Chapter 2:

Fourier series and the Fourier Transform

1. ***Orthogonal Signal space***
2. *Trigonometric Fourier Series*
3. *Exponential Fourier Series*
4. *The Fourier Transform*
5. *Properties of the Fourier Transform*

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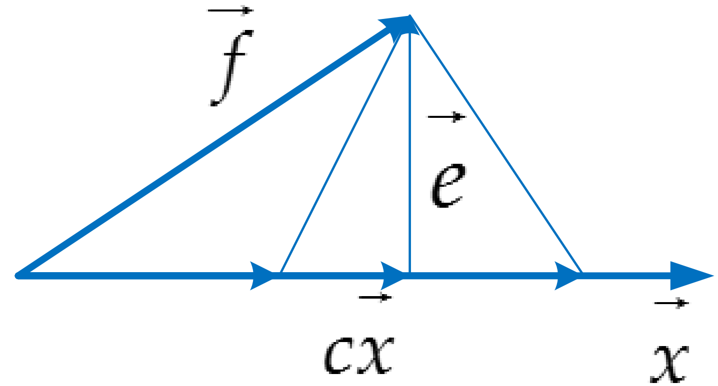
Component of a Vector:



$$\vec{f} = c\vec{x} + \vec{e} \Rightarrow \begin{cases} \vec{f} \approx c\vec{x} : \text{ component of } \vec{f} \text{ along } \vec{x} \\ \vec{e} = \vec{f} - c\vec{x} : \text{error} \end{cases}$$

- The error is minimized when: $c = \dfrac{1}{\left|\vec{x}\right|^2} \vec{f}.\vec{x}$

- The two vectors are **orthogonal** when:

$$c = 0 \Leftrightarrow \vec{f}.\vec{x} = 0$$

Component of a Signal:

- Approximating a real signal $f(t)$ in terms of other real signal $x(t)$ over an interval $[t_1, t_2]$:

$$f(t) = c.x(t) + e(t)$$

$$\Rightarrow \begin{cases} f(t) \approx c.x(t): \text{ component of } f(t) \text{ in term of } x(t) \\ e(t) = f(t) - c.x(t): \text{error} \end{cases}$$

- The error is minimized when its energy is minimum.

$$E_e = \int_{t_1}^{t_2} e^2(t)dt = \min \Leftrightarrow c = \frac{\int_{t_1}^{t_2} f(t).x(t)dt}{\int_{t_1}^{t_2} x^2(t)dt} = \frac{1}{E_x}\int_{t_1}^{t_2} f(t).x(t)dt$$

Orthogonality:

- We define the **real** signals $x_1(t)$ and $x_2(t)$ to be **orthogonal** over the interval $[t_1, t_2]$ if:

$$\int_{t_1}^{t_2} x_1(t).x_2(t)dt = 0$$

- **Orthogonality** in **complex** signals:

$$\int_{t_1}^{t_2} x_1^*(t).x_2(t)dt = 0 \quad or \quad \int_{t_1}^{t_2} x_1(t).x_2^*(t)dt = 0$$

Orthogonality:

- *Example 2.01:* How many of the following pairs of functions are orthogonal over [0, 3]?

$$a.\ \cos 2\pi t \quad \text{and} \quad \sin 2\pi t$$

$$b.\ \cos 2\pi t \quad \text{and} \quad \cos 4\pi t$$

$$c.\ \cos 2\pi t \quad \text{and} \quad \sin \pi t$$

$$d.\ \cos 2\pi t \quad \text{and} \quad e^{j2\pi t}$$

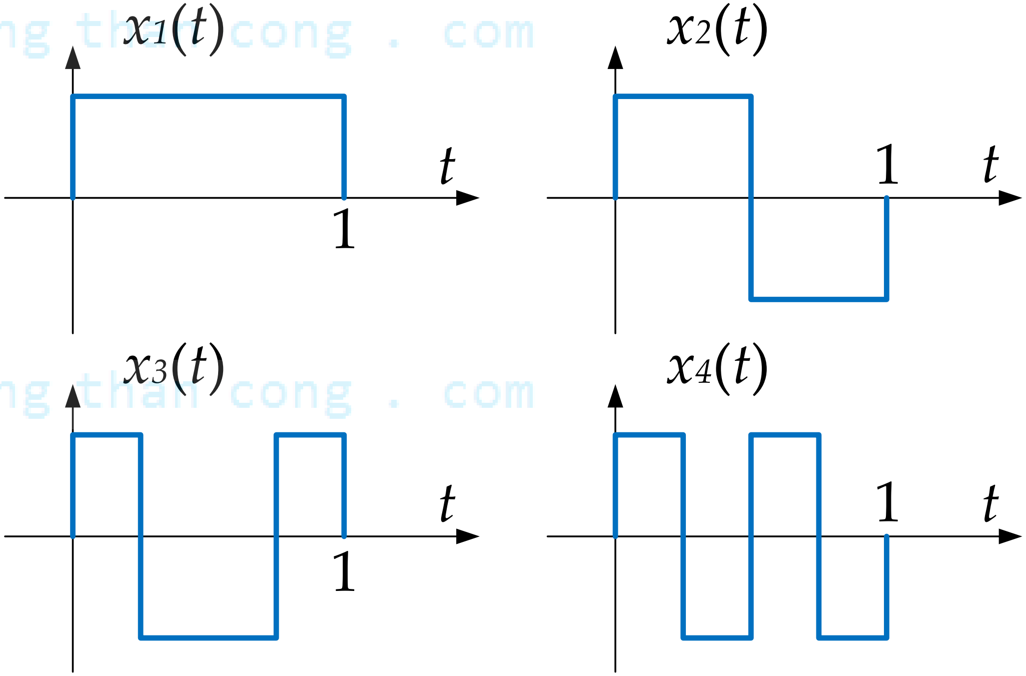
- *Answers:* a ✓ b ✓ c ✗ d ✗

Orthogonal signal set:

- A **signal set** $x_1(t)$, $x_2(t)$,…, $x_n(t)$ is **orthogonal** over the interval $[t_1, t_2]$ if:

$$\int_{t_1}^{t_2} x_m(t).x_n(t)dt = \begin{cases} 0 & m \neq n \\ E_m & m = n \end{cases}$$

Example 2.02: show that $\{x_1, x_2, x_3, x_4\}$ is an orthogonal signal set.



Orthogonal signal set:

- Approximate a signal $f(t)$ over $[t_1, t_2]$ by orthogonal signal set:

$$f(t) \approx c_1 x_1(t) + c_2 x_2(t) + \ldots + c_N x_N(t) = \sum_{n=1}^{N} c_n x_n(t)$$

- The error: $$e(t) = f(t) - \sum_{n=1}^{N} c_n x_n(t)$$

- The error's energy E_e is minimized when:

$$c_n = \frac{\int_{t_1}^{t_2} f(t) x_n(t) dt}{\int_{t_1}^{t_2} x_n^2(t) dt} = \frac{1}{E_n} \int_{t_1}^{t_2} f(t) x_n(t) dt$$

Complete orthogonal signal set - generalized Fourier series:

- The minimum error signal energy:

$$E_e = \int_{t_1}^{t_2} f^2(t)dt - \sum_{n=1}^{N} c_n^2 E_n$$

- E_e will decreases if N increases, and $\to 0$ as $N \to \infty$. When $N \to \infty$, we have a **complete orthogonal signal set**. And then:

$$f(t) = c_1 x_1(t) + \dots + c_n x_n(t) + \dots = \sum_{n=1}^{\infty} c_n x_n(t) \quad t_1 \le t \le t_2$$

- This series is called the **generalized Fourier series** of $f(t)$ with respect to the set $\{x_n(t)\}$.

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Complete orthogonal trigonometric set:

- Consider a signal set:

$$\{1, \cos\omega_0 t, \cos 2\omega_0 t, ..., \cos n\omega_0 t, ...; \sin\omega_0 t, \sin 2\omega_0 t, ..., \sin n\omega_0 t, ...\}$$

  - ω_0: fundamental frequency
  - n: positive integer
  - $n\omega_0$: nth harmonic
  - $T_0 = 2\pi/\omega_0$: period of the fundamental.

- The signal set above is **complete ortthogonal set** over any interval of duration T_0, and called **trigonometric set**.

Trigonometric Fourier series:

- Express a signal $f(t)$ by a **trigonometric Fourier series** over the interval $[t_1, t_1 + T_0]$ as:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$a_0 = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} f(t)dt$$

$$a_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t)\cos n\omega_0 t dt$$

$$b_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t)\sin n\omega_0 t dt$$

Compact trigonometric Fourier series:

$$a_n \cos n\omega_0 t + b_n \sin n\omega_0 t = C_n \cos\left(n\omega_0 t + \theta_n\right)$$

$$C_n = \sqrt{a_n^2 + b_n^2}$$

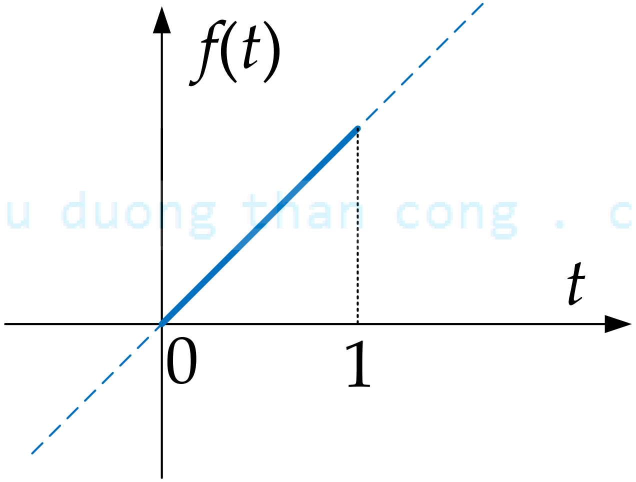
$$\theta_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right)$$

- **Compact form**:

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos\left(n\omega_0 t + \theta_n\right) \quad t_1 \le t \le t_1 + T_0$$

Trigonometric Fourier series:

- *Example 2.03:* Find the trigonometric Fourier series for the signal $f(t) = t$ over the interval [0, 1].



$$f(t) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin 2\pi nt}{n\pi}\ ;\ 0 \le t \le 1$$

Periodicity of the Trigonometric Fourier series:

- The Fourier series $\varphi(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$

is a periodic function with period T_0. and equal to $f(t)$ over only one interval $[t_1, t_1 + T_0]$, outside this interval $f(t) \neq \varphi(t)$.

- If $f(t)$ is a **periodic signal** with period T_0, then a Fourier series representing $f(t)$ over an interval T_0 will also represent $f(t)$ for all t.

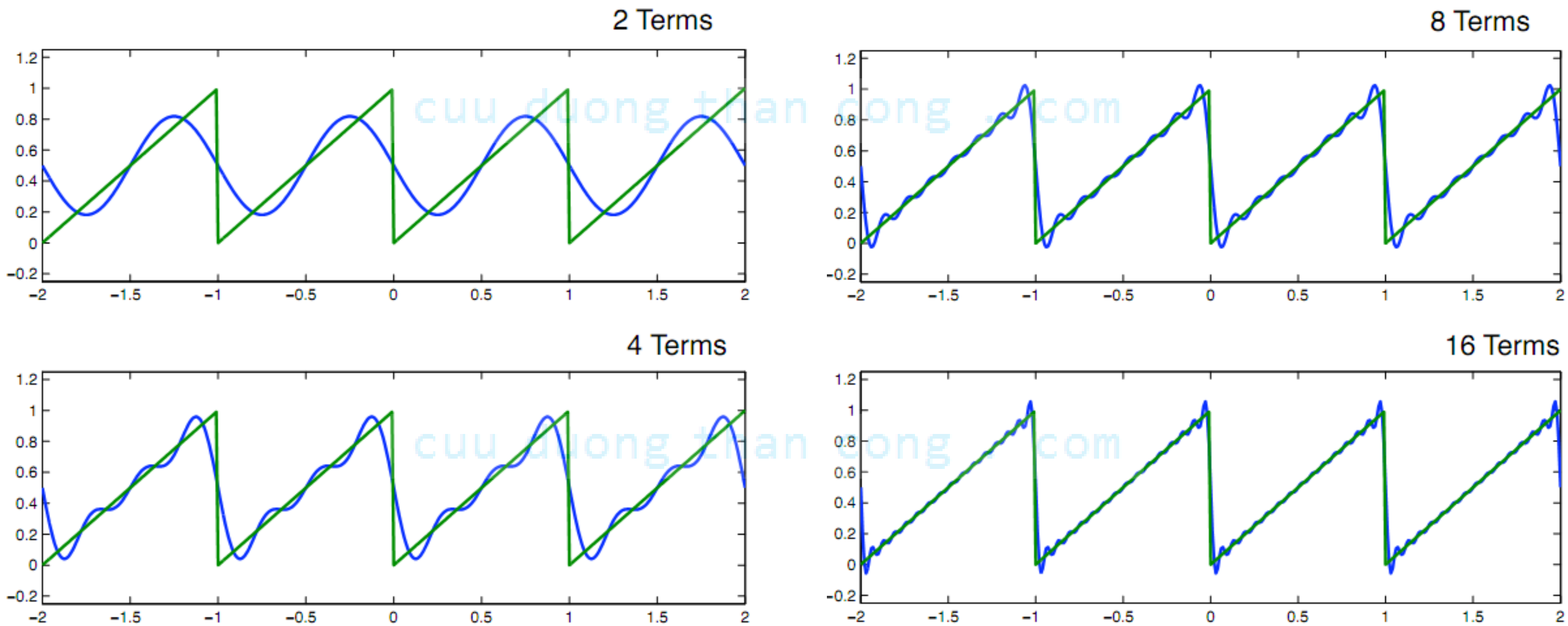
- *Example 2.04:*

$$\begin{cases} f(t) = t & 0 \leq t < 1 \\ f(t+1) = f(t) \end{cases} \Rightarrow f(t) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin 2\pi n t}{n\pi} \quad \text{for all } t$$

Periodicity of the Trigonometric Fourier series:

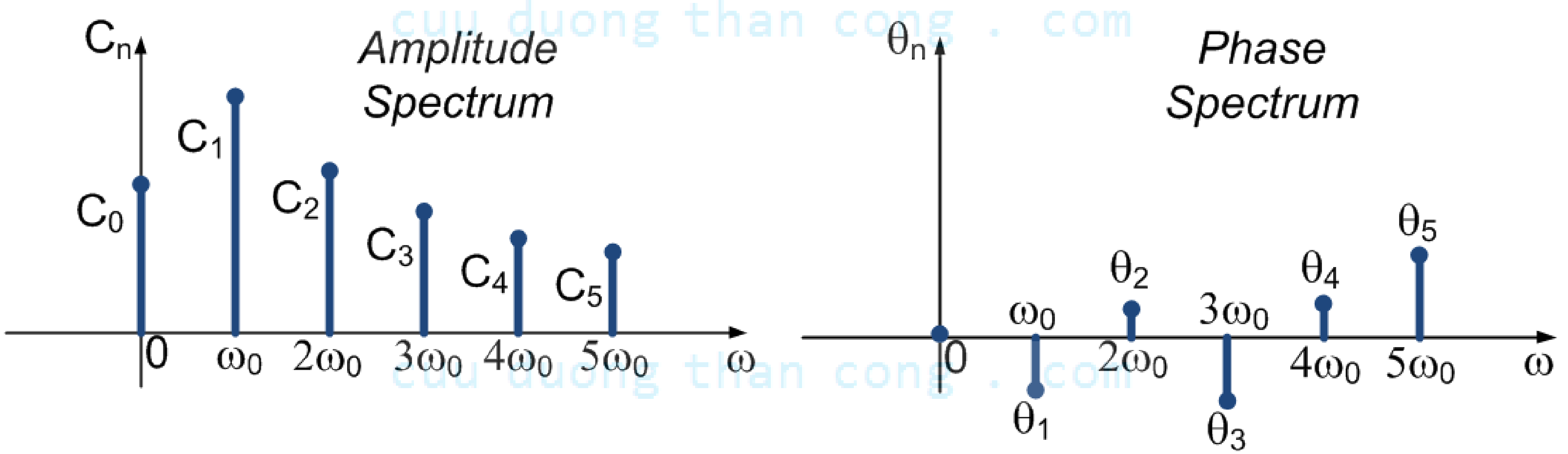
- *Example 2.04:*

$$\begin{cases} f(t) = t & 0 \le t < 1 \\ f(t+1) = f(t) \end{cases} \Rightarrow f(t) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin 2\pi n t}{n\pi} \quad \text{for all } t$$

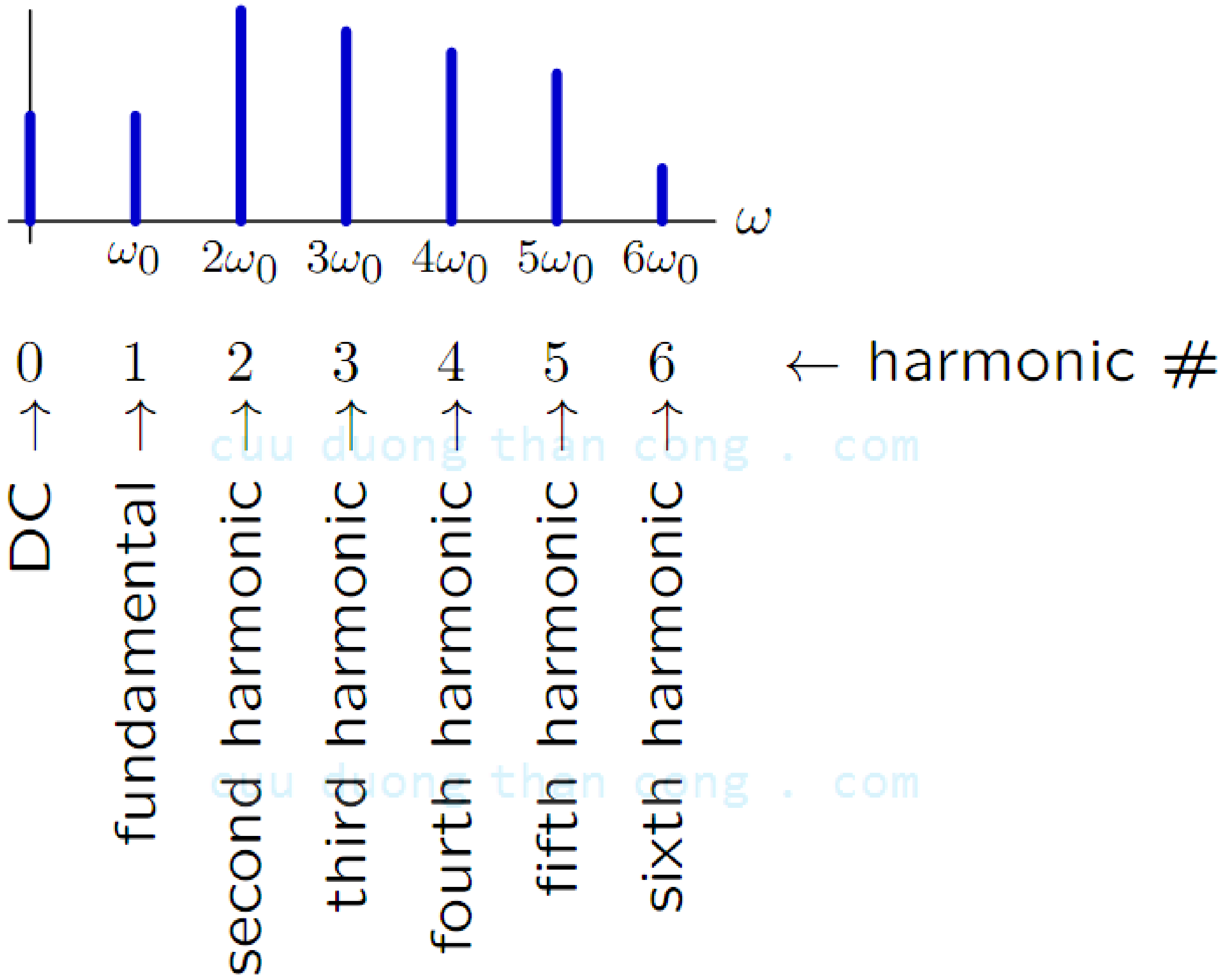


The Fourier spectrum:

- Consider the **compact form**: $f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos\left(n\omega_0 t + \theta_n\right)$
- **Amplitude spectrum:** the plot of Cn vs. ω
- **Phase spectrum:** the plot of θ_n vs. ω



The Fourier spectrum:



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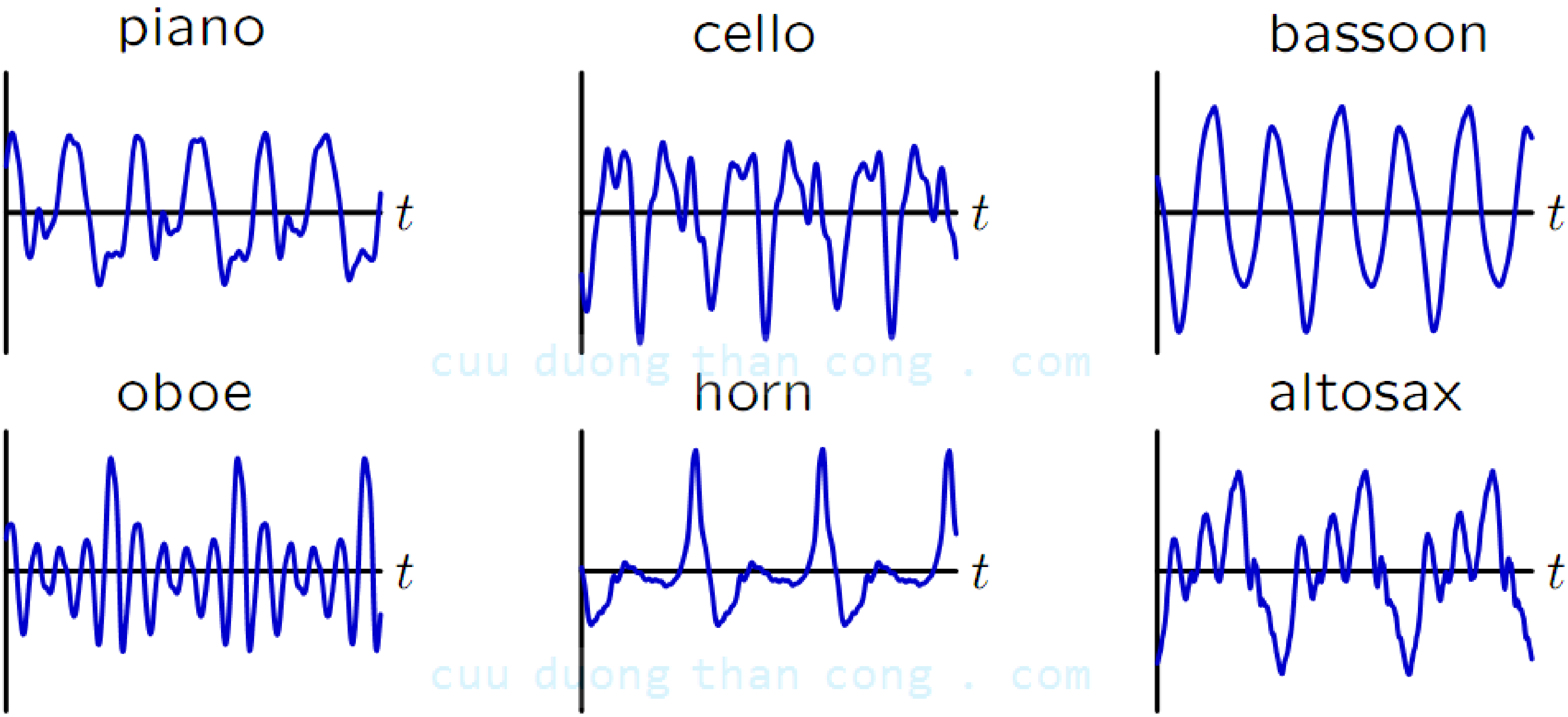
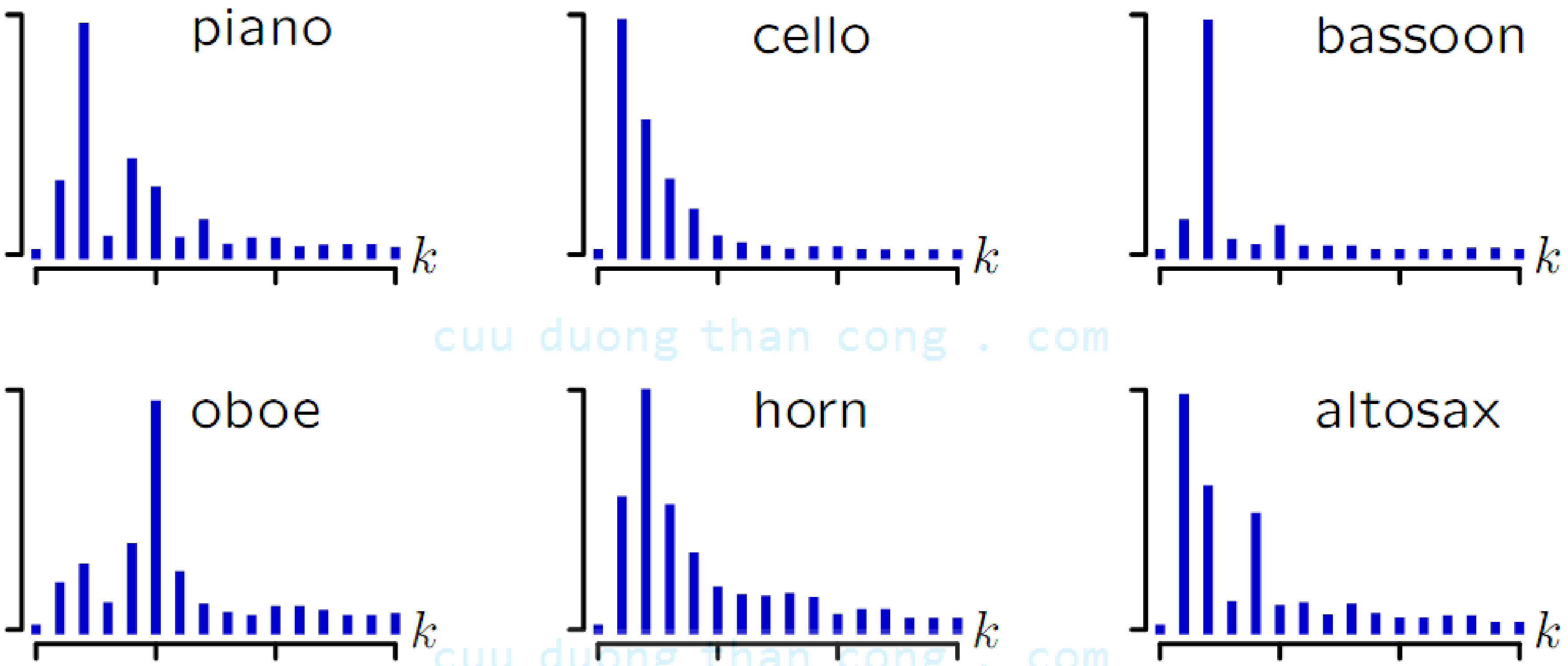
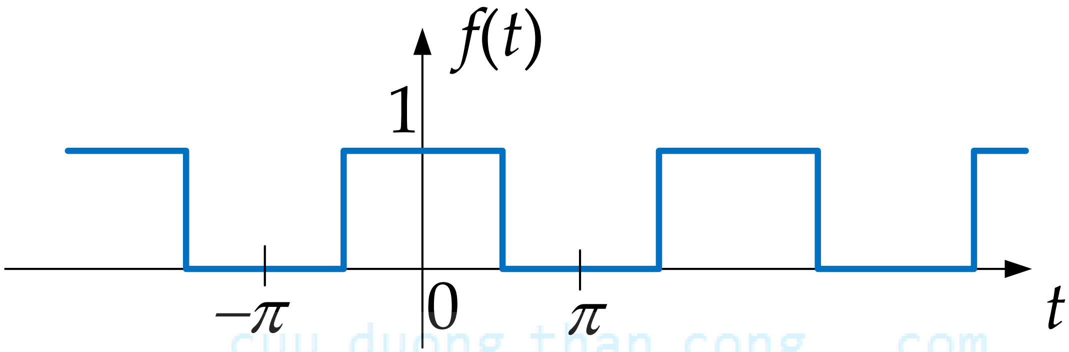
In time domain



Spectrum (frequency domain):



Example 2.05: Plot the amplitude and phase spectrum for the periodic square f(t):



$$f(t)=\frac{1}{2}+\frac{2}{\pi}\left[\cos t+\frac{1}{3}\cos\left(3t-\pi\right)+\frac{1}{5}\cos 5t+\frac{1}{7}\cos\left(7t-\pi\right)+\ldots\right]$$

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Complete orthogonal exponential set:

- Consider the signal set: $\left\{ e^{jn\omega_0 t} \right\} \; (n = 0, \pm 1, \pm 2 \ldots)$
- Since:

$$\int_{T_0} e^{jn\omega_0 t} \left(e^{jm\omega_0 t} \right)^* dt = \begin{cases} 0 & n \neq m \\ T_0 & n = m \end{cases}$$

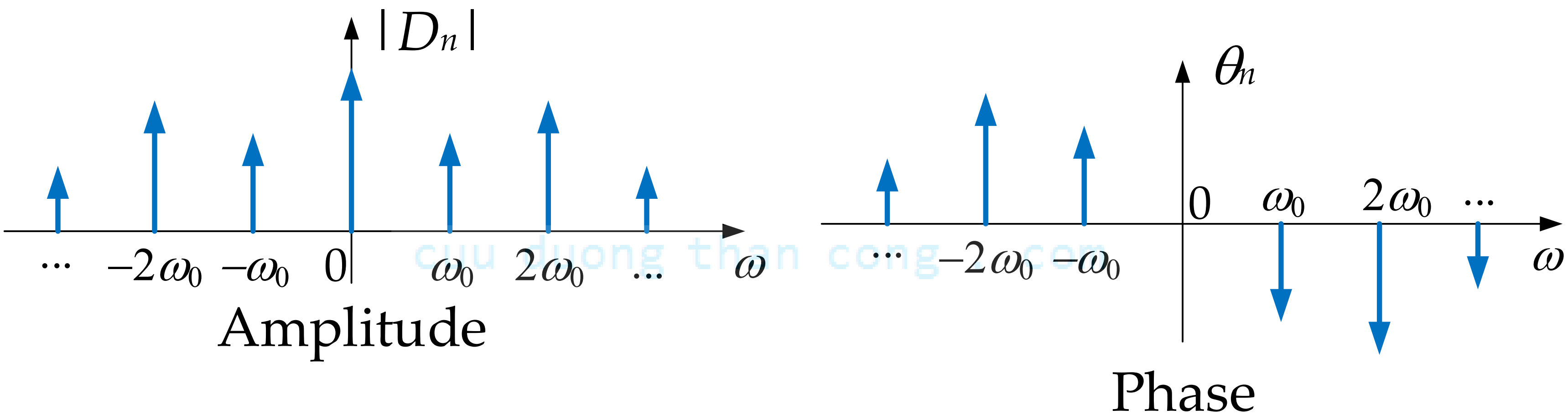
Then the signal set above is a complete orthogonal set.

- The **exponential Fourier series**:

$$\boxed{\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \\ D_n &= \frac{1}{T_0} \int_{T_0} f(t) e^{-jn\omega_0 t} dt \end{aligned}}$$

Exponential Fourier spectrum:

- Consider the signal set: $D_n = |D_n| \angle \theta_n$



- *Quiz:* Compare the trigonometric spectrum and exponential spectrum?

Parseval's Theorem:

- The **power** of a **periodic** signal is equal to the sum of the powers of its Fourier components.

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos\left(n\omega_0 t + \theta_n\right) \Rightarrow P_f = C_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} C_n^2$$

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \Rightarrow P_f = \sum_{n=-\infty}^{\infty} \left|D_n\right|^2 = D_0^2 + 2\sum_{n=1}^{\infty} \left|D_n\right|^2$$

- *Example 2.06:* Determine the exponential Fourier series of $f(t) = t^2$ $(-\pi \le t < \pi)$, $T_0 = 2\pi$.
  - Plot the exponential spectrum.
  - Use Parseval's Theorem, prove that: $\dfrac{\pi^4}{90} = \sum_{n=1}^{\infty} \dfrac{1}{n^4}$

Properties of Exponential Fourier Series:

- Linearity

$$\left.\begin{aligned} f_1(t) &= \sum_{n=-\infty}^{\infty} D_{1n} e^{jn\omega_0 t} \\ f_2(t) &= \sum_{n=-\infty}^{\infty} D_{2n} e^{jn\omega_0 t} \end{aligned}\right\} \Rightarrow k_1 f_1(t) + k_2 f_2(t) = \sum_{n=-\infty}^{\infty} \left(k_1 D_{1n} + k_2 D_{2n}\right) e^{jn\omega_0 t}$$

- Time-shifting

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \Rightarrow f(t-t_0) = \sum_{n=-\infty}^{\infty} e^{-jn\omega_0 t_0} D_n e^{jn\omega_0 t}$$

Properties of Exponential Fourier Series:

- Time-inversion

$$f(t)=\sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \Rightarrow f(-t)=\sum_{n=-\infty}^{\infty} D_{-n} e^{jn\omega_0 t}$$

- Time-scaling

$$f(t)=\sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \Rightarrow f(at)=\sum_{n=-\infty}^{\infty} D_n e^{jna\omega_0 t}$$

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Aperiodic Signal Representation by Fourier Integral:

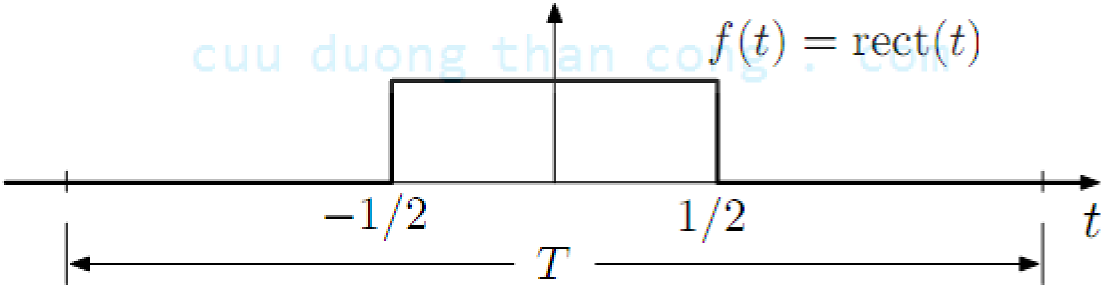
- We can expand *f*(*t*) as a Fourier series in [-*T*/2; *T*/2]:

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

$$D_n = \frac{1}{T_0}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt$$

$$\omega_0 = \frac{2\pi}{T}$$

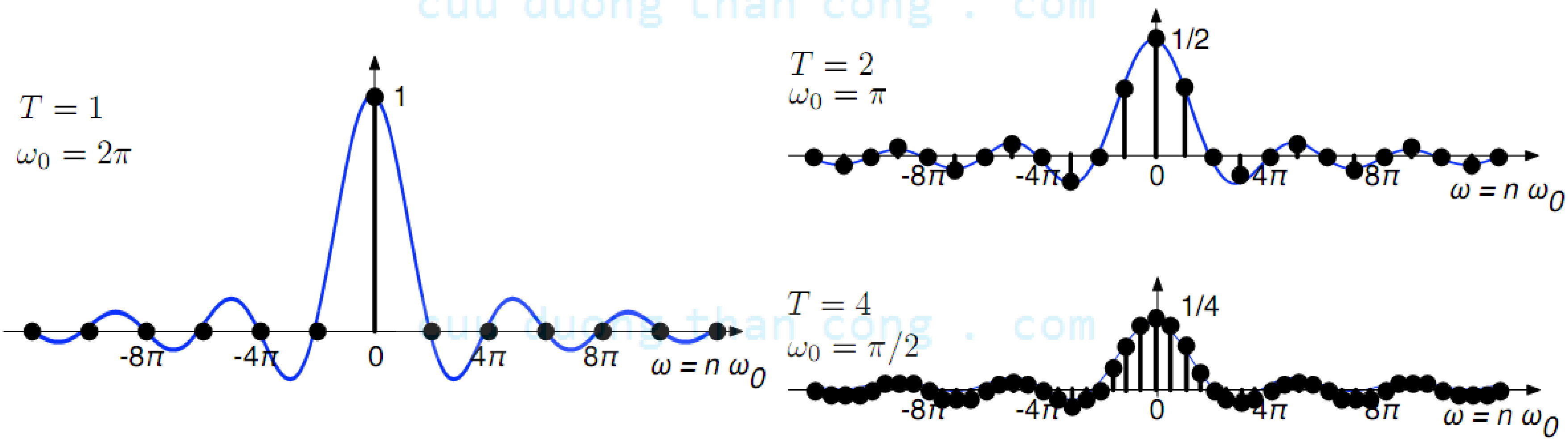
- For example, assume *y*(*t*) = rect(*t*):



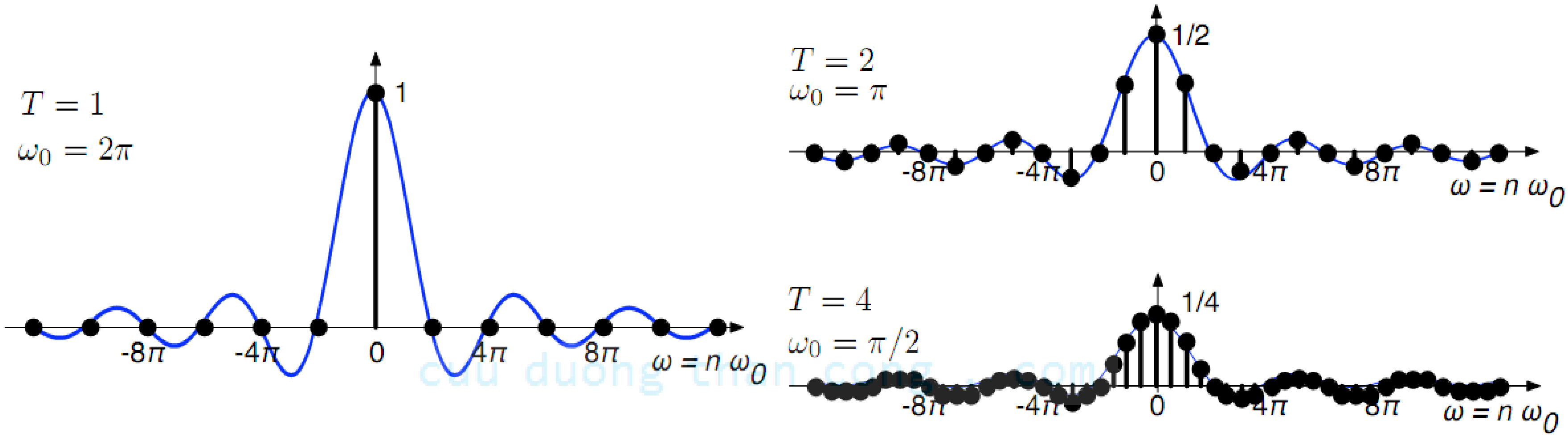
Aperiodic Signal Representation by Fourier Integral:

$$f(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \operatorname{sinc}\left(\frac{n\pi}{T}\right) e^{jn\omega_0 t}$$

- We will plot the Fourier series coefficients as a function of $\omega = n\omega_0$ for $T = 1, 2$ and 4.



<u>Some observation:</u>



- More densely sampled, decreased amplitude $|D_n|$
- If we plot $T.D_n$ instead of D_n, the envelope will be the same sinc().
- If $T \to \infty$, $T.D_n$ will be continuous function.

Some observation:

- In general, define the truncated Fourier transform:

$$F_T(j\omega) = \int_{-T/2}^{T/2} f(t)e^{-j\omega t}dt$$

so that:

$$D_n = \frac{1}{T}F_T(jn\omega_0)$$

- The Fourier series is then (note that $f_T(t) = f(t)$ over [-T/2; T/2] only):

$$f_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T}F_T(jn\omega_0)e^{jn\omega_0 t}$$

- As $T \rightarrow \infty$, then $\omega_0 \rightarrow 0$ and $n\omega_0 \rightarrow \omega$, where ω is a continuous variable.

Some observation:

- The limit of the truncated Fourier transform is:

$$F(j\omega) = \lim_{T\to\infty} F_T(j\omega) = \lim_{T\to\infty} \int_{-T/2}^{T/2} f(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

- Taking limit in Fourier series for $f(t)$ with $\Delta\omega = 2\pi/T$ and $\omega = 2\pi n/T = n\Delta\omega$:

$$f(t) = \lim_{T\to\infty} f_T(t) = \lim_{\Delta\omega\to 0} \sum_{n=-\infty}^{\infty} \frac{1}{T} F_T(jn\Delta\omega)e^{jn\Delta\omega t}$$

$$= \lim_{\Delta\omega\to 0} \sum_{n=-\infty}^{\infty} F_T(jn\Delta\omega)e^{jn\Delta\omega t} \frac{\Delta\omega}{2\pi}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$$

The Fourier transform:

- The **Fourier integral** theorem:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \qquad \textit{Fourier transform}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \qquad \textit{Invert Fourier transform}$$

Comments:

- There are technical conditions which must be satisfied for the integrals: **Dirichlet conditions**.
- As with Fourier series, inverse transform formula generally gives $f(t)$ accurately at point where $f(t)$ is continuous, but yields the midpoint when $f(t)$ has jumps.

Dirichlet conditions:

- *Weak condition*: $$\int_{-\infty}^{\infty} |f(t)| dt < \infty$$

- *Strong condition*: *f*(*t*) have only a finite number of maxima, minima and only a finite number of finite discontinuities over any finite interval.

- *For example:* we can not find the Fourier transform of the signal:

$$f(t) = e^{at} u(t) \quad a > 0$$

Transform of some useful functions:

i. $f(t) = \delta(t)$:

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1 \quad \Rightarrow \quad \boxed{\delta(t) \Leftrightarrow 1}$$

ii. $f(t) = e^{-at}u(t)$ with $a > 0$:

$$F(\omega) = \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt = \frac{1}{a + j\omega} \quad \Rightarrow \quad \boxed{e^{-at}u(t) \Leftrightarrow \frac{1}{a + j\omega}}$$

iii. $f(t) = e^{j\omega_0 t}$:

Because the inverse Fourier transform of $\delta(\omega - \omega_0)$ is

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} dt = \frac{1}{2\pi} e^{j\omega_0 t} \quad \Rightarrow \quad \boxed{e^{j\omega_0 t} \Leftrightarrow 2\pi\delta(\omega - \omega_0)}$$

Transform of some useful functions:

iv. $f(t) = u(t)$:

We consider $u(t)$ as a decaying exponential $e^{-at}u(t)$ in the limit as $a \to 0$

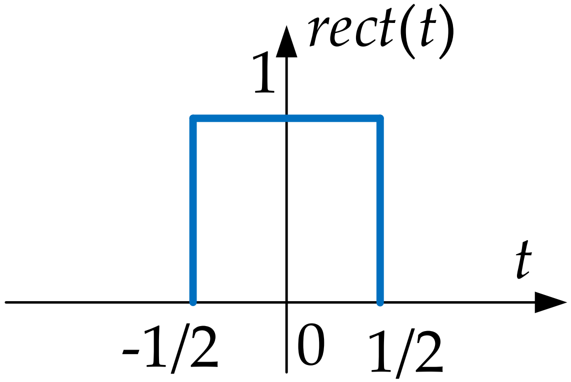
$$u(t) = \lim_{a\to 0}\left[e^{-at}u(t)\right]$$

$$U(\omega) = \lim_{a\to 0}\left[\frac{1}{a+j\omega}\right] = \lim_{a\to 0}\left[\frac{a}{a^2+\omega^2}\right] + \frac{1}{j\omega} = \pi\delta(\omega) + \frac{1}{j\omega}$$

$$\boxed{u(t) \Leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}}$$

Transform of some useful functions:

v. Unit gate function:



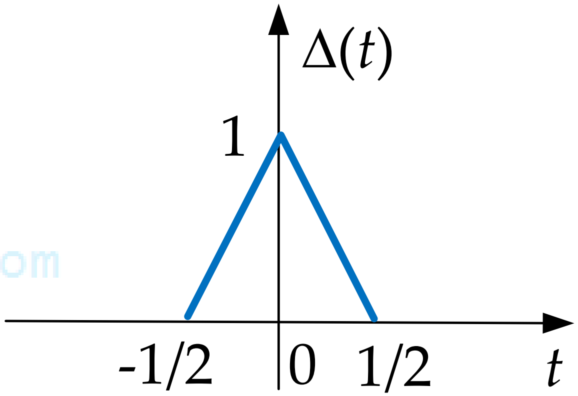
$$f(t) = rect(t) = \begin{cases} 0 & |t| > \dfrac{1}{2} \\ \dfrac{1}{2} & |t| = \dfrac{1}{2} \\ 1 & |t| < \dfrac{1}{2} \end{cases} \Rightarrow F(\omega) = \int_{-1/2}^{1/2} e^{-j\omega t} dt = \sin c\left(\frac{\omega}{2}\right)$$

$$\boxed{rect(t) \Leftrightarrow \sin c\left(\frac{\omega}{2}\right)}$$

Transform of some useful functions:

vi. $\Delta(t)$:

$$f(t) = \Delta(t) = \begin{cases} 0 & |t| > \dfrac{1}{2} \\ 2t+1 & -\dfrac{1}{2} \le t < 0 \\ -2t+1 & 0 \le t \le \dfrac{1}{2} \end{cases}$$



$$\boxed{\Delta(t) \Leftrightarrow \frac{1}{2}\operatorname{sinc}^2\left(\frac{\omega}{4}\right)}$$

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Some properties of the Fourier transform:

i. Symmetry property:

$$f(t) \Leftrightarrow F(\omega) \Rightarrow F(t) \Leftrightarrow 2\pi f(-\omega)$$

ii. Linearity:

$$c_1 f_1(t) + c_2 f_2(t) \Leftrightarrow c_1 F_1(\omega) + c_2 F_2(\omega)$$

iii. The scaling property:

$$f(t) \Leftrightarrow F(\omega) \Rightarrow f(at) \Leftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

iv. The time-shifting property:

$$f(t) \Leftrightarrow F(\omega) \Rightarrow f(t - t_0) \Leftrightarrow F(\omega) e^{-j\omega t_0}$$

Some properties of the Fourier transform:

v. The frequency-shifting property:

$$f(t) \Leftrightarrow F(\omega) \Rightarrow f(t)e^{j\omega_0 t} \Leftrightarrow F(\omega - \omega_0)$$

vi. Convolution:

$$f_1(t) * f_2(t) \Leftrightarrow F_1(\omega)F_2(\omega) \qquad \textit{time convolution}$$

$$f_1(t) f_2(t) \Leftrightarrow \frac{1}{2\pi} F_1(\omega) * F_2(\omega) \qquad \textit{frequency convolution}$$

vii. Time differentiation:

$$\frac{d^n f(t)}{dt^n} \Leftrightarrow (j\omega)^n F(\omega)$$

Some properties of the Fourier transform:

viii. Time integration:

$$\int_{-\infty}^{t} f(\tau)d\tau \Leftrightarrow \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$$

ix. Frequency differentiation:

$$t^n f(t) \Leftrightarrow (j)^n \frac{d^n F(\omega)}{d\omega^n}$$

x. Fourier transform of even and odd functions:

$f(t)$ is even: $$F(\omega) = 2\int_{0}^{\infty} f(t)\cos\omega t\, dt$$

$f(t)$ is odd: $$F(\omega) = -2j\int_{0}^{\infty} f(t)\sin\omega t\, dt$$

Some properties of the Fourier transform:

xi. Fourier transform of periodic signal:

If $f(t)$ is periodic:

$$f(t)=\sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad D_n=\frac{1}{T}\int_T f(t)e^{-j\omega_0 t}dt$$

$$\Rightarrow f(t)\Leftrightarrow F(\omega)=\sum_{n=-\infty}^{\infty} 2\pi D_n \delta(\omega-n\omega_0)$$

Example:

$$f(t)=\sin\omega_0 t=\frac{1}{2j}e^{j\omega_0 t}-\frac{1}{2j}e^{-j\omega_0 t}\Rightarrow F(\omega)=\frac{\pi}{j}\delta(\omega-\omega_0)-\frac{\pi}{j}\delta(\omega+\omega_0)$$

$$f(t)=\cos\omega_0 t=\frac{1}{2}e^{j\omega_0 t}+\frac{1}{2}e^{-j\omega_0 t}\Rightarrow F(\omega)=\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$$

Some properties of the Fourier transform:

xii. Other properties:

Condition	Property
Real $f(t)$	$F(-\omega) = F^*(\omega)$
Real and even $f(t)$	Real and even $F(\omega)$
Real and odd $f(t)$	Imaginary and odd $F(\omega)$

Example 2.07: Show that, given a real signal $f(t)$, the inverse Fourier transform integral can be express as:

$$f(t) = \frac{1}{2\pi}\int_0^\infty 2\left|F(\omega)\right|\cos\left(\omega t + \angle F(\omega)\right)d\omega$$

Signal energy – Parseval's theorem:

$$E_f = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} f(t) f^*(t) dt$$

$$= \int_{-\infty}^{\infty} f(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) e^{-j\omega t} d\omega \right] dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) \left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) F^*(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

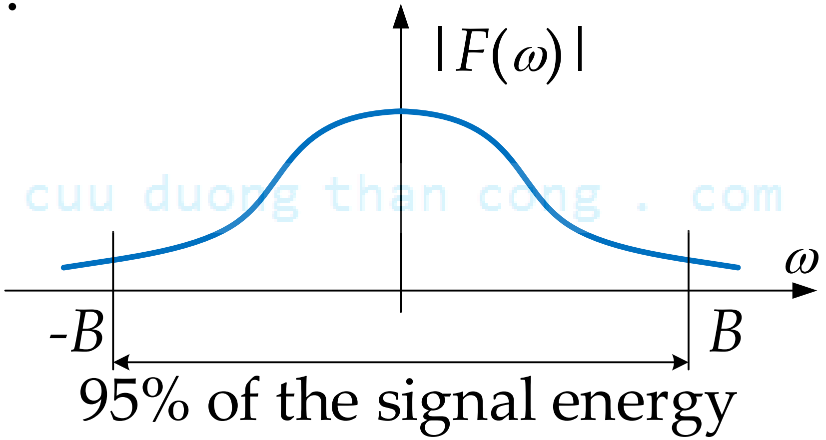
$$\boxed{E_f = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega}$$

- $|F(\omega)|^2$: **energy spectral density**

Example 2.08: Find the energy of $f(t) = \text{sinc}(t)$.

The bandwidth of a Signal:

- Most of the signal energy is contained within a certain band of B Hz.
- The **bandwidth B** is called the essential bandwidth of the signal.
- The criterion for selecting B depends on the error tolerance in a particular application.
- We may select B to be that band which contains 95% of the signal energy.



The bandwidth of a Signal:

- *Example 2.09:* $f(t) = e^{-at}u(t)$, $a > 0$, find the energy and the bandwidth of $f(t)$.

$$E_f = \frac{1}{\pi}\int_0^\infty \left(\frac{1}{a+j\omega}\right)^2 d\omega = \frac{1}{\pi}\int_0^\infty \frac{1}{a^2+\omega^2} d\omega$$

$$= \frac{1}{\pi a}\tan^{-1}\frac{\omega}{a}\Bigg|_0^\infty = \frac{1}{2a}$$

The signal bandwidth:

$$\frac{1}{\pi}\int_0^B \left(\frac{1}{a+j\omega}\right)^2 d\omega = 0.95E_f$$

$$\Leftrightarrow \frac{1}{\pi a}\tan^{-1}\frac{B}{a} = \frac{0.95}{2a} \Leftrightarrow B = 12.706a(rad/s)$$