Chapter 5:
System analysis using transfer function

1. ***Transfer function***
2. *Block diagram*
3. *System described by differential equation*
4. *System realization*
5. *Feedback and Control systems*

Created and edited by: **Nguyen Phuoc Bao Duy**
HCMC University of Technology

Transfer function:

$$f(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = f(t) * h(t)$$

$$F(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s) = F(s).H(s)$$

- $h(t)$: impulse response of system.

$$H(s) = Laplace\ transform\ of\ h(t)$$

- ***H(s): transfer function***
- *Example 5.01:* determine transfer function of the system

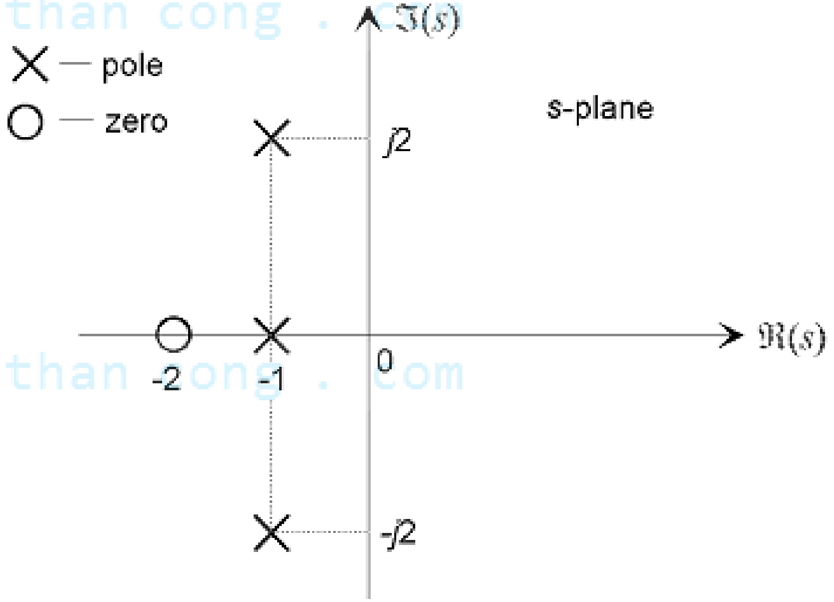
$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} f(\tau) d\tau$$

Zero-pole pattern:

$$H(s) = \frac{P(s)}{Q(s)} = K\frac{(s-z_1)\ldots(s-z_m)}{(s-p_1)\ldots(s-p_n)}$$

- Zeros (z_i): roots of the equation $P(s) = 0$.
- Poles (p_i): roots of the equation $Q(s) = 0$.
- K: gain constant

$$H(s) = \frac{3(s+2)}{(s+1)(s^2+2s+5)}$$



Created and edited by: **Nguyen Phuoc Bao Duy**
HCMC University of Technology

Stability:

- A system is stable when

$$\int_{-\infty}^{\infty} \left| h(\tau) \right| d\tau < \infty$$

Quiz 1: stable or unstable?

a. $h(t) = c$ (constant) *b.* $h(t) = \cos(\omega_0 t + \varphi)$

Quiz 2: stable or unstable?

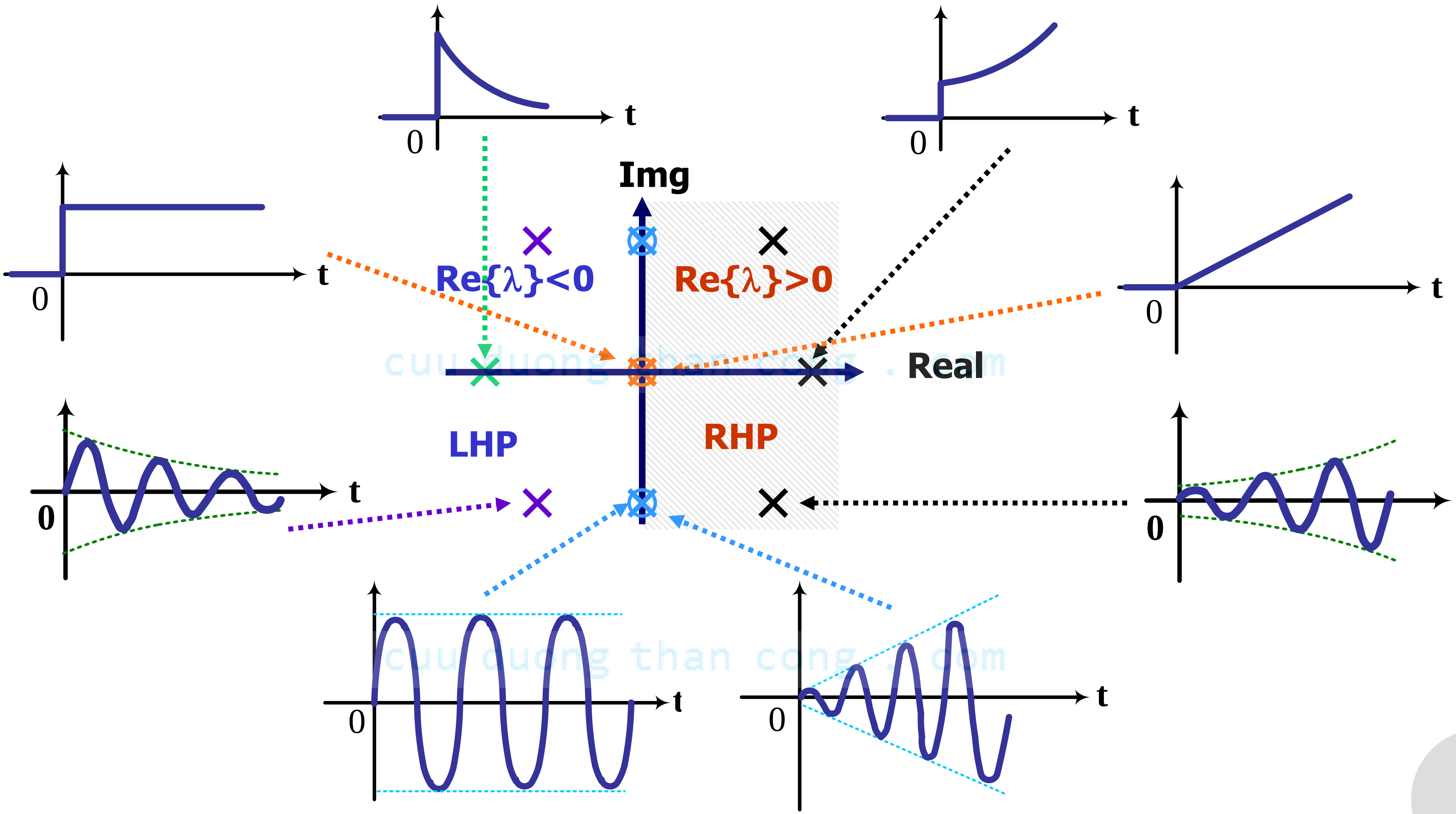
$$a.\ H(s) = \frac{s}{(s+1)(s^2+2s+5)}$$

$$b.\ H(s) = \frac{s^2-1}{s(s+1)^2}$$

$$c.\ H(s) = \frac{2s+1}{s^2(s^2+4s+5)}$$

$$d.\ H(s) = \frac{s^2+s+1}{(s+1)(s+2)(s-3)}$$

Stability:



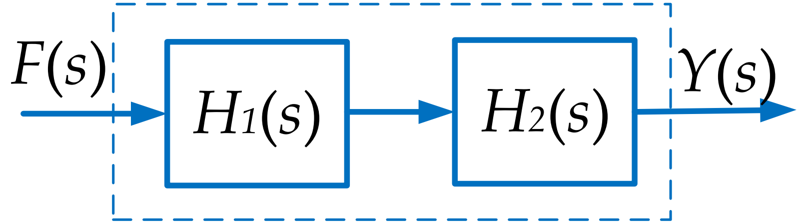
Created and edited by: **Nguyen Phuoc Bao Duy**
HCMC University of Technology

Chapter 5:
System analysis using transfer function

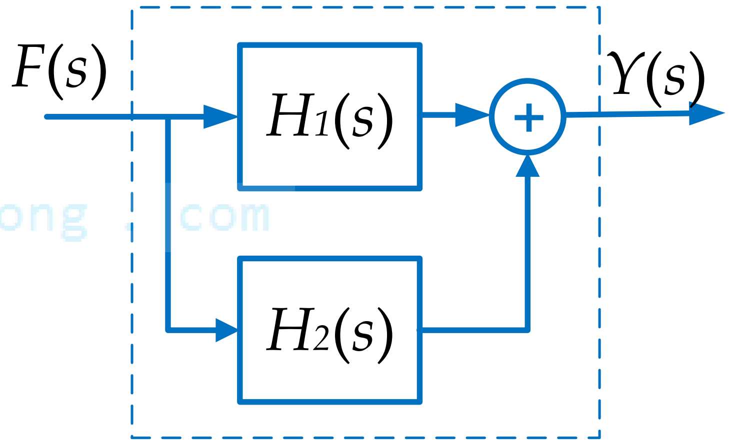
Created and edited by: **Nguyen Phuoc Bao Duy**
HCMC University of Technology

Block diagram:

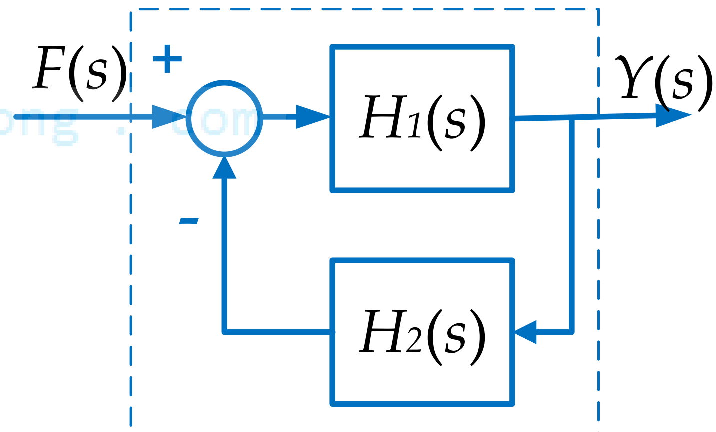
- In series: $H(s) = H_1(s).H_2(s)$



- In parallel: $H(s) = H_1(s) + H_2(s)$



- Feedback: $H(s) = \dfrac{H_1(s)}{1 + H_1(s)H_2(s)}$



Created and edited by: **Nguyen Phuoc Bao Duy**
HCMC University of Technology

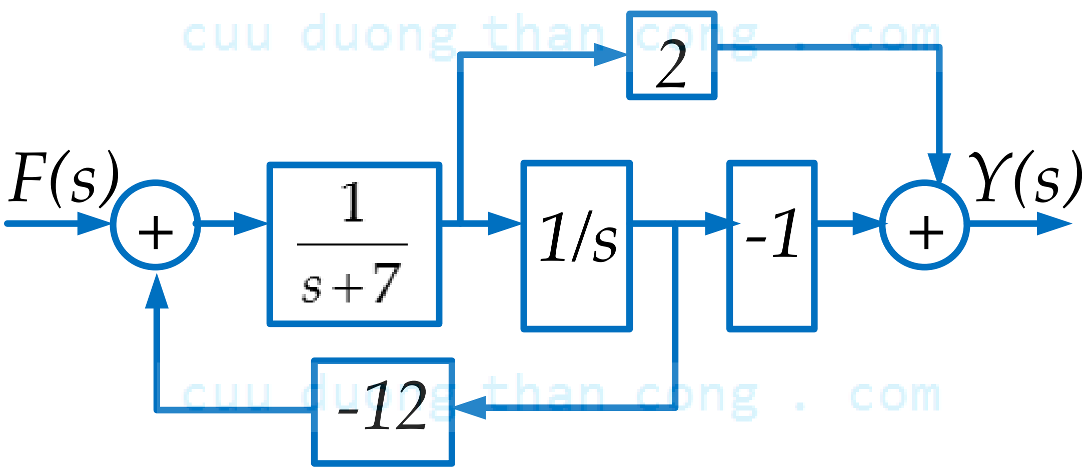
Block diagram:

- *Example 5.02:*

a. Find transfer function $H(s)$

b. Is that system stable or unstable?

c. Find the output $y(t)$ with the input $f(t) = e^{-2t}u(t)$



Chapter 5:
System analysis using transfer function

Created and edited by: **Nguyen Phuoc Bao Duy**
HCMC University of Technology

System described by differential equation:

- Systems that are described by a differential equation:

$$a_n y^{(n)}(t)+\ldots+a_1 y'(t)+a_0 y(t) = b_m f^{(m)}(t)+\ldots+b_1 f'(t)+b_0 f(t)$$

- In this course, we assume $m \le n$.
- Differential operator $D = d/dt$:

$$(D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0)y(t) = (b_m D^m + b_{m-1}D^{m-1} + \ldots + b_1 D + b_0)f(t)$$

$$\Leftrightarrow Q(D)y(t) = P(D)f(t)$$

- Laplace transform (assume that all initial conditions are zero):

$$D^n y(t) \leftrightarrow s^n Y(s)$$

$$D^m f(t) \leftrightarrow s^m F(s)$$

Transfer function and impulse response:

- Laplace transform of system equation:

$$Q(D)y(t) = P(D)f(t)$$
$$\leftrightarrow Q(s)Y(s) = P(s)F(s)$$

- Transfer function:

$$H(s) = \frac{Y(s)}{F(s)} = \frac{P(s)}{Q(s)}$$

- Impulse response:

$$h(t) = \mathcal{L}^{-1}\{H(s)\}$$

- *Example 5.03:* Find the impulse response of the system

$$(D^3 + 7D^2 + 19D + 13)y(t) = (D - 1)f(t)$$

Chapter 5:
System analysis using transfer function

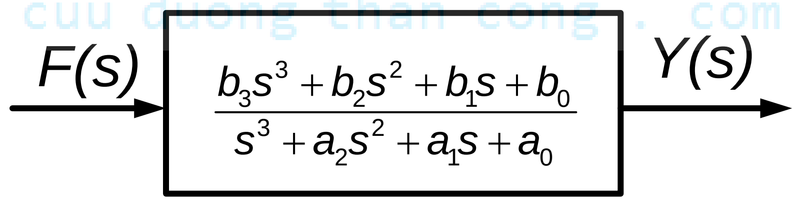
1. *Transfer function*
2. *Block diagram*
3. *System described by differential equation*
4. ***System realization***
5. *Feedback and Control systems*

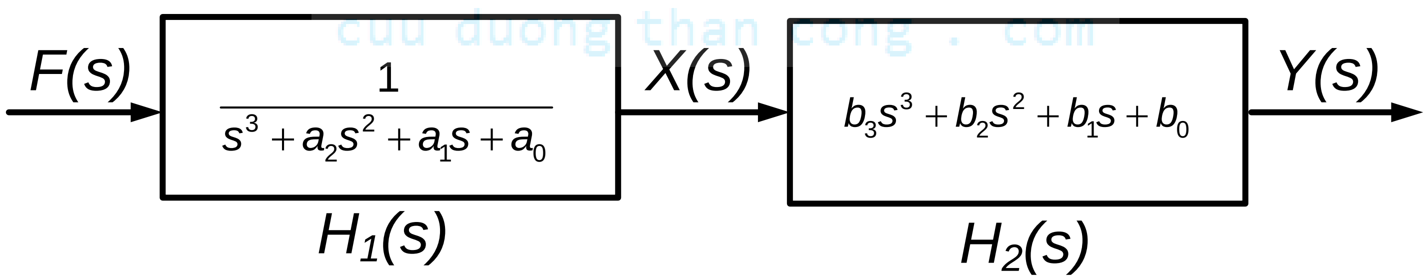
Created and edited by: **Nguyen Phuoc Bao Duy**
HCMC University of Technology

Canonical or Direct-Form Realization:

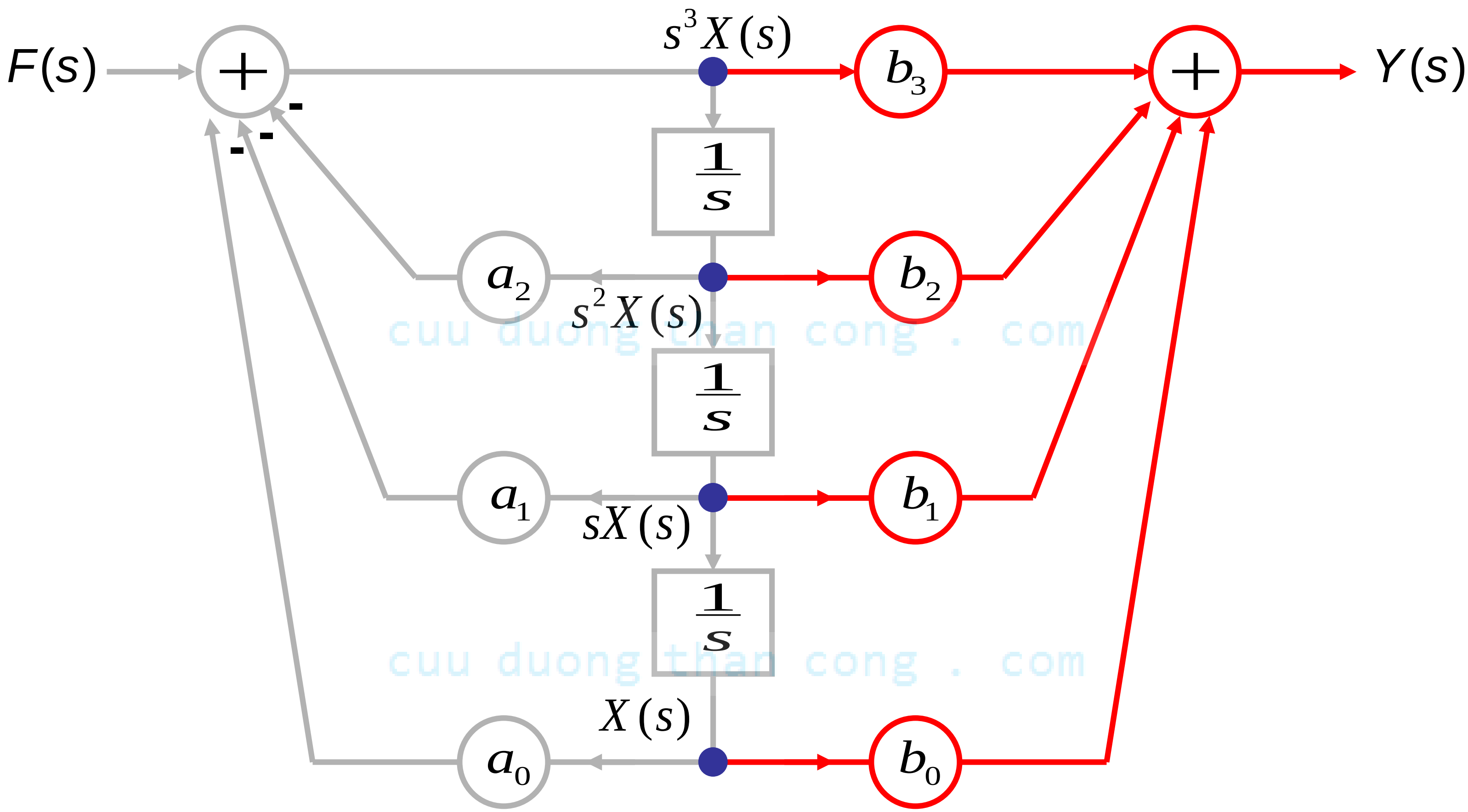
- For example, we consider a third-order sytem:

$$H(s)=\frac{b_3s^3+b_2s^2+b_1s+b_0}{s^3+a_2s^2+a_1s+a_0}$$





Canonical or Direct-Form Realization:



Created and edited by: **Nguyen Phuoc Bao Duy**
HCMC University of Technology

Canonical or Direct-Form Realization:

- *Example 5.04:* Find the canonical realization of the following transfer functions:

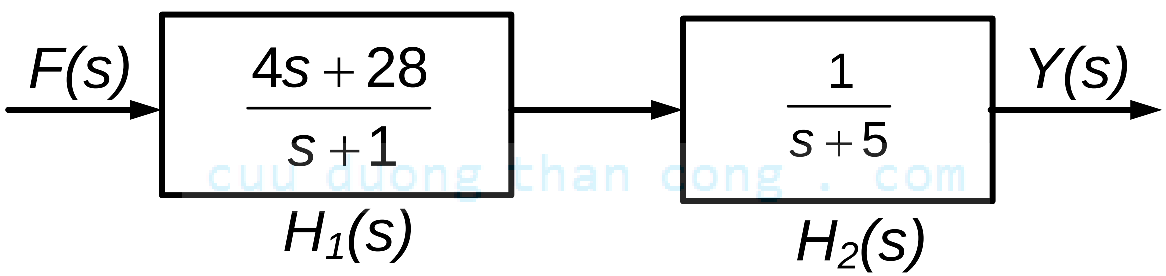
$$a.\ \frac{5}{s+2} \qquad b.\ \frac{s+5}{s+7}$$

$$c.\ \frac{s}{s+7} \qquad d.\ \frac{4s+28}{s^2+6s+5}$$

Cascade realization:

- An nth-order transfer function $H(s)$ can be expressed as a **product** of n first-order transfer functions.

$$H(s) = \frac{4s+28}{s^2+6s+5} = \underbrace{\left(\frac{4s+28}{s+1}\right)}_{H_1(s)}\underbrace{\left(\frac{1}{s+5}\right)}_{H_2(s)}$$

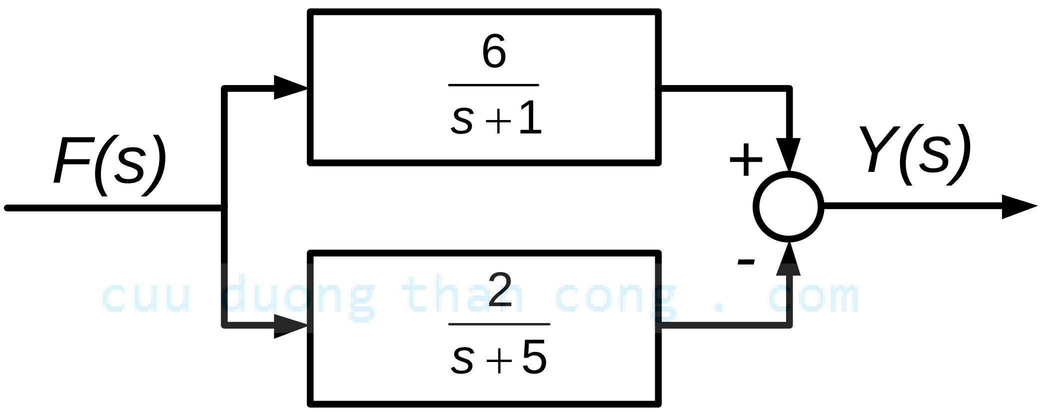


Created and edited by: **Nguyen Phuoc Bao Duy**
HCMC University of Technology

Parallel realization:

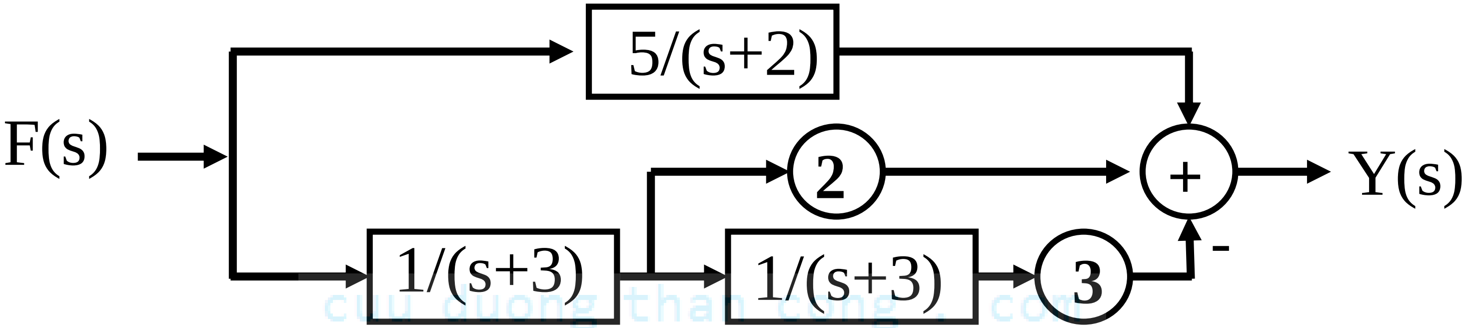
- An nth-order transfer function $H(s)$ can be expressed as a **sum** of n first-order transfer functions.

$$H(s) = \frac{4s+28}{s^2+6s+5} = \underbrace{\frac{6}{s+1}}_{H_3(s)} - \underbrace{\frac{2}{s+5}}_{H_4(s)}$$

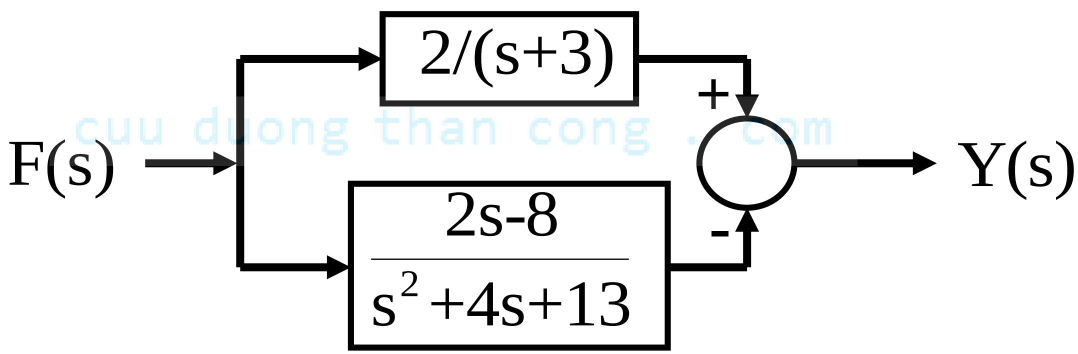


Some special cases:

$$H(s)=\frac{7s^2+37s+51}{(s+2)(s+3)^2}=\frac{5}{s+2}+\frac{2}{s+3}-\frac{3}{(s+3)^2}$$



$$H(s)=\frac{10s+50}{(s+3)(s^2+4s+13)}=\frac{2}{s+3}-\frac{2s-8}{s^2+4s+13}$$



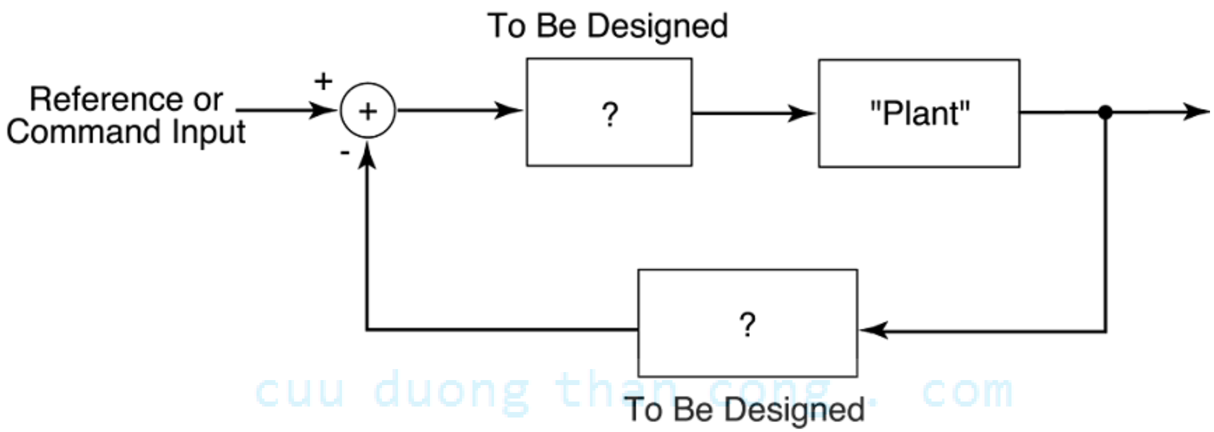
Chapter 5:
System analysis using transfer function

1. *Transfer function*
2. *Block diagram*
3. *System described by differential equation*
4. *System realization*
5. ***Feedback and Control systems***

cuu duong than cong . com

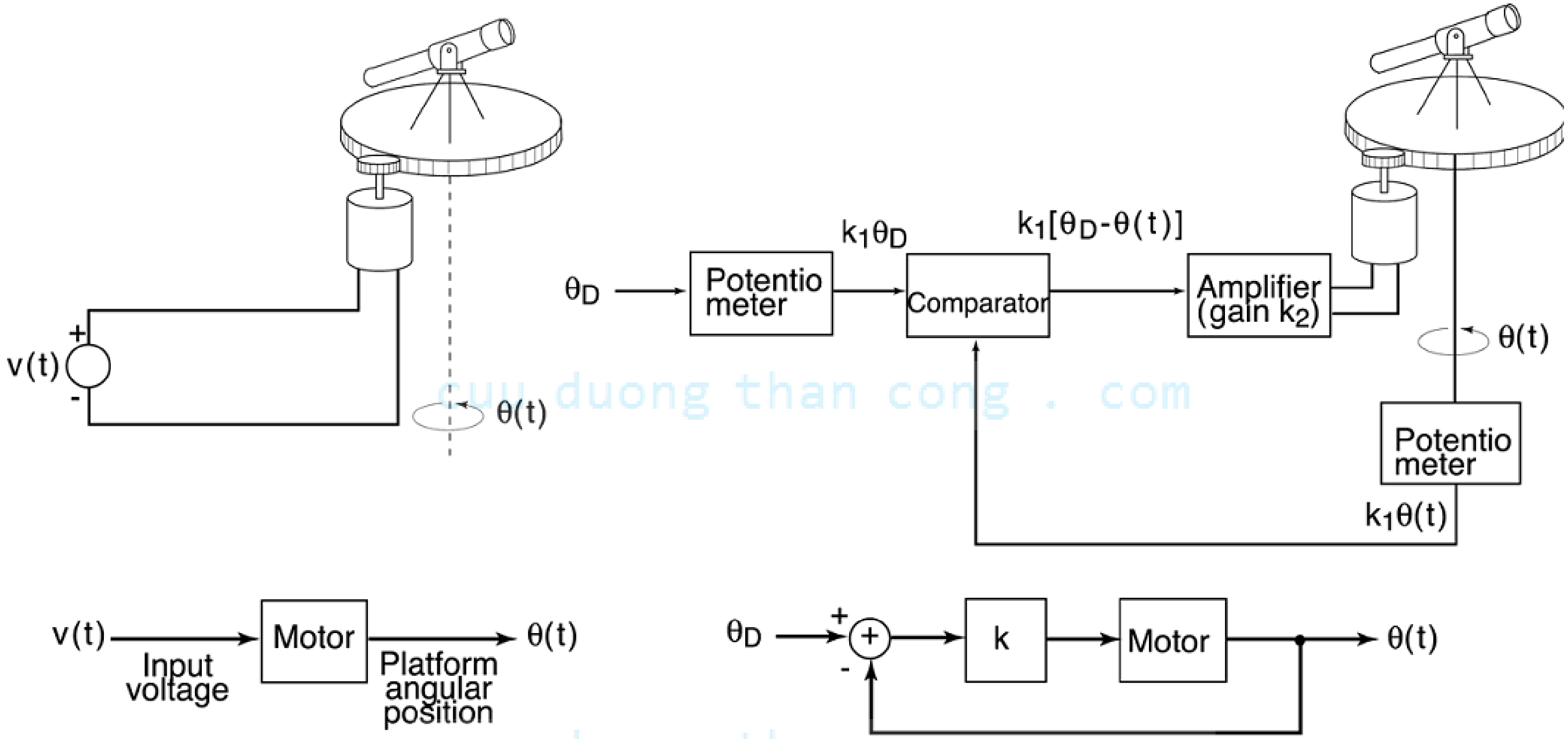
Created and edited by: **Nguyen Phuoc Bao Duy**
HCMC University of Technology

A typical feedback system:



- Why use Feedback?
    - Reducing Effects of Nonidealities and Disturbances
    - Reducing Sensitivity to Uncertainties and Variability
    - Stabilizing Unstable Systems
    - Tracking

A typical feedback system:

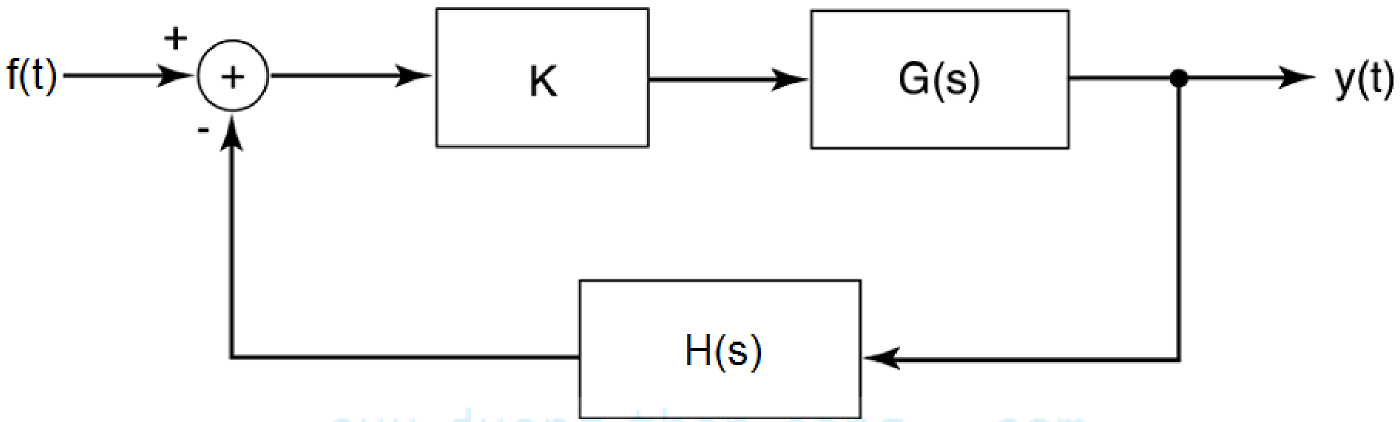


Open-Loop System

Closed-Loop Feedback System

Created and edited by: **Nguyen Phuoc Bao Duy**
HCMC University of Technology

The Use of Feedback to Compensate for Nonidealities



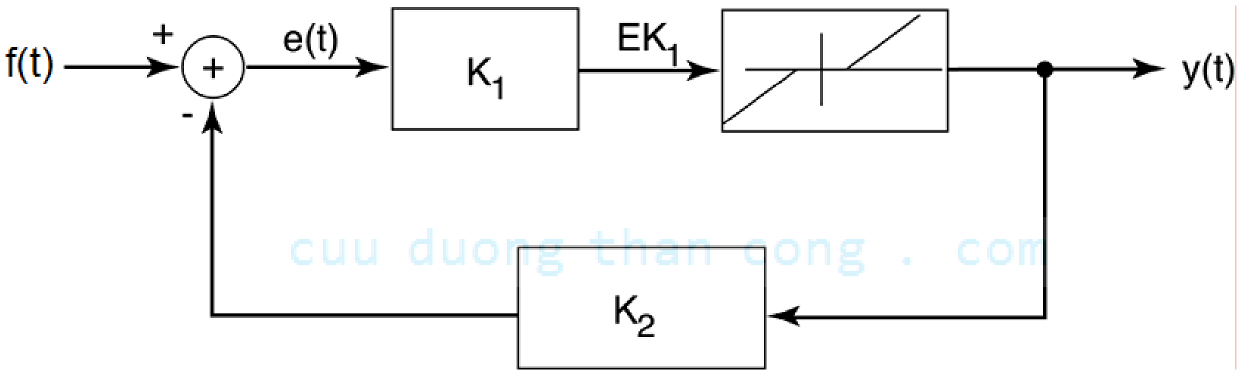
➢ If $|KG(s)H(s)| >> 1$:

$$T(s) = \frac{Y(s)}{F(s)} = \frac{KG(s)}{1 + KG(s)H(s)} \approx \frac{1}{H(s)}$$

Independent of $G(s)$!!

Compensation for Nonlinearities

- Amplifier with a Dead-zone: cause distortion



➢ If the loop gain (K_1, K_2) is large enough, the input-output response ≈ $1/K_2$.

Reduced Sensitivity

- Suppose $KG(s) = 1000, \;\; H(s) = 0.099$

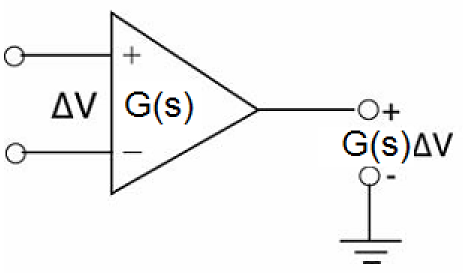
$$T(s) = \frac{1000}{1 + (1000)(0.099)} = 10$$

- Suppose $KG(s) = 500, \;\; H(s) = 0.099$ (50% gain change)

$$T(s) = \frac{500}{1 + (500)(0.099)} \approx 9.9 \quad (1\% \; gain \; change)$$

Improving the Dynamics of Systems

- Operational Amplifier 741:

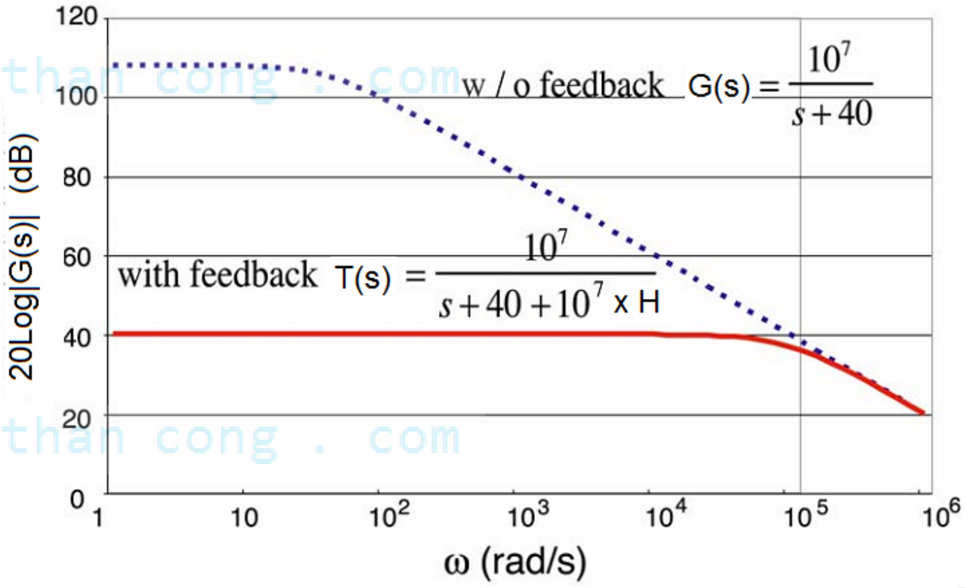


$$G(s) = \frac{10^7}{s+40}$$

- With feedback:

$$T(s) = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{10^7}{s+40+10^7 H(s)}$$



Much broader bandwidth!

Created and edited by: **Nguyen Phuoc Bao Duy**
HCMC University of Technology

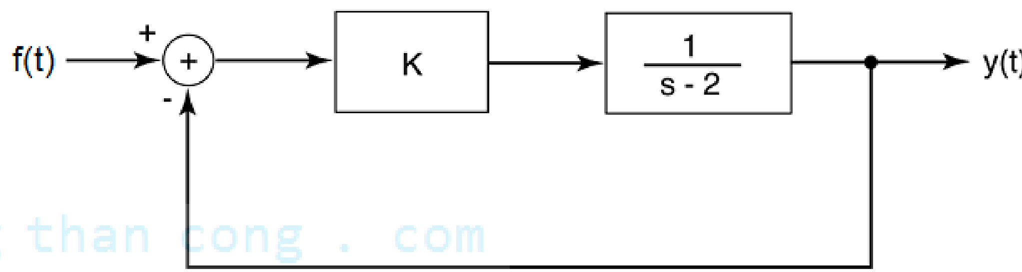
Stabilization of Unstable Systems

- If $G(s)$ is unstable, design K, $H(s)$ so that the closed-loop system is stable

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

- **Poles** of $T(s)$ = **roots** of $1 + KG(s)H(s)$ in LHP

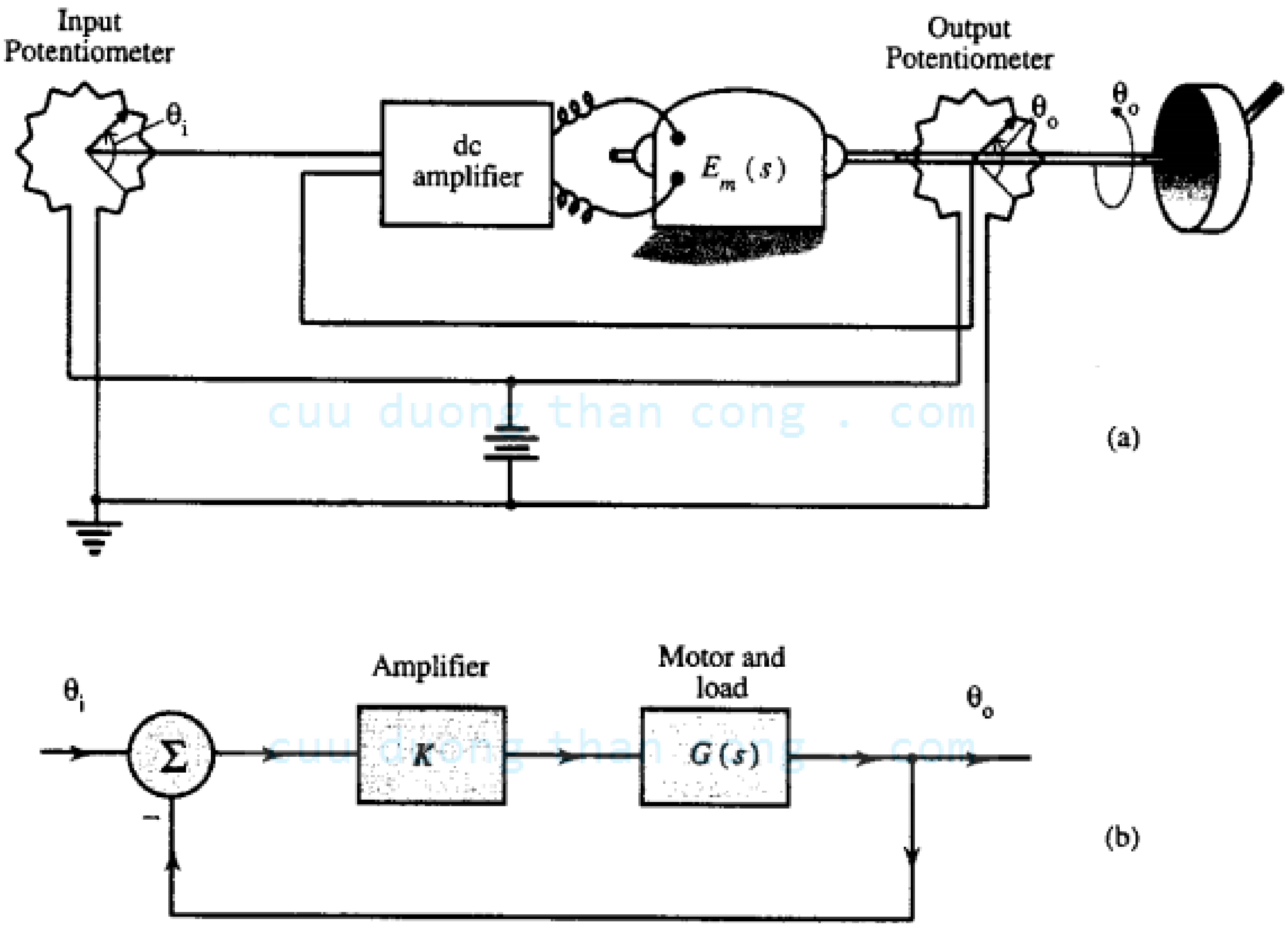
Example:



$$G(s) = \frac{1}{s-2}: \text{ unstable}$$

$$T(s) = \frac{K}{s-2+K}: \text{ stable as long as K} > 2$$

Analysis of a Simple Control System



Analysis of a Simple Control System

- The motor transfer function: $G(s) = \dfrac{1}{s(s+8)}$

$$\Rightarrow T(s) = \frac{K}{s^2 + 8s + K} \Rightarrow \theta_o(s) = \frac{K}{s^2 + 8s + K}\theta_i(s)$$

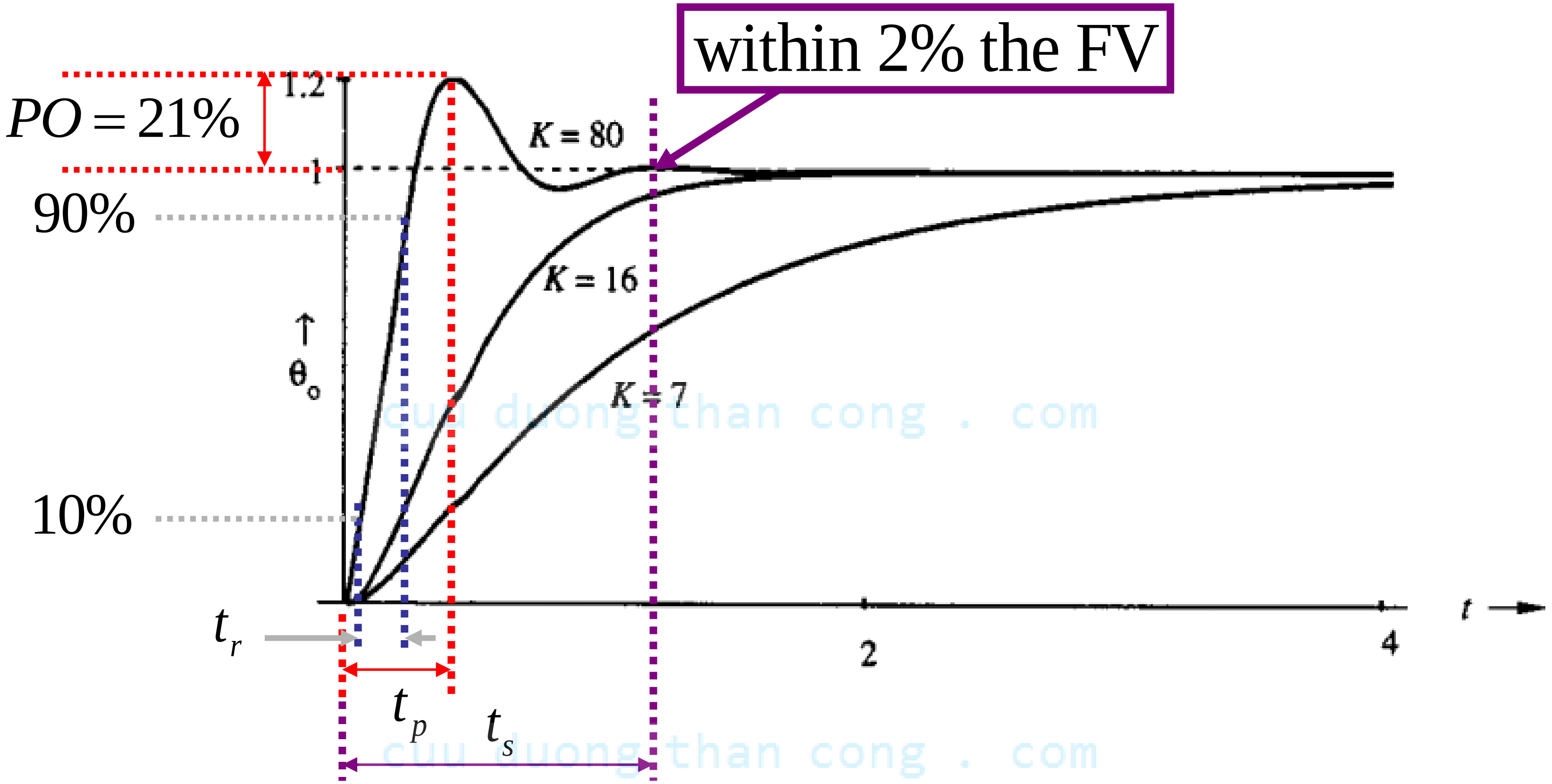
- Step input: $\theta_i(t) = u(t) \;\; \Rightarrow \theta_o(s) = \dfrac{K}{s\left(s^2 + 8s + K\right)}$

$K = 7 \quad \Rightarrow \theta_o(t) = \left(1 - \dfrac{7}{6}e^{-t} + \dfrac{1}{6}e^{-7t}\right)u(t)$ overdamped

$K = 16 \;\; \Rightarrow \theta_o(t) = \left[1 - (4t+1)e^{-4t}\right]u(t)$ critically damped

$K = 80 \;\; \Rightarrow \theta_o(t) = \left[1 - \dfrac{\sqrt{5}}{2}e^{-4t}\cos(8t - 0.4636)\right]u(t)$ underdamped

Analysis of a Simple Control System



PO: percent overshoot

t_r: rise time

t_p: peak time

t_s: settling time

Design Specifications:

1. Transient Response:
   - Specified overshoot to step input
   - Specified rise time t_r and/or delay time t_d
   - Specified settling time t_s

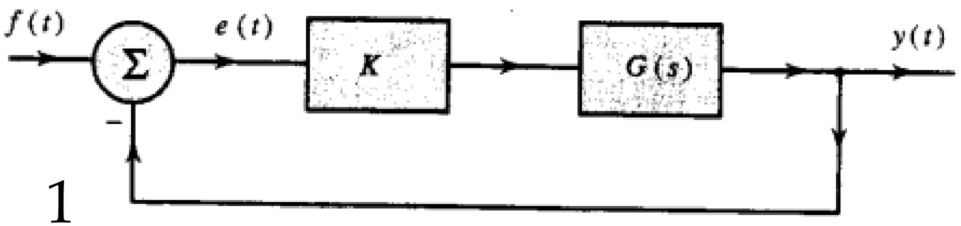
2. Steady-State Error
   - Specified steady-state error to certain expected inputs such as step, ramp, or parabolic inputs.
   - Notes: $e(t) = f(t) - y(t) \Rightarrow E(s) = F(s) - Y(s) = F(s)[1 - T(s)]$

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0}\left[sE(s)\right]$$

Steady-State Error:

For the system with unity feedback: $T(s)=\dfrac{KG(s)}{1+KG(s)H(s)}$

$$E(s)=\frac{1}{1+KG(s)}F(s)$$



a. Step input $f(t) = u(t)$:

$$e_s=\lim_{s\to 0}\left[s\frac{1}{1+KG(s)}\frac{1}{s}\right]=\frac{1}{1+K_p};$$

$$K_p=\lim_{s\to 0}\left[KG(s)\right]: \text{positional error constant}$$

b. Ramp input $f(t) = tu(t)$:

$$e_r=\lim_{s\to 0}\left[s\frac{1}{1+KG(s)}\frac{1}{s^2}\right]=\frac{1}{K_v};$$

$$K_v=\lim_{s\to 0}\left[sKG(s)\right]: \text{velocity error constant}$$

Created and edited by: **Nguyen Phuoc Bao Duy**
HCMC University of Technology

Steady-State Error:

c. Parabolic input $f(t) = t^2u(t)$:

$$e_s = \lim_{s\to 0}\left[s\frac{1}{1+KG(s)}\frac{1}{s^3}\right] = \frac{1}{K_a};$$

$$K_a = \lim_{s\to 0}\left[s^2 KG(s)\right]: \text{acceleration error constant}$$

For example, if

$$G(s) = \frac{1}{s(s+8)} \Rightarrow \begin{cases} K_p = \infty, & e_s = 0 \\ K_v = \dfrac{K}{8}, & e_r = \dfrac{8}{K} \\ K_a = 0, & e_p = \infty \end{cases}$$

Such a system can track position of an object with zero error; in tracking an object moving with constant velocity, it yields a constant error; it can not track a constant acceleration object.

Analysis of a Second-Order System

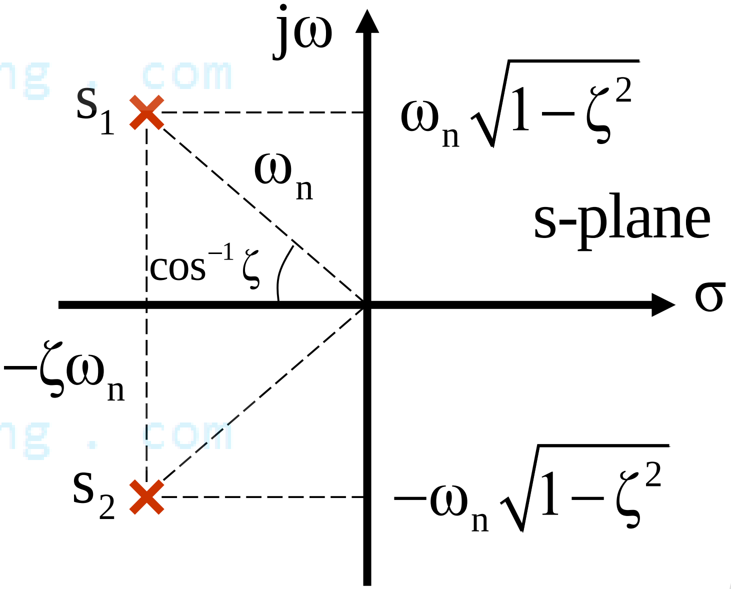
Let's consider a second-order transfer fucntion:

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}; \ \zeta - damping\ ratio$$

The poles of $T(s)$ are:

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

- $\zeta < 1$: underdamped
- $\zeta = 1$: critically damped
- $\zeta > 1$: overdamped



Created and edited by: **Nguyen Phuoc Bao Duy**
HCMC University of Technology

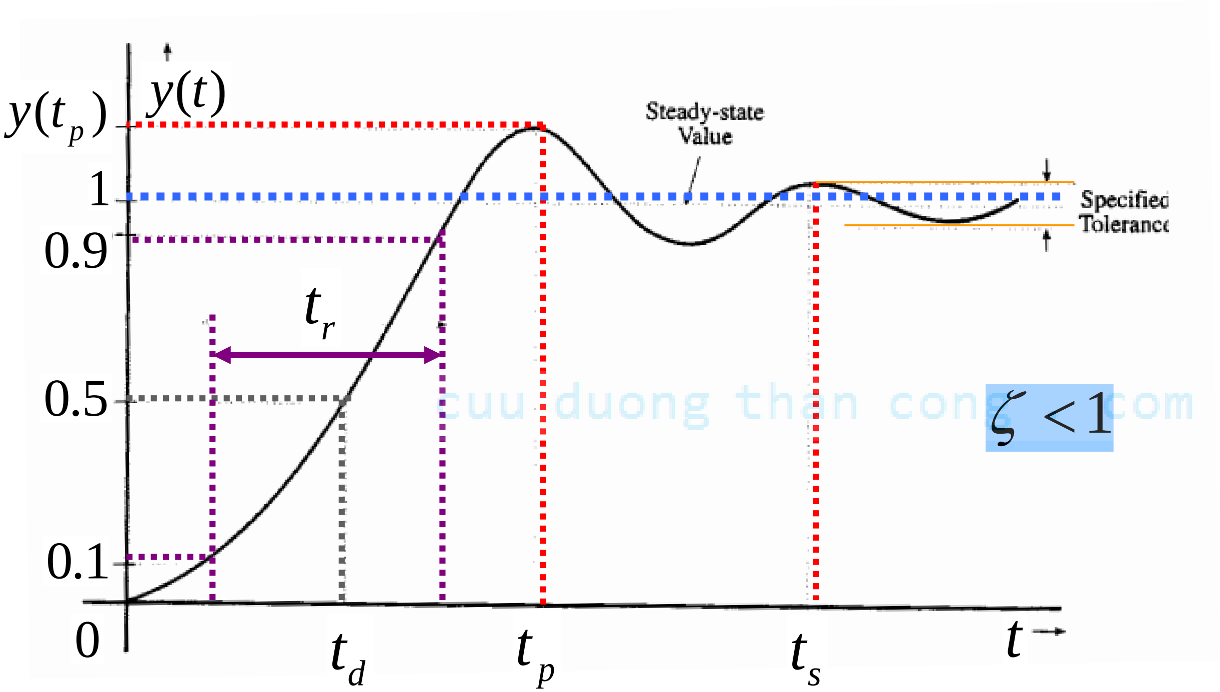
Analysis of a Second-Order System

➢ For the step input $F(s) = 1/s$, the response for the underdamp case ($\zeta < 1$) is:

$$Y(s) = \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\Rightarrow y(t) = \left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n t\sqrt{1-\zeta^2} + \cos^{-1}\zeta\right)\right] u(t)$$

Analysis of a Second-Order System



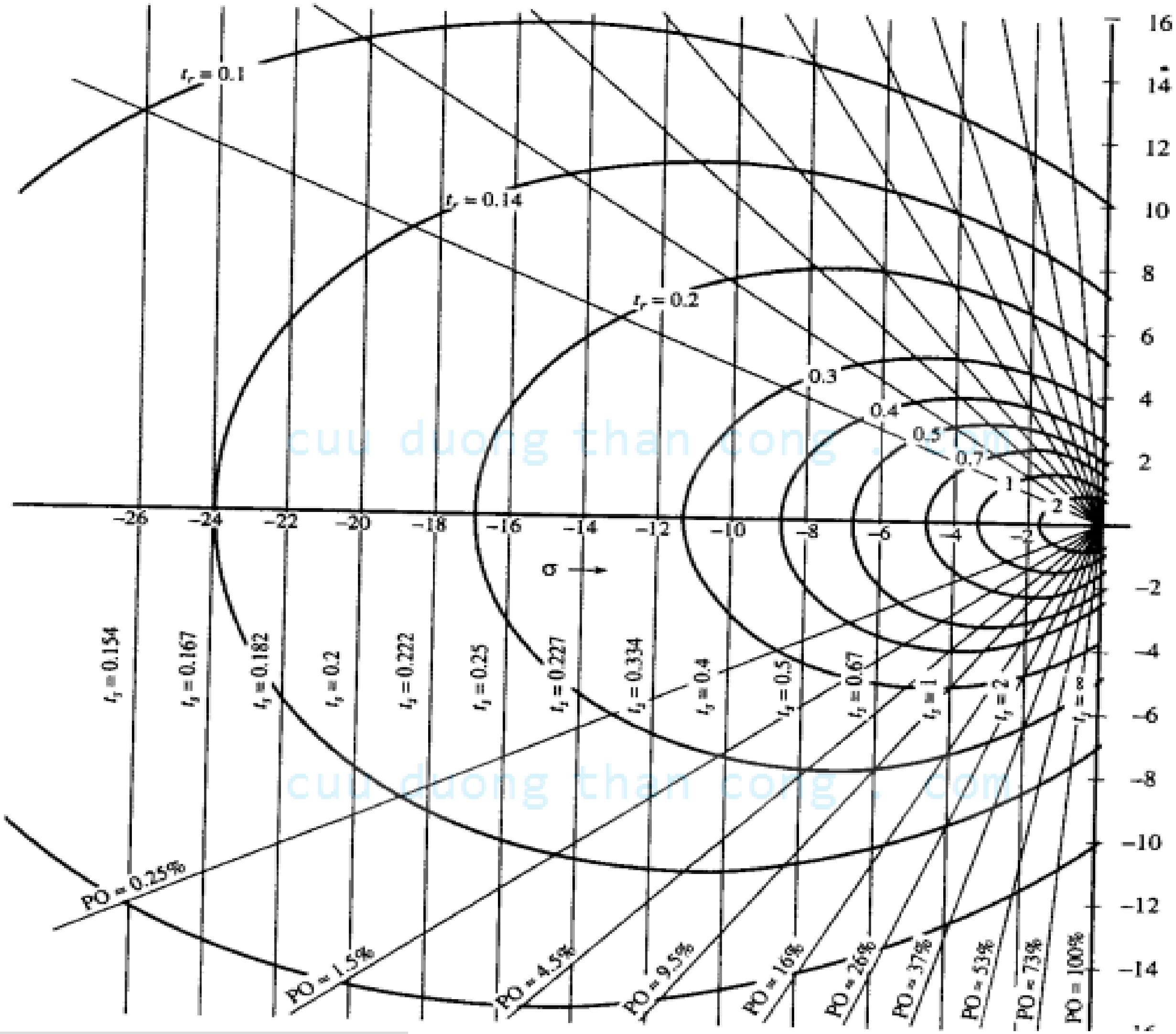
$$t_s = \frac{4}{\zeta\omega_n}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$PO = 100e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

$$t_r \approx \frac{1-0.4167\zeta+2.917\zeta^2}{\omega_n}; \quad t_d \approx \frac{1.1+0.125\zeta+0.469\zeta^2}{\omega_n}$$

Analysis of a Second-Order System



Analysis of a Second-Order System

- For example, let the transient specifications for the system

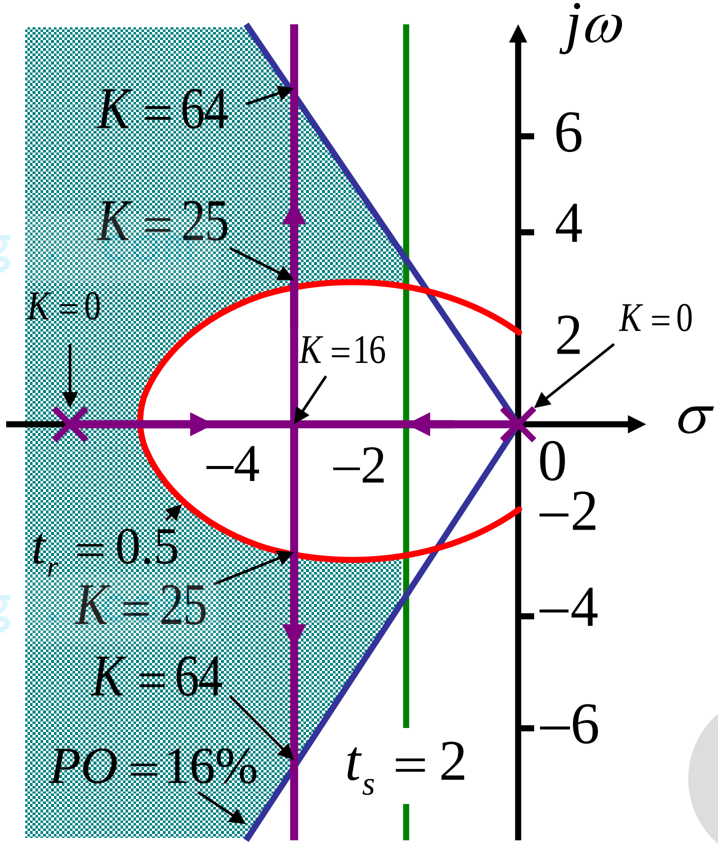
$$T(s) = \frac{K}{s^2 + 8s + K}$$

is $PO < 16\%$, $t_r < 0.5$s, $t_s < 2$s.

- Poles:

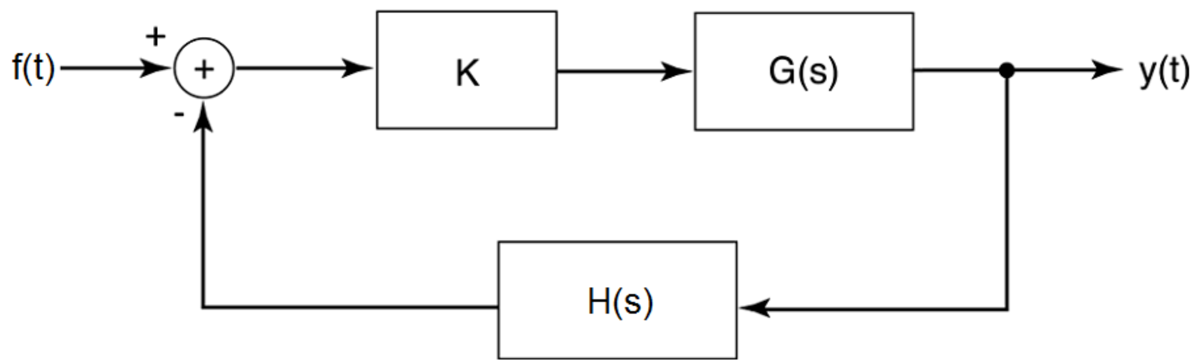
$$s_{1,2} = -4 \pm \sqrt{16 - K}$$

- System will meet the given specifications for $25 \le K \le 64$.



Created and edited by: **Nguyen Phuoc Bao Duy**
HCMC University of Technology

Root locus



$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

- We shall consider the paths of the roots of $1 + KG(s)H(s) = 0$ as K varies from 0 to ∞.

Rules for Plotting Root Locus

$$G(s)H(s) = -\frac{1}{K}$$

- End points:
  - At $K = 0$, $G(s_0)H(s_0) = \infty \Rightarrow s_0$ are *poles* of $G(s)H(s)$.
  - At $K = \infty$, $G(s_0)H(s_0) = 0 \Rightarrow s_0$ are *zeros* of $G(s)H(s)$.

- **Rule #1:**

A root locus starts (at K = 0) from a pole of $G(s)H(s)$ and ends (at $K = \infty$) at a zero of $G(s)H(s)$ or at $\pm\infty$.

Notes: in s-plane, the poles are denoted as ×, the zeros are denote as o.

If $G(s)H(s)$ has n poles p_i and m zeros z_i, m loci terminate on the zeros, the other $n - m$ loci terminate at ∞.

Rules for Plotting Root Locus

➢ Rule #2:

A real axis segment is a part of the root locus if the sum of the poles and zeros of $G(s)H(s)$ that lie to the right of the segment is odd.

The root locus are symmetric about real axis.

➢ Rule #3:

The centroid of the asymptotes of $(n - m)$ loci that terminate at ∞ is:

$$s = \sigma = \frac{\sum_{n} p_i - \sum_{m} z_i}{n - m}$$

at the angles:

$$\phi = \frac{(2k+1)\pi}{n-m}$$

Rules for Plotting Root Locus

➢**Rule #4:**

Real axis break-in and break away points:

For each $s = \sigma$ on a real-axis segment of the root locus:

$$KG(\sigma)H(\sigma) = -1 \Rightarrow K = -\frac{1}{G(\sigma)H(\sigma)}$$

Real axis break-in & breakaway points are the real values of σ for which:

$$\frac{dK(\sigma)}{d\sigma} = 0$$

Example: Sketch the root locus for a system with:

$$KG(s)H(s) = \frac{K}{s(s+2)(s+4)}$$

Rules for Plotting Root Locus