Chapter 7:

Frequency response of LTI system and Analog filter design

1. ***Frequency Response – Bode Plots***
2. *Analog filter design*
   - *Butterworth Filters*
   - *Chebyshev Filters*
   - *Frequency Transformation*

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Frequency Response of an LITC System:

- The input-output pair of an LTIC system:

$$f(t) = e^{st} \Rightarrow y(t) = H(s)e^{st}$$

- Is the system is stable, the ROC include the imaginary axis $s = j\omega$:

$$f(t) = e^{j\omega t} \Rightarrow y(t) = H(j\omega)e^{j\omega t}$$

$$f(t) = e^{-j\omega t} \Rightarrow y(t) = H(-j\omega)e^{-j\omega t}$$

$$f(t) = \cos(\omega \mathrm{t}) \Rightarrow y(t) = \frac{1}{2}\left[H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t}\right]$$

$$= \mathrm{Re}\left[H(j\omega)e^{j\omega t}\right]$$

$$= \left|H(j\omega)\right|\cos\left[\omega t + \angle H(j\omega)\right]$$

Frequency Response of an LITC System:

$$f(t)=\cos(\omega t)\Rightarrow y(t)=\left|H(j\omega)\right|\cos\left[\omega t+\angle H(j\omega)\right]$$

- Sinusoidal input of frequency $\omega \Rightarrow$ sinusoidal output of the same frequency ω.
- The amplitude of output sinusoidal is $|H(j\omega)|$ times the input amplitude, $|H(j\omega)|$ is the **amplitude response**.
- The phase of the output sinusoidal is shifted by $\angle H(j\omega)$ with respect to the input phase, $\angle H(j\omega)$ is the **phase response**.
- **Plots of $|H(j\omega)|$ and $\angle H(j\omega)$** versus ω are called **frequency response** of the system (Bode plots).

Bode plot:

- Let's consider a system with the transfer function:

$$H(s) = \frac{K(s+a_1)(s+a_2)}{s(s+b_1)(s^2+b_2s+b_3)}$$

where $(s^2 + b_2 s + b_3)$ is assumed to have complex conjugate roots.

$$H(s) = \frac{Ka_1a_2}{b_1b_3}\frac{\left(\frac{s}{a_1}+1\right)\left(\frac{s}{a_2}+1\right)}{s\left(\frac{s}{b_1}+1\right)\left(\frac{s^2}{b_3}+\frac{b_2}{b_3}s+1\right)}$$

Bode plot:

$$H(j\omega)=\frac{Ka_1a_2}{b_1b_3}\frac{\left(\frac{j\omega}{a_1}+1\right)\left(\frac{j\omega}{a_2}+1\right)}{j\omega\left(\frac{j\omega}{b_1}+1\right)\left(\frac{(j\omega)^2}{b_3}+\frac{b_2}{b_3}j\omega+1\right)}$$

$$\Rightarrow\begin{cases}\left|H(j\omega)\right|=\dfrac{Ka_1a_2}{b_1b_3}\dfrac{\left|\frac{j\omega}{a_1}+1\right|\left|\frac{j\omega}{a_2}+1\right|}{\left|j\omega\right|\left|\frac{j\omega}{b_1}+1\right|\left|\frac{(j\omega)^2}{b_3}+\frac{b_2}{b_3}j\omega+1\right|}\\ \angle H(j\omega)=\angle\left(\frac{j\omega}{a_1}+1\right)+\angle\left(\frac{j\omega}{a_2}+1\right)-\angle j\omega-\\ \qquad\angle\left(\frac{j\omega}{b_1}+1\right)-\angle\left(\frac{(j\omega)^2}{b_3}+\frac{b_2}{b_3}j\omega+1\right)\end{cases}$$

Bode plot:

- The **phase** function consists of the addition of only three kinds of terms:
  - ✓ The phase of jω (= 90°)
  - ✓ The phase of the first-order term 1 + $j\omega/a$
  - ✓ The phase of the second order term $\left[\frac{(j\omega)^2}{b_3}+\frac{b_2}{b_3}j\omega+1\right]$
- The **log amplitude** is

$$20\log\left|H(j\omega)\right| = 20\log\frac{Ka_1a_2}{b_1b_3} + 20\log\left|\frac{j\omega}{a_1}+1\right| + 20\log\left|\frac{j\omega}{a_2}+1\right| - 20\log\left|j\omega\right|$$

$$-20\log\left|\frac{j\omega}{b_1}+1\right| - 20\log\left|\frac{(j\omega)^2}{b_3}+\frac{b_2}{b_3}j\omega+1\right|$$

Bode plot:

- The log amplitude is a sum of four basis terms:
  - ✓ A constant
  - ✓ A pole or zero at the origin ($20\log|j\omega|$)
  - ✓ A first-order pole or zero
  - ✓ Complex conjugate poles or zeros
  - ✓ The logarithmic unit is the decibel (dB)

1. Constant Ka_1a_2/b_1b_3

- $20\log\dfrac{Ka_1a_2}{b_1b_3}$ is a constant, phase is zero

Bode plot:

2. Pole or Zero at the origin

- Log Magnitude:

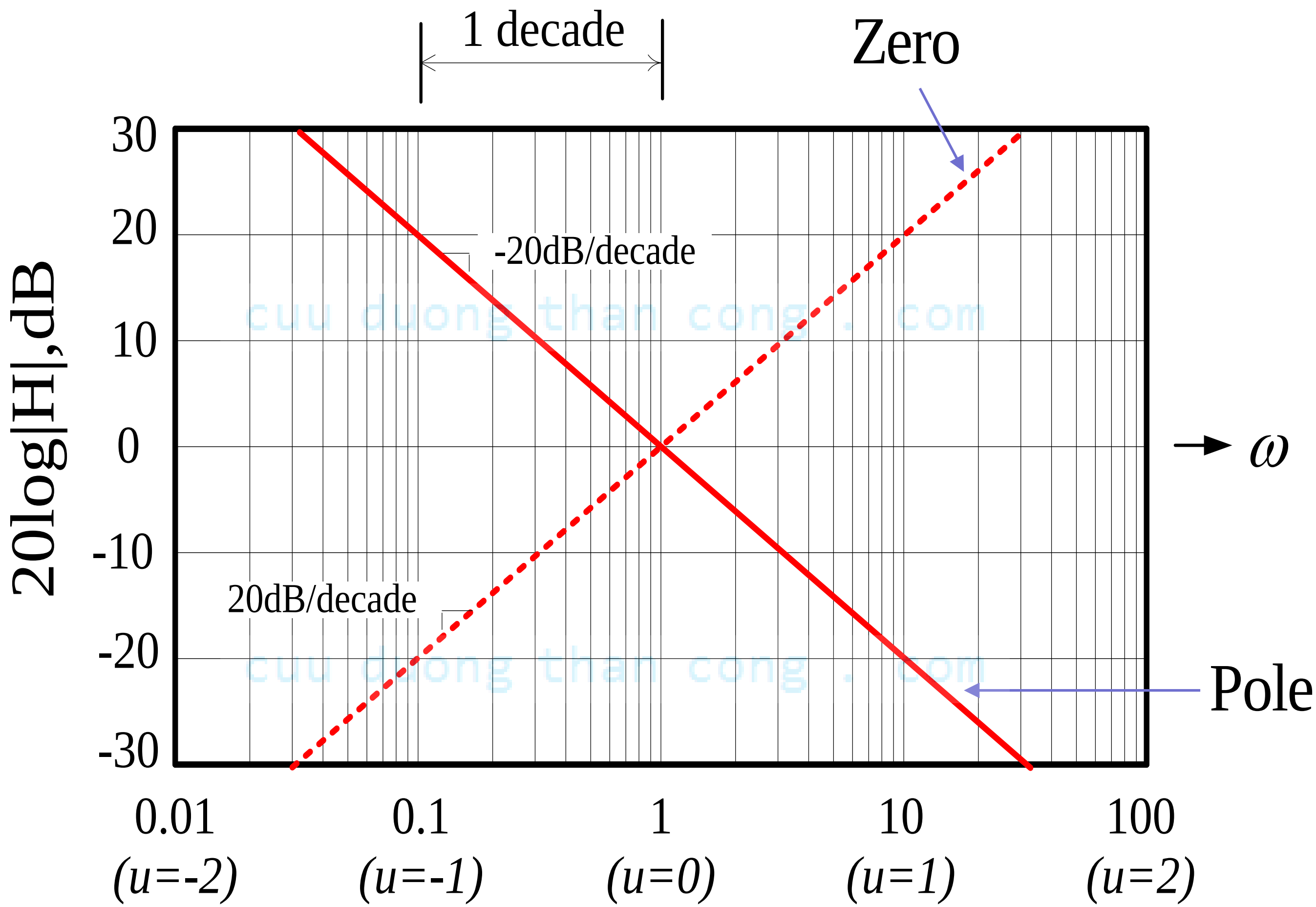
$$\begin{cases} u = \log \omega \\ Pole: -20\log|j\omega| = -20\log|\omega| = -20u \\ Zero: +20\log|j\omega| = +20\log|\omega| = +20u \end{cases}$$

- Phase:

$$\begin{cases} Pole: \quad -\angle j\omega = -90^o \\ Zero: \quad +\angle j\omega = +90^o \end{cases}$$

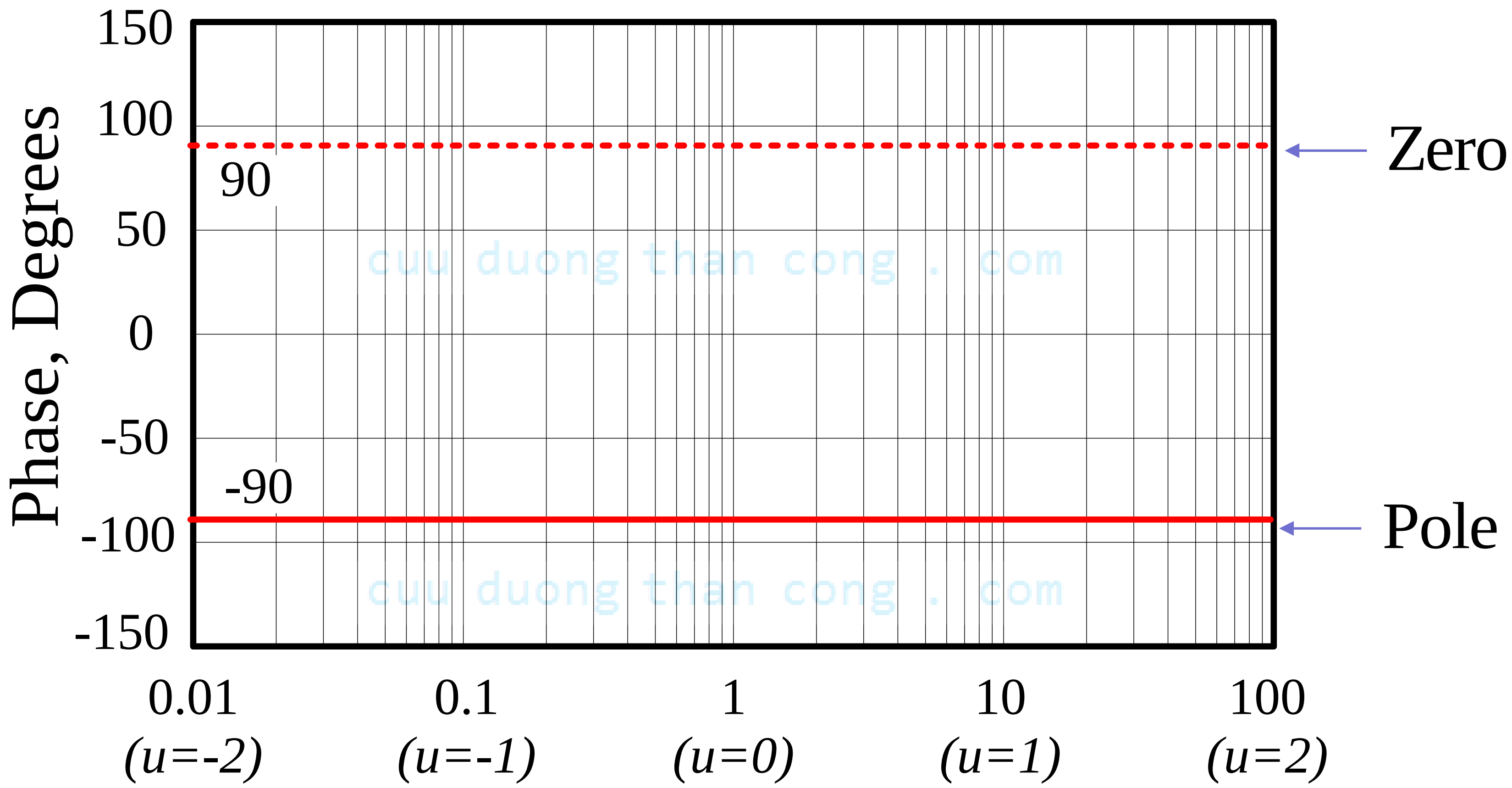
Bode plot:

2. Pole or Zero at the origin



Bode plot:

2. Pole or Zero at the origin



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Bode plot:

3. First order Pole or Zero

- Log Magnitude:

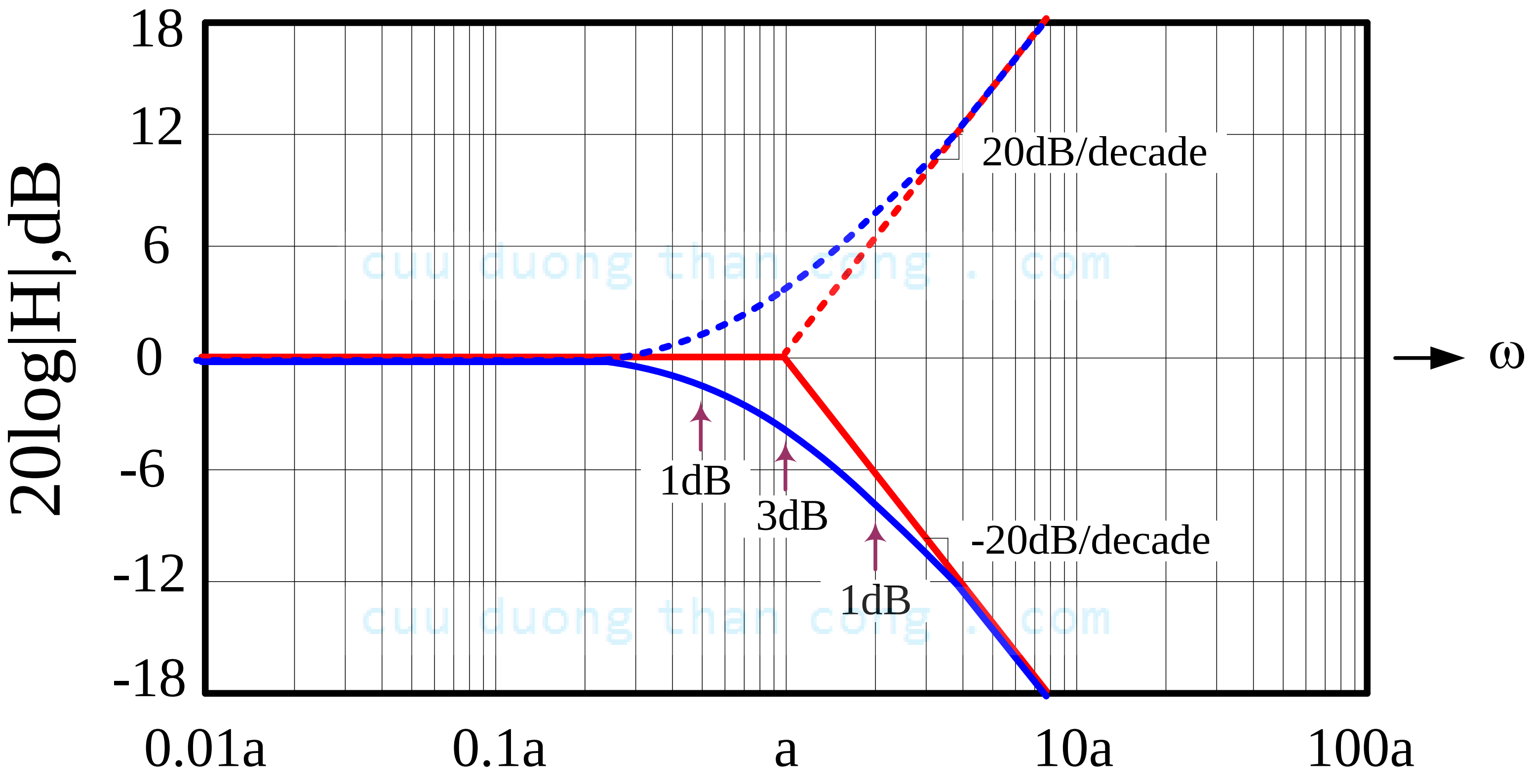
$$\begin{cases} Pole: -20\log\left|\frac{j\omega}{a}+1\right| \approx \begin{bmatrix} 0 & for\ \omega \ll a \\ -20u+20\log a & for\ \omega \gg a \end{bmatrix} \\ Zero: +20\log\left|\frac{j\omega}{a}+1\right| \approx \begin{bmatrix} 0 & for\ \omega \ll a \\ +20u-20\log a & for\ \omega \gg a \end{bmatrix} \end{cases}$$

- Phase:

$$\begin{cases} Pole: \quad -\angle\left(\frac{j\omega}{a}+1\right) = -\tan^{-1}\left(\frac{\omega}{a}\right) \approx \begin{bmatrix} 0 & for\ \omega \ll a \\ -90^o & for\ \omega \gg a \end{bmatrix} \\ Zero: \quad +\angle\left(\frac{j\omega}{a}+1\right) = +\tan^{-1}\left(\frac{\omega}{a}\right) \approx \begin{bmatrix} 0 & for\ \omega \ll a \\ +90^o & for\ \omega \gg a \end{bmatrix} \end{cases}$$

Bode plot:

3. First order Pole or Zero



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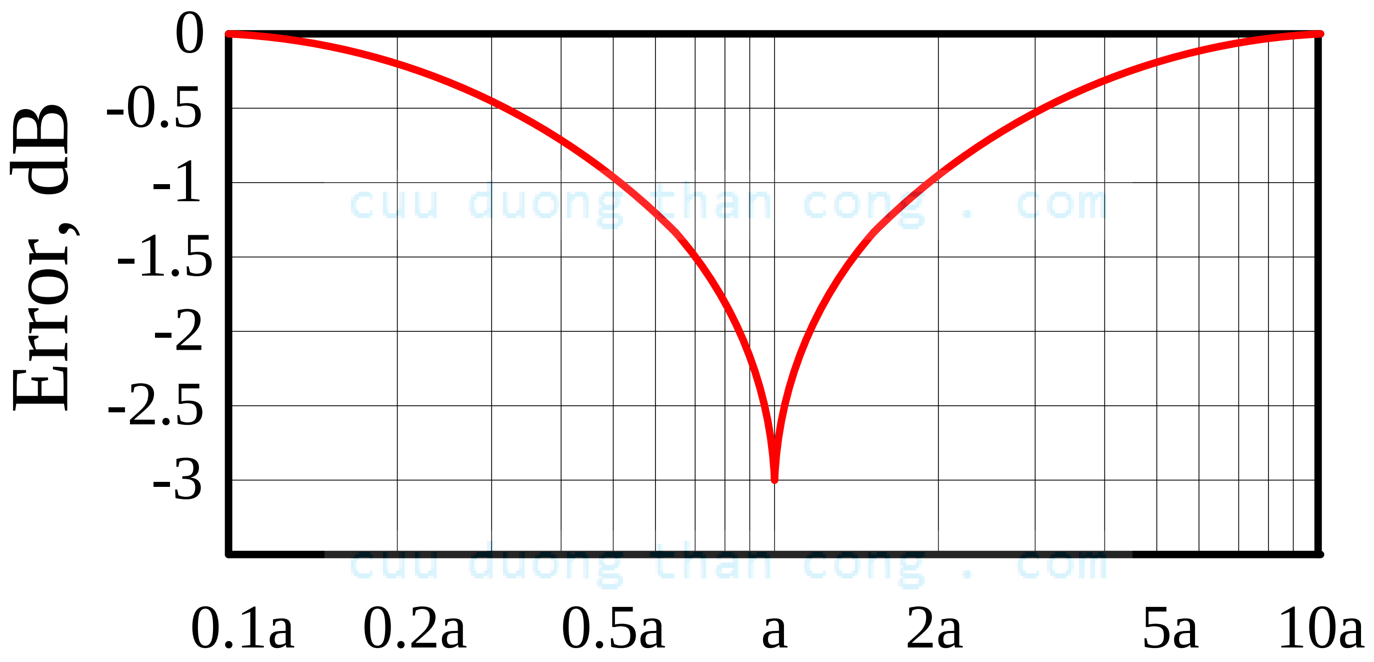
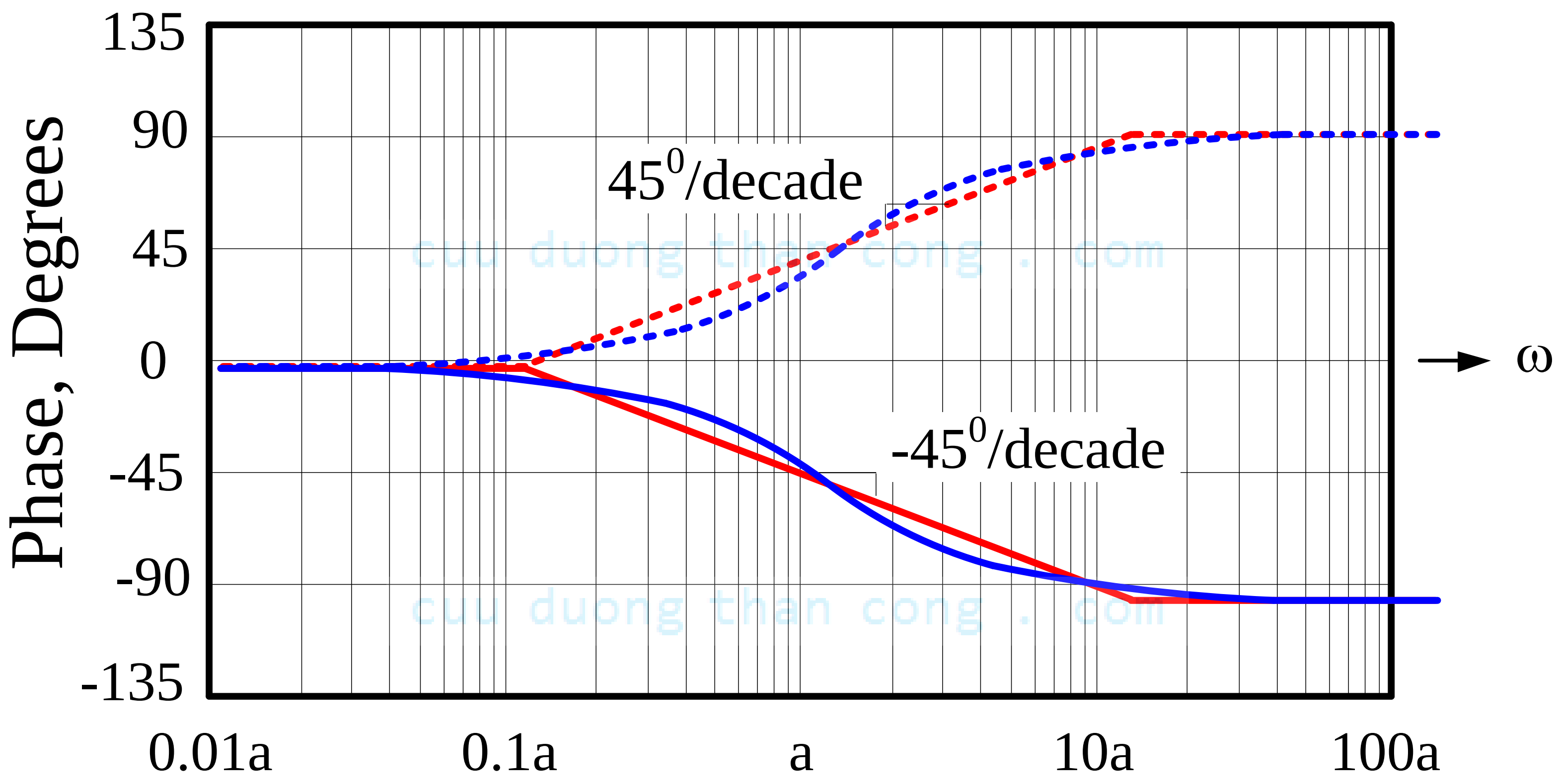
Bode plot:

3. First order Pole or Zero



Bode plot:

3. First order Pole or Zero



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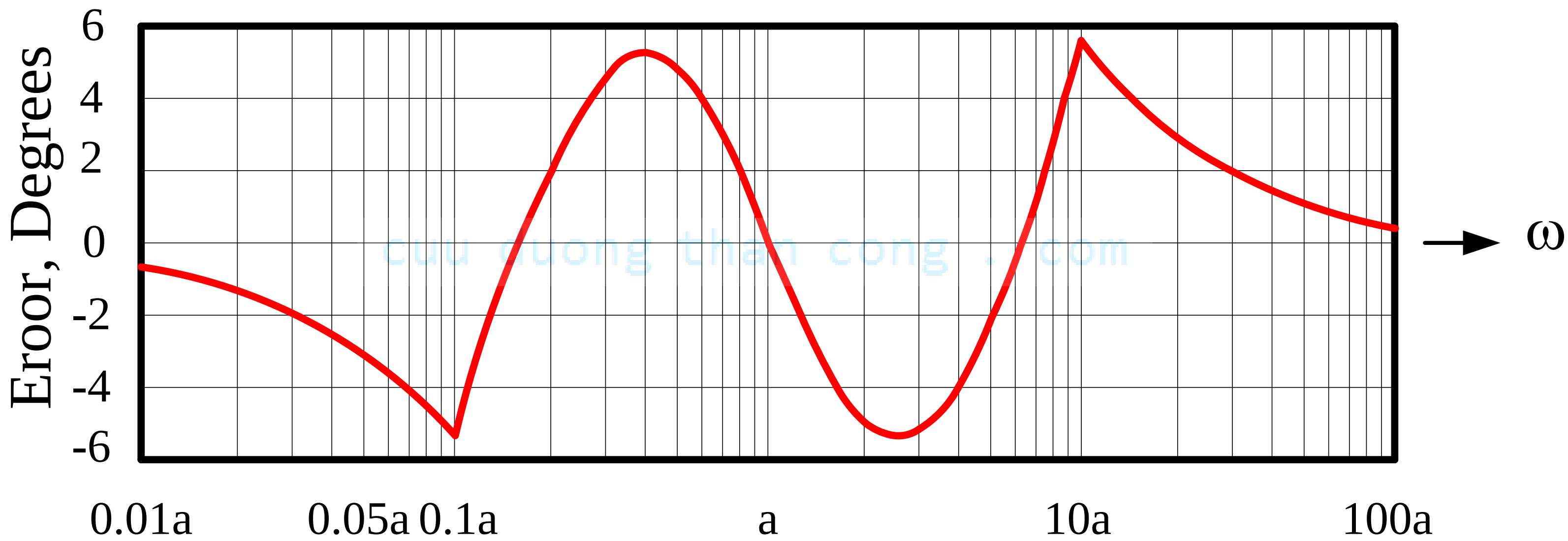
Bode plot:

3. First order Pole or Zero



Bode plot:

4. Second-order Pole or Zero $\pm 20\log\left|\left(\frac{j\omega}{\omega_n}\right)^2 + 2j\zeta\frac{\omega}{\omega_n} + 1\right|$

- Log Magnitude:

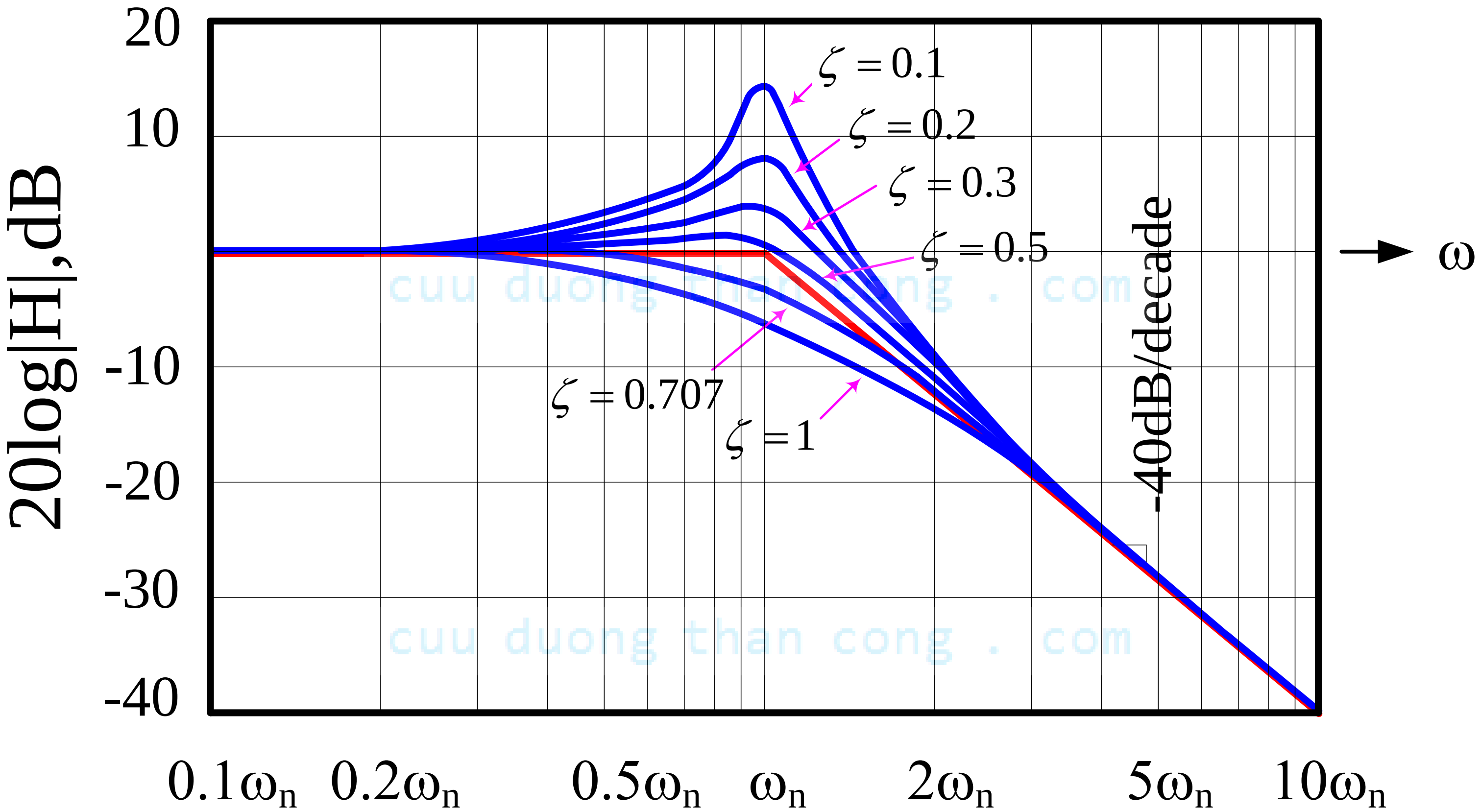
$$\begin{cases} Pole: -20\log\left|\left(\frac{j\omega}{\omega_n}\right)^2 + 2j\zeta\frac{\omega}{\omega_n} + 1\right| \approx \begin{bmatrix} 0 & for\ \omega \ll \omega_n \\ -40u + 40\log\omega_n & for\ \omega \gg \omega_n \end{bmatrix} \\ Zero: +20\log\left|\left(\frac{j\omega}{\omega_n}\right)^2 + 2j\zeta\frac{\omega}{\omega_n} + 1\right| \approx \begin{bmatrix} 0 & for\ \omega \ll \omega_n \\ +40u - 40\log\omega_n & for\ \omega \gg \omega_n \end{bmatrix} \end{cases}$$

- Phase:

$$\begin{cases} Pole: \quad -\angle(\ldots) = -\tan^{-1}\left(2\zeta\left(\frac{\omega}{\omega_n}\right) \Big/ 1 - \left(\frac{\omega}{\omega_n}\right)^2\right) \approx \begin{bmatrix} 0 & for\ \omega \ll \omega_n \\ -180^o & for\ \omega \gg \omega_n \end{bmatrix} \\ Zero: \quad +\angle(\ldots) = +\tan^{-1}\left(2\zeta\left(\frac{\omega}{\omega_n}\right) \Big/ 1 - \left(\frac{\omega}{\omega_n}\right)^2\right) \approx \begin{bmatrix} 0 & for\ \omega \ll \omega_n \\ +180^o & for\ \omega \gg \omega_n \end{bmatrix} \end{cases}$$

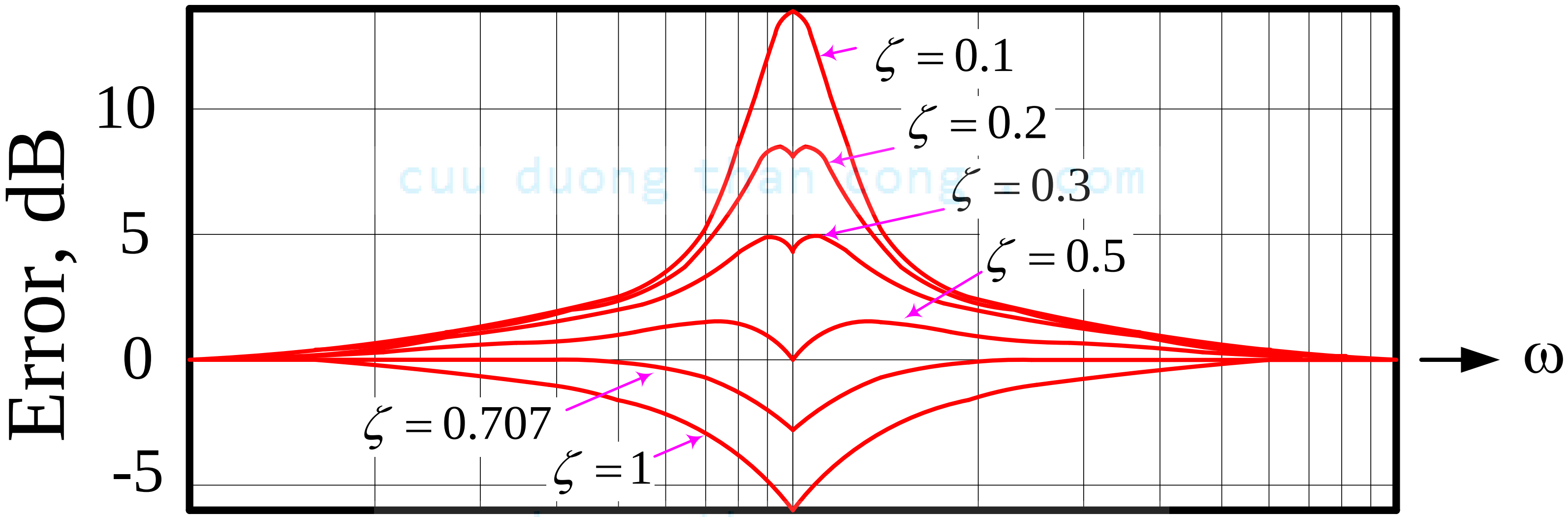
Bode plot:

4. Second-order Pole or Zero



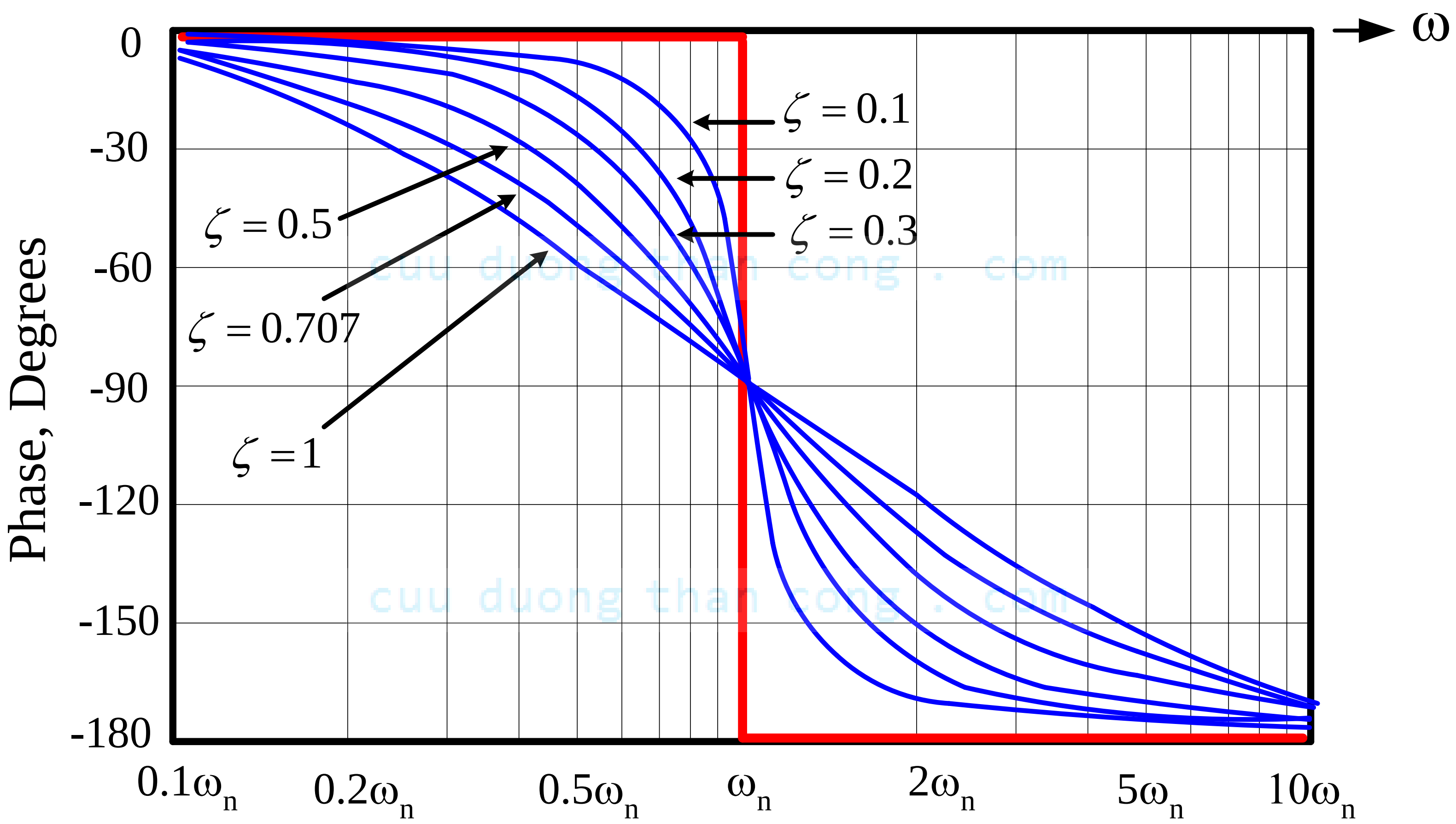
Bode plot:

4. Second-order Pole or Zero



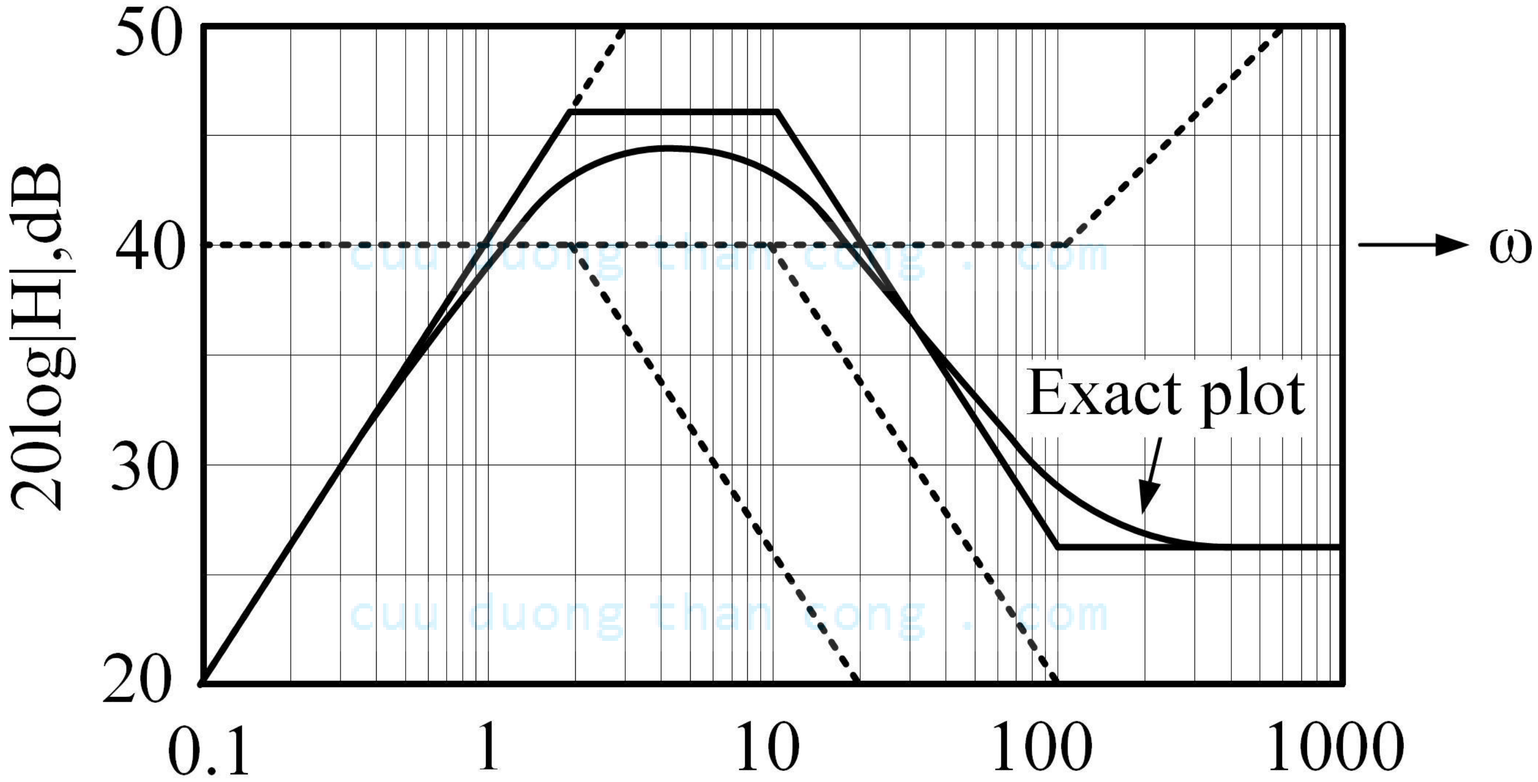
Bode plot:

4. Second-order Pole or Zero



Bode plot:

Example 7.01: $$H(s)=\frac{20s(s+100)}{(s+2)(s+10)}$$



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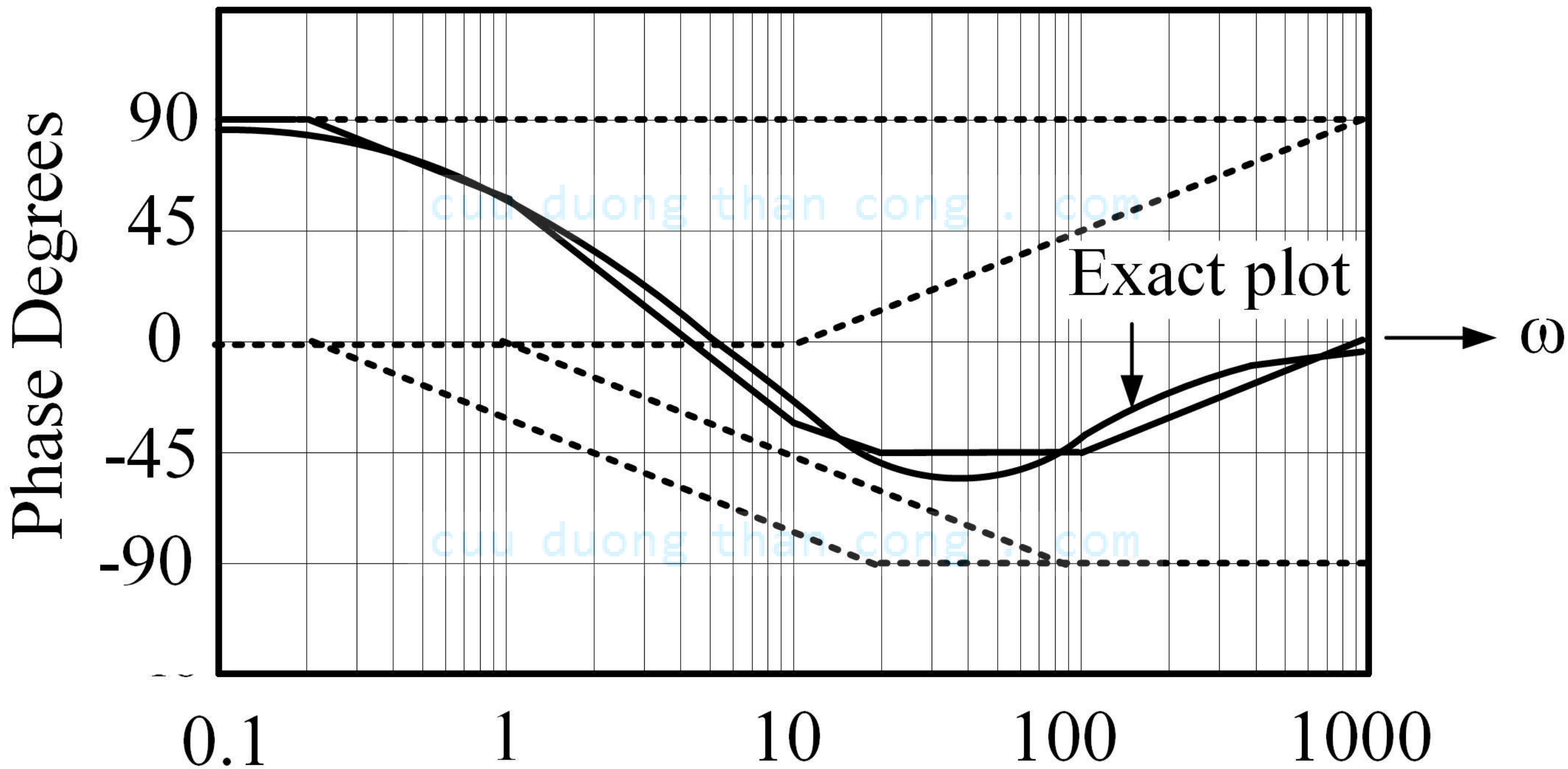
Bode plot:

Example 7.01: $$H(s)=\frac{20s(s+100)}{(s+2)(s+10)}$$



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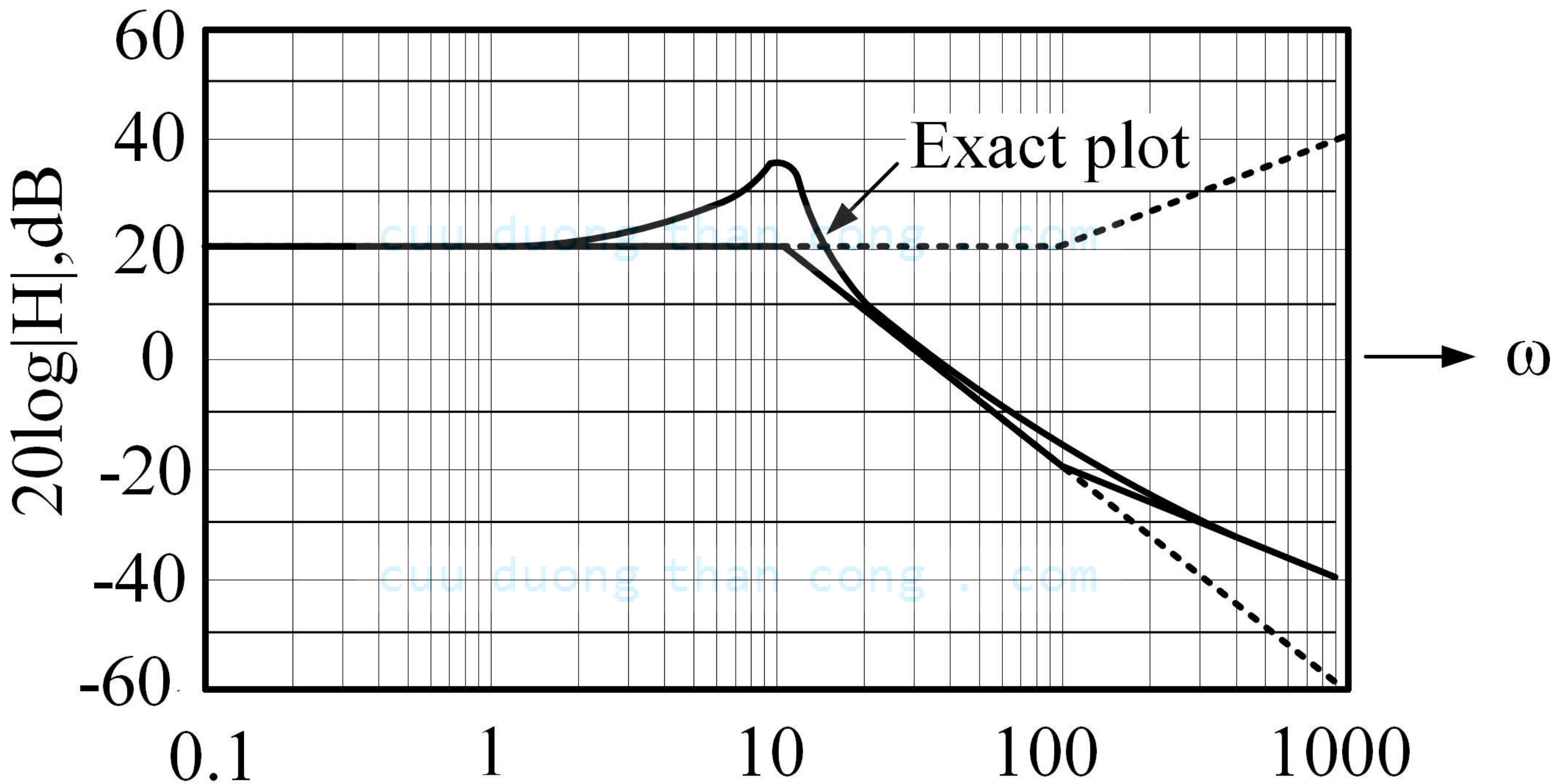
Bode plot:

Example 7.02:

$$H(s) = \frac{10(s+100)}{s^2 + 2s + 100}$$



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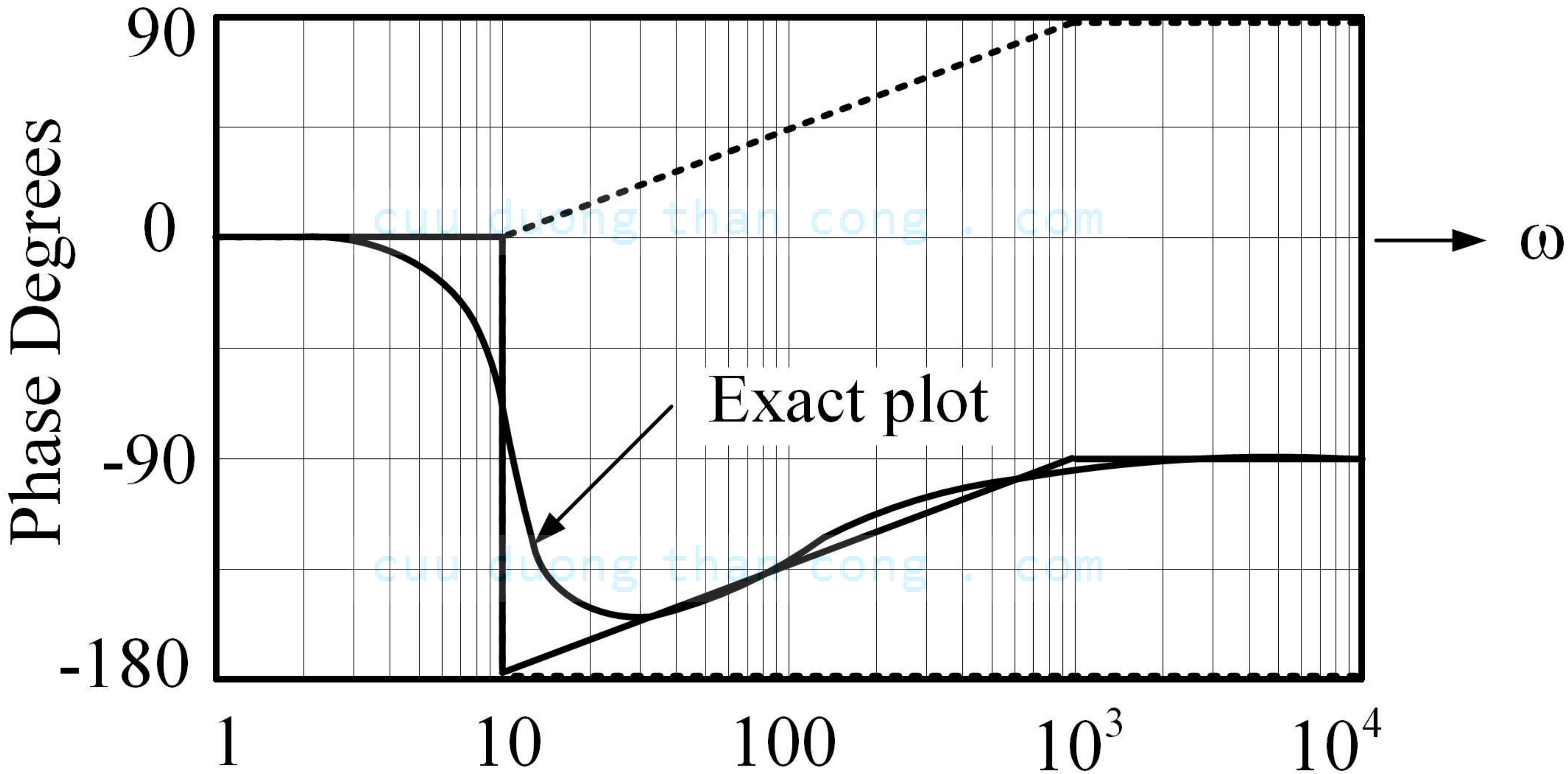
Bode plot:

Example 7.02: $$H(s) = \frac{10(s+100)}{s^2 + 2s + 100}$$



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Filter Design by Placement of Poles and Zeros:

- Rational transfer function $H(s)$ in factored form:

$$H(s) = \frac{P(s)}{Q(s)} = b_n \frac{(s-z_1)(s-z_2)\ldots(s-z_n)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

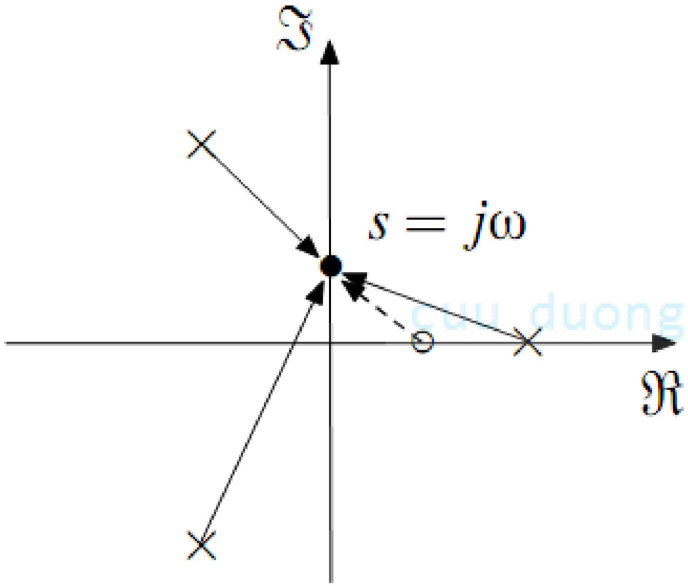
- Frequency response: $s = j\omega$.
- Bode plots of frequency response:

$$20\log\left|H(j\omega)\right| = 20\log\left|b_n\right| + \sum_{i=1}^{n} 20\log\left|j\omega - z_i\right| - 20\log\left|j\omega - p_i\right|$$

$$\angle H(j\omega) = \angle b_n + \sum_{i=1}^{n} \angle(j\omega - z_i) - \angle(j\omega - p_i)$$

Filter Design by Placement of Poles and Zeros:

- Graphical interpretation: $|H(j\omega)|$



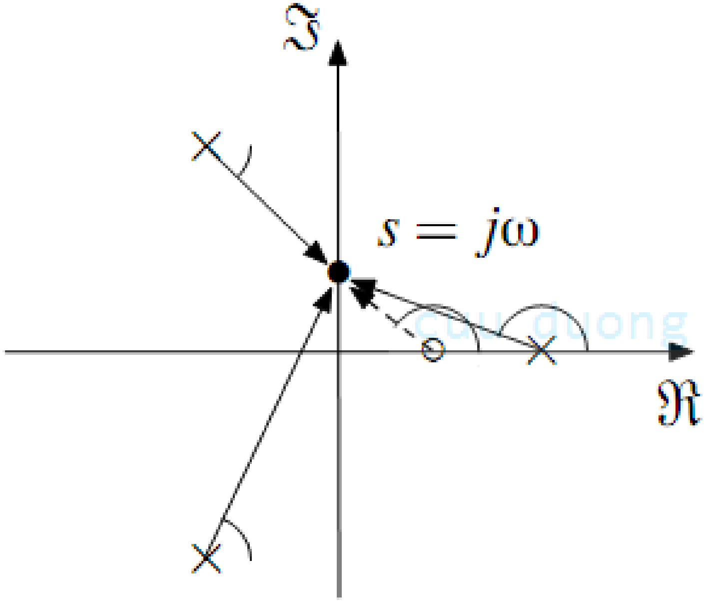
$$|H(j\omega)| = |b_n| \frac{\prod_{i=1}^{n} dist(j\omega, z_i)}{\prod_{i=1}^{n} dist(j\omega, p_i)}$$

$$where \quad dist(u,v) = |u - v|$$

- $|H(j\omega)|$ gets big when $j\omega$ is near a pole
- $|H(j\omega)|$ gets small when $j\omega$ is near a zero

Filter Design by Placement of Poles and Zeros:

- Graphical interpretation: $\angle H(j\omega)$



$$\angle H(j\omega) = \angle b_n + \sum \angle(j\omega - z_i)$$
$$- \sum \angle(j\omega - p_i)$$

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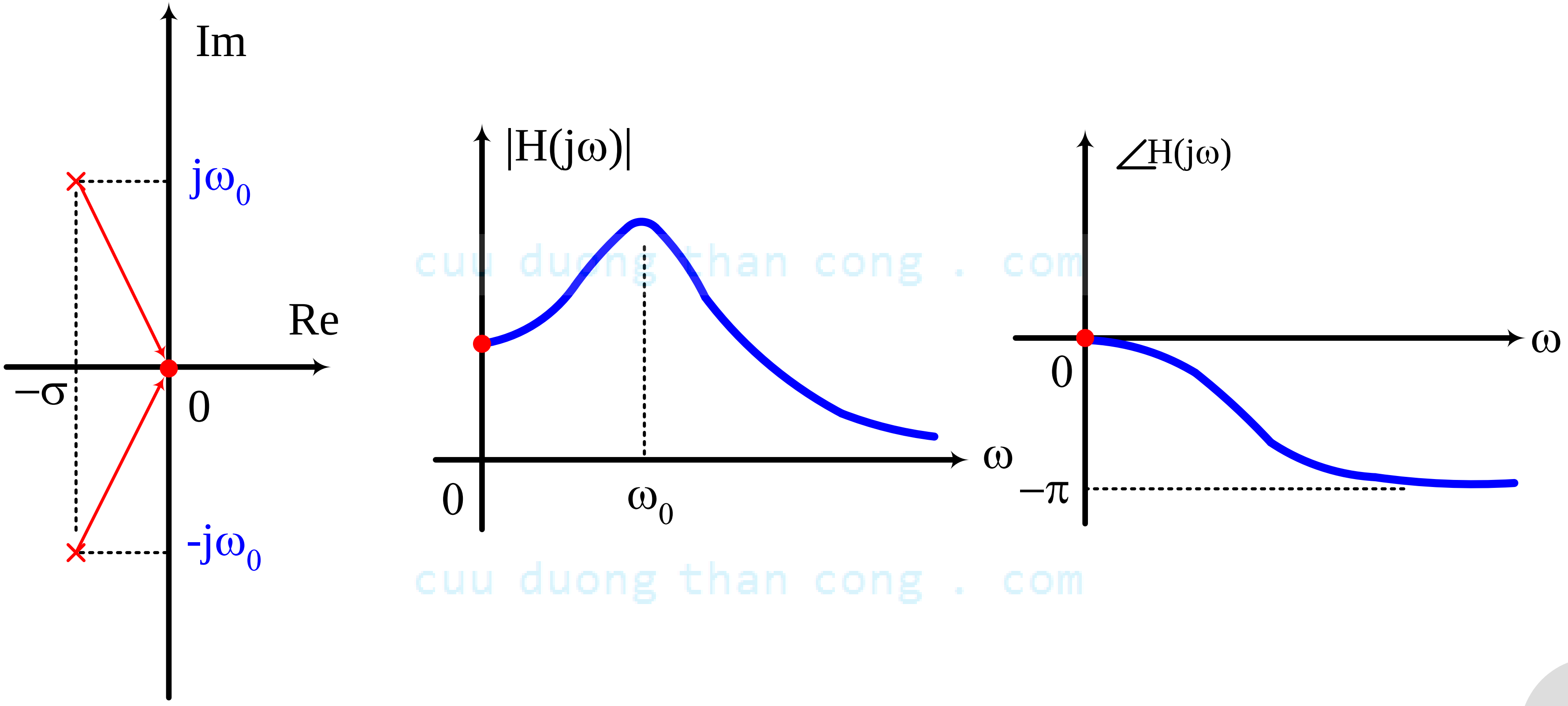
- $\angle H(j\omega)$ changes rapidly when a pole or zero is near $j\omega$

Filter Design by Placement of Poles and Zeros:

- Gain enhancement by a Pole



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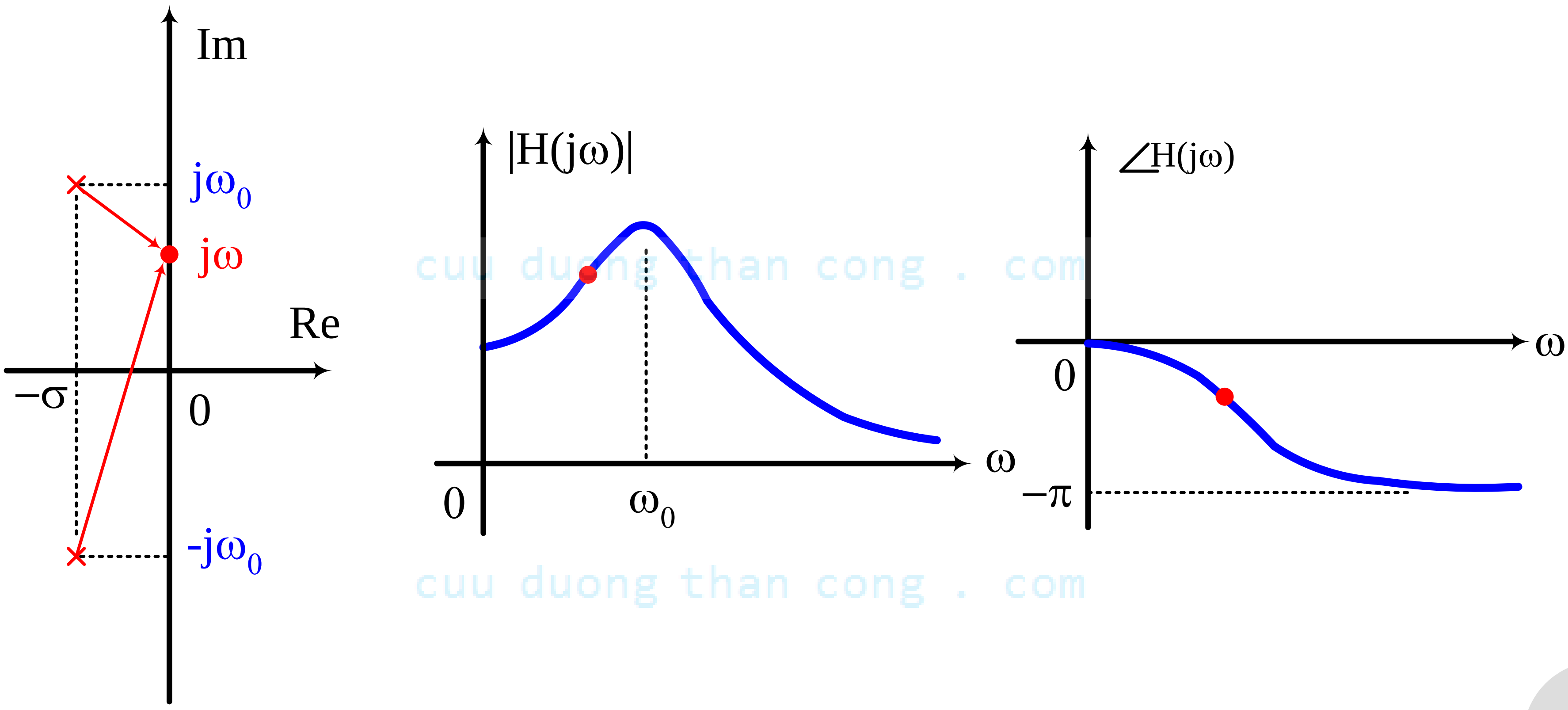

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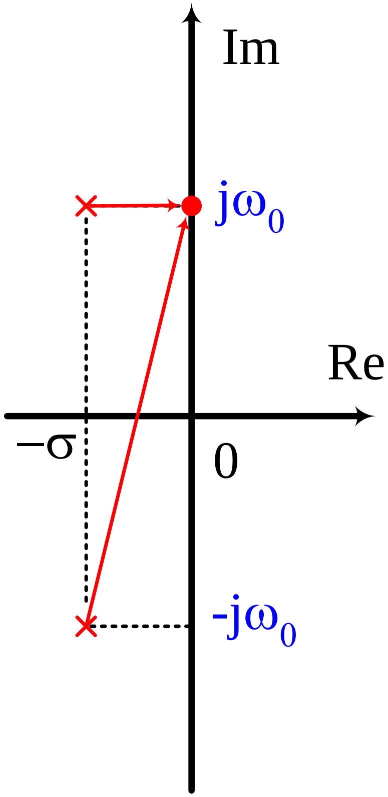

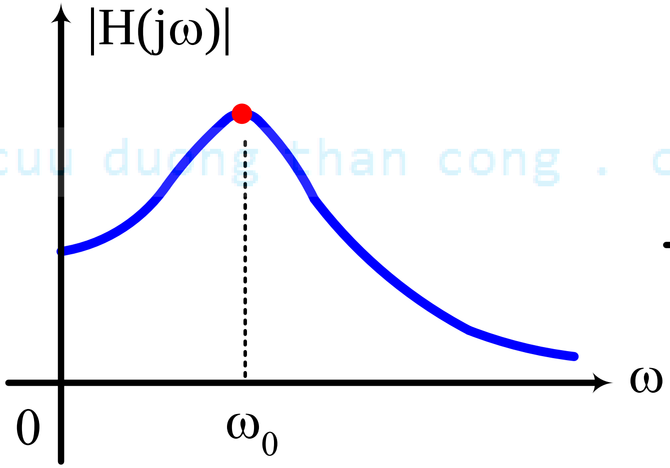

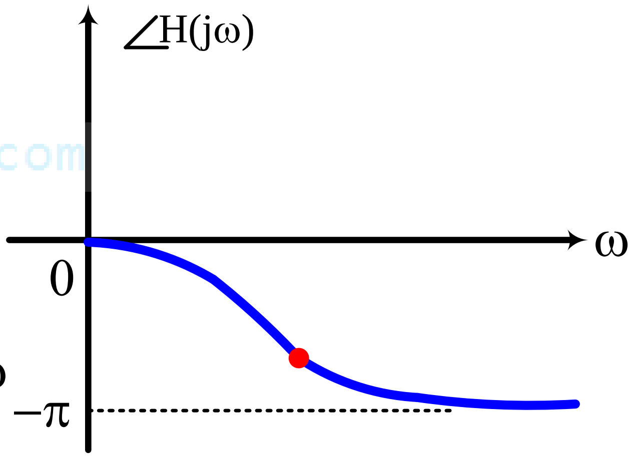

Filter Design by Placement of Poles and Zeros:

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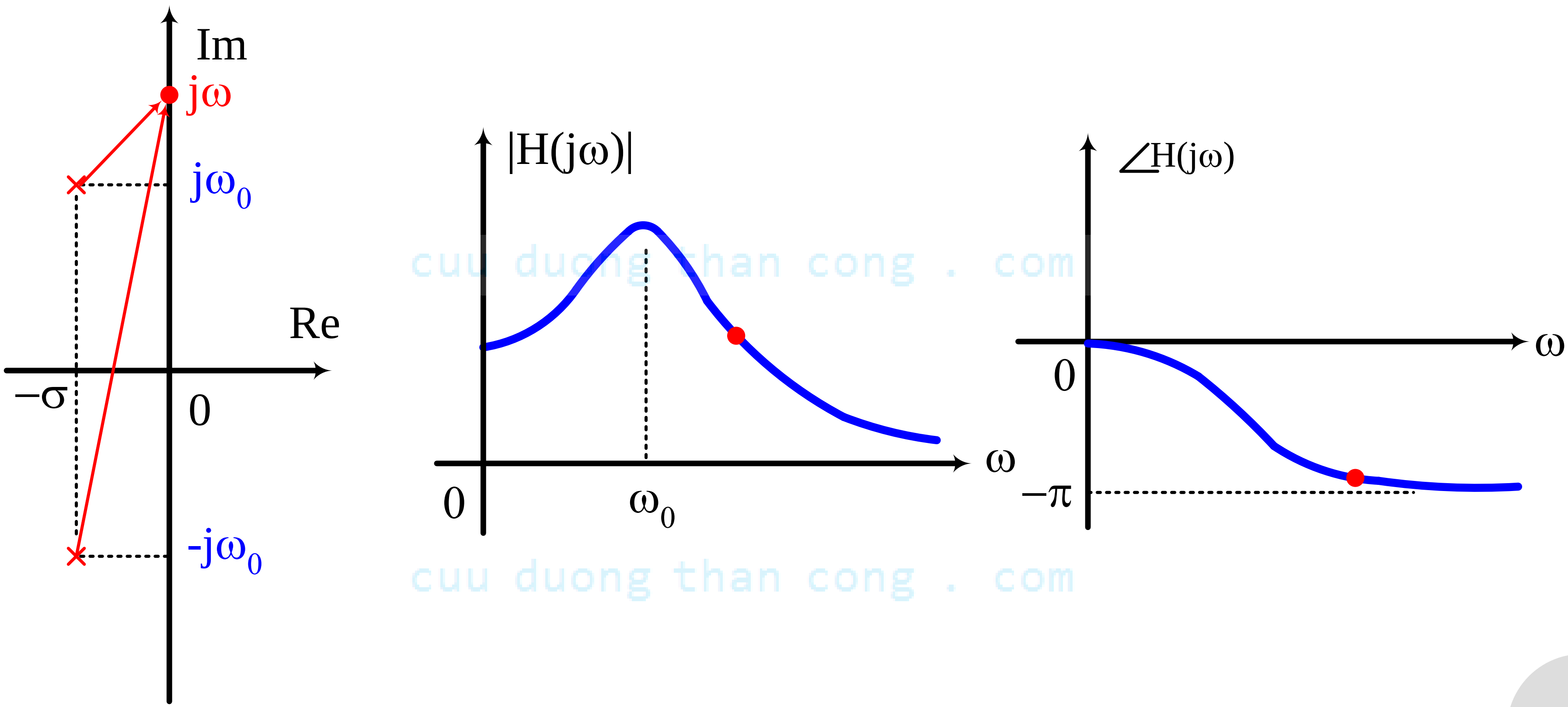

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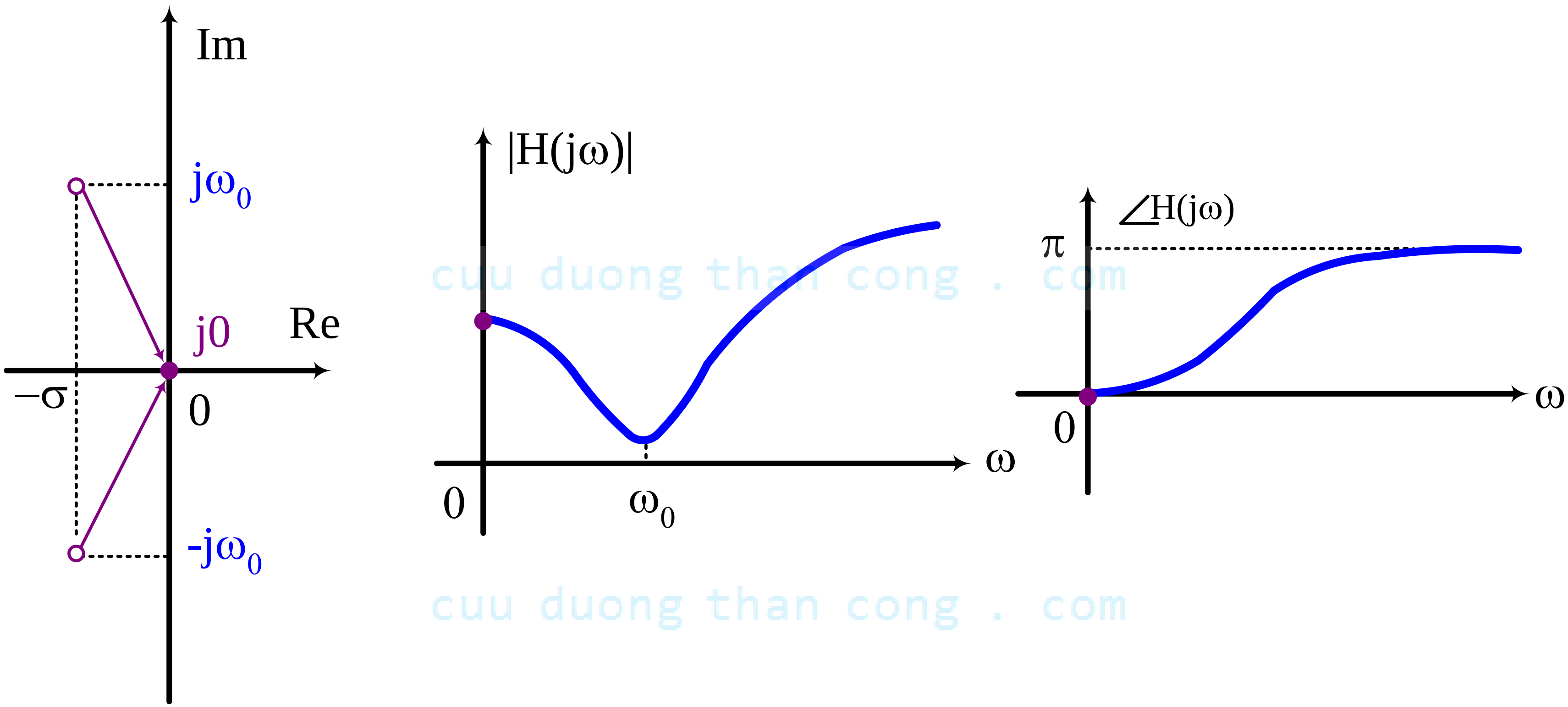
Filter Design by Placement of Poles and Zeros:

- Gain Suppression by a Zero



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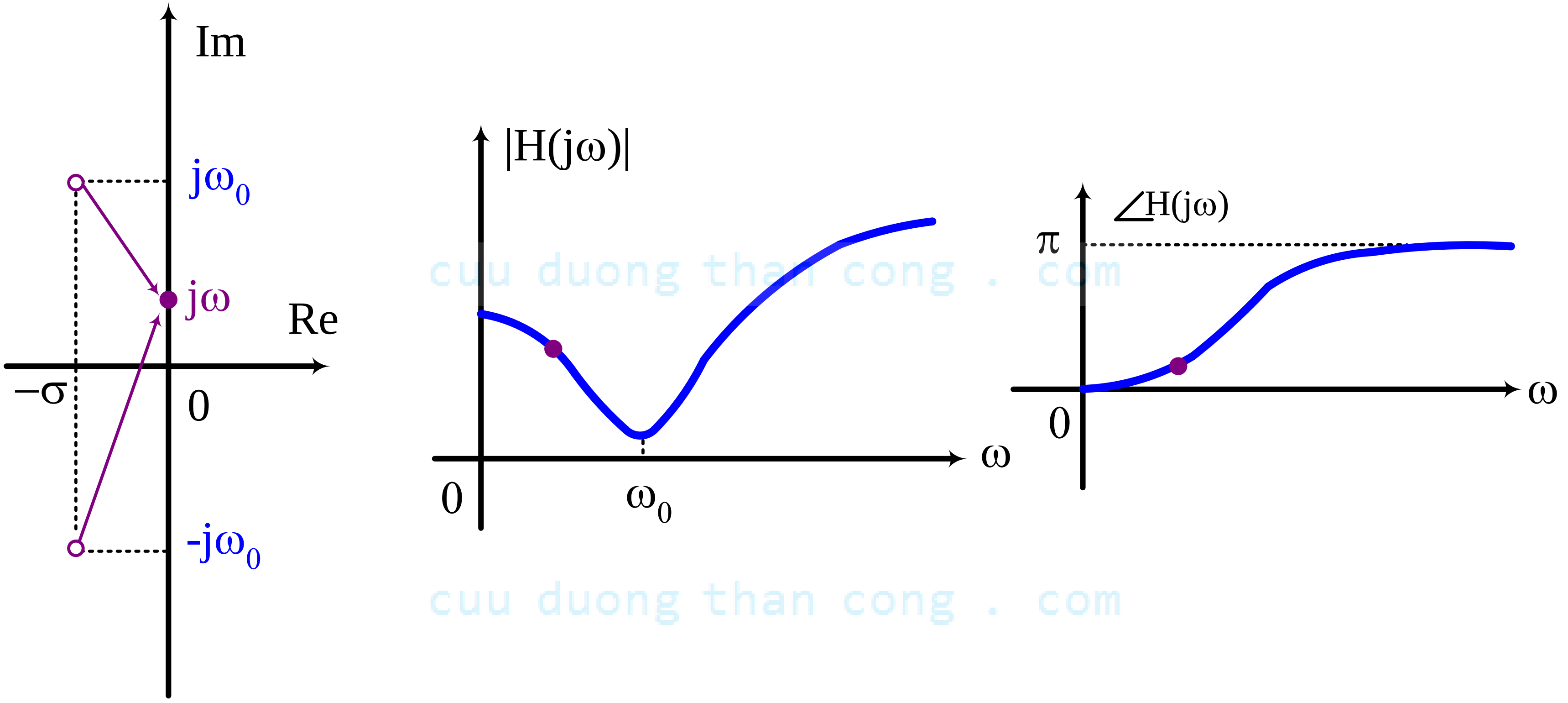

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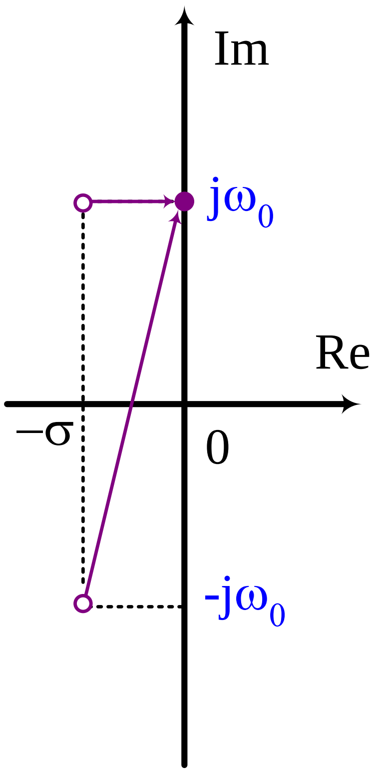

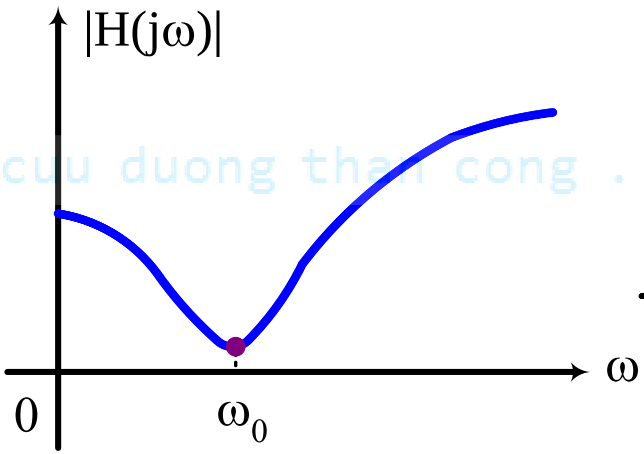

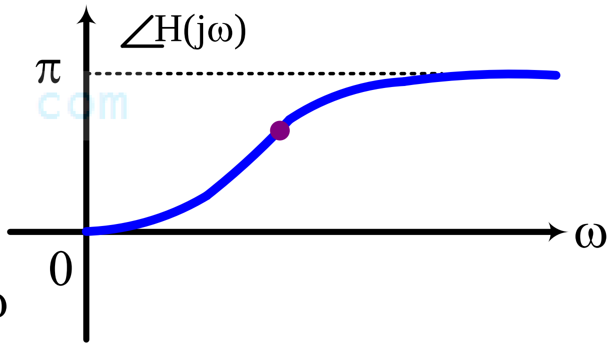

Filter Design by Placement of Poles and Zeros:

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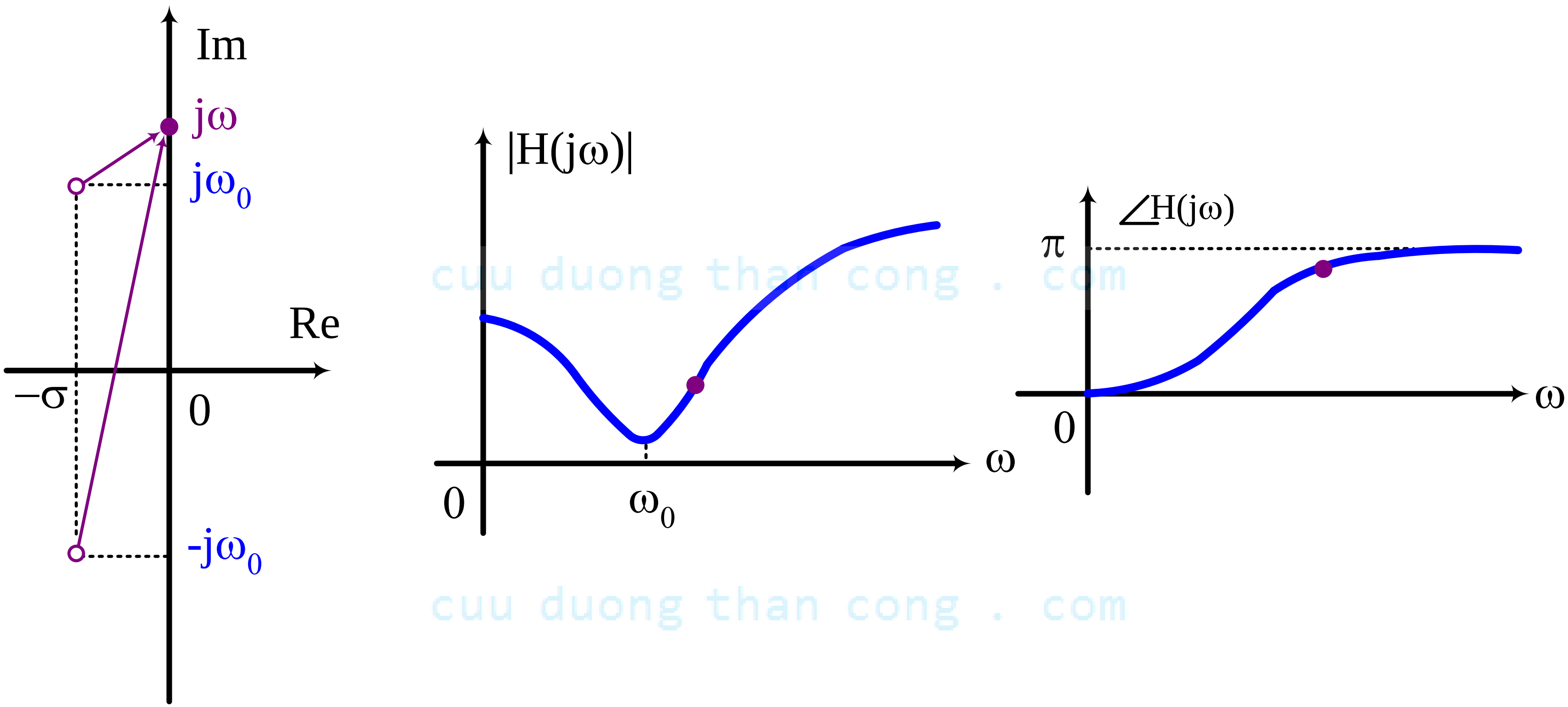
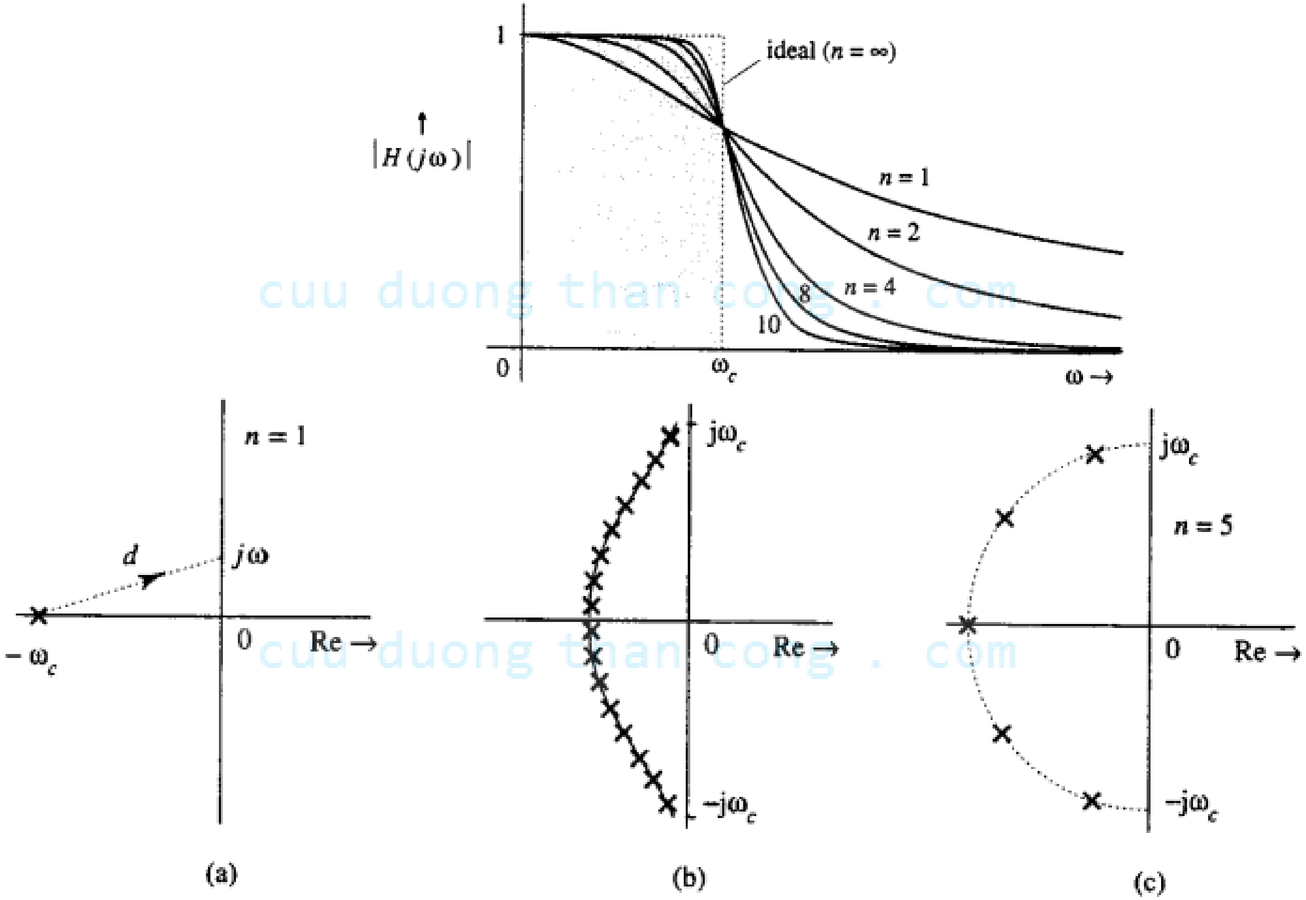


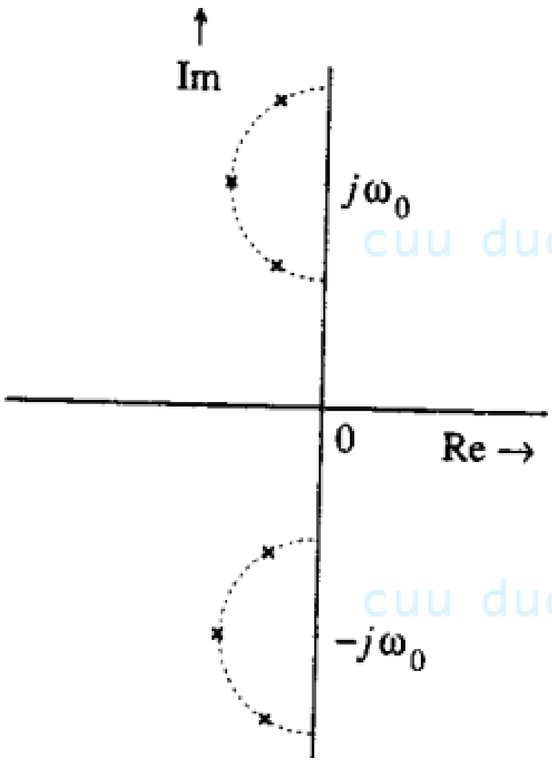
Filter Design by Placement of Poles and Zeros:

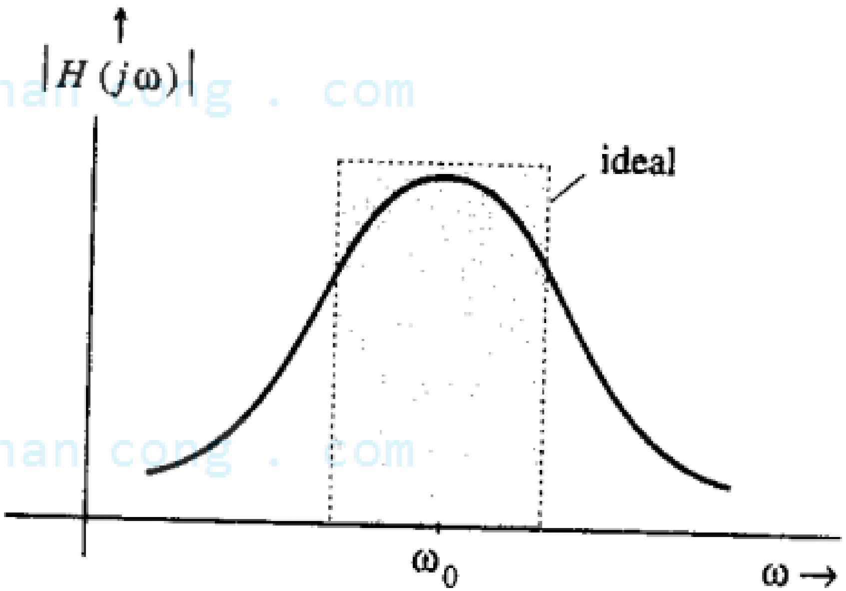
- Lowpass filter



Filter Design by Placement of Poles and Zeros:

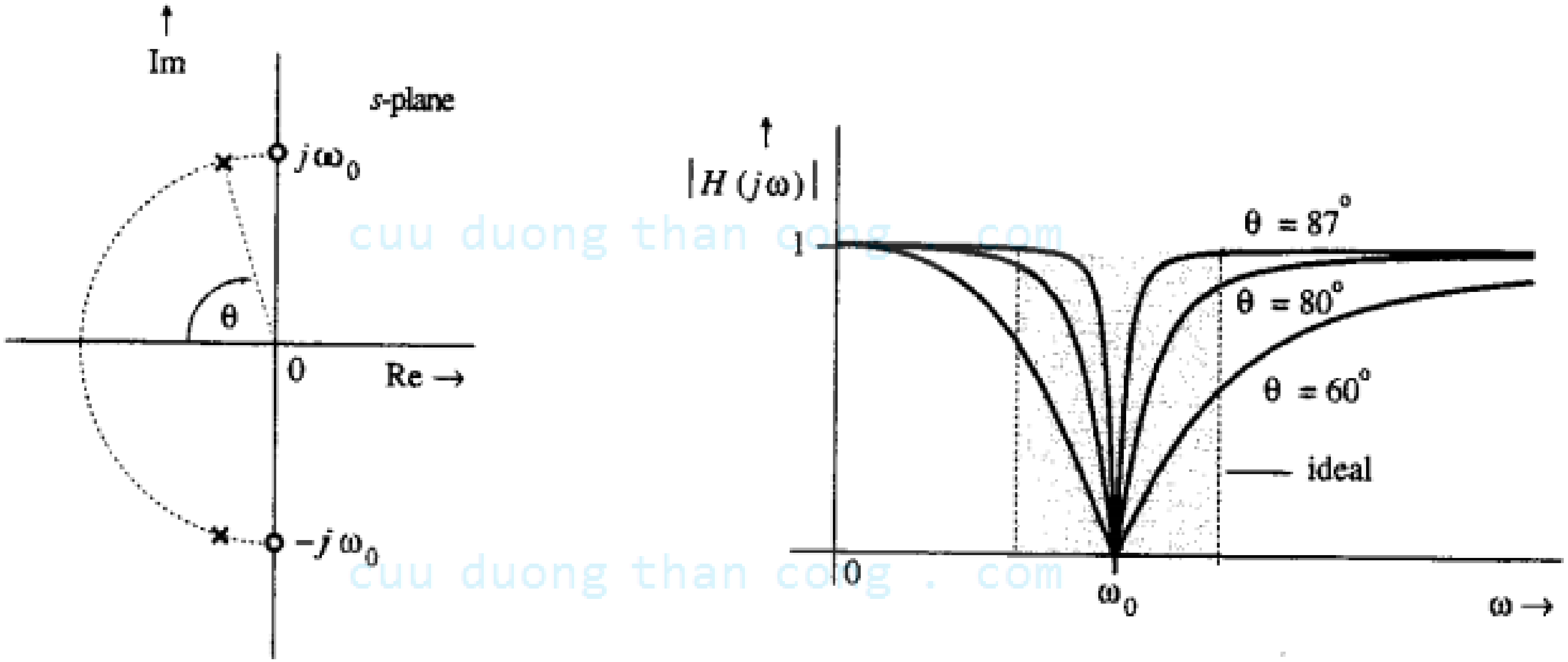
- Bandpass filter





Filter Design by Placement of Poles and Zeros:

- Bandstop filter



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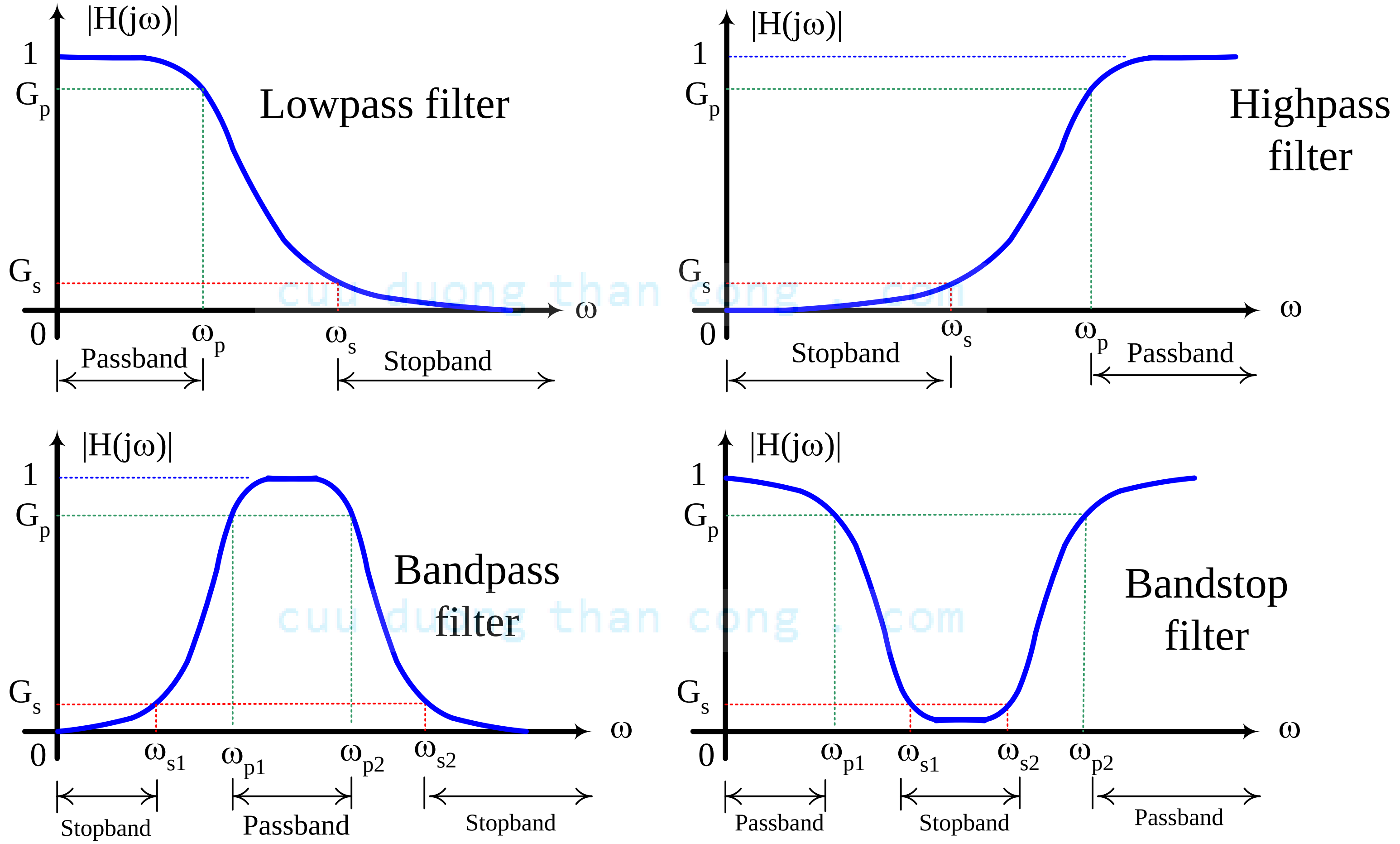
Filter Design by Placement of Poles and Zeros:

- Practical filters:



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Filter Design by Placement of Poles and Zeros:

- Practical Filters:
  - ✓ **No true stopband**: we define a stopband to be a band over which the gain is below some small number Gs
  - ✓ **No true passband**: we define a passband to be a band over which the gain is between 1 and Gp (Gp < 1).
  - ✓ There's always **a transition band**.

- The highpass, bandpass and bandstop filters can be obtained from a **basic lowpass filter** by simple frequency transformations.

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Butterworth Filters:

- The amplitude response $|H(j\omega)|$ of an nth-order Butterworth lowpass filter is given by

$$\left|H(j\omega)\right| = \frac{1}{\sqrt{1+\left(\dfrac{\omega}{\omega_c}\right)^{2n}}}$$

ω_c: the 3dB-cutoff frequency (or half-power frequency).

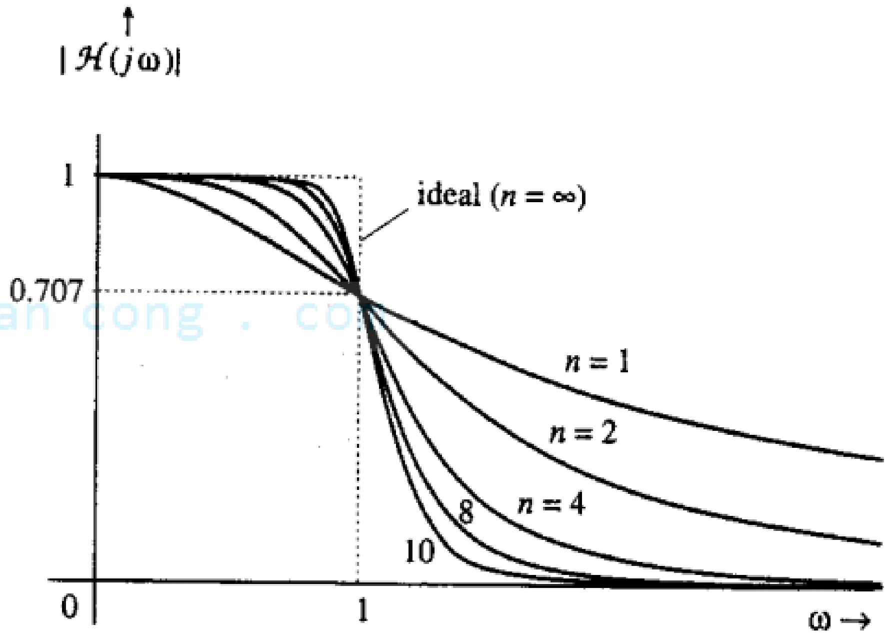
- Normalized Filters:

$$\left|H_N(j\omega)\right| = \frac{1}{\sqrt{1+\omega^{2n}}}$$

Butterworth Filters:

$$H_N(j\omega)H_N(-j\omega)=\frac{1}{1+\omega^{2n}}$$

$$\Rightarrow H_N(s)H_N(-s)=\frac{1}{1+\left(s/j\right)^{2n}}$$



- The poles of $H_N(s)H_N(-s)$:

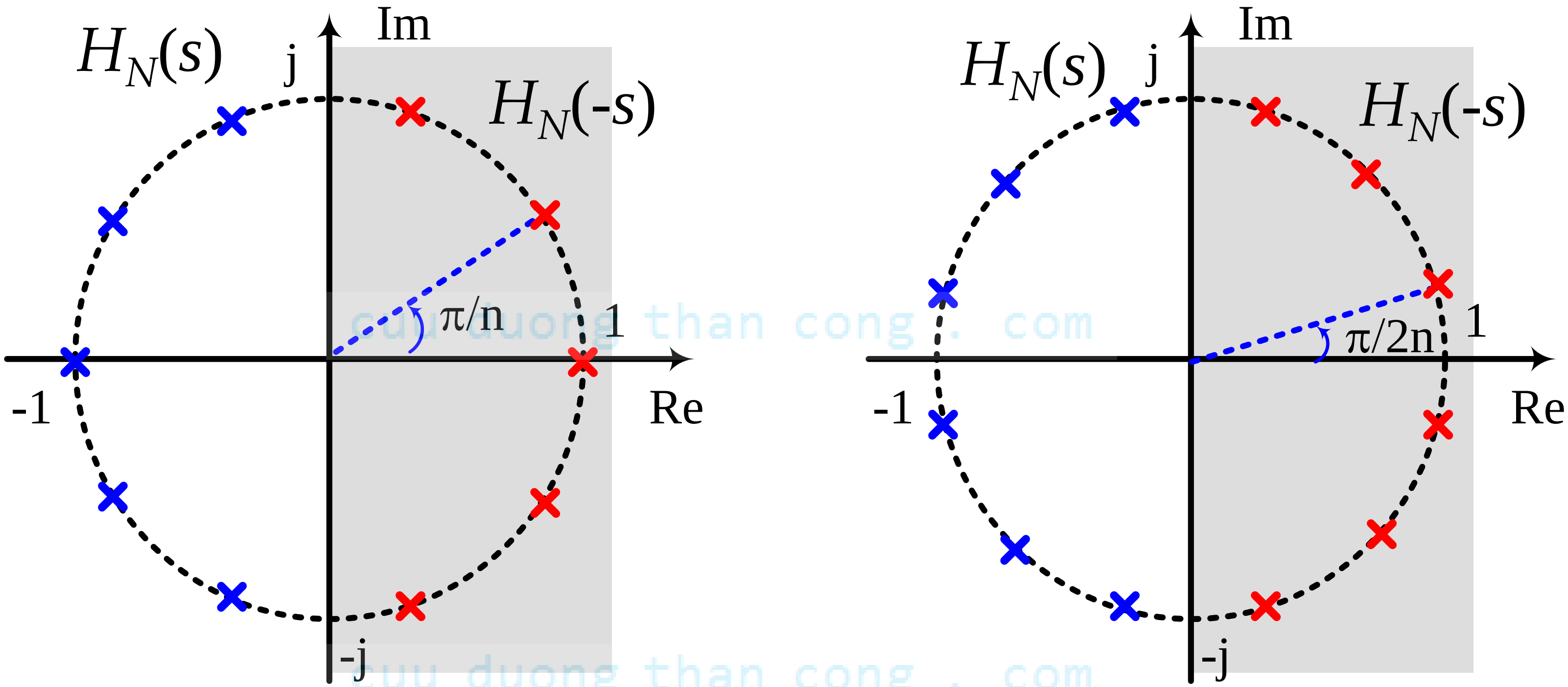
$$s^{2n}=-(j)^{2n}$$

$$=e^{j\pi(2k-1)}\left(e^{j\pi/2}\right)^{2n}$$

$$=e^{j\pi(2k-1+n)}$$

$$\Rightarrow s_k=e^{\frac{j\pi}{2n}(2k+n-1)}\qquad k=1,2,3,\ldots,2n$$

Butterworth Filters:



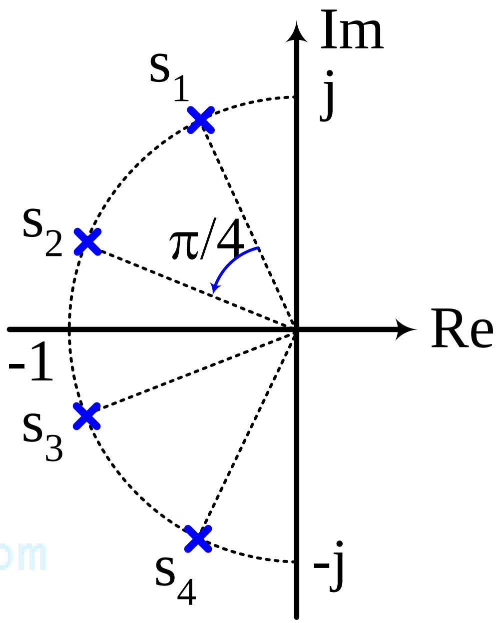
- The poles of $H_N(s)$: $s_k = e^{\frac{j\pi}{2n}(2k+n-1)}$ $\quad k = 1,2,3,\dots,n$

Butterworth Filters:

- $H_N(s)$ is given by:

$$H_N(s) = \frac{1}{(s-s_1)(s-s_2)\ldots(s-s_n)}$$

- *Example*: for $n = 4$



$$\begin{cases} s_1 = e^{j\frac{5\pi}{8}} \simeq -0.3827 + j0.9239; \ s_2 = e^{j\frac{7\pi}{8}} \simeq -0.9239 + j0.3827 \\ s_3 = e^{j\frac{9\pi}{8}} \simeq -0.9239 - j0.3827; \ s_4 = e^{j\frac{11\pi}{8}} \simeq -0.3827 - j0.9239 \end{cases}$$

$$H_N(s) = \frac{1}{s^4 + 2.6131s^3 + 3.4142s^2 + 2.6131s + 1}$$

Butterworth Filters:

- In general: $$H_N(s)=\frac{1}{B(s)}=\frac{1}{s^n+a_{n-1}s^{n-1}+\ldots+a_1s+1}$$

$B(s)$: Butterworth polynomial of the n^{th}-order.

Frequency Scaling:

- Normalize Butterworth filters have $\omega_c = 1$.
- By simply replacing s by s/ω_c, the results can be extended to any value of ω_c.

Examples:

For $n = 2$
$$\begin{cases} H_N(s)=\dfrac{1}{s^2+\sqrt{2}s+1} & ; \quad \omega_c = 1 \\ H(s)=\dfrac{1}{(s/100)^2+\sqrt{2}(s/100)+1} & ; \quad \omega_c = 100 \end{cases}$$

Butterworth Filters:

- Coefficients of $B_n(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + 1$

n	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
2	1.41421356								
3	2.00000000	2.00000000							
4	2.61312593	3.41421356	2.61312593						
5	3.23606798	5.23606798	5.23606798	3.23606798					
6	3.86370331	7.46410162	9.14162017	7.46410162	3.86370331				
7	4.49395921	10.09783468	14.59179389	14.59179389	10.09783468	4.49395921			
8	5.12583090	13.13707118	21.84615097	25.68835593	21.84615097	13.13707118	5.12583090		
9	5.75877048	16.58171874	31.16343748	41.98638573	41.98638573	31.16343748	16.58171874	5.75877048	
10	6.39245322	20.43172909	42.80206107	64.88239627	74.23342926	64.88239627	42.80206107	20.43172909	6.39245322

Butterworth Filters:

- Butterworth Polynomial in Factorized Form

n	$B_n(s)$
1	$s+1$
2	$s^2+1.41421356s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.76536686s+1)(s^2+1.84775907s+1)$
5	$(s+1)(s^2+0.61803399s+1)(s^2+1.931803399s+1)$
6	$(s^2+0.51763809s+1)(s^2+1.41421356s+1)(s^2+1.93185165s+1)$
7	$(s+1)(s^2+0.44504187s+1)(s^2+1.24697960s+1)(s^2+1.80193774s+1)$
8	$(s^2+0.39018064s+1)(s^2+1.11114047s+1)(s^2+1.66293922s+1)(s^2+1.96157056s+1)$
9	$(s+1)(s^2+0.34729636s+1)(s^2+s+1)(s^2+1.53208889s+1)(s^2+1.87938524s+1)$
10	$(s^2+0.31286893s+1)(s^2+0.90798100s+1)(s^2+1.41421356s+1)(s^2+1.78201305s+1)(s^2+1.97537668s+1)$

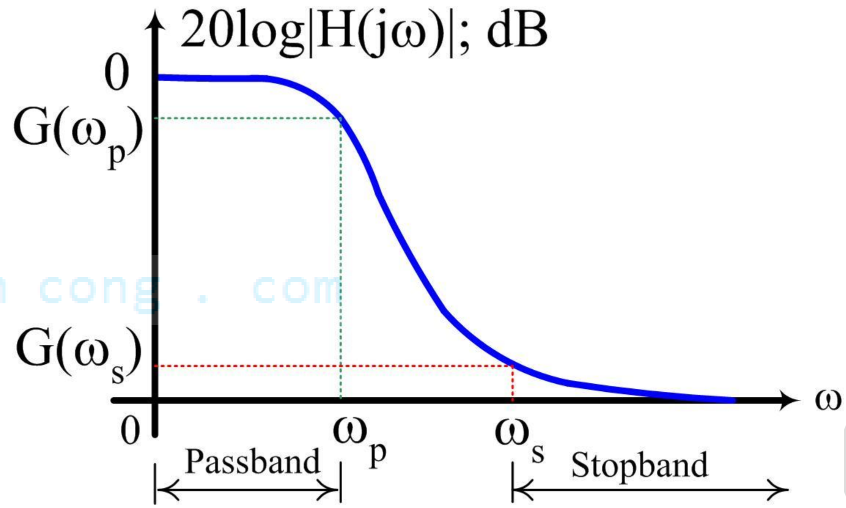
Butterworth Filters:

Determination of the Filter Order

- If G_x is the gain in dB units at $\omega = \omega_x$:

$$G_x = 20\log\left|H(j\omega_x\right| = -10\log\left[1+\left(\frac{\omega_x}{\omega_c}\right)^{2n}\right]$$

$$\Rightarrow \begin{cases} G_p = -10\log\left[1+\left(\dfrac{\omega_p}{\omega_c}\right)^{2n}\right] \\ G_s = -10\log\left[1+\left(\dfrac{\omega_s}{\omega_c}\right)^{2n}\right] \end{cases}$$



Butterworth Filters:

$$\Rightarrow \begin{cases} \left(\dfrac{\omega_p}{\omega_c}\right)^{2n} = 10^{-\frac{G_p}{10}} - 1 \\ \left(\dfrac{\omega_s}{\omega_c}\right)^{2n} = 10^{-\frac{G_s}{10}} - 1 \end{cases} \Rightarrow \left(\dfrac{\omega_s}{\omega_p}\right)^{2n} = \dfrac{10^{-\frac{G_s}{10}} - 1}{10^{-\frac{G_p}{10}} - 1}$$

$$\Rightarrow n \geq \dfrac{\log\left[\left(10^{-\frac{G_s}{10}} - 1\right) / \left(10^{-\frac{G_p}{10}} - 1\right)\right]}{2\log\left(\omega_s / \omega_p\right)} \quad and \quad \begin{cases} \omega_c \geq \dfrac{\omega_p}{\left(10^{-\frac{G_p}{10}} - 1\right)^{1/2n}} \\ \omega_c \leq \dfrac{\omega_s}{\left(10^{-\frac{G_s}{10}} - 1\right)^{1/2n}} \end{cases}$$

Butterworth Filters:

- *Example*: Design a Butterworth lowpass filter:
  - ✓ Passband gain lie between 1 and $G_p = 0.794$ (-2dB) for $\omega \in [0; 10)$
  - ✓ Stopband gain not to exceed $G_s = 0.1$ (-20dB) for $\omega \geq 20$

- **Step 1**: Determine n

  Here $\omega_p = 10$, $\omega_s = 20$

$$n \geq \frac{\log\left[\left(10^{-\frac{G_s}{10}} - 1\right) / \left(10^{-\frac{G_p}{10}} - 1\right)\right]}{2\log\left(\omega_s / \omega_p\right)} = 3.701 \Rightarrow n = 4$$

Butterworth Filters:

- **Step 2**: Determine ω_c

$$\begin{cases} \omega_c \geq \dfrac{\omega_p}{\left(10^{-\frac{G_p}{10}} - 1\right)^{1/2n}} = 10.693 \\ \omega_c \leq \dfrac{\omega_s}{\left(10^{-\frac{G_s}{10}} - 1\right)^{1/2n}} = 11.261 \end{cases} \Rightarrow \omega_c = 11$$

$(any\ real\ value\ between\ 10.693\ and\ 11.261$

Butterworth Filters:

- **Step 2**: Determine normalize transfer fucntion $H_N(s)$:

$$H_N(s)=\frac{1}{\left(s^2+0.76536686s+1\right)\left(s^2+1.84775907s+1\right)}$$

- **Step 4:** Determine final transfer function $H(s)$:

$$H(s)=\frac{1}{\left[\left(\frac{s}{11}\right)^2+0.76536686\left(\frac{s}{11}\right)+1\right]\left[\left(\frac{s}{11}\right)^2+1.84775907\left(\frac{s}{11}\right)+1\right]}$$

$$=\frac{14641}{(s^2+8.41903546s+121)(s^2+20.32534977s+121)}$$

Chapter 7:

Frequency response of LTI system and Analog filter design

1. *Frequency Response – Bode Plots*
2. *Analog filter design*
   - *Butterworth Filters*
   - ***Chebyshev Filters***
   - *Frequency Transformation*

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Chebyshev Filters

- The amplitude response $|H(j\omega)|$ of Chebyshev lowpass filter is given by

$$|H(j\omega)| = \frac{1}{\sqrt{1+\varepsilon^2 C_n^2\left(\dfrac{\omega}{\omega_p}\right)}}$$

- Normalized Filters:

$$|H_N(j\omega)| = \frac{1}{\sqrt{1+\varepsilon^2 C_n^2(\omega)}}$$

- $C_n(\omega)$ is the nth-order Chebyshev polynomial, given by:

$$\begin{cases} C_n(\omega) = \cos\left(n\cos^{-1}\omega\right) & for \ |\omega| < 1 \\ C_n(\omega) = \cosh\left(n\cosh^{-1}\omega\right) & for \ |\omega| > 1 \end{cases}$$

Chebyshev Filters

- The Chebyshev polynomial has the property:

$$\begin{cases} C_n(\omega) = 2\omega C_{n-1}(\omega) - C_{n-2}(\omega) & n > 2 \\ C_0(\omega) = 1 \\ C_1(\omega) = \omega \end{cases}$$

- Thus, we can construct $C_n(\omega)$ for any value of n. For example:

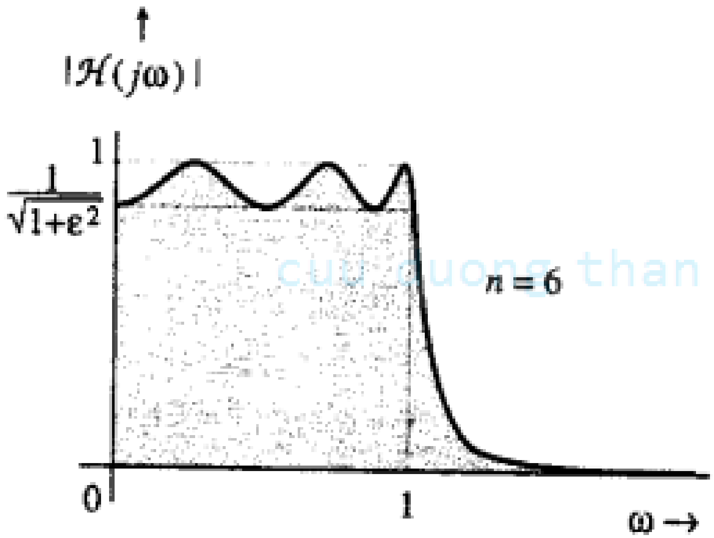
$$C_2(\omega) = 2\omega C_1(\omega) - C_0(\omega) = 2\omega^2 - 1$$

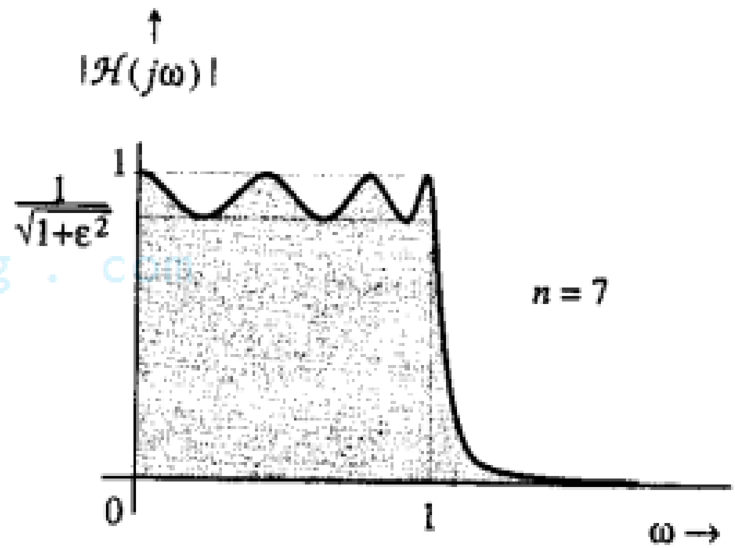
Chebyshev Filters

n	$C_n(\omega)$
0	1
1	ω
2	$2\omega^2 - 1$
3	$4\omega^3 - 3\omega$
4	$8\omega^4 - 8\omega^2 + 1$
5	$16\omega^5 - 20\omega^3 + 5\omega$
6	$32\omega^6 - 48\omega^4 + 18\omega^2 - 1$
7	$64\omega^7 - 112\omega^5 + 56\omega^3 - 7\omega$
8	$128\omega^8 - 256\omega^6 + 160\omega^4 - 32\omega^2 + 1$
9	$256\omega^9 - 576\omega^7 + 432\omega^5 - 120\omega^3 + 9\omega$
10	$512\omega^{10} - 1280\omega^8 + 1120\omega^6 - 400\omega^4 + 50\omega^2 - 1$

Chebyshev Filters

- Amplitude response of normalized lowpass Chebyshev filters:





- ***Some conclusions:***

✓ The Chebyshev amplitude response has ripples in the passband and is smooth in the stopband. There is a total of n maxima and minima over the passband.

Chebyshev Filters

- The DC gain is:

$$|H_N(0)| = \begin{cases} 1 & n \text{ odd} \\ \dfrac{1}{\sqrt{1+\varepsilon^2}} & n \text{ even} \end{cases}$$

- The ripple factor r – the ratio of the maximum gain to the minimum gain is:

$$r = \sqrt{1+\varepsilon^2}$$

$$\hat{r} = 20\log\sqrt{1+\varepsilon^2} = 10\log\left(1+\varepsilon^2\right) \Rightarrow \varepsilon^2 = 10^{\frac{\hat{r}}{10}} - 1$$

- At $\omega = 1$, the amplitude response is $1/r$.

Chebyshev Filters

✓ The ripple *r* takes the place of *Gp* in Butterworth filters.
✓ If we reduce the ripple (*r* is decreased), the gain in the stopband increases, and vice-versa.
✓ The Chebyshev filter has a sharper cutoff (smaller transition band) than the same order Butterworth filter.

- ***Determination of filter order n***

✓ The gain *G* in dB:

$$G = -10\log\left[1+\varepsilon^2 C_n^2\left(\frac{\omega}{\omega_p}\right)\right] \xrightarrow{at\ \omega_s} G_s = -10\log\left[1+\varepsilon^2 C_n^2\left(\frac{\omega_s}{\omega_p}\right)\right]$$

Chebyshev Filters

$$G_s = -10\log\left[1+\varepsilon^2 C_n^2\left(\frac{\omega_s}{\omega_p}\right)\right] \Leftrightarrow \varepsilon^2 C_n^2\left(\frac{\omega_s}{\omega_p}\right) = 10^{-\frac{G_s}{10}} - 1$$

$$\begin{cases}\varepsilon^2 = 10^{\frac{\hat{r}}{10}} - 1 \\ C_n(\omega) = \cosh\left(n\cosh^{-1}\omega\right)\end{cases} \Rightarrow \cosh\left[n\cosh^{-1}\left(\frac{\omega_s}{\omega_p}\right)\right] = \left[\frac{10^{-\frac{G_s}{10}}-1}{10^{\frac{\hat{r}}{10}}-1}\right]^{\frac{1}{2}}$$

$$\Rightarrow \boxed{n = \frac{1}{\cosh^{-1}\left(\frac{\omega_s}{\omega_p}\right)}\cosh^{-1}\left[\frac{10^{-\frac{G_s}{10}}-1}{10^{\frac{\hat{r}}{10}}-1}\right]^{\frac{1}{2}}}$$

Chebyshev Filters

- ***Determination of ε***

$$\boxed{\frac{\sqrt{10^{-G_s/10}-1}}{\cosh[n\cosh^{-1}(\omega_s/\omega_p)]} \le \varepsilon \le \sqrt{10^{r/10}-1}}$$

- ***Chebyshev filter transfer function***

✓ The chebyshev filter poles are:

$$s_k = -\sin\frac{(2k-1)\pi}{2n}\sinh x + j\cos\frac{(2k-1)\pi}{2n}\cosh x$$

$$k = 1,2,3,...,n$$

✓ The poles lie on a semiellipse of the major and minor semiaxes coshx and sinhx:

Chebyshev Filters

$$x = \frac{1}{n}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)$$

$$a = \sinh x$$

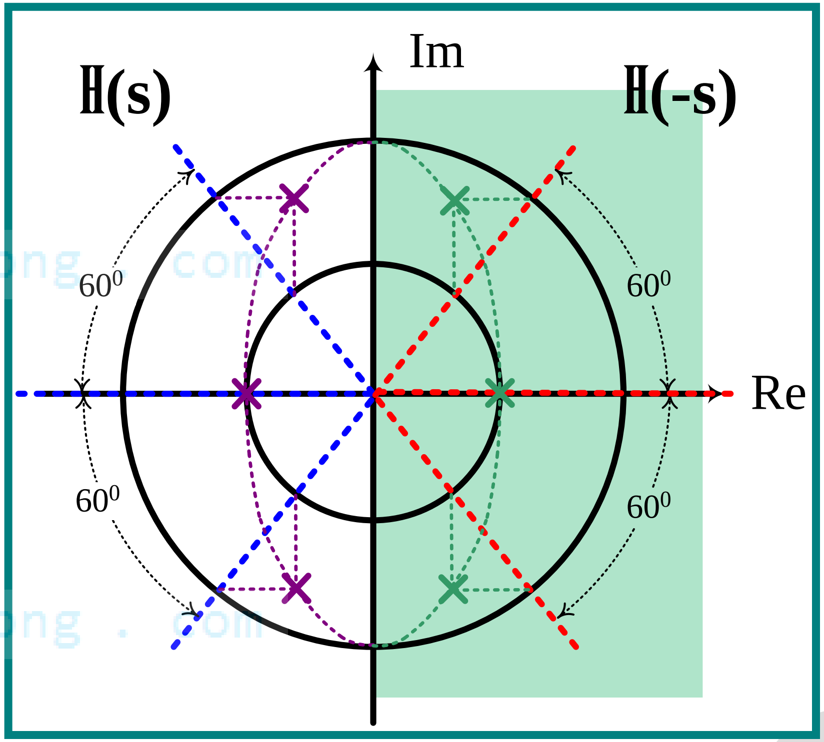
$$b = \cosh x$$

✓ The transfer function $H_N(s)$:

$$H_N(s) = \frac{K_n}{(s-s_1)(s-s_2)\ldots(s-s_n)}$$

$$= \frac{K_n}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

$$= \frac{K_n}{C_n^{'}(s)}$$



Chebyshev Filters

- The constant K_n is selected to have proper DC gain:

$$K_n = \begin{cases} a_0 & n \text{ odd} \\ \dfrac{a_0}{\sqrt{1+\varepsilon^2}} & n \text{ even} \end{cases}$$

- *Example:* Design a Chebyshev lowpass filter to satisfy the following criteria:
  - ✓ The ripple $r \le 2$dB over a passband $0 \le \omega \le 10$ ($\omega_p = 10$)
  - ✓ The stopband gain $G_s \le -20$dB for $\omega > 20$ ($\omega_s = 20$)

Chebyshev Filters

Table for Chebyshev Filter Coefficients of the Denominator Polynomial

$$C_n' = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_1 s + a_0$$

Chebyshev Filters

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6
1	2.8627752						
2	1.5162026	1.4256245					
3	0.7156938	1.5348954	1.2529130				
4	0.3790506	1.0254553	1.7168662	1.1973856			
5	0.1789234	0.7525181	1.3095747	1.9373675	1.1724909		
6	0.0947626	0.4323669	1.1718613	1.5897635	2.1718446	1.1591761	
7	0.0447309	0.2820722	0.7556511	1.6479029	1.8694079	2.4126510	1.1512176

0.5 dB ripple

$r = 0.5dB$

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6
1	1.9652267						
2	1.1025103	1.0977343					
3	0.4913067	1.2384092	0.9883412				
4	0.2756276	0.7426194	1.4539248	0.9528114			
5	0.1228267	0.5805342	0.9743961	1.6888160	0.9368201		
6	0.0689069	0.3070808	0.9393461	1.2021409	1.9308256	0.9282510	
7	0.0307066	0.2136712	0.5486192	1.3575440	1.4287930	2.1760778	0.9231228

1 dB ripple

$r = 1dB$

Chebyshev Filters

2 dB ripple

$r = 2dB$

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6
1	1.3075603						
2	0.8230604	0.8038164					
3	0.3268901	1.0221903	0.7378216				
4	0.2057651	0.5167981	1.2564819	0.7162150			
5	0.0817225	0.4593491	0.6934770	1.4995433	0.7064606		
6	0.0514413	0.2102706	0.7714618	0.8670149	1.7458587	0.7012257	
7	0.0204228	0.1660920	0.3825056	1.1444390	1.0392203	1.9935272	0.6978929

3 dB ripple

$r = 3dB$

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6
1	1.0023773						
2	0.7079478	0.6448996					
3	0.2505943	0.9283480	0.5972404				
4	0.1769869	0.4047679	1.1691176	0.5815799			
5	0.0626391	0.4079421	0.5488626	1.4149874	0.5744296		
6	0.0442467	0.1634299	6990977	6906098	1.6628481	0.5706979	
7	0.0156621	0.1461530	0.3000167	1.0518448	0.8314411	1.9115507	0.5684201

Chebyshev Filters

Table for Chebyshev Filter Pole Locations

Chebyshev Filters

n	$r = 0.5dB$	$r = 1dB$	$r = 2dB$	$r = 3dB$
1	-2.8628	-1.9652	-1.3076	-1.0024
2	$-0.7128 \pm j1.0040$	$-0.5489 \pm j0.8951$	$-0.4019 \pm j0.8133$	$-0.3224 \pm j0.7772$
3	-0.6265	-0.4942	-0.3689	-0.2986
	$-0.3132 \pm j1.0219$	$-0.2471 \pm j0.9660$	$-0.1845 \pm j0.9231$	$-0.1493 \pm j0.9038$
4	$-0.1754 \pm j1.0163$	$-0.1395 \pm j0.9834$	$-0.1049 \pm j0.9580$	$-0.0852 \pm j0.9465$
	$-0.4233 \pm j0.4209$	$-0.3369 \pm j0.4073$	$-0.2532 \pm j0.3968$	$-0.2056 \pm j0.3920$
5	-0.3623	-0.2895	-0.2183	-0.1775
	$-0.1120 \pm j1.0116$	$-0.0895 \pm j0.9901$	$-0.0675 \pm j0.9735$	$-0.0549 \pm j0.9659$
	$-0.2931 \pm j0.6252$	$-0.2342 \pm j0.6119$	$-0.1766 \pm j0.6016$	$-0.1436 \pm j0.5970$
6	$-0.0777 \pm j1.0085$	$-0.0622 \pm j0.9934$	$-0.0470 \pm j0.9817$	$-0.0382 \pm j0.9764$
	$-0.2121 \pm j0.7382$	$-0.1699 \pm j0.7272$	$-0.1283 \pm j0.7187$	$-0.1044 \pm j0.7148$
	$-0.2898 \pm j0.2702$	$-0.2321 \pm j0.2662$	$-0.1753 \pm j0.2630$	$-0.1427 \pm j0.2616$

Chebyshev Filters

n	$r = 0.5dB$	$r = 1dB$	$r = 2dB$	$r = 3dB$
7	-0.2562	-0.2054	-0.1553	-0.1265
	$-0.0570 \pm j1.0064$	$-0.0457 \pm j0.9953$	$-0.0346 \pm j0.9866$	$-0.0281 \pm j0.9827$
	$-0.1597 \pm j0.8071$	$-0.1281 \pm j0.7982$	$-0.0969 \pm j0.7912$	$-0.0789 \pm j0.7881$
	$-0.2308 \pm j0.4479$	$-0.1851 \pm j0.4429$	$-0.1400 \pm j0.4391$	$-0.1140 \pm j0.4373$
8	$-0.0436 \pm j1.0050$	$-0.0350 \pm j0.9965$	$-0.0265 \pm j0.9898$	$-0.0216 \pm j0.9868$
	$-0.1242 \pm j0.8520$	$-0.0997 \pm j0.8447$	$-0.0754 \pm j0.8391$	$-0.0614 \pm j0.8365$
	$-0.1859 \pm j0.5693$	$-0.1492 \pm j0.5644$	$-0.1129 \pm j0.5607$	$-0.0920 \pm j0.5590$
	$-0.2193 \pm j0.1999$	$-0.1760 \pm j0.1982$	$-0.1332 \pm j0.1969$	$-0.1085 \pm j0.1962$
9	-0.1984	-0.1593	-0.1206	-0.0983
	$-0.0345 \pm j1.0040$	$-0.0277 \pm j0.9972$	$-0.0209 \pm j0.9919$	$-0.0171 \pm j0.9896$
	$-0.0992 \pm j0.8829$	$-0.0797 \pm j0.8769$	$-0.0603 \pm j0.8723$	$-0.0491 \pm j0.8702$
	$-0.1520 \pm j0.6553$	$-0.1221 \pm j0.6509$	$-0.0924 \pm j0.6474$	$-0.0753 \pm j0.6459$
	$-0.1864 \pm j0.3487$	$-0.1497 \pm j0.3463$	$-0.1134 \pm j0.3445$	$-0.0923 \pm j0.3437$
10	$-0.0279 \pm j1.0033$	$-0.0224 \pm j0.9978$	$-0.0170 \pm j0.9935$	$-0.0138 \pm j0.9915$
	$-0.0810 \pm j0.9051$	$-0.1013 \pm j0.7143$	$-0.0767 \pm j0.7113$	$-0.0401 \pm j0.8945$
	$-0.1261 \pm j0.7183$	$-0.0650 \pm j0.9001$	$-0.0493 \pm j0.8962$	$-0.0625 \pm j0.7099$
	$-0.1589 \pm j0.4612$	$-0.1277 \pm j0.4586$	$-0.0967 \pm j0.4567$	$-0.0788 \pm j0.4558$
	$-0.1761 \pm j0.1589$	$-0.1415 \pm j0.1580$	$-0.1072 \pm j0.1574$	$-0.0873 \pm j0.1570$

Chebyshev Filters

- *Example:* Design a Chebyshev lowpass filter to satisfy the following criteria:
  - ✓ The ripple $r \leq 2$dB over a passband $0 \leq \omega \leq 10$ ($\omega_p = 10$)
  - ✓ The stopband gain $G_s \leq -20$dB for $\omega > 20$ ($\omega_s = 20$)

➢ ***Step 1:*** Determine n

$$n \geq \frac{1}{\cosh^{-1}(2)} \cosh^{-1}\left(\frac{10^2 - 1}{10^{0.2} - 1}\right)^{1/2} = 2.473 \Rightarrow n = 3$$

➢ ***Step 2:*** Determine ε

$$\frac{\sqrt{10^2 - 1}}{\cosh[3\cosh^{-1}(2)]} \leq \varepsilon \leq \sqrt{10^{0.2} - 1} \Leftrightarrow 0.382 \leq \varepsilon \leq 0.764$$

Chebyshev Filters

We choose $\varepsilon = 0.763$ for $r = 2$dB.

➢ ***Step 3:*** Determine $H_N(s)$

$$H_N(s) = \frac{0.3269}{s^3 + 0.7378s^2 + 1.0222s + 0.3269}$$

➢ ***Step 4:*** Determine $H(s)$

$$H(s) = \frac{0.3269}{\left(\frac{s}{10}\right)^3 + 0.7378\left(\frac{s}{10}\right)^2 + 1.0222\left(\frac{s}{10}\right) + 0.3269}$$

$$= \frac{326.9}{s^3 + 7.378s^2 + 102.22s + 326.9}$$

<u>*Chapter 7:*</u>

Frequency response of LTI system and Analog filter design

1. *Frequency Response – Bode Plots*
2. *Analog filter design*
   - *Butterworth Filters*
   - *Chebyshev Filters*
   - ***Frequency Transformation***

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Frequency Transformation

- Using certain frequency transformations, we can obtain transfer functions of highpass, bandpass and bandstop filters ***from a basic lowpass filter*** (prototype filter).

- The prototype filter may be of any kind (Butterworth, Chebyshev…), we ***first design a suitable lowpass filter*** $H_P(s)$, and ***then replace 's' with a proper transformation T(s)*** to obtain the desired filter.

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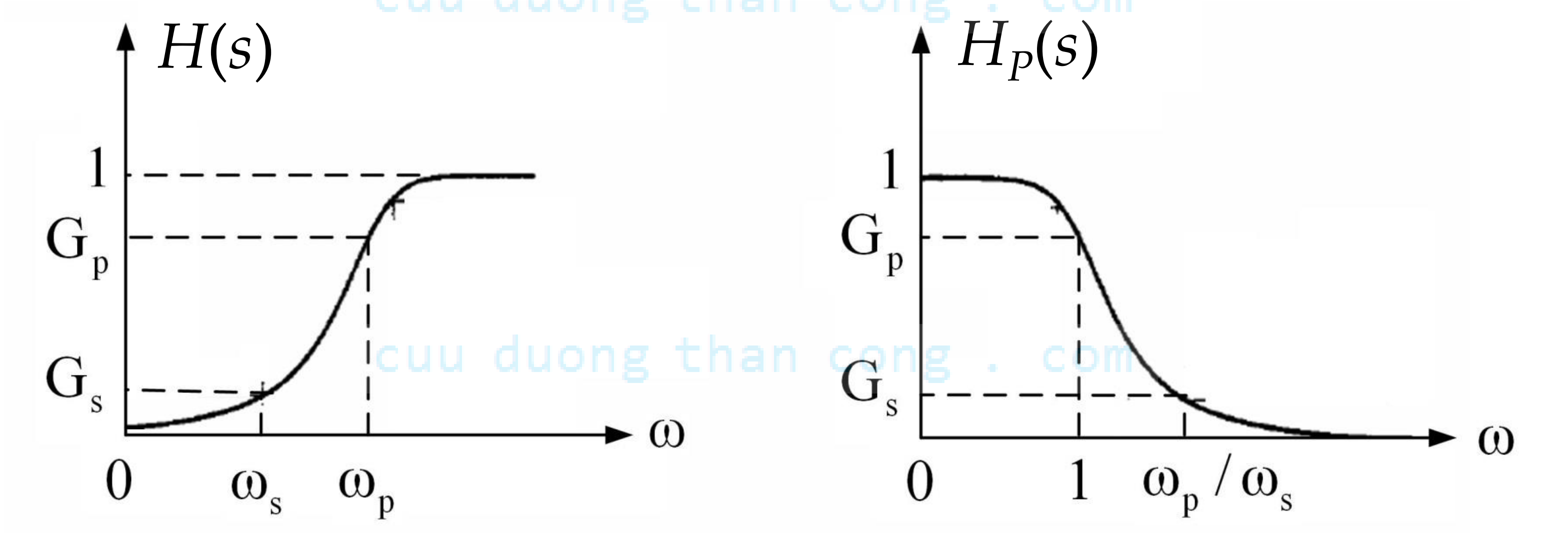
Frequency Transformation

Highpass Filters

- Determine prototype filter $H_P(s)$ with the passband $0 \le \omega \le 1$ and the stopband $\omega \ge \omega_p/\omega_s$.
- Replace s with $T(s) = \omega_p/s$.



Frequency Transformation

Bandpass Filters

- Determine prototype filter $H_P(s)$ with the passband $0 \le \omega \le 1$ and the stopband ω_s. where:

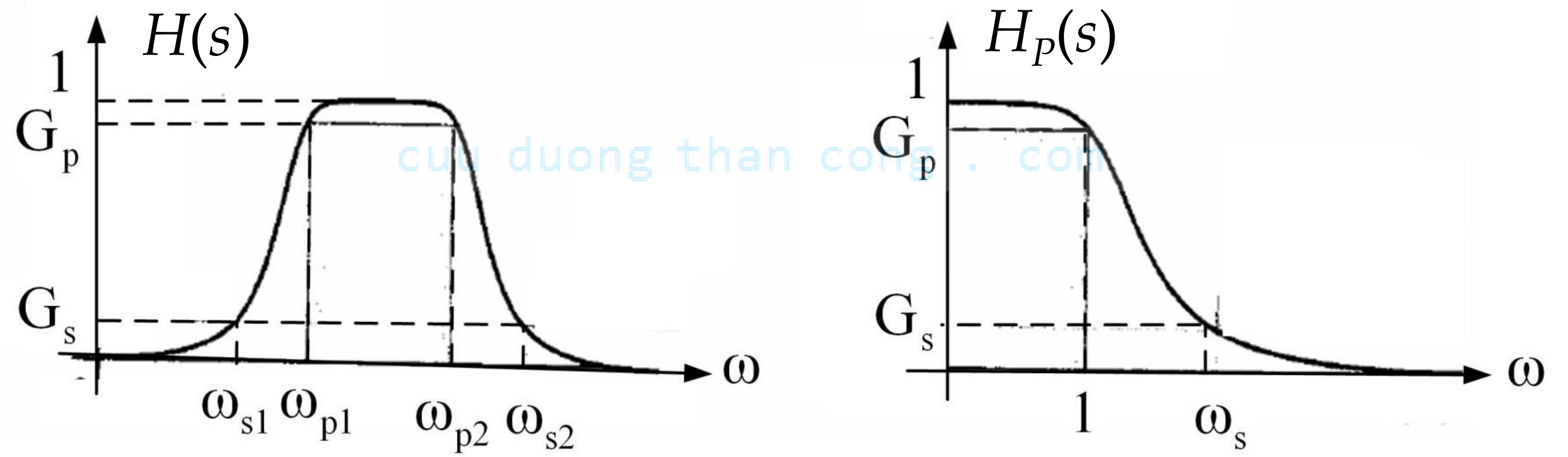
$$\omega_s = \min\left\{\frac{\omega_{p1}\omega_{p2} - \omega_{s1}^2}{\omega_{s1}\left(\omega_{p2} - \omega_{p1}\right)}; \frac{\omega_{s2}^2 - \omega_{p1}\omega_{p2}}{\omega_{s2}\left(\omega_{p2} - \omega_{p1}\right)}\right\}$$

- Replace s with $T(s)$.

$$T(s) = \frac{s^2 + \omega_{p1}\omega_{p2}}{(\omega_{p2} - \omega_{p1})s}$$

Frequency Transformation

Bandpass Filters



Frequency Transformation

Bandstop Filters

- Determine prototype filter $H_p(s)$ with the passband $0 \le \omega \le 1$ and the stopband ω_s. where:

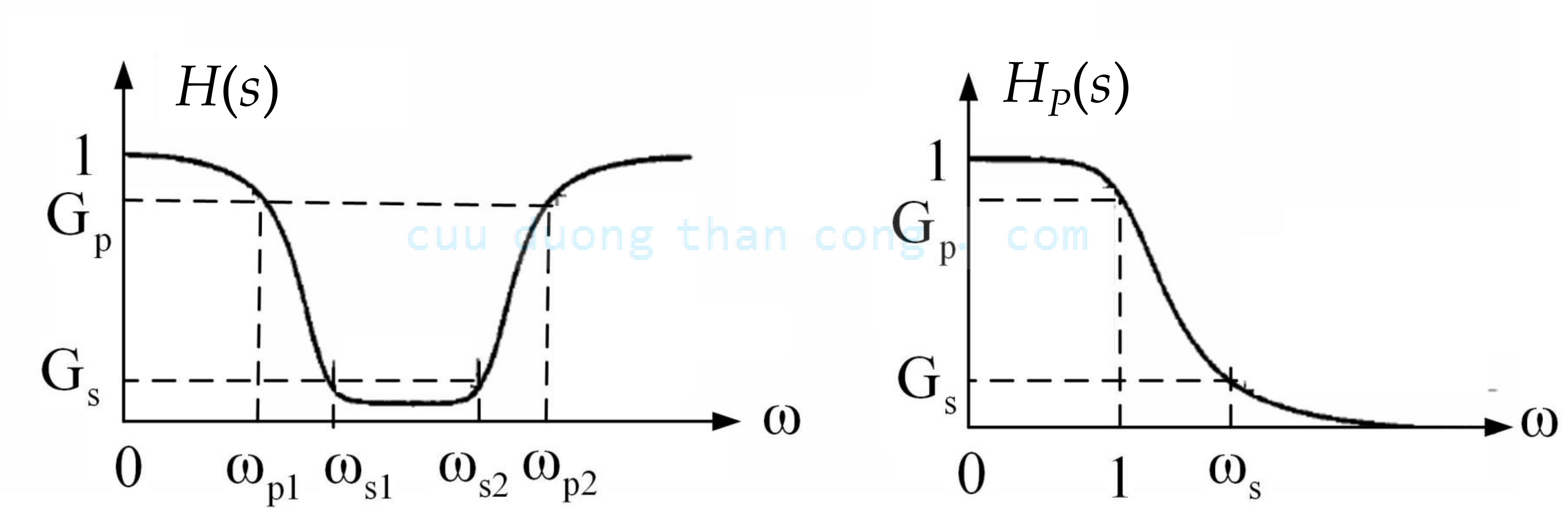
$$\omega_s = \min\left\{\frac{\omega_{s1}\left(\omega_{p2}-\omega_{p1}\right)}{\omega_{p1}\omega_{p2}-\omega_{s1}^2};\frac{\omega_{s2}\left(\omega_{p2}-\omega_{p1}\right)}{\omega_{s2}^2-\omega_{p1}\omega_{p2}}\right\}$$

- Replace s with $T(s)$.

$$T(s)=\frac{(\omega_{p2}-\omega_{p1})s}{s^2+\omega_{p1}\omega_{p2}}$$

Frequency Transformation

Bandstop Filters



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