Digital Signal Processing

405109

Chapter 5

z-Transform

Nguyen Thanh Tuan, M.Eng.
Department of Telecommunications (113B3)
Ho Chi Minh City University of Technology
Email: nttbk97@yahoo.com

- The z-transform is a tool for analysis, design and implementation of discrete-time signals and LTI systems.

- Convolution in time-domain ⇔ multiplication in the z-domain

Content

1. z-transform
2. Properties of the z-transform
3. Causality and Stability
4. Inverse z-transform

1. The z-transform

❖ The z-transform of a discrete-time signal x(n) is defined as the power series:

$$X(z)=\sum_{n=-\infty}^{\infty}x(n)z^{-n}=\cdots x(-2)z^{2}+x(-1)z+x(0)+x(1)z^{-1}+x(2)z^{-2}+\cdots$$

❖ The region of convergence (ROC) of X(z) is the set of all values of z for which X(z) attains a finite value.

$$ROC=\{z\in C \mid X(z)=\sum_{n=-\infty}^{\infty}x(n)z^{-n}\neq\infty\}$$

❖ The z-transform of impulse response h(n) is called the transform function of the filter:

$$H(z)=\sum_{n=-\infty}^{\infty}h(n)z^{-n}$$

Example 1

❖ Determine the z-transform of the following finite-duration signals

a) $x_1(n)=[1, 2, 5, 7, 0, 1]$

b) $x_2(n)=x_1(n-2)$

c) $x_3(n)=x_1(n+2)$

d) $x_4(n)=\delta(n)$

e) $x_5(n)=\delta(n-k)$, $k>0$

f) $x_6(n)=\delta(n+k)$, $k>0$

Example 2

❖ Determine the z-transform of the signal

a) $x(n)=(0.5)^n u(n)$

b) $x(n)=-(0.5)^n u(-n-1)$

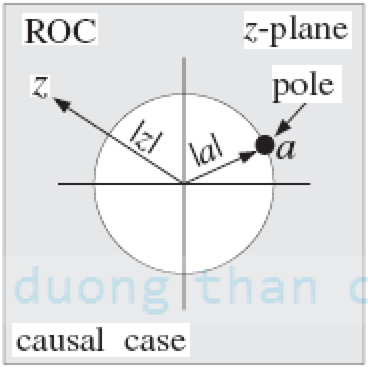
z-transform and ROC

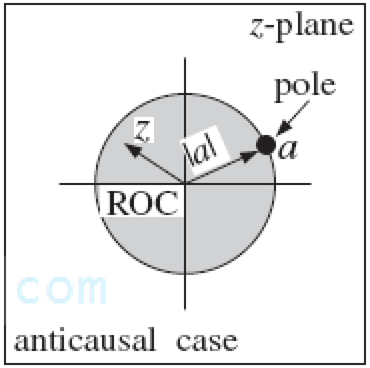
- It is possible for two different signal x(n) to have the same z-transform. Such signals can be distinguished in the z-domain by their region of convergence.
- z-transforms:

$$a^n u(n) \xleftrightarrow{z} \frac{1}{1 - az^{-1}}, \quad \text{with } |z| > |a|$$

$$-a^n u(-n-1) \xleftrightarrow{z} \frac{1}{1 - az^{-1}}, \quad \text{with } |z| < |a|$$

and their ROCs:





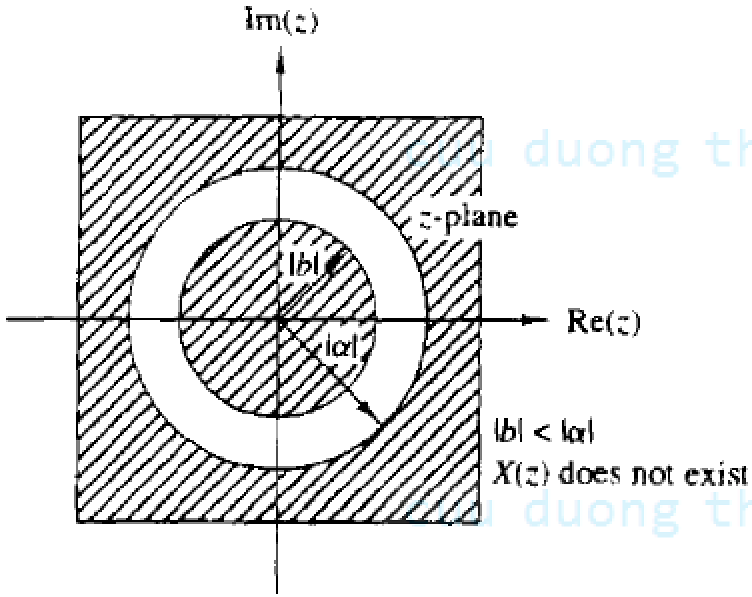
ROC of a causal signal is the exterior of a circle.

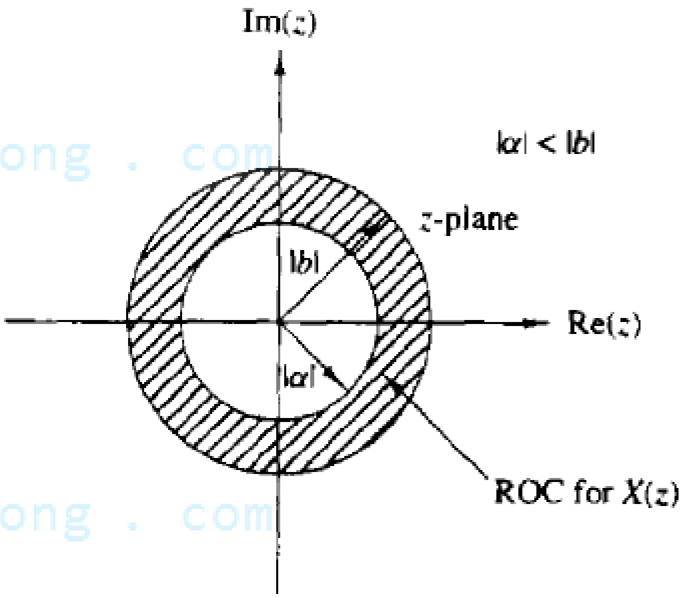
ROC of an anticausal signal is the interior of a circle.

Example 3

❖ Determine the z-transform of the signal

$$x(n) = a^n u(n) + b^n u(-n-1)$$





❖ **The ROC of two-sided signal is a ring (annular region).**

2. Properties of the z-transform

❖ **Linearity:**

if $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$ with ROC_1

and $x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$ with ROC_2

then

$$x(n) = x_1(n) + x_2(n) \overset{z}{\longleftrightarrow} X(z) = X_1(z) + X_2(z) \quad \text{with ROC} = \text{ROC}_1 \cap \text{ROC}_2$$

❖ Example: Determine the z-transform and ROC of the signals

a) $x(n)=[3(2)^n-4(3)^n]u(n)$

b) $x(n)=\cos(w_0 n)u(n)$

c) $x(n)=\sin(w_0 n)u(n)$

2. Properties of the z-transform

❖ Time shifting:

if $$x(n) \overset{z}{\longleftrightarrow} X(z)$$

then $$x(n-D) \overset{z}{\longleftrightarrow} z^{-D}X(z)$$

❖ The ROC of $z^{-D}X(z)$ is the same as that of X(z) except for z=0 if D>0 and z=∞ if D<0.

Example: Determine the z-transform of the signal $x(n)=2^n u(n-1)$.

❖ Convolution of two sequence:

if $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$

then $$x(n) = x_1(n) * x_2(n) \overset{z}{\longleftrightarrow} X(z) = X_1(z)X_2(z)$$

the ROC is, at least, the intersection of that for $X_1(z)$ and $X_2(z)$.

Example: Compute the convolution of x=[1 1 3 0 2 1] and h=[1, -2, 1] ?

2. Properties of the z-transform

❖ Time reversal:

if $x(n) \overset{z}{\longleftrightarrow} X(z) \quad \text{ROC}: r_1 \le |z| \le r_2$

then $x(-n) \overset{z}{\longleftrightarrow} X(z^{-1}) \quad \text{ROC}: \frac{1}{r_2} \le |z| \le \frac{1}{r_1}$

Example: Determine the z-transform of the signal x(n)=u(-n).

❖ Scaling in the z-domain:

if $x(n) \overset{z}{\longleftrightarrow} X(z) \quad \text{ROC}: r_1 \le |z| \le r_2$

then $a^n x(n) \overset{z}{\longleftrightarrow} X(a^{-1}z) \quad \text{ROC}: |a| r_1 \le |z| \le |a| r_2$

for any constant a, real or complex

Example: Determine the z-transform of the signal $x(n)=a^n\cos(w_0 n)u(n)$.

3. Causality and stability

❖ A causal signal of the form

$$x(n) = A_1 p_1^n u(n) + A_2 p_2^n u(n) + \cdots$$

will have z-transform

$$X(z) = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} + \cdots \quad \text{ROC} |z| > \max_i |p_i|$$

the ROC of causal signals are outside of the circle.

❖ A anticausal signal of the form

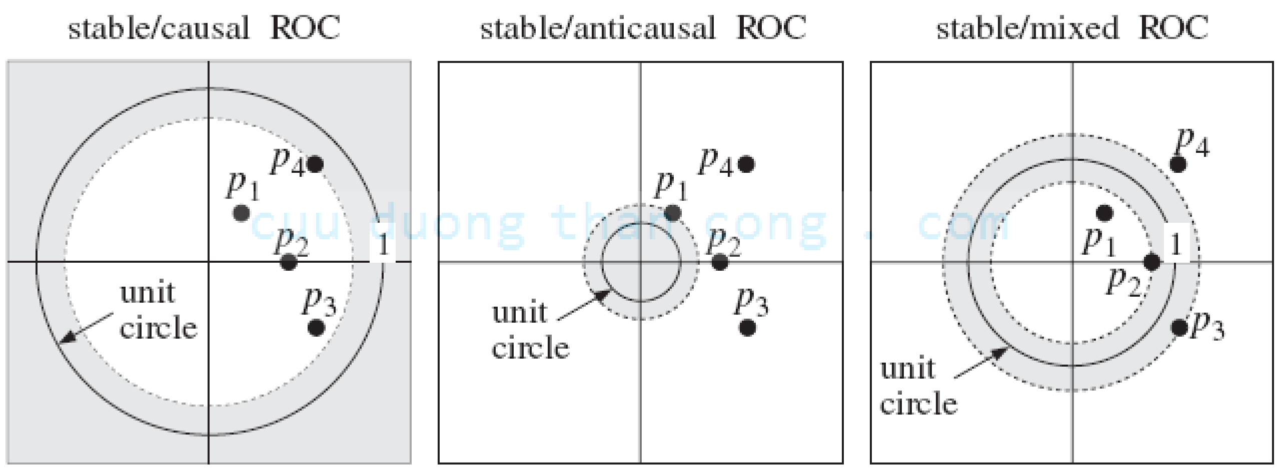
$$x(n) = -A_1 p_1^n u(-n-1) - A_2 p_2^n u(-n-1) + \cdots$$

$$X(z) = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} + \cdots \quad \text{ROC} |z| < \min_i |p_i|$$

the ROC of causal signals are inside of the circle.

3. Causality and stability

❖ Mixed signals have ROCs that are the annular region between two circles.



❖ It can be shown that a necessary and sufficient condition for the stability of a signal x(n) is that its ROC contains the unit circle.

4. Inverse z-transform

$$x(n) \xrightarrow{\text{z-transform}} X(z),\ \text{ROC}$$

$$X(z),\ \text{ROC} \xrightarrow{\text{inverse z-transform}} x(n)$$

$$x(n) \overset{z}{\longleftrightarrow} X(z), \text{ROC}$$

- In inverting a z-transform, it is convenient to break it into its partial fraction (PF) expression form, i.e., into a sum of individual pole terms whose inverse z transforms are known.

- Note that with $X(z) = \dfrac{1}{1-az^{-1}}$ we have

$$x(n) = \begin{cases} a^n u(n) & \text{if ROC } |z| > |a| \text{ (causal signals)} \\ -a^n u(-n-1) & \text{if ROC } |z| < |a| \text{ (anticausal signals)} \end{cases}$$

Partial fraction expression

❖ In general, the z-transform is of the form

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_N z^{-N}}{1 + a_0 z^{-1} + \cdots a_M z^{-M}}$$

❖ The poles are defined as the solutions of D(z)=0. There will be M poles, say at $p_1, p_2, \ldots, p_M$. Then, we can write

$$D(z) = (1 - p_1 z^{-1})(1 - p_2 z^{-1}) \cdots (1 - p_M z^{-1})$$

❖ If N < M and all M poles are single poles.

$$X(z) = \frac{N(z)}{D(z)} = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} + \cdots + \frac{A_M}{1 - p_M z^{-1}}$$

where

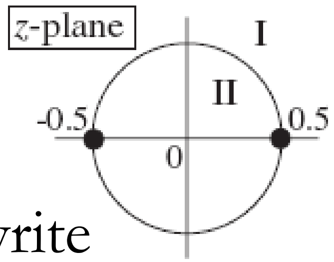
$$A_i = \left[(1 - p_i z^{-1}) X(z)\right]_{z=p_i} = \left[\frac{N(z)}{\prod_{j \neq i} (1 - p_j z^{-1})}\right]_{z=p_i}$$

Example 4

❖ Compute all possible inverse z-transform of

$$X(z) = \frac{6 + z^{-1}}{1 - 0.25z^{-2}}$$

Solution:



- Find the poles: $1\text{-}0.25z^{-2} = 0 \rightarrow p_1 = 0.5,\ p_2 = -0.5$

- We have N=1 and M=2, i.e., N < M. Thus, we can write

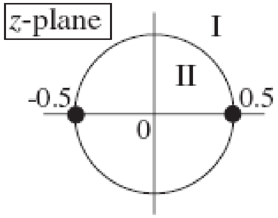
$$X(z) = \frac{6 + z^{-1}}{1 - 0.25z^{-2}} = \frac{6 + z^{-1}}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})} = \frac{A_1}{1 - 0.5z^{-1}} + \frac{A_2}{1 + 0.5z^{-1}}$$

where

$$A_1 = \left[\frac{6 + z^{-1}}{1 + 0.5z^{-1}}\right]_{z=0.5} = 4, \quad A_2 = \left[\frac{6 + z^{-1}}{1 - 0.5z^{-1}}\right]_{z=-0.5} = 2$$

Example 5

The two poles at ± 0.5 have the same magnitude and therefore divide the z-plane into two ROC regions I and II: $|z| > 0.5$ and $|z| < 0.5$. For the first ROC, both terms in the PF expansion are inverted causally giving:



$$x(n) = A_1 (0.5)^n u(n) + A_2 (-0.5)^n u(n)$$

Because this ROC also contains the unit circle the signal $x(n)$ will be stable. For the second ROC, both PF expansion terms are inverted anticausally giving:

$$x(n) = -A_1 (0.5)^n u(-n-1) - A_2 (-0.5)^n u(-n-1)$$

This answer is unstable, because the ROC does not contain the unit circle. □

Partial fraction expression

- If N=M

$$X(z) = \frac{N(z)}{D(z)} = \frac{N(z)}{(1-p_1z^{-1})(1-p_2z^{-1})\cdots(1-p_Mz^{-1})}$$

$$= A_0 + \frac{A_1}{1-p_1z^{-1}} + \frac{A_2}{1-p_2z^{-1}} + \cdots + \frac{A_M}{1-p_Mz^{-1}}$$

Where $A_0 = X(z)|_{z=0}$ and $A_i = [(1-p_iz^{-1})X(z)]_{z=p_i} = \left[\frac{N(z)}{\prod_{j\neq i}(1-p_jz^{-1})}\right]_{z=p_i}$ for i=1,…,M

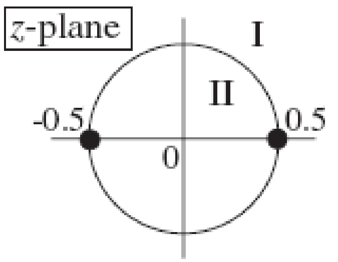
- If N> M

$$X(z) = \frac{N(z)}{D(z)} = \frac{Q(z)D(z)+R(z)}{D(z)} = Q(z) + \frac{R(z)}{D(z)}$$

Example 6

❖ Compute all possible inverse z-transform of

$$X(z) = \frac{10 + z^{-1} - z^{-2}}{1 - 0.25z^{-2}}$$



Solution:

- Find the poles: $1\text{-}0.25z^{-2} = 0 \rightarrow p_1 = 0.5,\ p_2 = -0.5$

- We have N=2 and M=2, i.e., N = M. Thus, we can write

$$X(z) = \frac{10 + z^{-1} - z^{-2}}{1 - 0.25z^{-2}} = \frac{10 + z^{-1} - z^{-2}}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})} = A_0 + \frac{A_1}{1 - 0.5z^{-1}} + \frac{A_2}{1 + 0.5z^{-1}}$$

where

$$A_0 = \left[\frac{10 + z^{-1} - z^{-2}}{1 - 0.25z^{-2}}\right]_{z=0} = \left[\frac{10z^2 + z - 1}{z^2 - 0.25}\right]_{z=0} = \frac{-1}{-0.25} = 4$$

$$A_1 = \left[\frac{10 + z^{-1} - z^{-2}}{1 + 0.5z^{-1}}\right]_{z=0.5} = 4, \qquad A_2 = \left[\frac{10 + z^{-1} - z^{-2}}{1 - 0.5z^{-1}}\right]_{z=-0.5} = 2$$

Example 6 (cont.)

Again, there are only two ROCs I and II: $|z| > 0.5$ and $|z| < 0.5$. For the first ROC, the A_1 and A_2 terms are inverted causally, and the A_0 term inverts into a simple $\delta(n)$:

$$x(n) = A_0\delta(n) + A_1(0.5)^n u(n) + A_2(-0.5)^n u(n)$$

For the second ROC, we have:

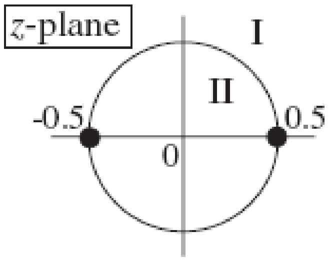
$$x(n) = A_0\delta(n) - A_1(0.5)^n u(-n-1) - A_2(-0.5)^n u(-n-1)$$

Only the first inverse is stable because its ROC contains the unit circle. □

Example 7 (cont.)

❖ Determine the causal inverse z-transform of

$$X(z) = \frac{6 + z^{-5}}{1 - 0.25z^{-2}}$$



Solution:

- We have N=5 and M=2, i.e., N > M. Thus, we have to divide the denominator into the numerator, giving

$$(6 + z^{-5}) = (1 - 0.25z^{-2})(-16z^{-1} - 4z^{-3}) + (6 + 16z^{-1})$$

where $(6 + 16z^{-1})$ is the remainder polynomial and $(-16z^{-1} - 4z^{-3})$ the quotient. Then,

$$X(z) = \frac{6 + z^{-5}}{1 - 0.25z^{-2}} = -16z^{-1} - 4z^{-3} + \frac{6 + 16z^{-1}}{1 - 0.25z^{-2}}$$

and expanding the last term in PF expansion:

$$X(z) = -16z^{-1} - 4z^{-3} + \frac{19}{1 - 0.5z^{-1}} - \frac{13}{1 + 0.5z^{-1}}$$

Partial fraction expression

❖ Complex-valued poles: since D(z) have real-valued coefficients, the complex-valued poles of X(z) must come in complex-conjugate pairs

$$X(z) = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_1^*}{1 - p_1^* z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} + \cdots$$

Considering the causal case, we have

$$x(n) = A_1 p_1^n u(n) + A_1^* p_1^{*n} u(n) + A_2 p_2^n u(n) + \cdots$$

Writing A_1 and p_1 in their polar form, say, $A_1 = B_1 e^{j\alpha_1}$ and $p_1 = R_1 e^{j\omega_1}$ with B_1 and $R_1 > 0$, and thus, we have

$$A_1 p_1^n + A_1^* p_1^{*n} = 2\mathrm{Re}[A_1 p_1^n] = 2B_1 R_1^n \cos(\omega_1 n + \alpha_1)$$

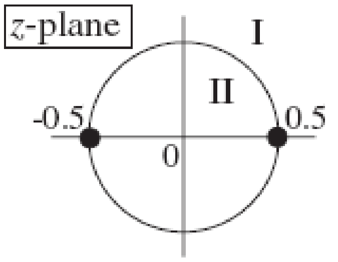
As a result, the signal in time-domain is

$$x(n) = 2B_1 R_1^n \cos(\omega_1 n + \alpha_1) u(n) + A_2 p_2^n u(n) + \cdots$$

Example 8

❖ Determine the causal inverse z-transform of

$$X(z) = \frac{4 - 3z^{-1} + z^{-2}}{1 + 0.25z^{-2}}$$



Solution:

We write

$$X(z) = \frac{4 - 3z^{-1} + z^{-2}}{1 + 0.25z^{-2}} = \frac{4 - 3z^{-1} + z^{-2}}{(1 - 0.5jz^{-1})(1 + 0.5jz^{-1})}$$

$$= A_0 + \frac{A_1}{1 - 0.5jz^{-1}} + \frac{A_1^*}{1 + 0.5jz^{-1}}$$

with the numerical values:

$$A_0 = \left[\frac{4 - 3z^{-1} + z^{-2}}{1 + 0.25z^{-2}}\right]_{z=0} = 4, \qquad A_1 = \left[\frac{4 - 3z^{-1} + z^{-2}}{1 + 0.5jz^{-1}}\right]_{z=0.5j} = 3j$$

Example 8 (cont.)

The causal ROC is $|z| > |0.5j| = 0.5$, resulting in

$$x(n) = 4\delta(n) + 3j(0.5j)^n u(n) - 3j(-0.5j)^n u(n)$$

Because the last two terms are complex conjugates of each other, we may write them as

$$x(n) = 4\delta(n) + 2\mathrm{Re}[3j(0.5j)^n u(n)] = 4\delta(n) + 6(0.5)^n u(n)\mathrm{Re}[j^{n+1}]$$

Writing $j^{n+1} = e^{j\pi(n+1)/2}$ and taking real parts we find

$$\mathrm{Re}[j^{n+1}] = \cos\left(\frac{\pi(n+1)}{2}\right) = -\sin\left(\frac{\pi n}{2}\right)$$

and

$$x(n) = 4\delta(n) - 6(0.5)^n \sin\left(\frac{\pi n}{2}\right) u(n)$$

Some common z-transform pairs

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
3	$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
4	$na^n u(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
5	$-a^n u(-n-1)$	$\dfrac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
6	$-na^n u(-n-1)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
7	$(\cos \omega_0 n)u(n)$	$\dfrac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
8	$(\sin \omega_0 n)u(n)$	$\dfrac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
9	$(a^n \cos \omega_0 n)u(n)$	$\dfrac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$
10	$(a^n \sin \omega_0 n)u(n)$	$\dfrac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$

Review

- Định nghĩa biến đổi z
- Ý nghĩa miền hội tụ của biến đổi z
- Mối liên hệ giữa miền hội tụ với đặc tính nhân quả và ổn định của tín hiệu/hệ thống-LTI rời rạc.
- Biến đổi z của một số tín hiệu cơ bản: $\delta(n)$, $a^n u(n)$, $a^n u(-n-1)$
- Một số tính chất cơ bản (tuyến tính, trễ, tích chập) của biến đổi z
- Phân chia đa thức và biến đổi z ngược

Homework 1

Show the z-transform of a triangular signal:

$$\sum_{n=-L}^{L}\left(1-\frac{|n|}{L}\right)z^{-n}=\frac{1}{L}\left[\frac{1-z^{-L}}{1-z^{-1}}\right]^{2}z^{L-1}$$

Homework 2

Consider the z-transform for $|z| > 1$:

$$X(z) = 1 - z^{-2} + z^{-4} - z^{-6} + z^{-8} - \cdots$$

Derive a rational expression for $X(z)$ in two ways: (a) by summing the above series, and (b) by showing that it satisfies the equation $X(z) = 1 - z^{-2}X(z)$.

Derive also the inverse z-transform $x(n)$ for all n.

Homework 3

Consider a transfer function $H(z) = N(z)/D(z)$, where the numerator and denominator polynomials have real-valued coefficients and degrees L and M in z^{-1}, and assume $L > M$. Show that $H(z)$ can be written in the form:

$$H(z) = Q(z) + \sum_{i=1}^{K} \frac{b_{i0} + z^{-1} b_{i1}}{1 + a_{i1} z^{-1} + a_{i2} z^{-2}}$$

where $Q(z)$ is a polynomial of degree $L - M$ in z^{-1} and the second-order sections have real coefficients. The number of sections K is related to M by $K = M/2$ if M is even and $K = (M - 1)/2$ if M is odd. This result forms the basis of the *parallel realization form* of $H(z)$.

Homework 4

Without using partial fractions, determine the causal inverse z-transforms of:

a. $X(z) = \dfrac{1}{1 + z^{-4}}$

b. $X(z) = \dfrac{1}{1 - z^{-4}}$

c. $X(z) = \dfrac{1}{1 + z^{-8}}$

d. $X(z) = \dfrac{1}{1 - z^{-8}}$

Homework 5

Using partial fractions or power series expansions, determine all possible inverse z-transforms of the following z-transforms, sketch their ROCs, and discuss their stability and causality properties:

a. $X(z) = \dfrac{3(1 + 0.3z^{-1})}{1 - 0.81z^{-2}}$

b. $X(z) = \dfrac{6 - 3z^{-1} - 2z^{-2}}{1 - 0.25z^{-2}}$

c. $X(z) = \dfrac{6 + z^{-5}}{1 - 0.64z^{-2}}$

d. $X(z) = \dfrac{10 + z^{-2}}{1 + 0.25z^{-2}}$

e. $X(z) = \dfrac{6 - 2z^{-1} - z^{-2}}{(1 - z^{-1})(1 - 0.25z^{-2})}$, ROC $|z| > 1$

f. $X(z) = -4 + \dfrac{1}{1 + 4z^{-2}}$

g. $X(z) = \dfrac{4 - 0.6z^{-1} + 0.2z^{-2}}{(1 - 0.5z^{-1})(1 + 0.4z^{-1})}$

Homework 6

❖ Tìm biến đổi z và miền hội tụ của các tín hiệu sau:

1) $\delta(n + 2) - \delta(n - 2)$
2) $u(n - 2)$
3) $u(n + 2)$
4) $u(n + 2) - u(n - 2)$
5) $u(-n)$
6) $u(n) + u(-n)$
7) $u(n) - u(-n)$
8) $u(1-n)$
9) $u(|n|)$
10) $2^n u(-n)$
11) $2^n u(n-1)$
12) $2^n u(1-n)$

Homework 7

❖ Tìm biến đổi z và miền hội tụ của các tín hiệu sau:

1) $\cos(\pi n)u(n)$
2) $\cos(\pi n/2)u(n)$
3) $\sin(\pi n/2)u(n)$
4) $\cos(\pi n/3)u(n)$
5) $\sin(\pi n/3)u(n)$
6) $\cos(\pi n)u(n-1)$
7) $\cos(\pi n)u(1-n)$
8) $\cos(\pi n)u(-n-1)$
9) $2^n\cos(\pi n/2)u(n)$
10) $2^n\sin(\pi n/2)u(n)$
11) $3^n\cos(\pi n/3)u(n)$
12) $3^n\sin(\pi n/3)u(n)$

Homework 8

❖ Liệt kê giá trị các mẫu (n=0, 1, 2, 3) của tín hiệu nhân quả có biến đổi z sau:

1) $2z^{-1}/(1 - 2z^{-1})$
2) $2z^{-1}/(1 + 2z^{-1})$
3) $2/(1 - 4z^{-2})$
4) $2/(1 + 4z^{-2})$
5) $2z^{-1}/(1 - 4z^{-2})$
6) $2z^{-1}/(1 + 4z^{-2})$
7) $2z^{-2}/(1 - 4z^{-2})$
8) $2z^{-2}/(1 + 4z^{-2})$
9) $2z^{-1}/(1 - z^{-1} - 2z^{-2})$
10) $2z^{-2}/(1 - z^{-1} - 2z^{-2})$
11) $2z^{-1}/(1 - 3z^{-1} + 2z^{-2})$
12) $2z^{-2}/(1 - 3z^{-1} + 2z^{-2})$