



Digital Signal Processing

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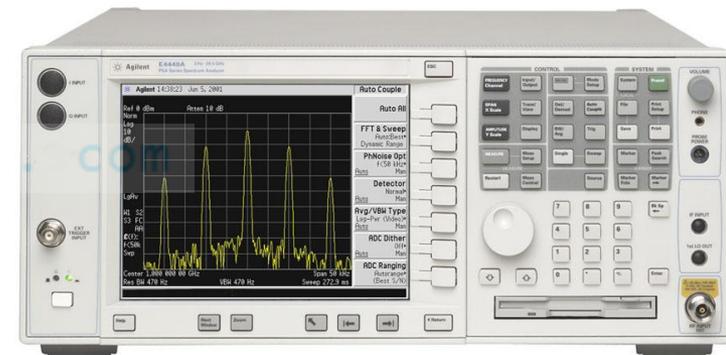
Chapter 7

Frequency Analysis of Signals and Systems

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- ❖ Frequency analysis of signal involves the resolution of the signal into its frequency (sinusoidal) components. The process of obtaining the spectrum of a given signal using the basic mathematical tools is known as **frequency or spectral analysis**.
- ❖ The term **spectrum** is used when referring the frequency content of a signal.
- ❖ The process of determining the spectrum of a signal in practice based on actual measurements of signal is called **spectrum estimation**.
- ❖ The instruments or software programs used to obtain spectral estimate of such signals are known as **spectrum analyzers**.



❖ The frequency analysis of signals and systems have three major uses in DSP:

- 1) The numerical computation of frequency spectrum of a signal.
- 2) The efficient implementation of convolution by the fast Fourier transform (FFT)
- 3) The coding of waves, such as speech or pictures, for efficient transmission and storage.

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1. Discrete time Fourier transform DTFT

2. Discrete Fourier transform DFT

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3. Fast Fourier transform FFT

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1. Discrete-time Fourier transform (DTFT)

- ❖ The Fourier transform of the finite-energy discrete-time signal $x(n)$ is defined as:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

where $\omega = 2\pi f / f_s$

- ❖ The spectrum $X(\omega)$ is in general a complex-valued function of frequency:

$$X(\omega) = |X(\omega)| e^{j\theta(\omega)}$$

where $\theta(\omega) = \arg(X(\omega))$ with $-\pi \leq \theta(\omega) \leq \pi$

- ❖ $|X(\omega)|$: is the magnitude spectrum
- ❖ $\theta(\omega)$: is the phase spectrum

❖ Determine and sketch the spectra of the following signal:

a) $x(n) = \delta(n)$

b) $x(n) = a^n u(n)$ with $|a| < 1$

❖ $X(\omega)$ is periodic with period 2π . tailieudientu.com

$$X(\omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega+2\pi k)n} = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = X(\omega)$$

The frequency range for discrete-time signal is unique over the frequency interval $(-\pi, \pi)$, or equivalently, $(0, 2\pi)$. tailieudientu.com

❖ Remarks: Spectrum of discrete-time signals is continuous and periodic.

- ❖ Given the frequency spectrum $X(\omega)$, we can find the $x(n)$ in time-domain as

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

which is known as inverse-discrete-time Fourier transform (IDTFT)

Example: Consider the ideal lowpass filter with cutoff frequency ω_c . Find the impulse response $h(n)$ of the filter.

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❖ **Symmetry:** if the signal $x(n)$ is real, it easily follows that

$$X^*(\omega) = X(-\omega)$$

or equivalently, $|X(-\omega)| = |X(\omega)|$ (even symmetry)

$$\arg(X(-\omega)) = -\arg(X(\omega)) \quad (\text{odd symmetry})$$

We conclude that the frequency range of real discrete-time signals can be limited further to the range $0 \leq \omega \leq \pi$, or $0 \leq f \leq f_s/2$.

❖ **Energy density of spectrum:** the energy relation between $x(n)$ and $X(\omega)$ is given by Parseval's relation:

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

$S_{xx}(\omega) = |X(\omega)|^2$ is called **the energy density spectrum** of $x(n)$

❖ The relationship of DTFT and z-transform: if $X(z)$ converges for $|z|=1$, then

$$X(z) \Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = X(\omega)$$

❖ **Linearity:** if $x_1(n) \xleftrightarrow{F} X_1(\omega)$

$$x_2(n) \xleftrightarrow{F} X_2(\omega)$$

then $a_1x_1(n) + a_2x_2(n) \xleftrightarrow{F} a_1X_1(\omega) + a_2X_2(\omega)$

❖ **Time-shifting:** if $x(n) \xleftrightarrow{F} X(\omega)$

$$x(n-k) \xleftrightarrow{F} e^{-j\omega k} X(\omega)$$

Properties of DTFT

❖ **Time reversal:** if $x(n) \xleftrightarrow{F} X(\omega)$

then $x(-n) \xleftrightarrow{F} X(-\omega)$

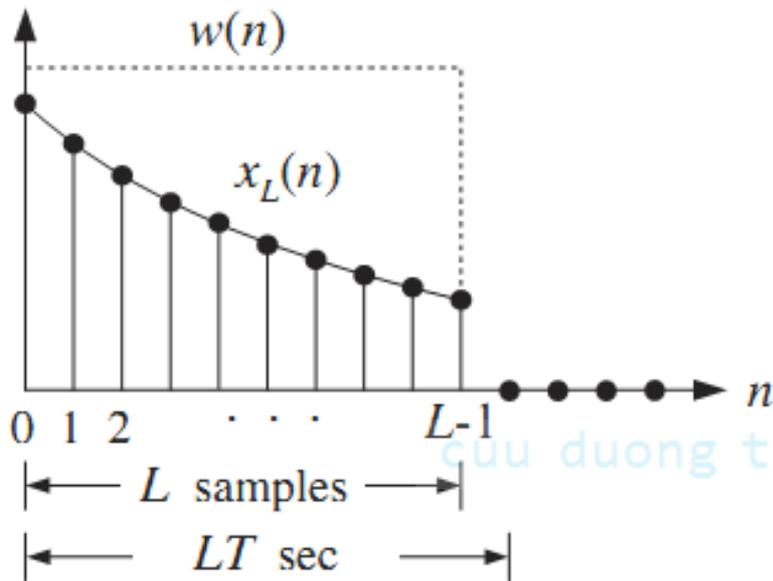
❖ **Convolution theory:** if $x_1(n) \xleftrightarrow{F} X_1(\omega)$

$$x_2(n) \xleftrightarrow{F} X_2(\omega)$$

then $x(n) = x_1(n) * x_2(n) \xleftrightarrow{F} X(\omega) = X_1(\omega)X_2(\omega)$

Example: Using DTFT to calculate the convolution of the sequences $x(n)=[1 \ 2 \ 3]$ and $h(n)=[1 \ 0 \ 1]$.

Frequency resolution and windowing



- ❖ The duration of the data record is:

$$T_L = LT$$

- ❖ The rectangular window of length L is defined as:

$$w(n) = \begin{cases} 1, & \text{if } 0 \leq n \leq L - 1 \\ 0, & \text{otherwise} \end{cases}$$

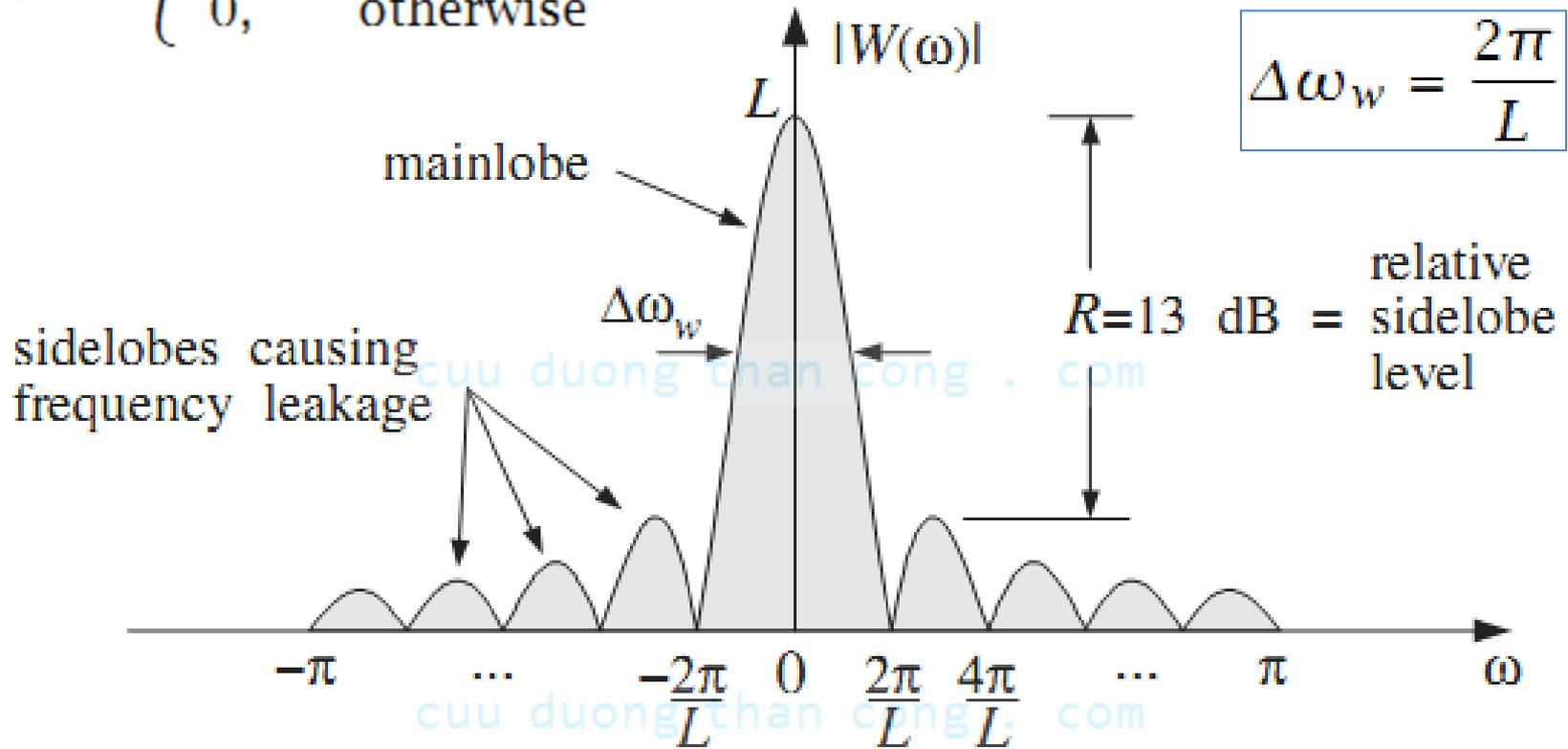
$$x_L(n) = x(n)w(n) \quad X_L(\omega) = \int_{-\pi}^{\pi} X(\omega')W(\omega - \omega') \frac{d\omega'}{2\pi}$$

- ❖ The windowing processing has two major effects: reduction in the **frequency resolution** and **frequency leakage**.

Rectangular window

$$w(n) = \begin{cases} 1, & \text{if } 0 \leq n \leq L - 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\Delta\omega_w = \frac{2\pi}{L}$$



$$W(\omega) = \frac{1 - e^{-jL\omega}}{1 - e^{-j\omega}} = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2}$$

Impact of rectangular window

- ❖ Consider a single analog complex sinusoid of frequency f_1 and its sample version:

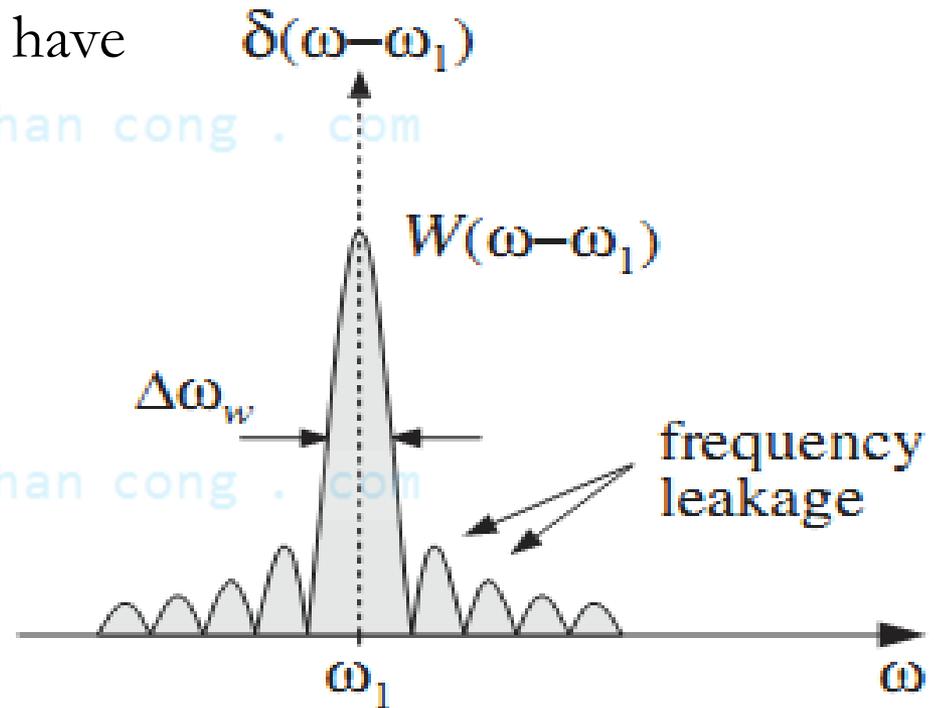
$$x(t) = e^{2\pi j f_1 t}, \quad -\infty < t < \infty \quad \Rightarrow \quad x(n) = e^{2\pi j f_1 n T} = e^{j\omega_1 n}, \quad -\infty < n < \infty$$

- ❖ With assumption $|f_1| \leq f_s/2$, we have

$$X(\omega) = \hat{X}(f) = \frac{1}{T} X(f) = \frac{1}{T} \delta(f - f_1)$$

$$X_L(\omega) = \int_{-\pi}^{\pi} X(\omega') W(\omega - \omega') \frac{d\omega'}{2\pi} =$$

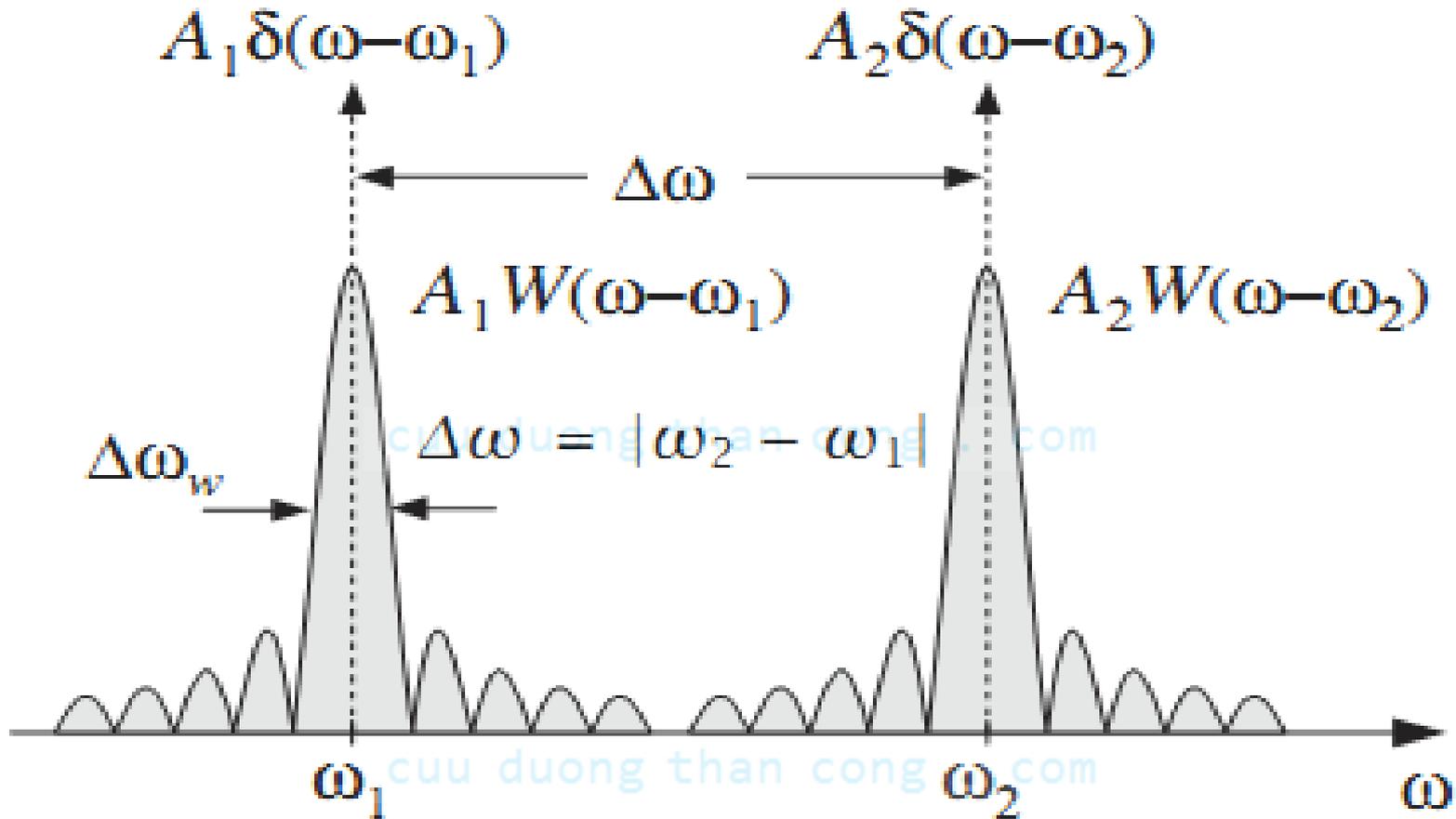
$$X_L(\omega) = W(\omega - \omega_1)$$



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Double sinusoids

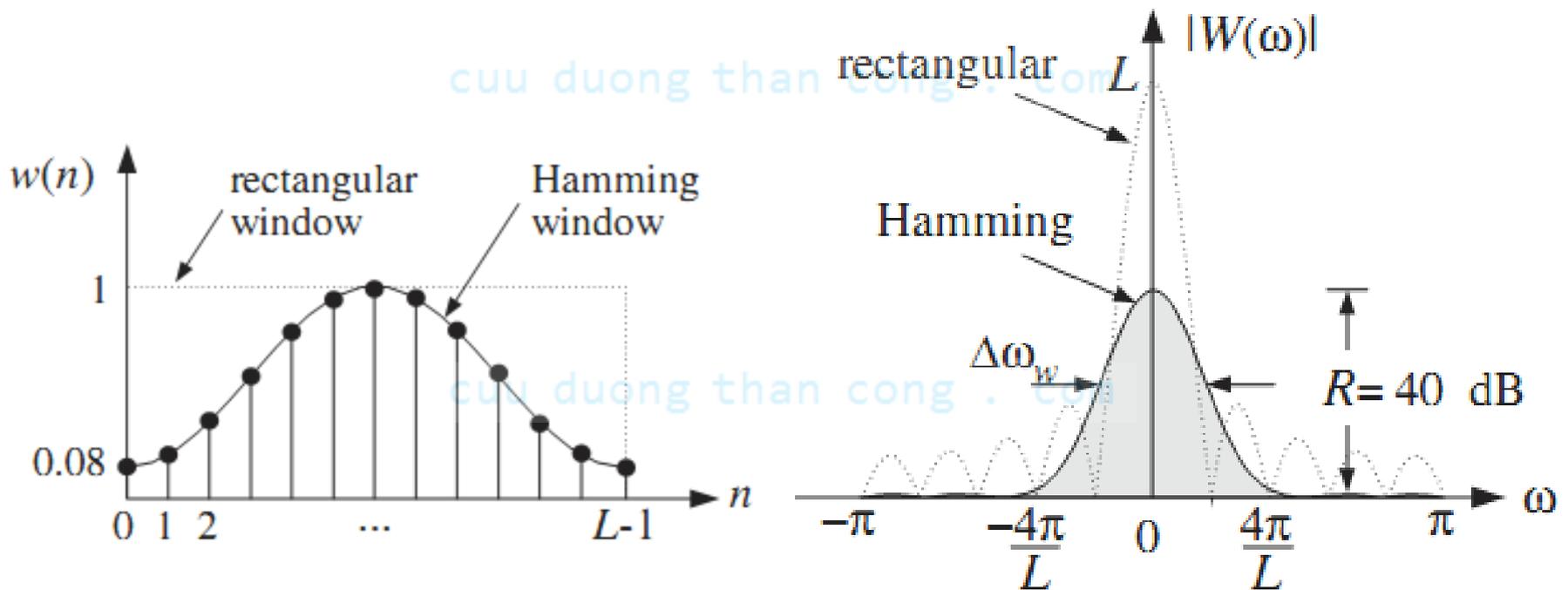


❖ Frequency resolution:

$$\Delta f \geq \Delta f_w = \frac{f_s}{L}$$

Hamming window

$$w(n) = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{L-1}\right), & \text{if } 0 \leq n \leq L-1 \\ 0, & \text{otherwise} \end{cases}$$



Non-rectangular window

- ❖ The standard technique for suppressing the sidelobes is to use a non-rectangular window, for example Hamming window.
- ❖ The main tradeoff for using non-rectangular window is that its mainlobe becomes wider and shorter, thus, reducing the frequency resolution of the windowed spectrum.
- ❖ The minimum resolvable frequency difference will be

$$\Delta f \geq \Delta f_w = c \frac{f_s}{L} = c \frac{1}{T_L}$$

where $c \geq 1$: $c=1$ for rectangular window and $c=2$ for Hamming window.

Example

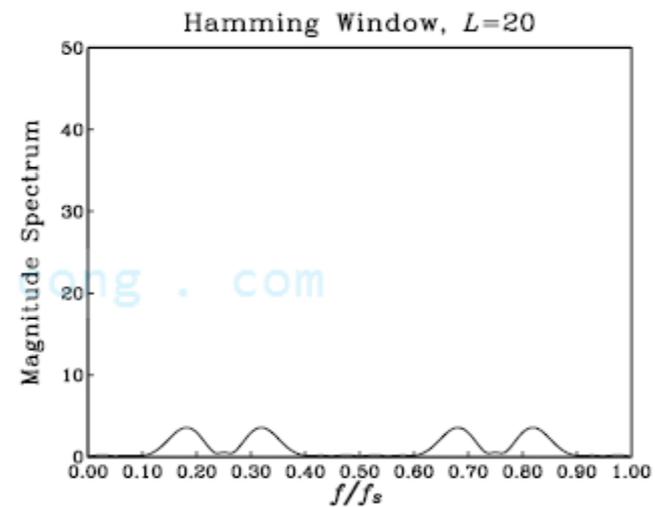
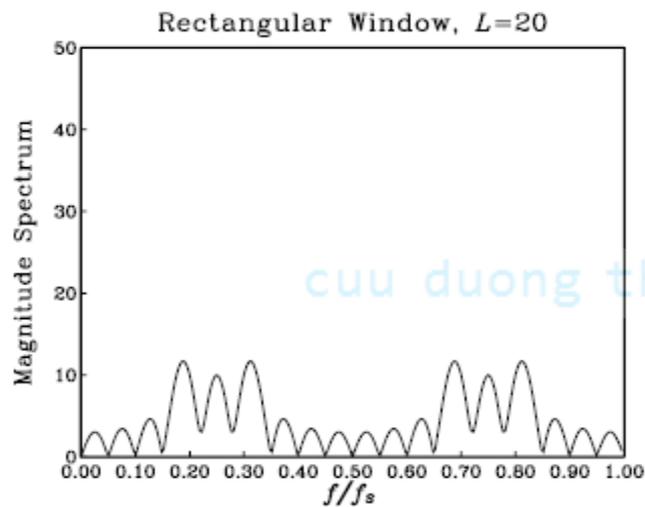
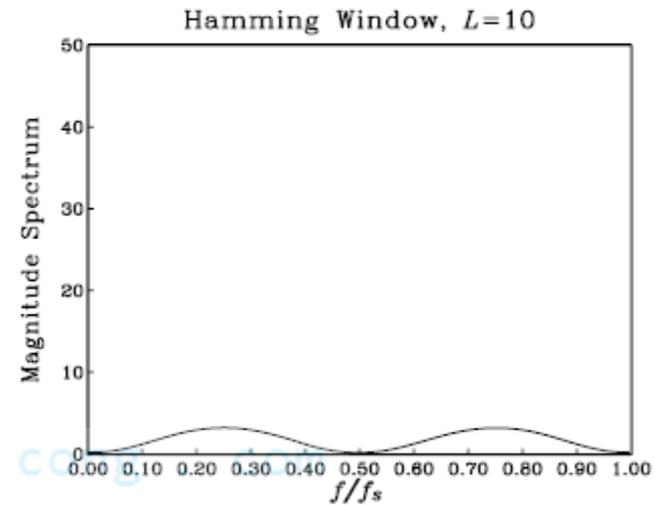
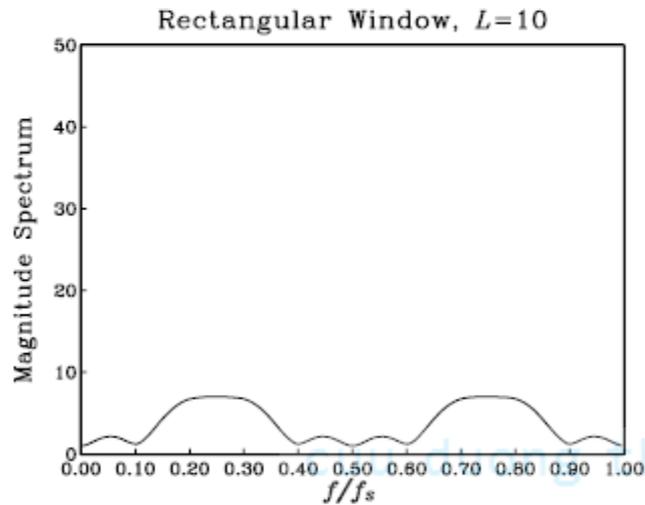
- ❖ The following analog signal consisting of three equal-strength sinusoids at frequencies $f_1 = 2$ kHz, $f_2 = 2.5$ kHz, and $f_3 = 3$ kHz:

$$x(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t) + \cos(2\pi f_3 t)$$

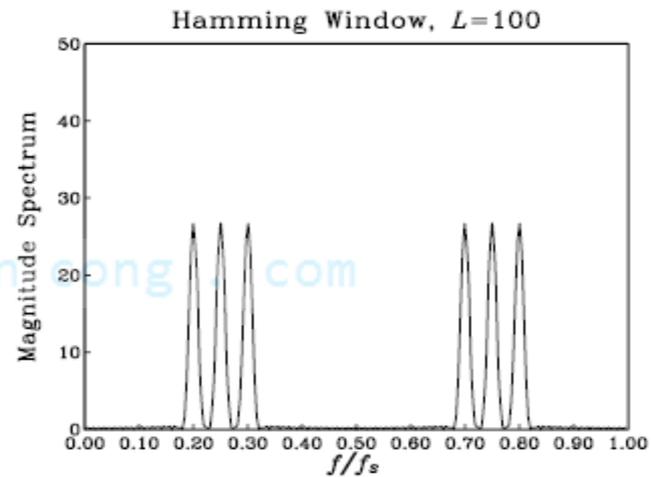
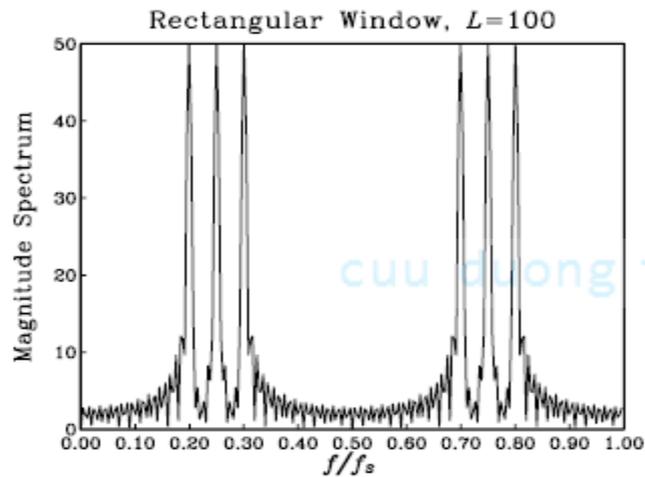
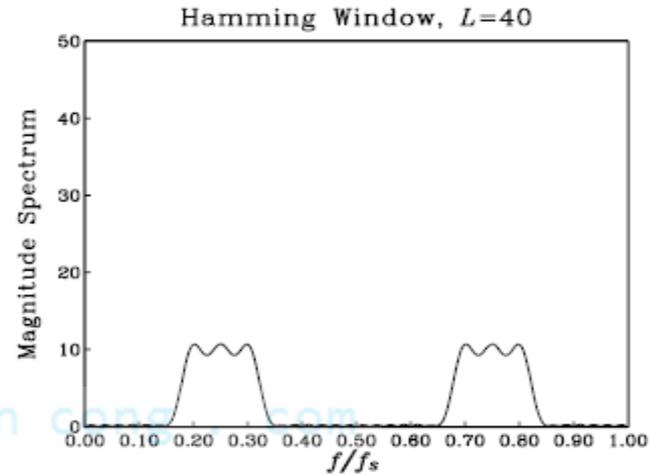
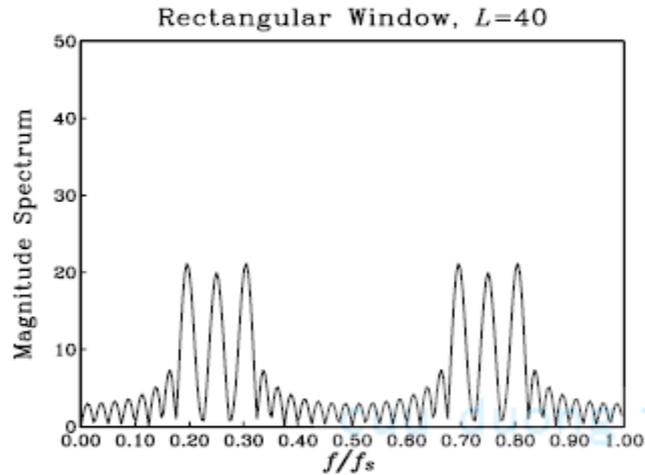
where t (ms), is sampled at a rate of 10 kHz. We consider four data records of $L=10, 20, 40,$ and 100 samples. They corresponding of the time durations of 1, 2, 4, and 10 msec.

- ❖ The minimum frequency separation is $\Delta f = 2.5 - 2 = 0.5$ kHz. Applying the formulation $\Delta f \geq \Delta f_w = c \frac{f_s}{L} = c \frac{1}{T_L}$, the minimum length L to resolve all three sinusoids show be 20 samples for the rectangular window, and $L = 40$ samples for the Hamming case.

Example



Example



2. Discrete Fourier transform (DFT)

- ❖ $X(\omega)$ is a **continuous function** of frequency and therefore, it is not a computationally convenient representation of the sequence $x(n)$.
- ❖ DFT will present $x(n)$ in a frequency-domain by samples of its spectrum $X(\omega)$.

- ❖ A finite-duration sequence $x(n)$ of length L has a Fourier transform:

$$X(\omega) = \sum_{n=0}^{L-1} x(n)e^{-j\omega n} \quad 0 \leq \omega \leq 2\pi$$

Sampling $X(\omega)$ at equally spaced frequency $\omega_k = \frac{2\pi k}{N}$, $k=0, 1, \dots, N-1$ where $N \geq L$, we obtain **N-point DFT** of length N -signal:

$$X(k) \equiv X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{L-1} x(n)e^{-j2\pi kn/N} \quad (\text{N-point DFT})$$

- ❖ DFT presents the discrete-frequency samples of spectra of discrete-time signals.

2. Discrete Fourier transform (DFT)



- ❖ With the assumption $x(n)=0$ for $n \geq L$, we can write

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, k = 0, 1, \dots, N-1. \quad (\text{DFT})$$

- ❖ The sequence $x(n)$ can recover from the frequency samples by inverse DFT (IDFT)

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N}, n = 0, 1, \dots, N-1. \quad (\text{IDFT})$$

Example: Calculate 4-DFT and plot the spectrum of $x(n)=[1, 1, 2, 1]$

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- ❖ By defining an Nth root of unity $W_N = e^{-j2\pi/N}$, we can rewrite DFT and IDFT as follows

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{kn}, k = 0, 1, \dots, N-1. \quad (\text{DFT})$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)W_N^{-kn}, n = 0, 1, \dots, N-1. \quad (\text{IDFT})$$

- ❖ Let us define:

$$\mathbf{x}_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \quad \mathbf{X}_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

The N-point DFT can be expressed in matrix form as: $\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N$

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & W_N & W_N^2 & \dots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \dots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \dots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

❖ Let us define:

$$\mathbf{x}_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \quad \mathbf{X}_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

The N-point DFT can be expressed in matrix form as: $\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N$

- ❖ Example: Determine the DFT of the four-point sequence $x(n)=[1 \ 1, \ 2 \ 1]$ by using matrix form.

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Properties of DFT



| Properties | Time domain | Frequency domain |
|--------------------------------------|---|---|
| ❖ Notation | $x(n)$ | $X(k)$ |
| ❖ Periodicity | $x(n + N) = x(n)$ | $X(k) = X(k + N)$ |
| ❖ Linearity | $a_1x_1(n) + a_2x_2(n)$ | $a_1X_1(k) + a_2X_2(k)$ |
| ❖ Circular time-shift | $x((n-l))_N$ | $e^{-j2\pi kl/N} X(k)$ |
| ❖ Circular convolution | $x_1(n) \circledast x_2(n)$ | $X_1(k)X_2(k)$ |
| ❖ Multiplication of two sequences | $x_1(n)x_2(n)$ | $\frac{1}{N} X_1(k) \circledast X_2(k)$ |
| ❖ Parseval's theorem | $E_x = \sum_{n=0}^N x(n) ^2 = \frac{1}{N} \sum_{k=0}^{N-1} X(k) ^2$ | |

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- ❖ The circular shift of the sequence can be represented as the index modulo N :

$$x'(n) = x(n - k, \text{ modulo } N) \equiv x((n - k))_N$$

For example, if $k = 2$ and $N = 4$, we have

$$x'(n) = x((n - 2))_4$$

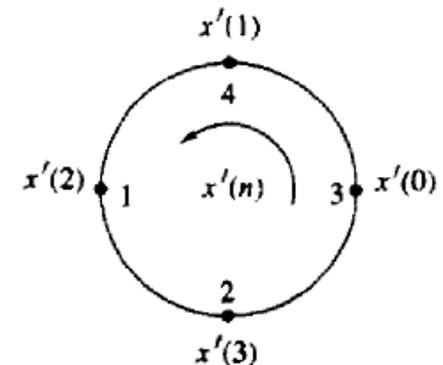
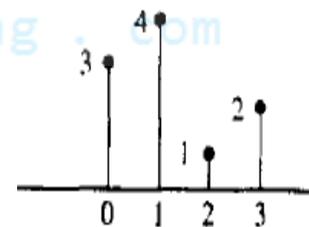
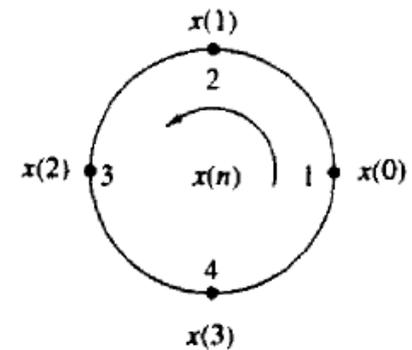
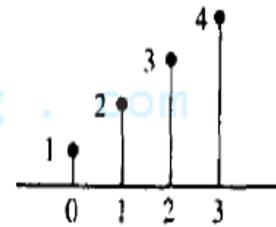
which implies that

$$x'(0) = x((-2))_4 = x(2)$$

$$x'(1) = x((-1))_4 = x(3)$$

$$x'(2) = x((0))_4 = x(0)$$

$$x'(3) = x((1))_4 = x(1)$$



- ❖ The circular convolution of two sequences of length N is defined as

$$x_3(m) = x_1(n) \textcircled{N} x_2(n)$$

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)x_2((m-n))_N \quad m = 0, 1, \dots, N-1$$

- ❖ Example: Perform the circular convolution of the following two sequence:

$$x_1(n) = [2, 1, 2, 1] \quad x_2(n) = [1, 2, 3, 4]$$

It can be shown from the below Fig, $x_3(m) = x_1(n) \textcircled{N} x_2(n)$

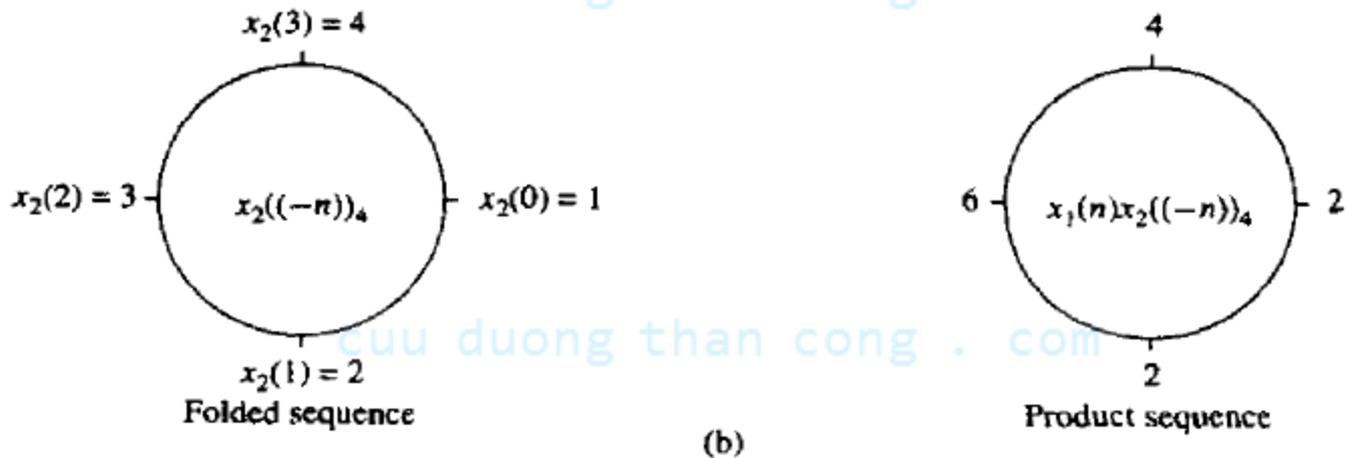
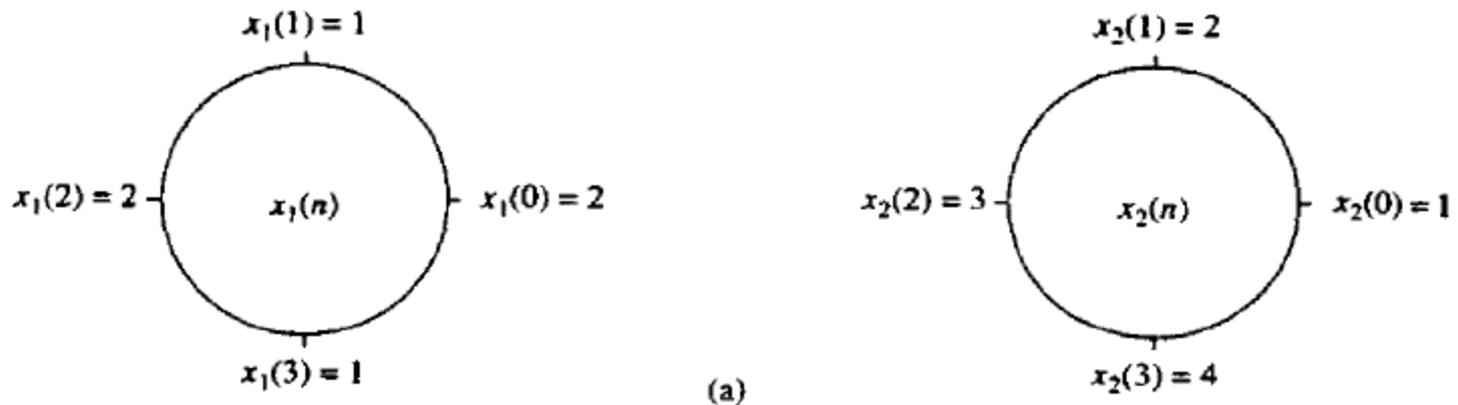
$$x_3(n) = \{14, 16, 14, 16\}$$

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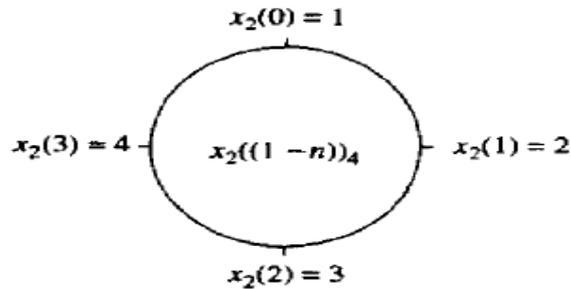
Circular convolution



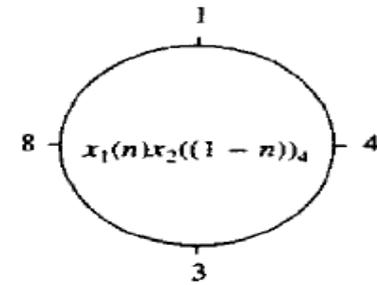
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Circular convolution

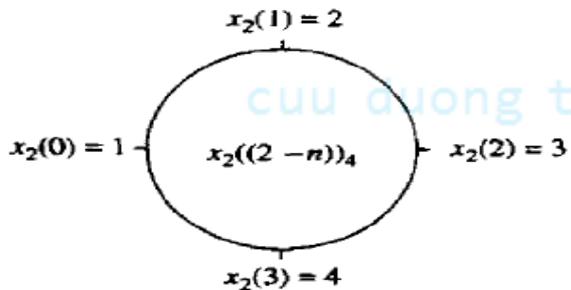


Folded sequence rotated by one unit in time

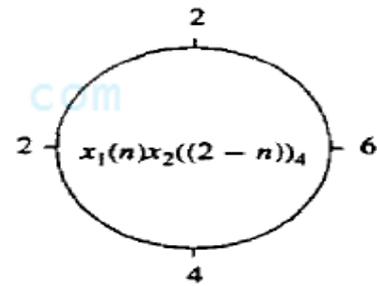


Product sequence

(c)

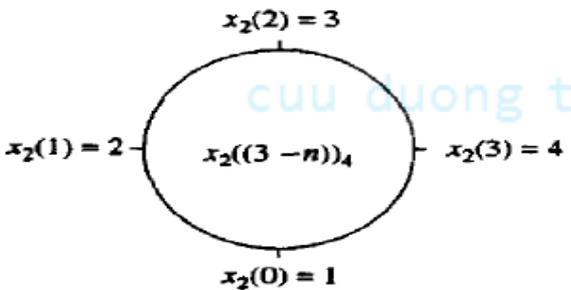


Folded sequence rotated by two units in time

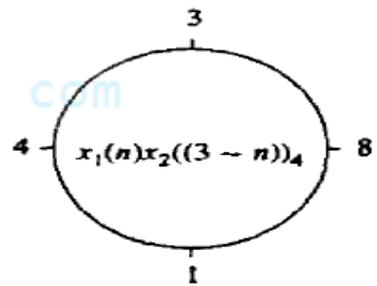


Product sequence

(d)



Folded sequence rotated by three units in time



Product sequence

(e)

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Use of the DFT in Linear Filtering

- ❖ Suppose that we have a finite duration sequence $x = [x_0, x_1, \dots, x_{L-1}]$ which excites the FIR filter of order M .
- ❖ The sequence output is of length $L_y = L + M$ samples.
- ❖ If $N \geq L + M$, N -point DFT is sufficient to present $y(n)$ in the frequency domain, i.e.,

$$Y(k) \equiv Y(\omega)|_{\omega=2\pi k/N} \quad k = 0, 1, \dots, N - 1$$

$$= X(\omega)H(\omega)|_{\omega=2\pi k/N} \quad k = 0, 1, \dots, N - 1$$

$$Y(k) = X(k)H(k) \quad k = 0, 1, \dots, N - 1$$

- ❖ Computation of the N -point IDFT must yield $y(n)$.
- ❖ Thus, with zero padding, the DFT can be used to perform linear filtering.

4. Fast Fourier transform (FFT)

- ❖ N-point DFT of the sequence of data $x(n)$ of length N is given by following formula:

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^k, \quad k = 0, 1, 2, \dots, N-1$$

where $W_N = e^{-j2\pi/N}$

- ❖ In general, the data sequence $x(n)$ is also assumed to be complex valued. To calculate all N values of DFT require **N^2 complex multiplications and $N(N-1)$ complex additions.**
- ❖ **FFT** exploits the symmetry and periodicity properties of the phase factor W_N to reduce the computational complexity.

- **Symmetry:** $W_N^{k+N/2} = -W_N^k$

- **Periodicity:** $W_N^{k+N} = W_N^k$

3. Fast Fourier transform (FFT)

- ❖ Based on decimation, leads to a factorization of computations.
- ❖ Let us first look at the classical radix 2 decimation in time.
- ❖ First we split the computation between odd and even samples:

$$X[k] = \sum_{n=0}^{N/2-1} x[2n] W_N^{-k2n} + \sum_{n=0}^{N/2-1} x[2n+1] W_N^{-k(2n+1)}$$

- ❖ Using the following property: $W_N^2 = W_{\frac{N}{2}}$

- ❖ The N-point DFT can be rewritten:

$$X[k] = \sum_{n=0}^{N/2-1} x[2n] W_{\frac{N}{2}}^{-kn} + W_N^{-k} \sum_{n=0}^{N/2-1} x[2n+1] W_{\frac{N}{2}}^{-kn}$$

for $k=0, 1, \dots, N-1$

❖ Using the property that:

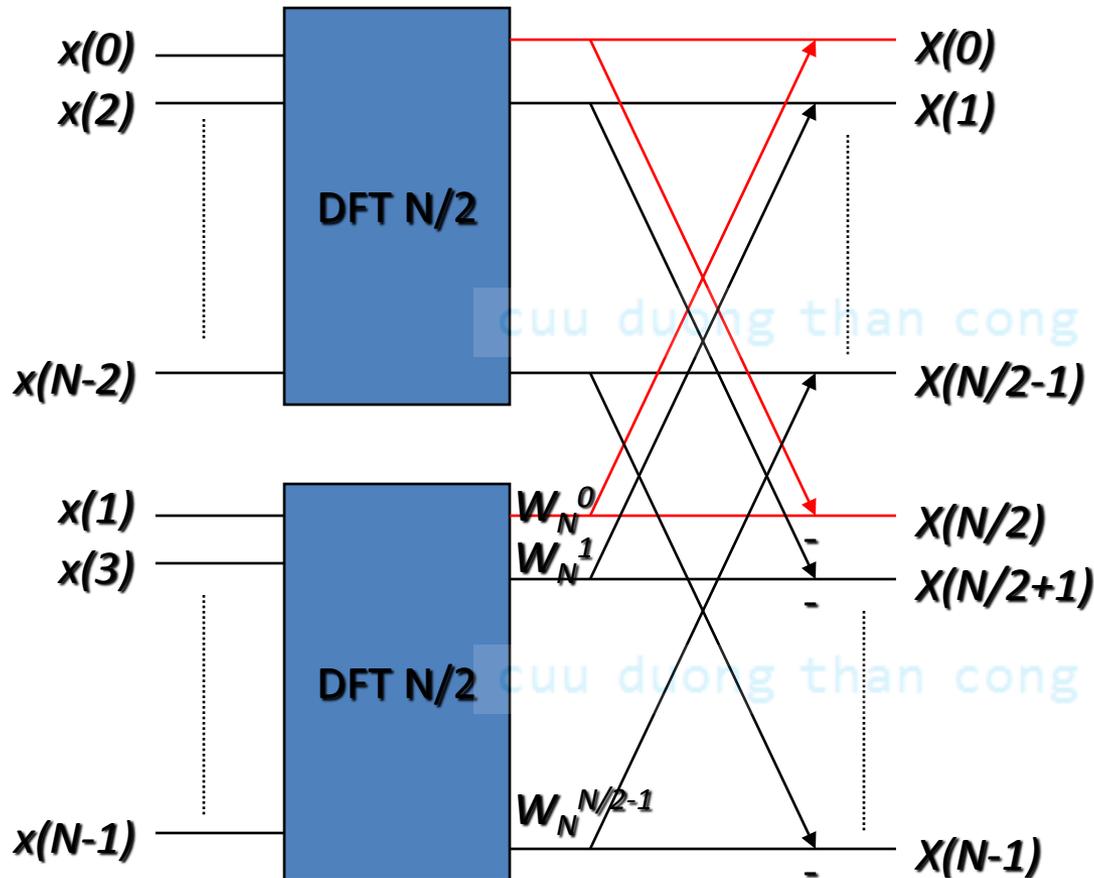
$$W_N^{k+\frac{N}{2}} = -W_N^k$$

❖ The entire DFT can be computed with only $k=0, 1, \dots, N/2-1$.

$$X[k] = \sum_{n=0}^{N/2-1} x[2n] W_{\frac{N}{2}}^{-kn} + W_N^{-k} \sum_{n=0}^{N/2-1} x[2n+1] W_{\frac{N}{2}}^{-kn}$$

$$X\left[k + \frac{N}{2}\right] = \sum_{n=0}^{N/2-1} x[2n] W_{\frac{N}{2}}^{-kn} - W_N^{-k} \sum_{n=0}^{N/2-1} x[2n+1] W_{\frac{N}{2}}^{-kn}$$

❖ This leads to basic building block of the FFT, the butterfly.

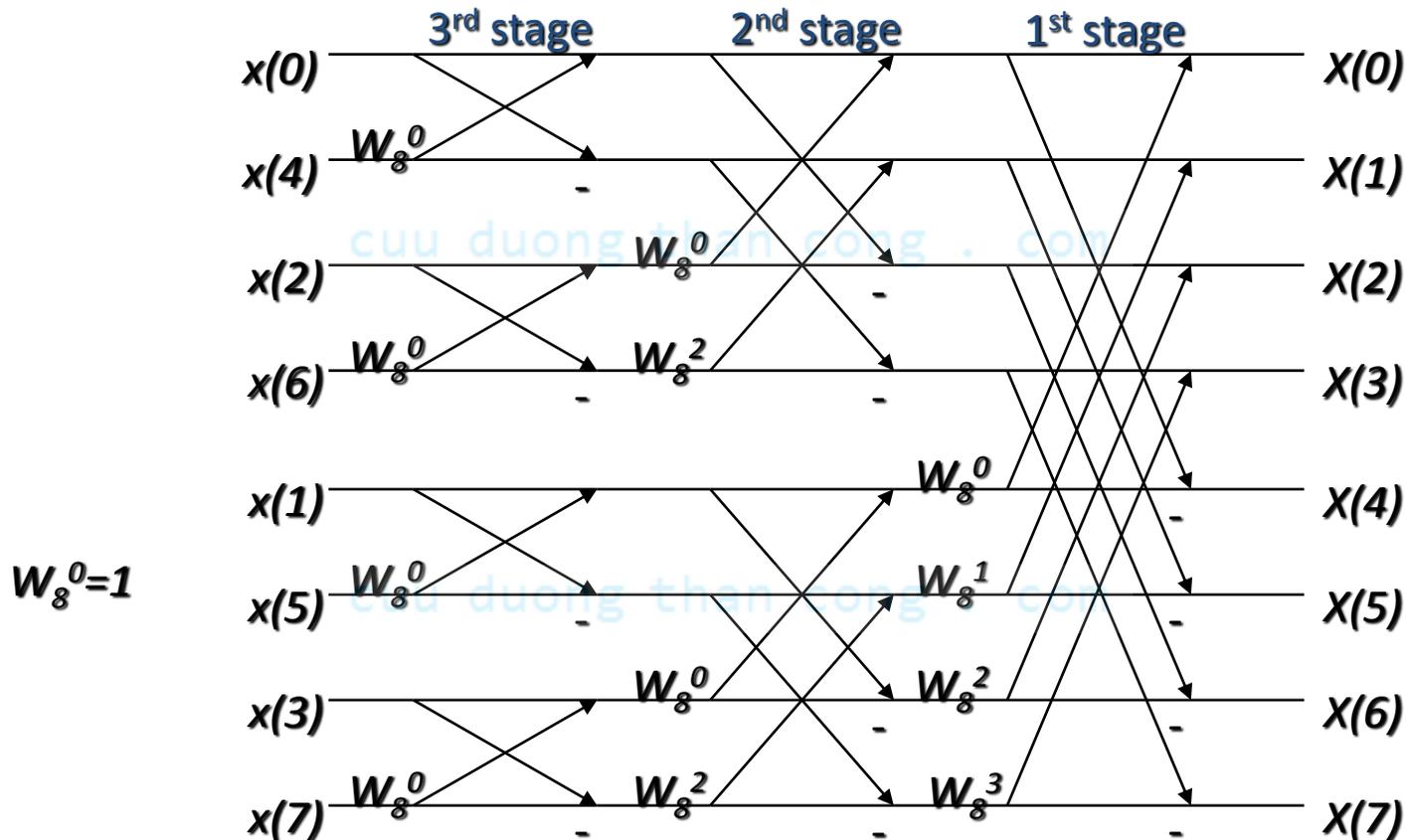


We need:

- $N/2(N/2-1)$ complex '+' for each $N/2$ DFT.
- $(N/2)^2$ complex '×' for each DFT.
- $N/2$ complex '×' at the input of the butterflies.
- N complex '+' for the butterflies.
- Grand total:
 $N^2/2$ complex '+'
 $N/2(N/2+1)$ complex '×'

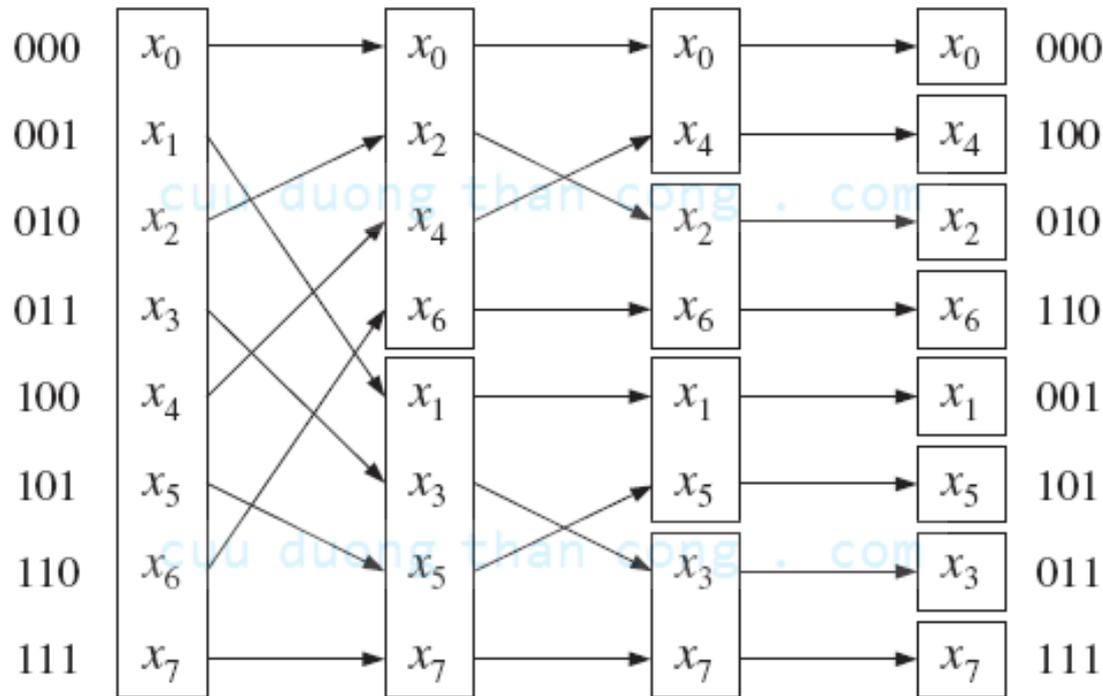
Recursion

- ❖ If $N/2$ is even, we can further split the computation of each DFT of size $N/2$ into two computations of half size DFT. When $N=2^r$ this can be done until DFT of size 2 (i.e. butterfly with two elements).



Shuffling the data, bit reverse ordering

- ❖ At each step of the algorithm, data are split between even and odd values. This results in scrambling the order.



Number of operations



❖ If $N=2^r$, we have $r=\log_2(N)$ stages. For each one we have:

- $N/2$ complex '×' (some of them are by '1').
- N complex '+'.

❖ Thus the grand total of operations is:

- $N/2 \log_2(N)$ complex '×'.
- $N \log_2(N)$ complex '+'

❖ Example: Calculate 4-point DFT of $x=[1, 3, 2, 3]$?

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Homework 1



- Tính DFT-4 điểm của tín hiệu $x(n) = \{1, 1, 2, 19, 11, 19, 11\}$.
- Tính IDFT-4 điểm của tín hiệu $X(k) = \{1, 1 + j, 16, 1 - j\}$.
- Vẽ sơ đồ thực hiện và tính FFT-4 điểm của tín hiệu $x(n) = \{1, 1 - j, 16, 1 + j\}$.
- Vẽ 1 sơ đồ tổng quát thực hiện FFT-8 điểm.

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Homework 2



- Tính DFT-4 điểm của tín hiệu $x(n) = \{1, 2, 8\}$.
- Vẽ sơ đồ thực hiện và tính FFT-4 điểm của tín hiệu $x(n) = \{1, 0, 1, 2\}$.
- Xác định giá trị của A và B trong tín hiệu $x(n) = \{-20, -8, 1, 2, A, B\}$ để DFT-4 điểm của tín hiệu trên có dạng $X(k) = \{5, 1 + j2, 1, 1 - j2\}$.

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Homework 3



- Tính DFT-4 điểm của tín hiệu $x(n) = \{ @, 8, 0, 5, 4, 0, 4, 1 \}$.
- Xác định giá trị của A và B trong tín hiệu $x(n) = \{1, 2, 3, 4, 5, 6, A, B\}$ để DFT-4 điểm của tín hiệu trên có dạng $X(k) = \{12, 1 - j, -2, 1 + j\}$.
- Vẽ sơ đồ thực hiện và tính FFT-4 điểm của tín hiệu $x(n) = \{ @, 8, 4, 6 \}$.
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- Vẽ sơ đồ thực hiện tính IFFT-4 điểm của tín hiệu $X(k) = \{ @, 8, 0, 5 \}$.

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Homework 4



- a) Tính DFT-4 điểm của tín hiệu $x(n) = \{1, 2, 1, 0, 1, 1, 1\}$.
- b) Xác định giá trị của A và B trong tín hiệu $x(n) = \{3, 1, 2, 0, A, B\}$ để DFT-4 điểm của tín hiệu trên có dạng $X(k) = \{9, 2 - j3, 3, 2 + j3\}$.
- c) Chứng minh và vẽ sơ đồ thực hiện tính DFT-4 điểm dựa trên các DFT-2 điểm. cuuduongthancong.com
- d) Chứng minh và vẽ sơ đồ thực hiện tính IDFT-4 điểm dựa trên DFT-4 điểm.

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