Chapter 4:

Linear Time-Invariant systems – Impulse response and Convolution integral

1. ***Impulse response***
2. *The convolution integral*

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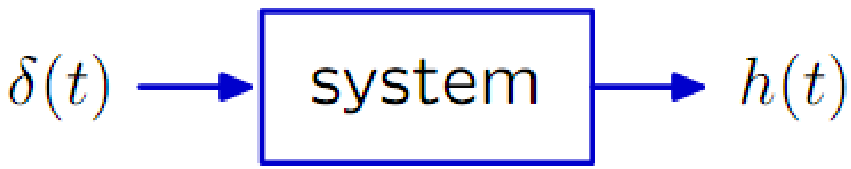
Impulse response $h(t)$:

- The impulse response $h(t)$ of a system is the output of that system when the unit impluse $\delta(t)$ is the input.



- *Example 4.01:* Find the impulse response of the following systems:

$$a.\ y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} f(\tau-1)d\tau$$

$$b.\ y'(t) + 5y(t) = f(t)$$

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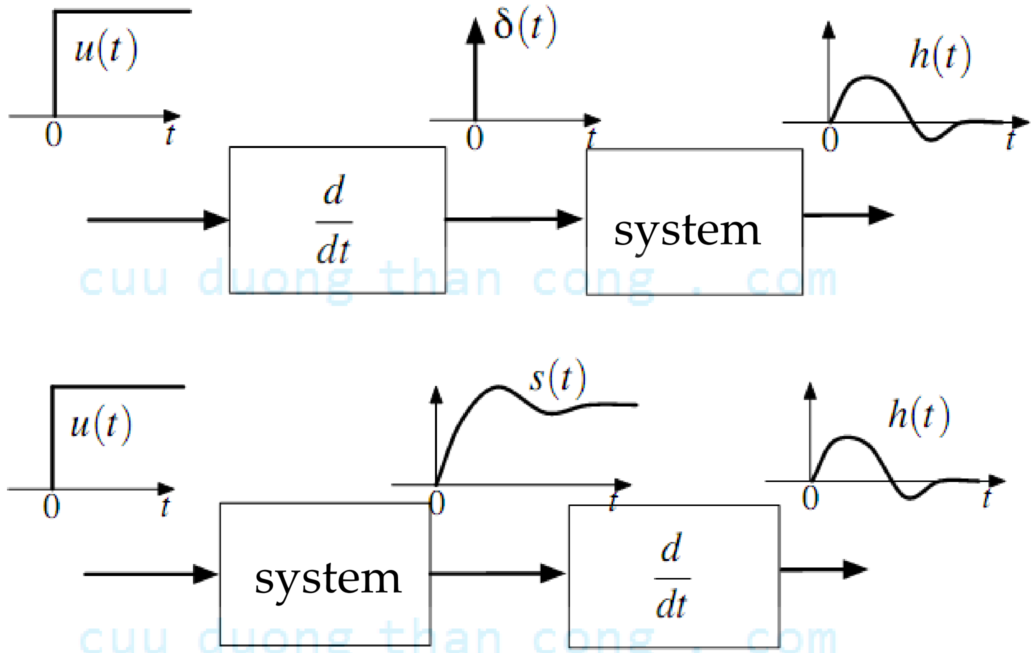
Impulse response *h*(*t*):

- In some cases, we can determine $h(t)$ using the step-response



- *Answer to Ex 4.01b:* step response = $\frac{1}{5}\left(1-e^{-5t}\right)u(t)$

$$\Rightarrow h(t)=e^{-5t}u(t)$$

Chapter 2:

Linear Time-Invariant systems – Impulse response and Convolution integral

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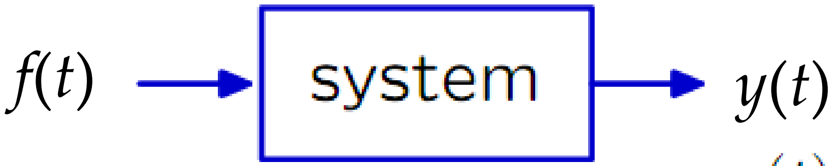
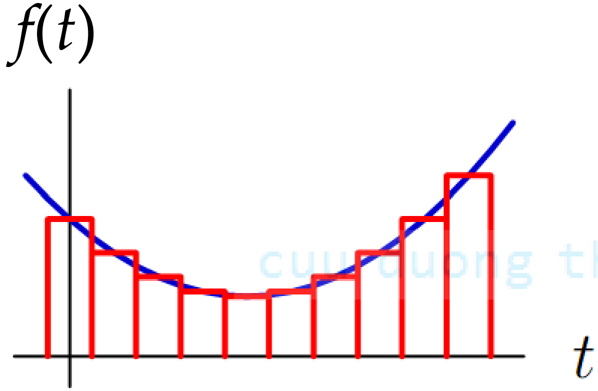
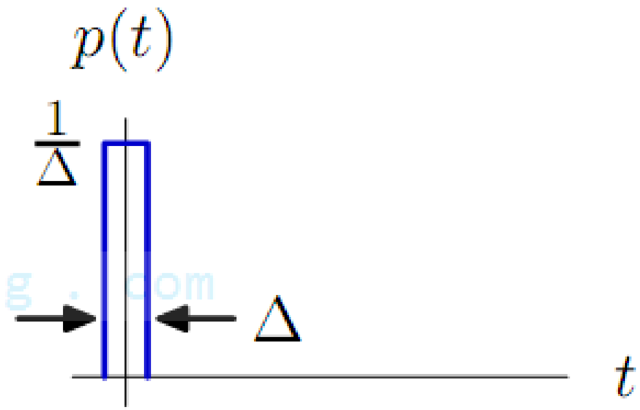
LTI system with CT-input signal:







$$f(t) = \lim_{\Delta \to 0} \sum_{k} f(k\Delta)p(t - k\Delta)\Delta$$

As $\Delta \to 0, k\Delta \to \tau, \Delta \to d\tau$ and $p(t) \to \delta(t)$

$$f(t) \to \int_{-\infty}^{\infty} f(\tau)\delta(t - \tau)d\tau$$

LTI system with CT-input signal:

$$\delta(t) \longrightarrow \boxed{\text{system}} \longrightarrow h(t)$$

$$\delta(t-\tau) \longrightarrow \boxed{\text{system}} \longrightarrow h(t-\tau)$$

$$f(\tau)\delta(t-\tau) \longrightarrow \boxed{\text{system}} \longrightarrow f(\tau)h(t-\tau)$$

$$f(t)=\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau \longrightarrow \boxed{\text{system}} \longrightarrow y(t)=\int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

Convolution

- Definition: $$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$
- Properties:

i. The Comutative Property:

$$f(t)*h(t) = h(t)*f(t)$$

ii. The Distributive Property:

$$f(t)*[h(t) + g(t)] = f(t)*h(t) + f(t)*g(t)$$

iii. The Associative Property:

$$f(t)*[h(t)*g(t)] = [f(t)*h(t)]*g(t)$$

iv. Time-Shift Property:

$$f(t)*h(t) = y(t)$$

$$f(t - T)*h(t) = y(t - T)$$

$$f(t - T_1)*h(t - T_2) = y(t - T_1 - T_2)$$

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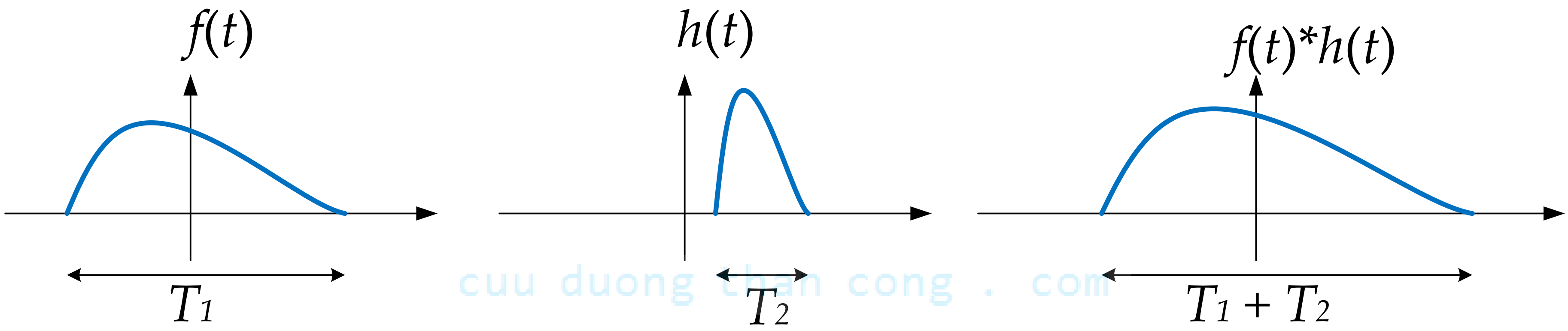
Convolution

v. Convolution with an Impulse:

$$f(t) * \delta(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau = f(t)$$

vi. The width property:

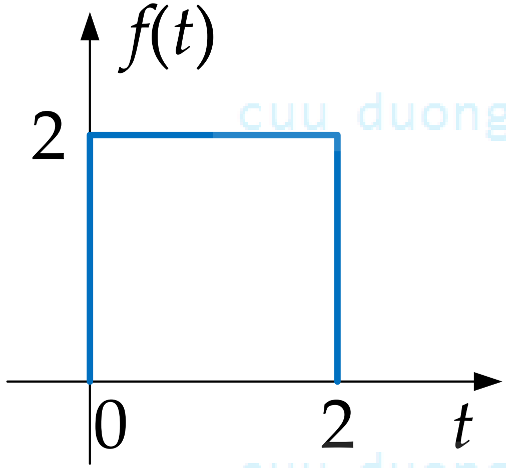


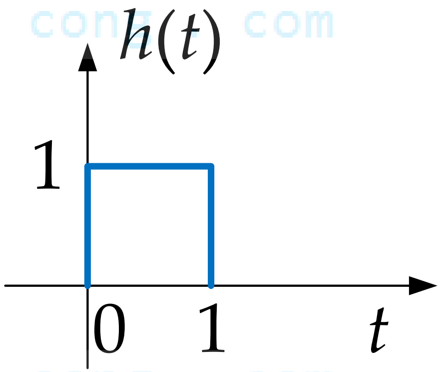
Convolution

Example 4.02: Find $y(t) = f(t)*h(t)$ with:

a. $f(t) = e^{-t}u(t)$; $h(t) = u(t)$

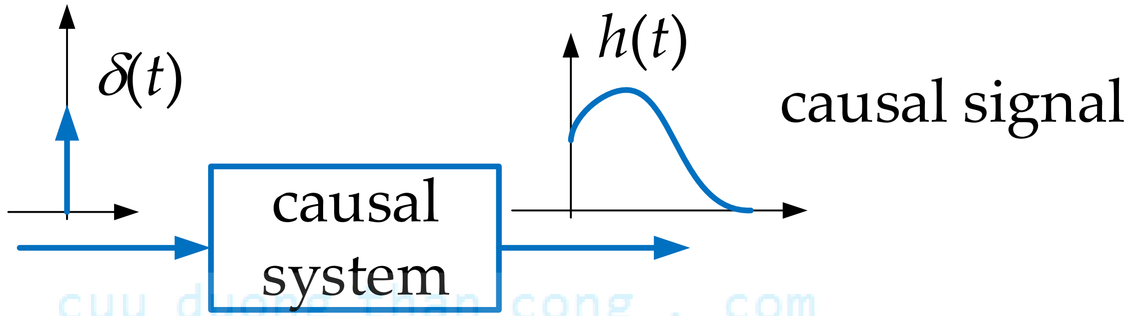
b. $f(t)$ and $h(t)$ as figure below:





Causal systems:

A causal system's unit impulse response *h(t)* is a causal signal.



$$y(t) = \int_{-\infty}^{\infty} f(\tau).\underbrace{h(t-\tau)}_{=0,\, if\, (t-\tau)<0}\, d\tau = \int_{-\infty}^{t} f(\tau).h(t-\tau)d\tau$$

If the input *f(t)* is also a causal signal:

$$y(t) = \begin{cases} \displaystyle\int_{0-}^{t} f(\tau)h(t-\tau)d\tau & t \geq 0 \\ 0 & t \leq 0 \end{cases}$$

Causal systems:

Example 4.03: For an LTIC system with $h(t) = e^{-2t}u(t)$, determine the response $y(t)$ for the input $f(t) = e^{-t}u(t)$.

Because both $f(t)$ and $h(t)$ are causal:

$$y(t) = \int_{0-}^{t} f(\tau)h(t-\tau)d\tau \quad t \geq 0$$

$$= \int_{0-}^{t} e^{-\tau}e^{-2(t-\tau)}d\tau = e^{-t} - e^{-2t} \quad t \geq 0$$

and $y(t) = 0$ when $t < 0 \Rightarrow y(t) = (e^{-t} - e^{-2t})u(t)$

Stable systems:

- Stable systems: bounded input $\rightarrow$ bounded output.
- LTI systems with bounded input $|f(t)| < B$:

$$y(t) = h(t) * y(t) = \int_{-\infty}^{\infty} h(\tau).f(t-\tau)d\tau$$

$$\Rightarrow |y(t)| \leq \int_{-\infty}^{\infty} |h(\tau)|.|f(t-\tau)|d\tau \leq B\int_{-\infty}^{\infty} |h(\tau)|d\tau$$

- Therefore, the system is stable if the impulse response is absolutely integrable:

$$\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty$$