CHƯƠNG 7: XOẮN THUẦN TÚY THANH THẲNG (PURE TORSION)

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I/ INTRODUCTION (GIỚI THIỆU)

- Torsion → the twisting of a structural member is loaded by couples that produce rotation about its longitudinal axis
- Categories:

  ✓ Pure torsion: → only shear stresses (τ)

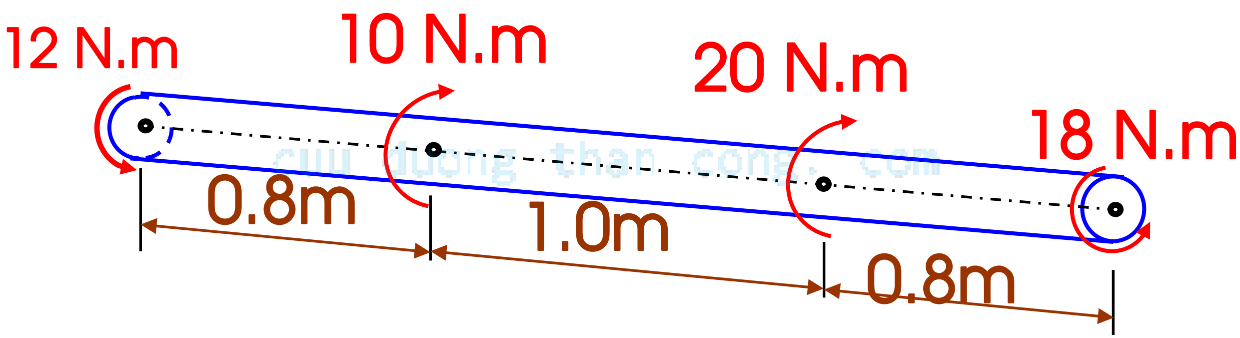
  Ex: Circular shaft, rectangular member subjected to end torques only..

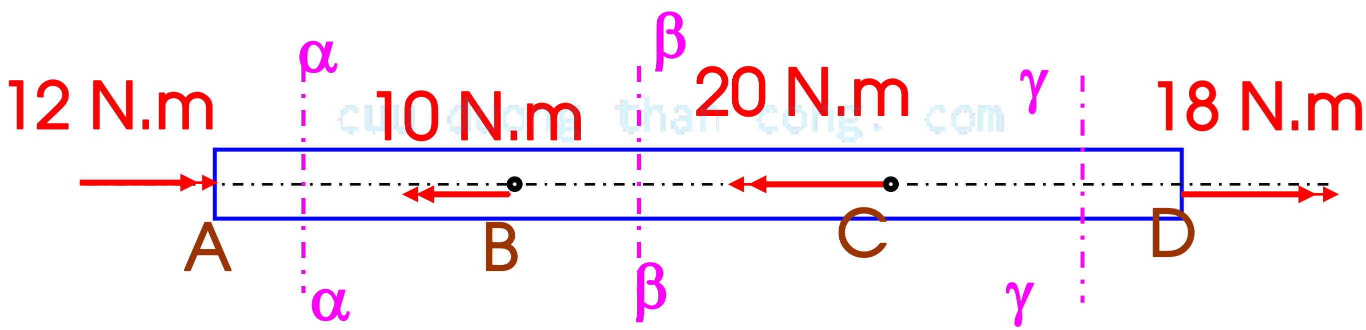
  ✓ Restrained torsion: $\tau \neq 0$; $\sigma \neq 0$ though both N and M are not present

  Ex: Non-circular members with one end fixed subjected to torques...

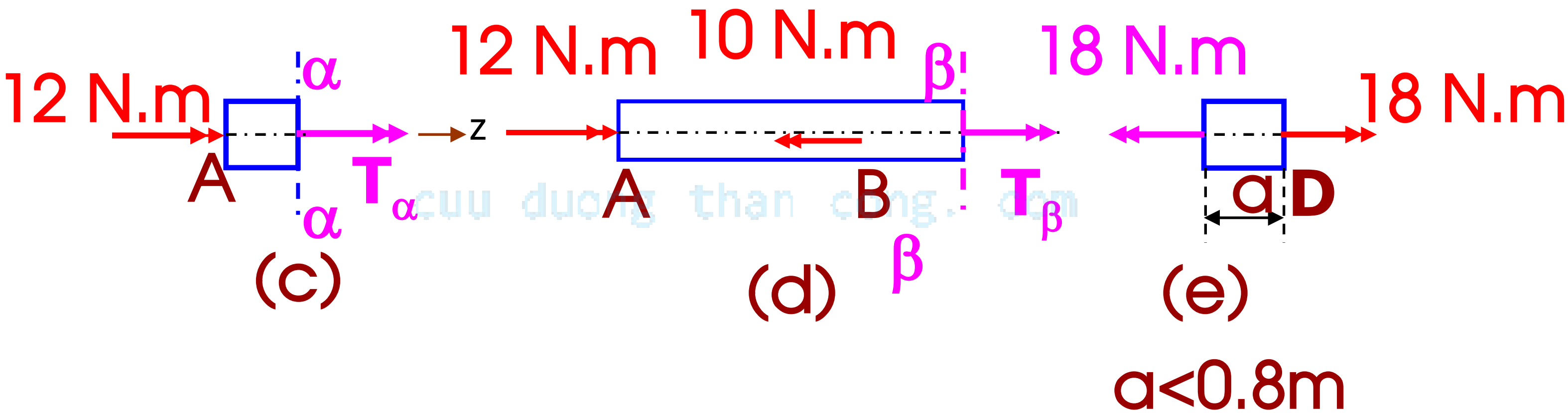
II/ TORSIONAL MOMENTS IN STATICALLY DETERMINATE MEMBERS

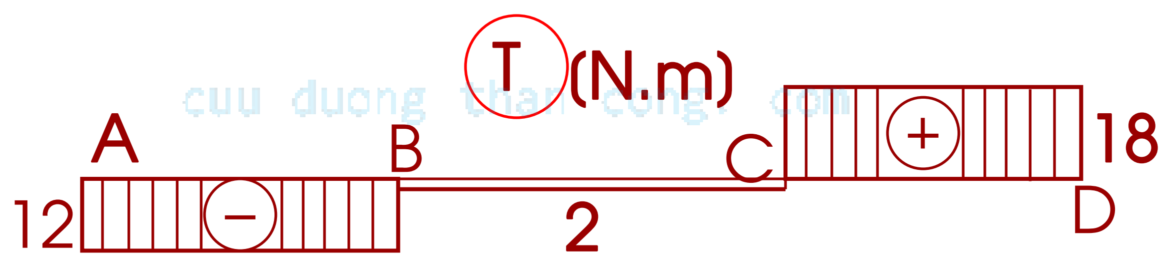
Example 1:





II/ TORSIONAL MOMENTS IN STATICALLY DETERMINATE MEMBERS (continued)





II/ TORSIONAL MOMENTS IN STATICALLY DETERMINATE MEMBERS (continued)

- Conventions (Quy ước):

  • The given external torsional moments and the internal torques can be expressed by vectors, by applying the right-hand rule (As in Fig. b)

  • The internal torque T is positive if its vector acts away from the respective cross section

II/ TORSIONAL MOMENTS IN STATICALLY DETERMINATE MEMBERS (continued)

- Internal torque expressions for each part:
  - Part AB: Ta +12 =0 → Ta = -12 (Fig.c)
  - Part BC: Tb +12 -10=0 → Tb = -2 (Fig.d)
  - Part CD: Tg - 18 = 0 → Tg = 18 (Fig.e)

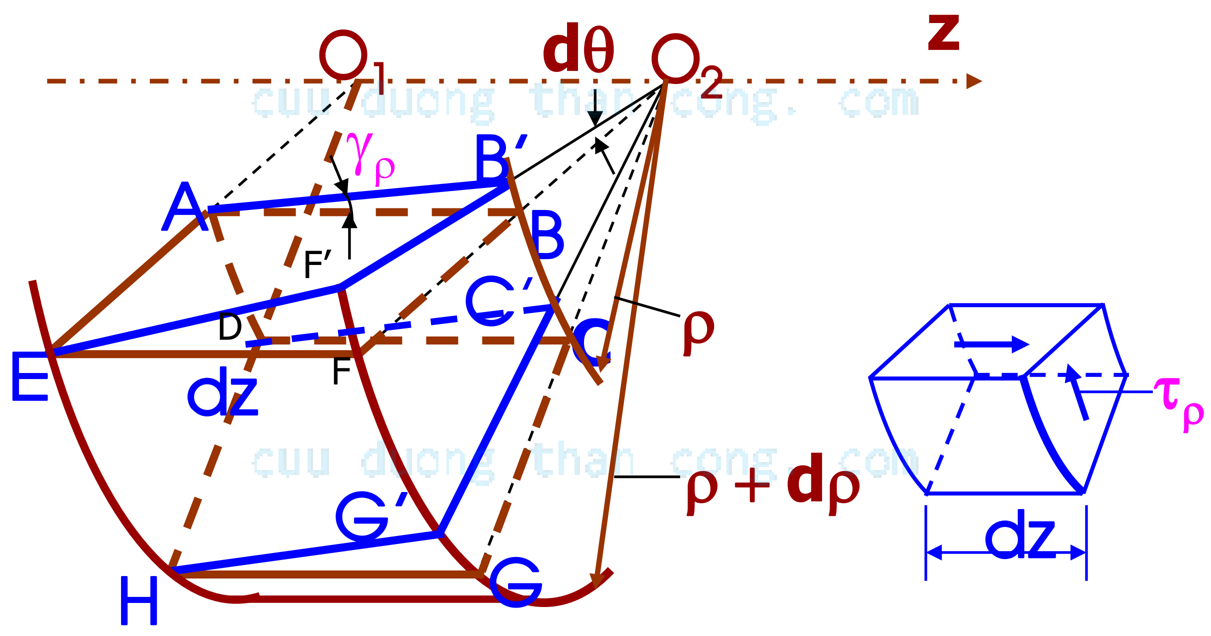
- The internal torque diagram is in Fig.f

III/ TORSIONAL MEMBERS OF CIRCULAR CROSS SECTION

- Assumptions:
  - ✓ Material: homogenous, isotrope, and linear elastic
  - ✓ Member: straight, uniform cross section
  - ✓ Radii: always straight & unchanged length during twisting
  - ✓ Cross sections: always plane during twisting

III/ TORSIONAL MEMBERS OF CIRCULAR CROSS SECTION (cont.)

- Formulation of stress



III/ TORSIONAL MEMBERS OF CIRCULAR CROSS SECTION (cont.)

- Consider an infinitesimal cubic element ABCDEFGH. After twisting →AB'C'DEF'G'H

- Deformation

$$BB' = \gamma_\rho dz = \rho d\theta$$

whence

$$\gamma_\rho = \rho \frac{d\theta}{dz}$$

- Hooke's law:

$$\tau = G\gamma = G\rho\frac{d\theta}{dz}$$

III/ TORSIONAL MEMBERS OF CIRCULAR CROSS SECTION (cont.)

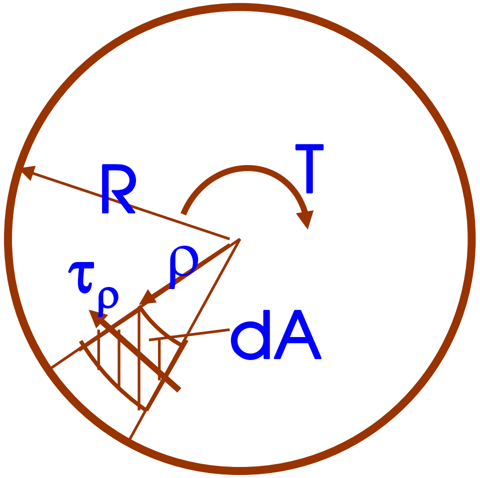
- Equilibrium: $T = \int_A \rho \tau_\rho dA$
- Developping:

$$T = \int_A G \frac{d\theta}{dz} \rho^2 dA = G \frac{d\theta}{dz} \int_A \rho^2 dA = GJ \frac{d\theta}{dz}$$

or $$\frac{d\theta}{dz} = \frac{T}{GJ}$$

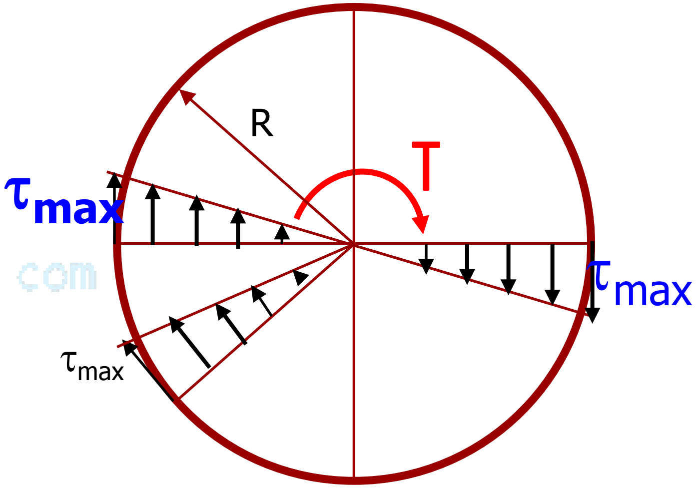
with $J = \int_A \rho^2 dA$

- polar moment of inertia



III/ TORSIONAL MEMBERS OF CIRCULAR CROSS SECTION (cont.)

- **Shear Stress:** $\tau_\rho = \frac{T}{J}\rho$



$\rho = R,\quad \tau_\rho \rightarrow \tau_{max} = \frac{T}{J}R = \frac{T}{Z_p}$

with Zp - Polar Modulus of Section

$$Z_p = \frac{J}{R} = \frac{\pi R^3}{2} = \frac{\pi D^3}{16} \approx 0.2D^3$$

III/ TORSIONAL MEMBERS OF CIRCULAR CROSS SECTION (cont.)

- The angle of twist between the 2 sections of length L:

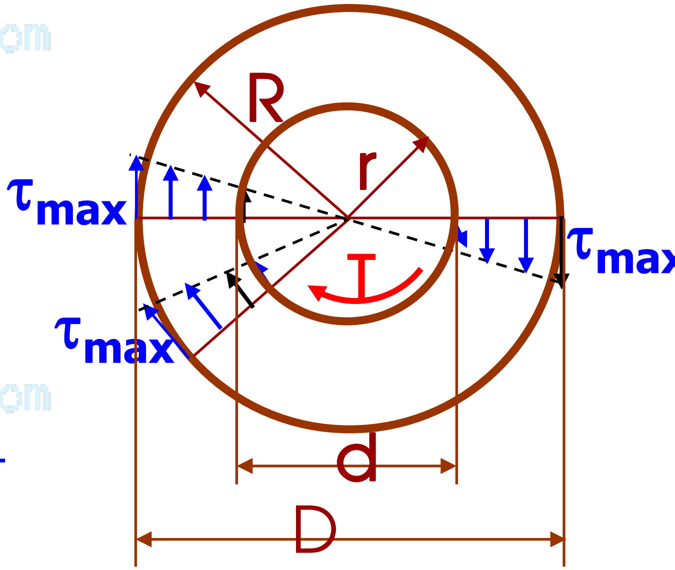
$$\theta = \int_0^L d\theta = \int_0^L \frac{T}{GJ} dz = \frac{TL}{GJ}$$

III/ TORSIONAL MEMBERS OF CIRCULAR CROSS SECTION (cont.)

- Notation:
  - For hollow circular section:

$$J = \frac{\pi\left(R^4 - r^4\right)}{2} = \frac{\pi\left(D^4 - d^4\right)}{32}$$

$$Z_p = \frac{\pi\left(R^4 - r^4\right)}{2R} = \frac{\pi\left(D^4 - d^4\right)}{16D}$$



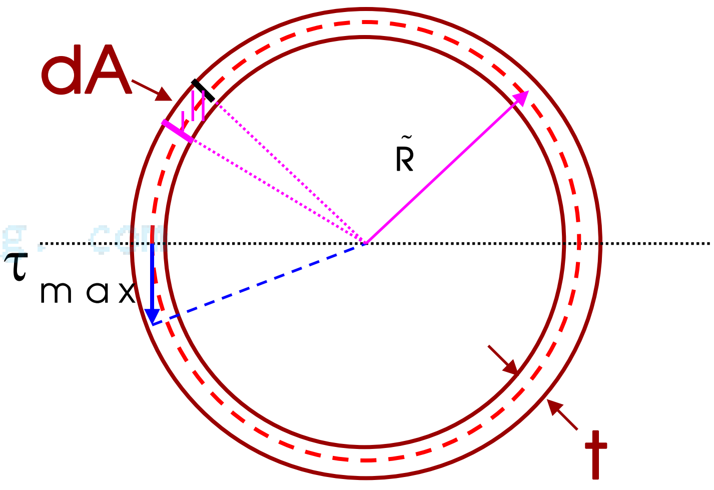
III/ TORSIONAL MEMBERS OF CIRCULAR CROSS SECTION (cont.)

➢ When the wall thickness of a shaft is small compared with its outer radius, the torsional properties of the hollow section can be approximatively as:

$$J = \int_A \rho^2 dA \cong \tilde{R}^2 A = 2\pi t \tilde{R}^3$$

$$Z_p = \frac{J}{A} = 2\pi t \tilde{R}^2$$

$$\tau \cong \tau_{max}$$

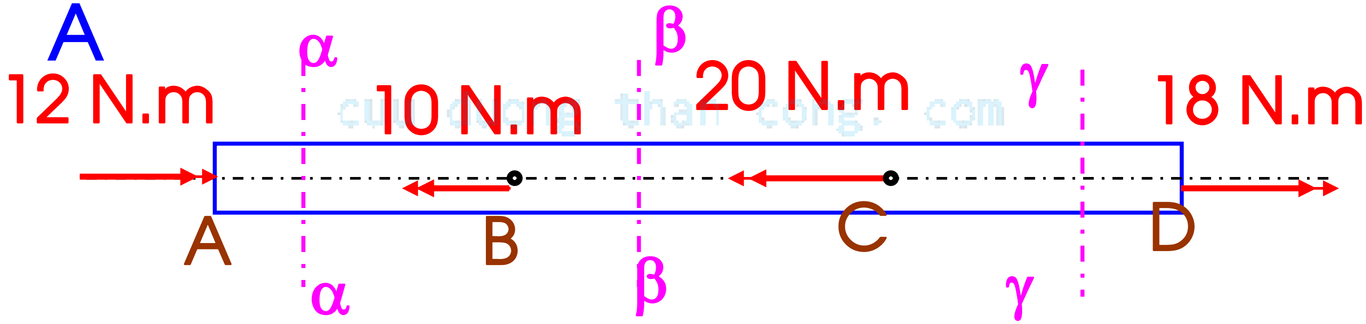


III/ TORSIONAL MEMBERS OF CIRCULAR CROSS SECTION (cont.)

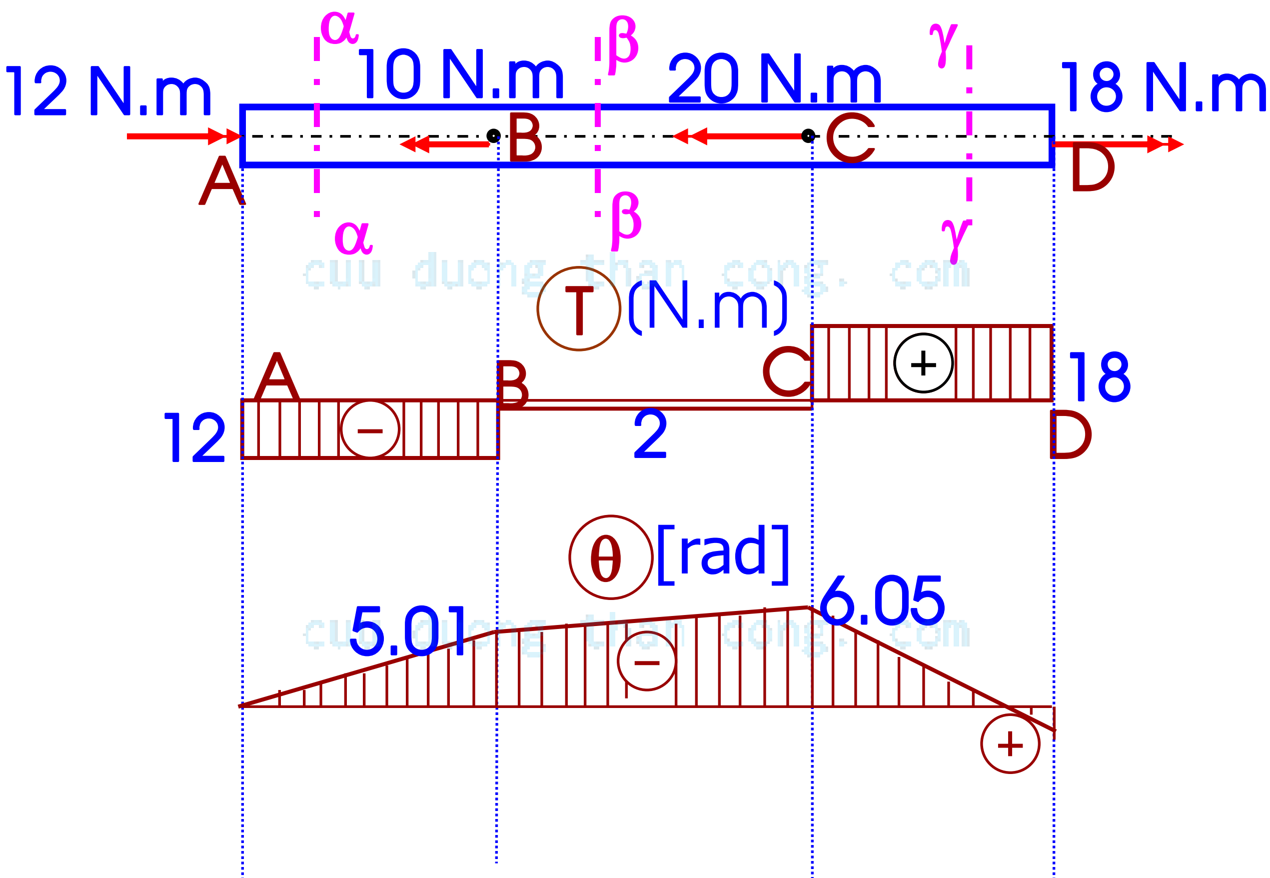
- Example 2: Consider the shaft in the Ex.1. Assume: $G = 80$ GPa, $[\tau] = 50$MPa.

  (a) Determine the minimum diameter [d]

  (b) Plot the diagram of angles of twist of particular cross sections with respect to A



III/ TORSIONAL MEMBERS OF CIRCULAR CROSS SECTION (cont.)



III/ TORSIONAL MEMBERS OF CIRCULAR CROSS SECTION (cont.)

- Maximum stress occurs at: $T_{max} = 18$ N.m

$$\tau_{max} = \frac{T_{max}}{Z_p} = \frac{16T_{max}}{\pi d^3} \le \tau_{all} \rightarrow d \ge \sqrt[3]{\frac{16T_{max}}{\pi \tau_{all}}} = 12.24 mm$$

- Twisting angle of section at z from A in AB:

$$\theta = \int_0^z \frac{T}{GJ} dz = \frac{Tz}{GJ} = 5.22 x 10^{-3} Tz$$

(T in N.m and z in m)

III/ TORSIONAL MEMBERS OF CIRCULAR CROSS SECTION (cont.)

$$\theta_B = (\theta)_{z=0.8} = -5.01x10^{-2} \text{radian}$$

$$\theta_C = \theta_{AC} = \theta_B + \theta_{BC} = \theta_B + \frac{-2x1}{GJ}$$

$$= -5.01x10^{-2} + 5.22x10^{-3}(-2)x1$$

$$= -6.05x10^{-2} \text{rad}$$

Similarly,

$$\theta_D = \theta_C + \int_{1.8}^{2.6} \frac{18dz}{GJ} = 1.47x10^{-2} \text{rad}$$

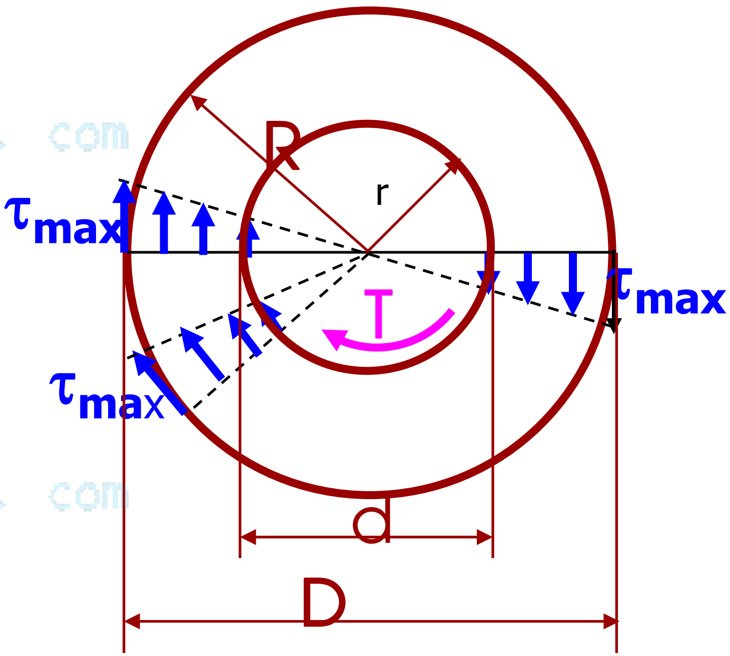
III/ TORSIONAL MEMBERS OF CIRCULAR CROSS SECTION (cont.)

- Example 3: A hollow section shaft in pure torsion

$T = 5kNm; \; \eta = \frac{d}{D} = 0.7$

$\tau_{all} = 60MPa$

Determine (D) and the weigth reduction compared to the solid section



III/ TORSIONAL MEMBERS OF CIRCULAR CROSS SECTION (cont.)

- SOLUTION
  - ✓ The polar modulus:

$$Z_p = \frac{\pi\left(D^4 - d^4\right)}{16D} = \frac{\pi}{16}D^3\left(1-\eta^4\right) = 0.149D^3$$

  - ✓ The strength condition :

$$\tau_{max} = \frac{T}{Z_p} = \frac{T}{0.149D^3} \leq \tau_{all}$$

III/ TORSIONAL MEMBERS OF CIRCULAR CROSS SECTION (cont.)

- ✓ Choice of section:

$$D \geq 1.886 \sqrt[3]{\frac{T}{\tau_{all}}} = 82.4 \times 10^{-3} m$$

- ✓ The hollow section area:

$$A_h = \frac{\pi D^2}{4}\left(1 - \eta^2\right) = \frac{\pi \times 82.4^2}{4}\left(1 - 0.7^2\right)$$

$$= 2717.8 mm^2$$

III/ TORSIONAL MEMBERS OF CIRCULAR CROSS SECTION (cont.)

- In case of solid section:

$$Z_p = \frac{\pi D_s^3}{16} \geq \frac{T}{\tau_{all}} \rightarrow D_s = 75.2mm$$

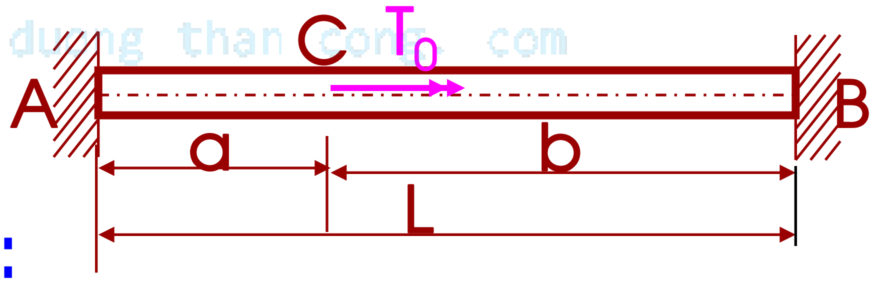
$$A_s = \frac{\pi D_s^2}{4} = 4441.5mm^2$$

- The weight reduction: $r = \frac{A_s - A_h}{A_s}100\% = 38.8\%$

→ the advantage of hollow section!

IV/ STATICALLY INDETERMINATE PROBLEMS OF TORSION

- Example 4: Steel shaft, d = 80mm, a=0.5m, b=1.0m, T_0=7.5kNm, G=80GPa.



Determine:

(i) Diagram of internal torque

(ii) Diagram of angles of twist

(iii) The maximum shear stress

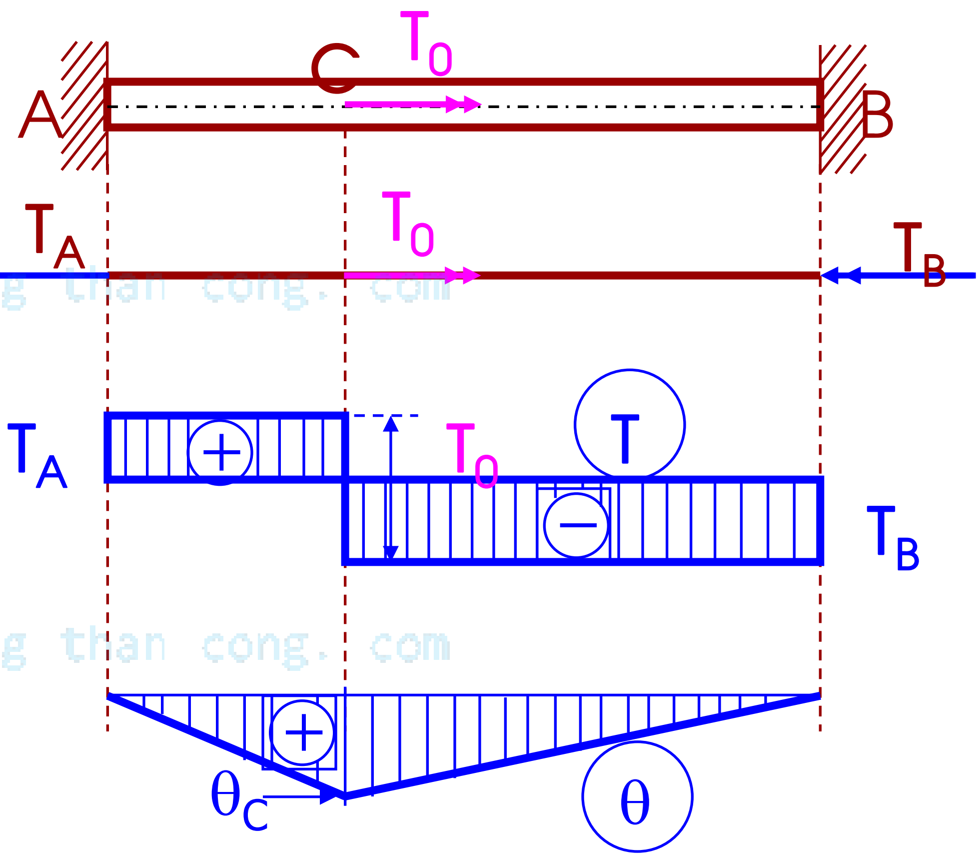
IV/ STATICALLY INDETERMINATE PROBLEMS OF TORSION (cont.)

a/ Statically Indeter-minate Structure

b/ Free-body diagram

c/ Torsional Moment Diagram

d/ Twisting angle diagram



IV/ STATICALLY INDETERMINATE PROBLEMS OF TORSION (cont.)

- Diagram of internal torque:
  - ✓ Equilibrium equation: (P/t cân bằng)

$$T_A + T_B - T_0 = 0 \quad (a)$$

  - ✓ Compatibility condition of deformation:(P/t tương thích của b/dạng

$$\theta_{B/A} = \int_0^L \frac{T dz}{GJ} = \int_0^a \frac{T_A dz}{GJ} + \int_a^L \frac{(-T_B) dz}{GJ} = \frac{T_A a}{GJ} - \frac{T_B b}{GJ} = 0 \quad (b)$$

IV/ STATICALLY INDETERMINATE PROBLEMS OF TORSION (cont.)

whence $T_A = \frac{b}{a} T_B$ (c)

✓ Resolution of (a) & (c) gives:

$$T_B = \frac{a}{a+b} T_0 = \frac{a}{L} T_0 = \frac{0.5}{1.5} 7.5 = 2.5 \, kN.m$$

(c) → $T_A = \frac{1}{0.5} 2.5 = 5 \, kN.m$

V/TRANSMISSION OF POWER BY CIRCULAR SHAFTS

- The use of circular shafts to transmit mechanical power from one device to another is very important
- The power is transmitted through the rotary motion of the shaft, and the amount of power transmitted depends upon the magnitude of the torque and the speed of rotation

V/TRANSMISSION OF POWER BY CIRCULAR SHAFTS (continued)

- Work done by a torque T on the angle α

  $W = T\alpha$, α - angle of rotation(in rad.)

  $[W] = N.m.rad$ (or Joules)

  If $\alpha = \omega t$, ω – angular speed (in rad/s)

  Then $W = T\omega t$

- Power: = time rate of doing work

$$P = \frac{dW}{dt} = T\omega \qquad (Nm.rad/s = J/s = Watt)$$

V/TRANSMISSION OF POWER BY CIRCULAR SHAFTS (continued)

- If:

n - number of revolutions per minute (rpm)

f – frequency of rotations, in Hz(hertz)

Then:

$$P = T\omega = T(2\pi f) = \frac{2\pi n}{60} T$$

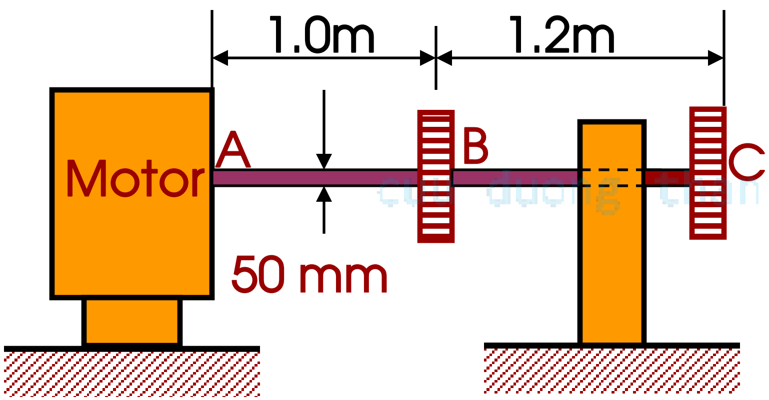
- Remark: When P is known together with n or f, the torque can be determined:

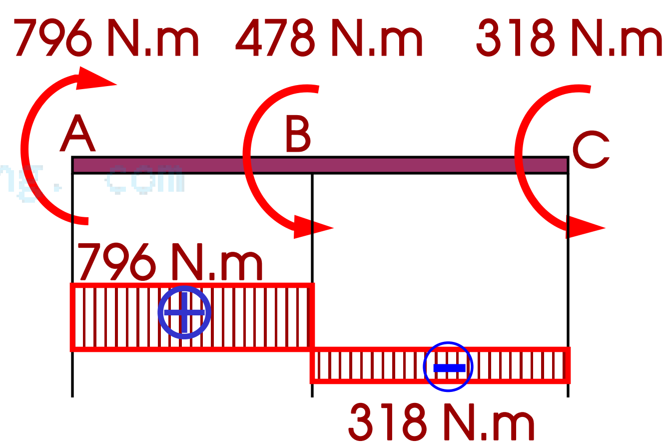
$$T = \frac{P}{\omega} = \frac{P}{2\pi f} = \frac{30P}{\pi n}$$

V/TRANSMISSION OF POWER BY CIRCULAR SHAFTS (cont.)-Example

- Example 1: Steel shaft ABC, diameter 50mm, motor at A transmits 50kW at 10Hz

  The gears at B and C require powers: 30kW & 20kW. Compute: τ_{max} & φ_{AC}





V/TRANSMISSION OF POWER BY CIRCULAR SHAFTS (cont.)-Example

- SOLUTION

  ✓ $P_A = 50kW$ → the torque transferred by the motor

$$T_a = \frac{P_A}{2\pi f} = \frac{50kW}{2\pi(10Hz)} = 796N.m$$

✓ $P_C = 20kW \rightarrow$ $T_c = \frac{P_c}{2\pi f} = \frac{20kW}{2\pi(10Hz)} = 318N.m$

✓ $P_C = 20kW \rightarrow$ $T_b = \frac{P_b}{2\pi f} = \frac{30kW}{2\pi(10Hz)} = 478N.m$

V/TRANSMISSION OF POWER BY CIRCULAR SHAFTS (cont.)-Example

- The shear stress and angle of twist in AB

$$\tau_{ab} = \frac{16T_{ab}}{\pi d^3} = \frac{16(796N.m)}{\pi(50mm)^3} = 32.4MPa$$

$$\varphi_{ab} = \frac{T_{ab}L_{ab}}{GI_p} = \frac{(796N.m)(1.0m)}{(80GPa)\left(\frac{\pi}{32}\right)(50mm)^4} = 0.0162\ rad$$

V/TRANSMISSION OF POWER BY CIRCULAR SHAFTS (cont.)-Example

- The shear stress and angle of twist in BC

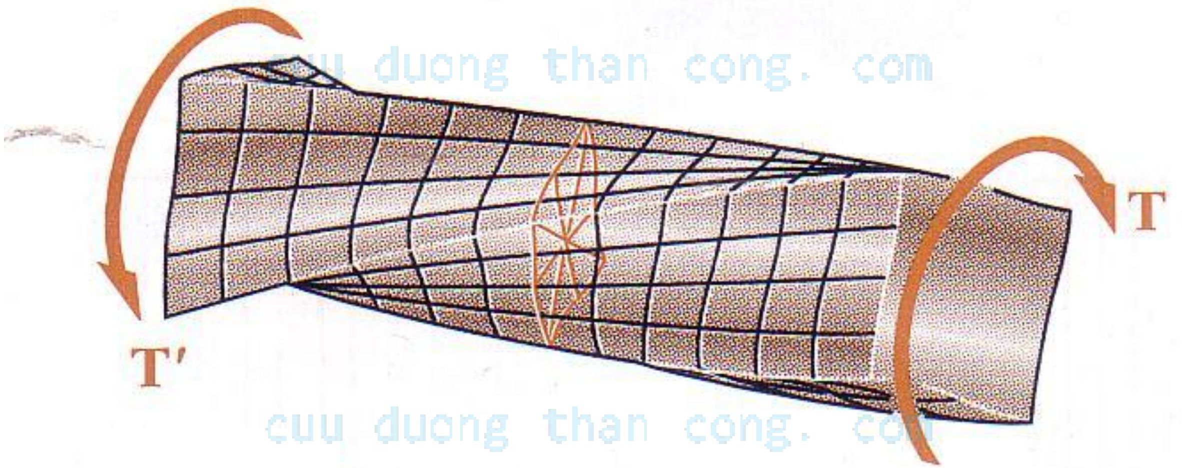
$$\tau_{bc} = \frac{16T_{bc}}{\pi d^3} = \frac{16(318N.m)}{\pi(50mm)^3} = 13.0\ MPa$$

$$\varphi_{bc} = \frac{T_{bc}L_{bc}}{GI_p} = \frac{(318N.m)(1.2m)}{(80GPa)\left(\frac{\pi}{32}\right)(50mm)^4} = 0.0078\ rad$$

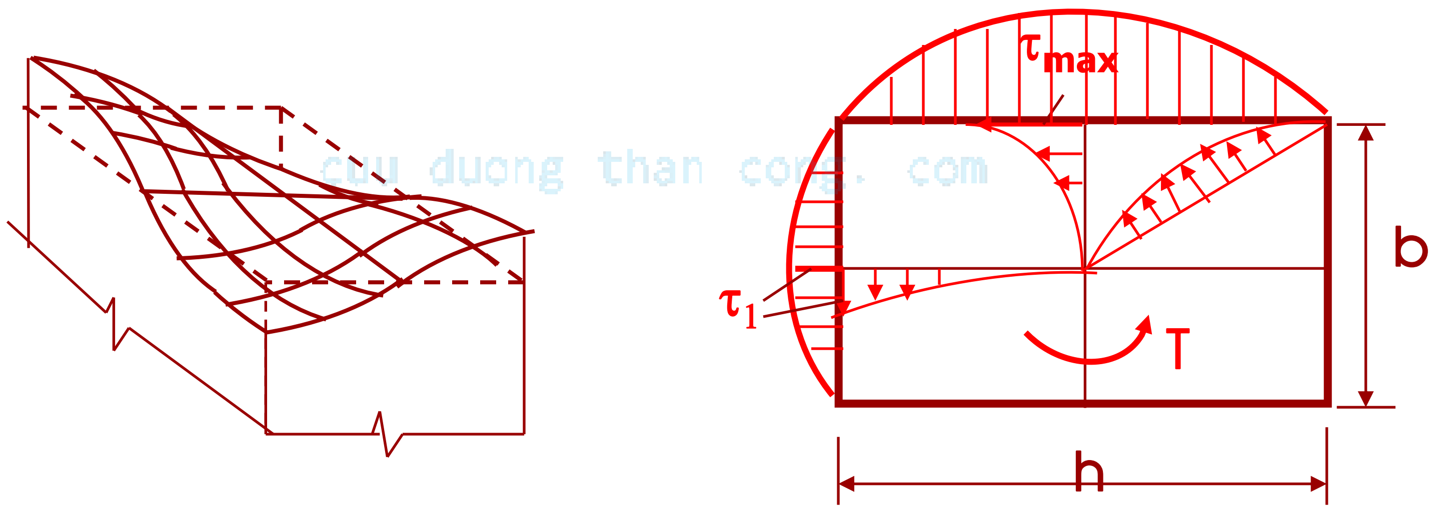
- The maximum shear stress: $\tau_{max} = 32.4MPa$
- The total angle of twist:

$\varphi_{AC} = \varphi_{AB} + \varphi_{BC} = 0.0162\ rad - 0.0078\ rad = 0.0084\ rad$

VI/ Pure Torsion of Rectangular Cross Section Members



VI/ Pure Torsion of Rectangular Cross Section Members (cont.)



a/ Warping Surface; b/ Shear Stress Distribution

VI/ Pure Torsion of Rectangular Cross Section Members (cont.)

- Characteristics:

  ✓ Cross-sections don't remain plane during deformation→ distorted laterally (warping surface) (H.a)

  ✓ The distribution of shear stress over the section is complicated: not proportional to the distance from the point to the centroid of the section as the case of the cross section (H.b)

VI/ Pure Torsion of Rectangular Cross Section Members (cont.)

- Formulae: From the Theory of Elasticity

$$\tau_{max} = \frac{T}{Z_t} \quad \text{and} \quad \theta = \frac{TL}{GJ_t}$$

where $Z_t = \alpha h b^2$ and $J_t = \beta h b^3$

with α, β - coefficients depending on the ratio h/b given in the table

VI/ Pure Torsion of Rectangular Cross Section Members (cont.)

h/b	1.0	1.5	2.0	3.0	4.0	6.0	8.0	10.0	∞
α	0.208	0.231	0.246	0.267	0.282	0.299	0.307	0.313	1/3
β	0.104	0.196	0.229	0.263	0.281	0.299	0.307	0.313	1/3

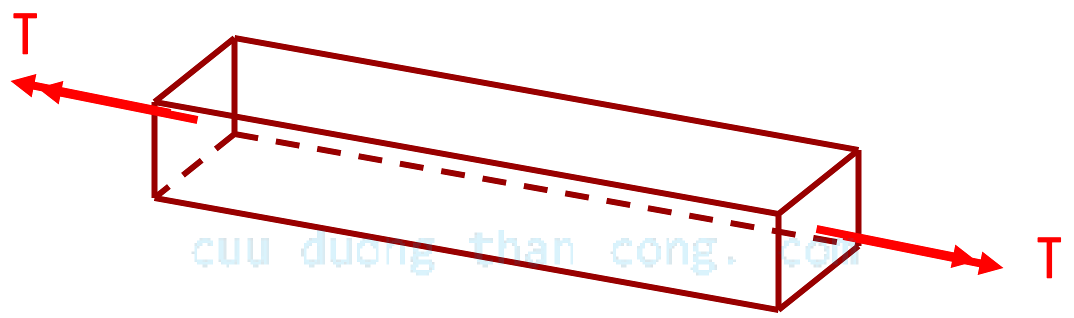
- Strength & Rigidity Conditions

$$\tau_{max} = \frac{M_t}{W_t} = \frac{M_t}{\alpha h b^2} \leq [\tau]$$

$$\theta = \frac{T}{G I_t} = \frac{T}{G \beta h b^3} \leq [\theta]$$

VI/ Pure Torsion of Rectangular Cross Section Members (cont.)

- Example: A timber member: L=3m, rectangular section 100x100 mm, torque T=200N.m, G=700 MPa. Compute τ_{max}? θ?



VI/ Pure Torsion of Rectangular Cross Section Members (cont.)

- Solution

$h/b = 1 \rightarrow \alpha = 0.208,\ \beta = 0.14$

$W_t = 0.208 x 100 x 100^2 = 20.8 x 10^{-5}\ m^3$

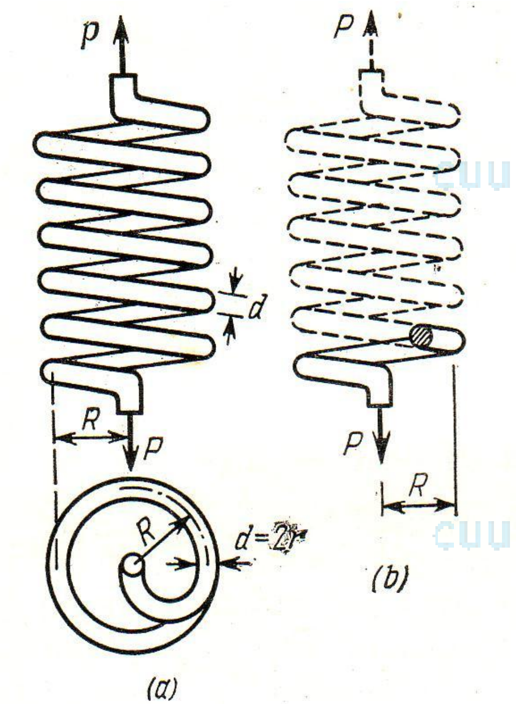
$I_t = 0.140 x 100 x 100^3 = 14 x 10^{-6}\ m^4$

$$\tau_{max} = \frac{200 x 10^{-4}}{20.8 x 10^{-5}} = 0.962\ MPa$$

$$\theta = \frac{200 x 3}{100 x 10^{6} x 14 x 10^{-6}} = 0.0613 rad = 3^{0} 30' 40''$$

VII/ STRESS AND STRAIN IN CLOSED-COILED HELICAL SPRINGS

- Description



(a)Closed-coiled Helical Spring

Closed-coiled helical springs: distance between adjacent coils is small

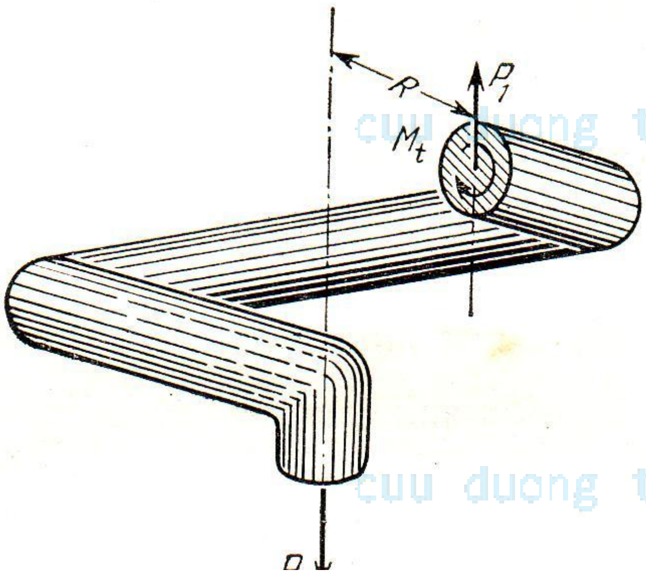
R: radius of the spring helix

d = 2r: diameter of the spring wire

n: number of turns in the spring

G: shear modulus of the spring matarial

VII/ STRESS AND STRAIN IN CLOSED-COILED HELICAL SPRINGS (continued)

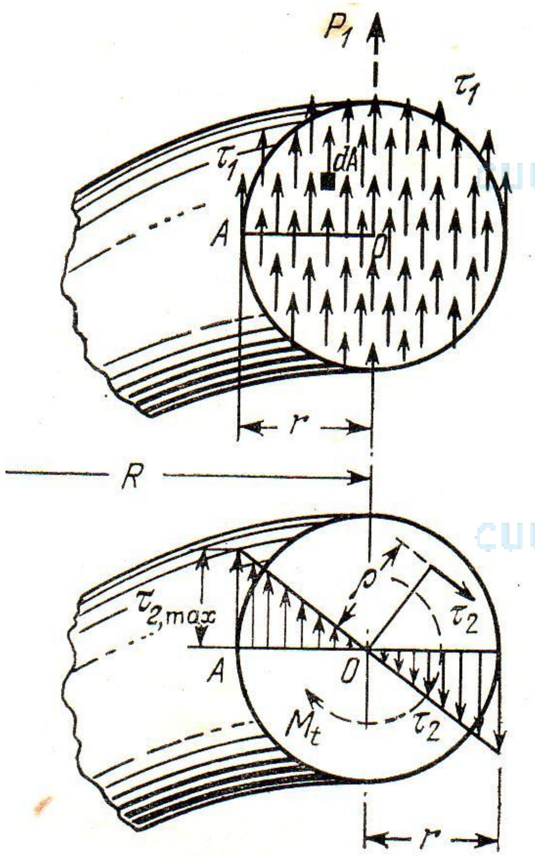


(b) Internal Forces

- Internal Forces:
  - ✓ Shear force: $P_1 = P$
  - ✓ Torque: $M_t = PR$

VII/ STRESS AND STRAIN IN CLOSED-COILED HELICAL SPRINGS (continued)

- Shear Stresses



$$\tau_{max} = \tau_1 + \tau_{2,max}$$

$$\tau_1 = \frac{P}{A} = \frac{4P}{\pi d^2}$$

$$\tau_{2,max} = \frac{M_t}{W_t} = \frac{16PR}{\pi d^3}$$

$$\rightarrow \tau_{max} = \frac{16PR}{\pi d^3}\left(1 + \frac{d}{4R}\right)$$

With $\frac{d}{2R} \approx 0 \rightarrow \tau_{max} \approx \frac{16PR}{\pi d^3}$

VII/ STRESS AND STRAIN IN CLOSED-COILED HELICAL SPRINGS (continued)

- In practice, when designing springs

  → a correction factor k (taking into account of other factors: bending of the spring wire, longitudinal deformation,..)

$$\max \tau = k \frac{M_t}{W_p} = k \frac{2PR}{\pi r^3} = k \frac{8PD}{\pi d^3}$$

R/r	4	5	6	7	8	9	10	11	12	15
k	1.42	1.31	1.25	1.21	1.18	1.16	1.14	1.12	1.11	1.09

VII/ STRESS AND STRAIN IN CLOSED-COILED HELICAL SPRINGS (continued)

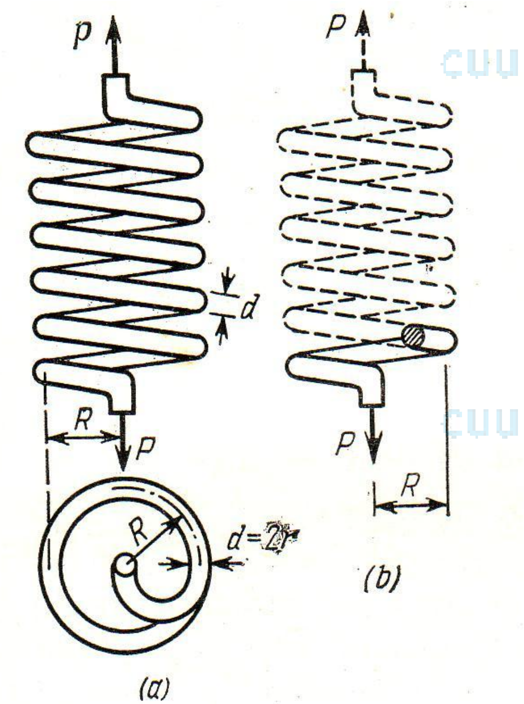
- Elongation (or compression)
  - ✓ Principle of energy conservation:
    - Work done by external force $W = \frac{1}{2}P.\lambda$
    - Potential energy of torsion $U = \frac{M_t^2 L}{2GJ}$

✓ $W = U \rightarrow \lambda = \frac{8PD^3 n}{Gd^4} = \frac{4PR^3 n}{Gr^4} = \frac{P}{C}$

with $C = \frac{Gd^4}{8D^3 n} = \frac{Gr^4}{4R^3 n}$

VII/ STRESS AND STRAIN IN CLOSED-COILED HELICAL SPRINGS (continued)

- Example: Calculate the maximum stress and elongation of the cylindrical spring



If spring radius R = 100 mm
spring wire diameter d=20mm
Number of turns: n = 10
Tensile force: P = 2200 N
Shear modulus: G = 85 GPa

VII/ STRESS AND STRAIN IN CLOSED-COILED HELICAL SPRINGS (continued)

- Solution
  - ✓ Maximum shear stress:

As the ratio R/r = 10, the correction factor : k = 1.14

$$\tau_{max} = k\frac{2PR}{\pi r^3} = 1.14\frac{2x2200x100}{\pi x10^3} = 159.66MPa$$

  - ✓ Elongation of the spring:

$$\lambda = \frac{4PR^3}{Gr^4} = \frac{4x2200x100^3}{85x10^3x10^4} = 10.4mm$$