

c) Tìm giá trị mẫu ngõ ra y(n = @) khi ngõ vào x(n) = 2δ(n).

d) Tìm giá trị mẫu ngõ ra y(n = @) khi ngõ vào x(n) = δ(n–2).

e) Tìm giá trị mẫu ngõ ra y(n = @) khi ngõ vào x(n) = δ(n)–δ(n–2).

f) Tìm giá trị mẫu ngõ ra y(n = @) khi ngõ vào x(n) = u(n)–u(n-2).

g) Tìm giá trị mẫu ngõ ra y(n = @) khi ngõ vào x(n) = u(n).

h) Tìm giá trị mẫu ngõ ra y(n = @) khi ngõ vào x(n) = u(–n).

i) Tìm giá trị mẫu ngõ ra y(n = @) khi ngõ vào x(n) = u(–n–1).

j) Tìm giá trị mẫu ngõ ra y(n = @) khi ngõ vào x(n) = 1.

Kiểm tra tính chất tuyến tính, bất biến, nhân quả, ổn định, tĩnh của các hệ thống rời rạc sau:

1) $y(n) = x(n) + 2$.
2) $y(n) = 2 - x(n)$.
3) $y(n) = x(2 - n)$.
4) $y(n) = x^2(n)$.
5) $y(n) = x(n^2)$.
6) $y(n) = x(2n)$.
7) $y(n) = x(2n + 1)$.
8) $y(n) = nx(n)$.
9) $y(n) = x(2^{|n|})$.
10) $y(n) = 2^{x(n)}$.
11) $y(n) = 2^n x(n)$.
12) $y(n) = 2^{-n} x(n)$.

Kiểm tra tính chất tuyến tính, bất biến, nhân quả, ổn định, tĩnh của các hệ thống rời rạc sau:

1) $y(n) = \cos\{x(n)\}$.
2) $y(n) = \cos\{x(2n)\}$.
3) $y(n) = \cos\{x^2(n)\}$.
4) $y(n) = \cos^2\{x(n)\}$.
5) $y(n) = \cos(n)x(n)$.
6) $y(n) = \cos\{nx(n)\}$.
7) $y(n) = \cos(n) + x(n)$.
8) $y(n) = x(n) + 2x(n-3) - 3x(n+2)$.
9) $y(n) = 2x(n) + y(n-1)$.
10) $y(n) = x(n) + 2y(n-1)$.
11) $y(n) = x(n) + y(n-1)/2$.
12) $y(n) = y(n-1) - y(n-2)$.

Xác định và vẽ tín hiệu ngõ ra tương ứng với tín hiệu ngõ vào x(n) = {− @, 0, 1, 2, 3↑} của các hệ thống rời rạc sau:

1) y(n) = nx(n).
2) y(n) = x(n − 2).
3) y(n) = x(n + 2).
4) y(n) = x(n) + 2.
5) y(n) = x(2n).
6) y(n) = x(2n − 1).
7) y(n) = x(− n).
8) y(n) = x(2 − n).
9) y(n) = x^2(n).
10) y(n) = x(n) + x(n + 2).
11) y(n) = x(n) − x(n − 2).
12) y(n) = x(n) + x(− n).

Xác định và vẽ tín hiệu ngõ ra tương ứng với tín hiệu ngõ vào x(n) = {0↑, 4, 5, @} của các hệ thống rời rạc sau:

1) y(n) = nx(n).
2) y(n) = x(n − 2).
3) y(n) = x(n + 2).
4) y(n) = x(n) + 2.
5) y(n) = x(2n).
6) y(n) = x(2n − 1).
7) y(n) = x(− n).
8) y(n) = x(2 − n).
9) y(n) = $x^2(n)$.
10) y(n) = x(n) + x(n + 2).
11) y(n) = x(n) − x(n − 2).
12) y(n) = x(n) + x(−n).

Xác định và vẽ tín hiệu ngõ ra tương ứng với tín hiệu ngõ vào x(n) = {− @, 0, 1, 2, 3↑, 4, 5, @} của các hệ thống rời rạc sau:

1) y(n) = nx(n).
2) y(n) = x(n − 2).
3) y(n) = x(n + 2).
4) y(n) = x(n) + 2.
5) y(n) = x(2n).
6) y(n) = x(2n − 1).
7) y(n) = x(− n).
8) y(n) = x(2 − n).
9) y(n) = x^2(n).
10) y(n) = x(n) + x(n + 2).
11) y(n) = x(n) − x(n − 2).
12) y(n) = x(n) + x(− n).

Xác định và vẽ tín hiệu ngõ ra tương ứng với tín hiệu ngõ vào x(n) = @δ(n) + 2δ(n – 2) – 3δ(n + 3) của các hệ thống rời rạc sau:

1) y(n) = nx(n).
2) y(n) = x(n – 2).
3) y(n) = x(n + 2).
4) y(n) = x(n) + 2.
5) y(n) = x(2n).
6) y(n) = x(2n – 1).
7) y(n) = x(– n).
8) y(n) = x(2 – n).
9) y(n) = x^2(n).
10) y(n) = x(n) + x(n + 2).
11) y(n) = x(n) – x(n – 2).
12) y(n) = x(n) + x(–n).

Vẽ sơ đồ khối thực hiện các hệ thống rời rạc sau:
1) y(n) = x(n) + 2x(n − 1) − 3x(n − 3).
2) y(n) = 2x(n − 1) + y(n − 1).
3) y(n) = x(n − 1) + 2y(n − 1).
4) y(n) = x(n − 1) + y(n − 1)/2.
5) y(n) = y(n − 1) − y(n − 2).
6) y(n) = x(n − 1) − y(n − 2).
7) y(n) = x(n − 2) − y(n − 2).
8) y(n) = x(n − 2) − y(n − 1).
9) y(n) = 2x(n) − y(n − 2).
10) y(n) = 0.5{2x(n) − y(n − 2)}.
11) y(n) = x(n) + 2x(n − 1) − 3y(n − 2).
12) y(n) = x(n) + 2x(n − 2) − 3y(n − 2).

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Vẽ dạng sóng của các tín hiệu rời rạc sau:
1) x(n) = δ(n) – δ(n – 2).
2) x(n) = 2δ(n – 2) – δ(n + 2).
3) x(n) = u(n) – u(n – 2).
4) x(n) = u(–n).
5) x(n) = u(2 – n).
6) x(n) = u(2 + n).
7) x(n) = u(n) + u(–n).
8) x(n) = u(– n) – u(–n – 1).
9) x(n) = nu(n).
10) x(n) = nu(–n – 1).
11) x(n) = u(n) – 1.
12) x(n) = 1 – u(–n – 1).

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