Chapter 3:
The Laplace Transform

1. ***The Laplace Transform***
2. *The inverse Laplace Transform.*

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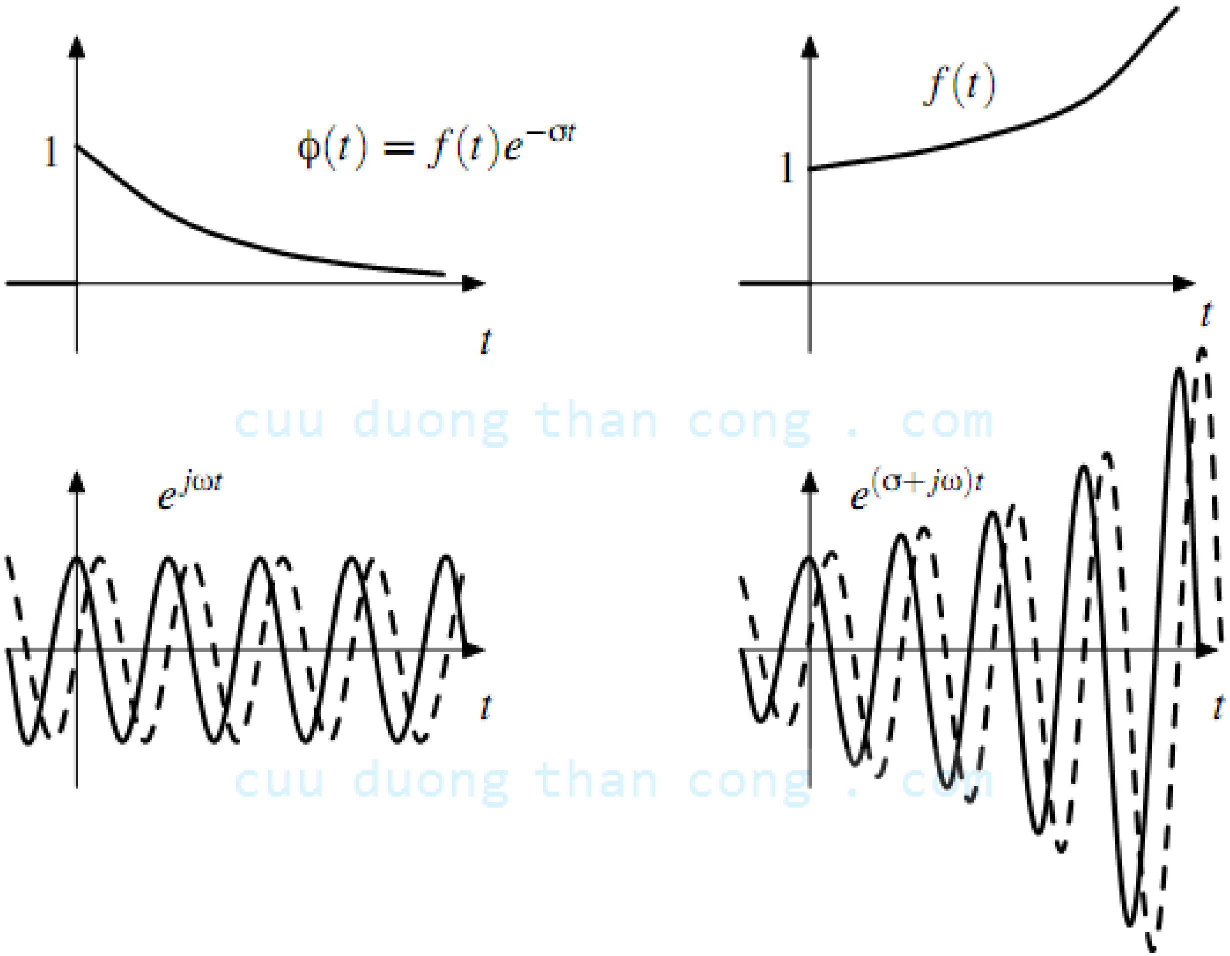
Limitations of the Fourier Transform:

- To be useful, Fourier transform must be exist, or be defined in a generalized sense. For many areas, this will be all you will need (communications, optics, image processing).
- For many signals and systems the Fourier transform is not enough:
  - Signals that grow with time (do not satisfy the Dirichlet conditions)
  - Systems that are unstable (many mechanical or electrical systems)

Limitations of the Fourier Transform:

- Consider the signal $f(t) = e^{2t}$, which doesn't have a Fourier transform.
- We can create a new function $\phi(t) = f(t)e^{-\sigma t}$, it does have a Fourier transform (with $\sigma > 2$).
- The Fourier transform represents of $\phi(t)$ in terms of spectral components $e^{j\omega t}$.
- $f(t) = \phi(t)e^{-\sigma t}$ so $f(t)$ can be represented by spectral components $e^{\sigma t}e^{j\omega t}$.
- We can choose many value of σ for which $f(t)e^{-\sigma t}$ goes to zero (have a Fourier transform), this means the spectrum of $f(t)$ is not unique.

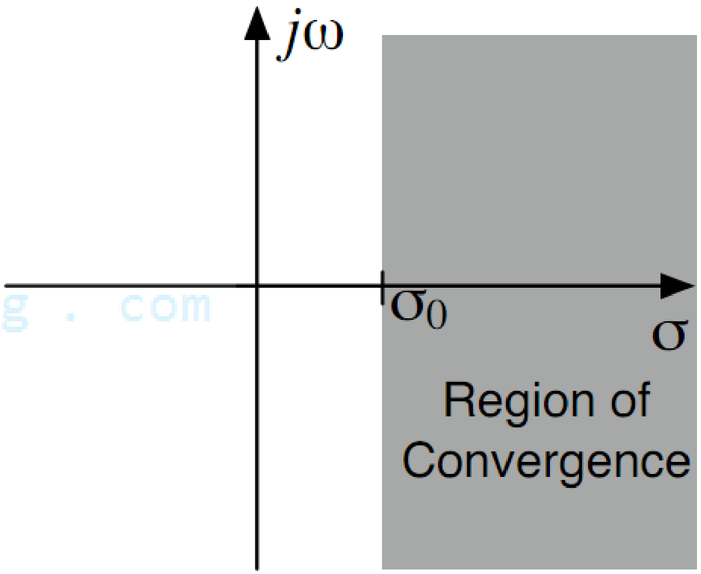
Limitations of the Fourier Transform:



Limitations of the Fourier Transform:

- For $f(t) = e^{2t}$, any $\sigma > 2$ will produce a decaying, Fourier transformable signal.
- If σ_0 is the smallest value for which $f(t)e^{-\sigma t}$ goes to zero, then any $\sigma > \sigma_0$ will do.
- The part of the complex plane $\sigma + j\omega$ where the spectrum exists is the *region of convergence* (ROC).

$$\text{ROC: } \sigma = \text{Re}\{s\} > \sigma_0$$



Bilateral Laplace transform:

- The Fourier transform of $f(t)e^{-\sigma t}$ is:

$$f(t)e^{-\sigma t} \leftrightarrow \int_{-\infty}^{\infty} f(t)e^{-\sigma t}e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(t)e^{-(\sigma+j\omega)t}dt$$

$$= \int_{-\infty}^{\infty} f(t)e^{-st}dt = F(s)$$

where s is the complex frequency $s = \sigma + j\omega$, includes:

- The oscillation component $j\omega$ that we're used to, plus
- A decay/growth component σ.

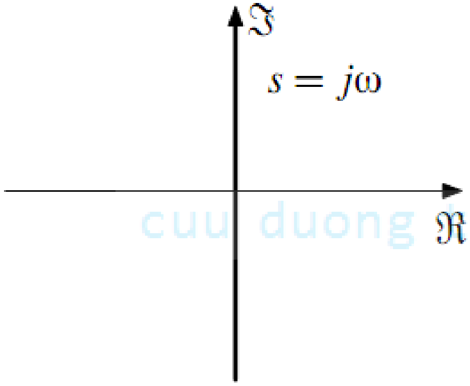
$$bilateral\ \ Laplace\ transform: \boxed{f(t) \leftrightarrow F(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt}$$

Bilateral Laplace transform:

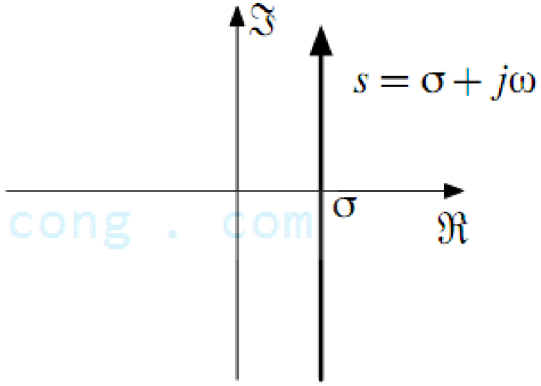
- The Fourier transform is a special case of the Laplace transform, provided the $j\omega$ axis is in the ROC:

$$F(\omega) = F(s)\big|_{s=j\omega}$$

- The Fourier transform is the Laplace transform evaluated along the $j\omega$ axis, it is also often called the *frequency response*.



Fourier Transform



Laplace Transform

Bilateral Laplace transform:

- The same bilateral Laplace transform can correspond to different signals (causal, anti-causal, or infinite extent) depending on the ROC. The $e^{\sigma t}$ factor that makes the integral converge or a causal signal can mate the integral for an anti-causal signal blow up.

- *Example 3.01:*

$$f(t) = e^{-at}u(t); a > 0 \leftrightarrow F(s) = \frac{1}{s+a}; \; ROC : \text{Re}\{s\} > -a$$

$$f(t) = -e^{-at}u(-t); a > 0 \leftrightarrow F(s) = \frac{1}{s+a}; \; ROC : \text{Re}\{s\} < -a$$

Unilateral Laplace transform:

- We'll be interested in causal signals, this is the common application for Laplace transform.
- The bilateral Laplace transform of a causal signal $f(t)u(t)$ is:

$$F(s) = \int_{-\infty}^{\infty} f(t)u(t)e^{-st}dt = \int_{0-}^{\infty} f(t)e^{-st}dt$$

$$unilateral\ Laplace\ transform: \boxed{f(t) \leftrightarrow F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt}$$

- The lower limit 0- indicates that we include impulse at the origin.

- The **unilateral Laplace transform** is just the bilateral transform for **causal signals**.

Unilateral Laplace transform:

- If we restrict ourselves to the unilateral transform, the Laplace transform is (almost) unique, and we can ignore the region of convergence.
- Some common (unilateral) Laplace Transform:

$$f(t)=\delta(t) \quad \leftrightarrow \quad F(s)=1$$

$$f(t)=t^n u(t) \quad \leftrightarrow \quad F(s)=\frac{n!}{s^{n+1}}$$

$$f(t)=\cos(bt)u(t) \quad \leftrightarrow \quad F(s)=\frac{s}{s^2+b^2}$$

$$f(t)=\sin(bt)u(t) \quad \leftrightarrow \quad F(s)=\frac{b}{s^2+b^2}$$

Unilateral Laplace transform:

Example 3.02: Similar Fourier and Laplace Transform

- Consider the Laplace transform of $f(t) = e^{-at}u(t)$ with $a > 0$:

$$F(s) = \int_0^\infty e^{-at}e^{-st}dt = \frac{1}{s+a}$$

The ROC is then $\sigma > -a$ or $\mathrm{Re}\{s\} > -a$.

- The Fourier transform of $f(t) = e^{-at}u(t)$ for $a > 0$:

$$F(\omega) = \frac{1}{j\omega + a}$$

- The Laplace transform is the Fourier transform with $j\omega$ replaced by s. This is true only when the ROC includes the $j\omega$ axis (in this case, when $a > 0$).

Unilateral Laplace transform:

Example 3.03: Different Fourier and Laplace Transform

- Consider the Laplace transform of $f(t) = \cos\omega_0 t$:

$$f(t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) \leftrightarrow F(s) = \int_0^\infty \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)e^{-st}dt$$

$$= \frac{1}{2}\left(\frac{1}{s - j\omega_0} + \frac{1}{s + j\omega_0}\right)$$

$$f(t) = \cos\omega_0 t \leftrightarrow F(s) = \frac{s}{s^2 + \omega_0{}^2}; \quad ROC: \mathrm{Re}\{s\} > 0$$

- Here the Fourier transform and the Laplace transform differ, because the $j\omega$ axis is not in the ROC for the Laplace transform.

Properties of the Laplace transform:

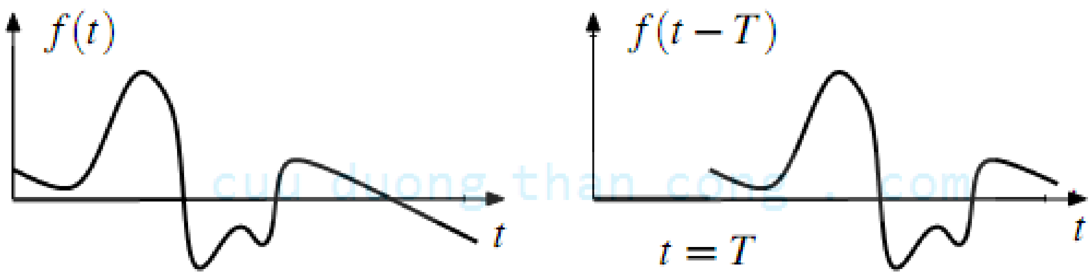
i. Linearity: $af(t) + bg(t) \leftrightarrow aF(s) + bG(s)$

ii. Time scaling: $a > 0$

$$f(at) \leftrightarrow \frac{1}{a} F\left(\frac{s}{a}\right)$$

iii. Frequency Shifting: $f(t)e^{s0t} \leftrightarrow F(s - s_0)$

iv. Time delay: $f(t - T)u(t - T) \leftrightarrow e^{-sT}F(s)$



Properties of the Laplace transform:

iv. Convolution: $f(t)*g(t) \leftrightarrow F(s)G(s)$

$$f(t)g(t) \leftrightarrow \frac{1}{2\pi j}\left[F(s)*G(s)\right]$$

v. Time integration:

$$\int_0^t f(\tau)d\tau \leftrightarrow \frac{1}{s}F(s)$$

vi. Time differentiation:

$$\frac{d^n f(t)}{dt^n} \leftrightarrow s^n F(s) - s^{n-1} f(0-) - s^{n-2} f^{(1)}(0-) - \dots - sf^{n-2}(0-) - f^{(n-1)}(0-)$$

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Properties of the Laplace transform:

vii. Multiplication by t:

$$t.f(t) \leftrightarrow -\frac{dF(s)}{ds}$$

viii. Frequency differentiation:

$$\frac{f(t)}{t} \leftrightarrow \int_s^{\infty} F(x)dx$$

ix. Initial value:

$$f(0+) = \lim_{s\to\infty}\left[sF(s)\right]$$

x. Final value:

$$f(\infty) = \lim_{s\to 0}\left[sF(s)\right]$$

Chapter 2:
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Inversion of the Laplace transform:

- The inverse Laplace transform is given by:

$$f(t)=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}F(s)e^{st}ds$$

- Where σ is large enough that $F(s)$ is defined for $\mathrm{Re}\{s\}\geq c$.

- Simpler approach: rewrite a rational Laplace transform into simple terms we recognize, and can invert by inspection (*partial fraction expansion*).

$$F(s)=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_1 s+b_0}{a_n s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0};\ m<n$$

Inversion of the Laplace transform:

Case 1: no poles of $F(s)$ are repeated.

- Then we can write $F(s)$ in the form (called partial fraction expansion of $F(s)$):

$$F(s) = \frac{k_1}{s-\lambda_1} + \ldots + \frac{k_n}{s-\lambda_n}$$

  - $\lambda_1,\ldots,\lambda_n$ are the poles of $F(s)$
  - The number $k_1,\ldots,k_n$ are called the residues

- *Example 3.04*:

$$F(s) = \frac{s-1}{s^2+3s+2}$$

Inversion of the Laplace transform:

Case 2: repeated poles

$$F(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{(s-\lambda_1)^r (s-\lambda_2)\ldots(s-\lambda_i)}$$

$$= \frac{k_{1,r}}{(s-\lambda_1)^r} + \frac{k_{1,r-1}}{(s-\lambda_1)^{r-1}} + \ldots + \frac{k_{1,1}}{(s-\lambda_1)} + \frac{k_2}{s-\lambda_2} + \ldots + \frac{k_i}{s-\lambda_i}$$

Example 3.05:

$$F(s) = \frac{1}{s^2(s+1)}$$

Inversion of the Laplace transform:

Find residues $k_1,\dots,k_n$

- *Method 1:* **Clear Fraction, Solve linear equations**

- *Method 2:* **Heaviside "Cover-up" Procedure**
  - Case 1: single poles:

$$k_r = (s-\lambda_r)F(s)\Big|_{s=\lambda_r}$$

  - Case 2: repeated poles:

$$k_{i,r-j} = \frac{1}{j!}\frac{d^j}{ds^j}\left[(s-\lambda_i)^r F(s)\right]\Bigg|_{s=\lambda_i}$$