COMBINED LOADINGS (CHỊU LỰC PHỨC TẠP)

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I/ INTRODUCTION

- Biaxial Flexure (or Unsymmetrical Bending)
- Traction (Compression) & Bending
- Torsion and bending
- General loadings

I/ INTRODUCTION (cont.)

- PRINCIPLE OF SUPERPOSITION

  "The internal forces, stresses, strains, displacements,.., due to multiple causes are the sum of these quantities due to each one acting independently"

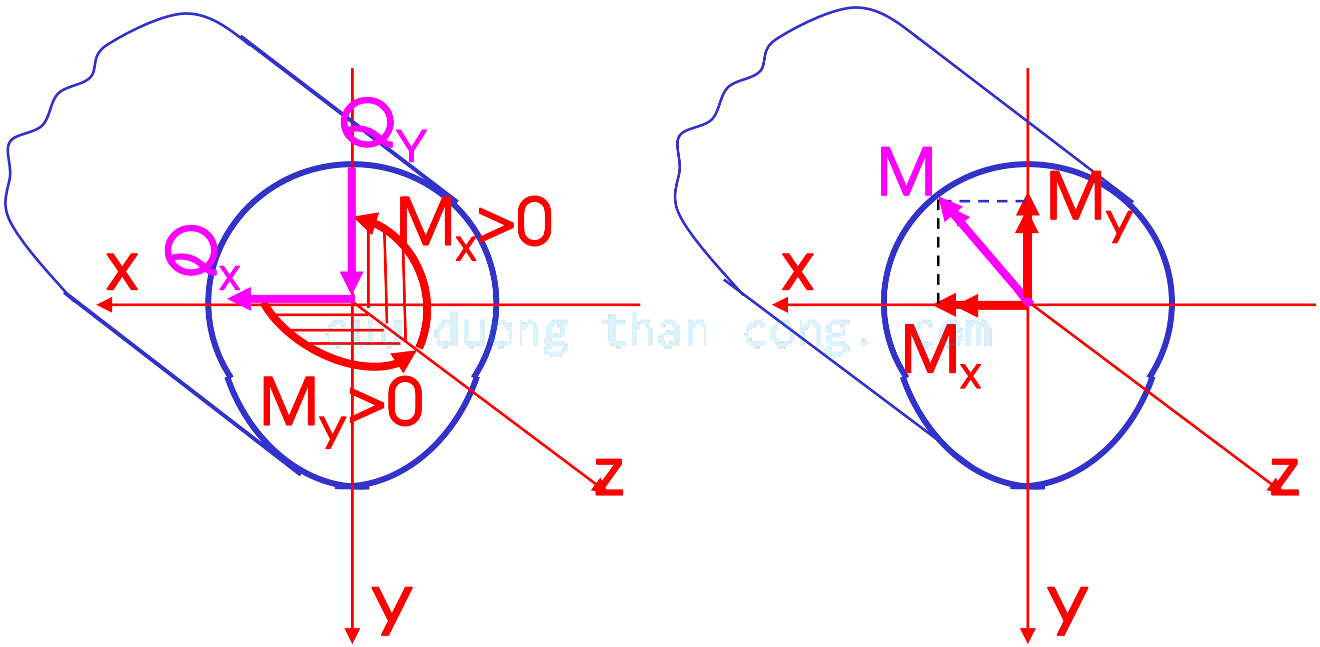
$$S(P_1,P_2,..,P_n,\Delta,.) = S(P_1)+S(P_2)+..+S(P_n)+..S(\Delta)+.$$

I/ INTRODUCTION (cont.)

- Condition of application (Điều kiện áp dụng)
  - ✓ Material is linearly elastic
  - ✓ Small displacements

II/ BIAXIAL FLEXURE (UNSYMETRICAL BENDING)

- Definition: when there are two bending moments and two shear forces in two principal planes M_x, M_y, Q_x, Q_y



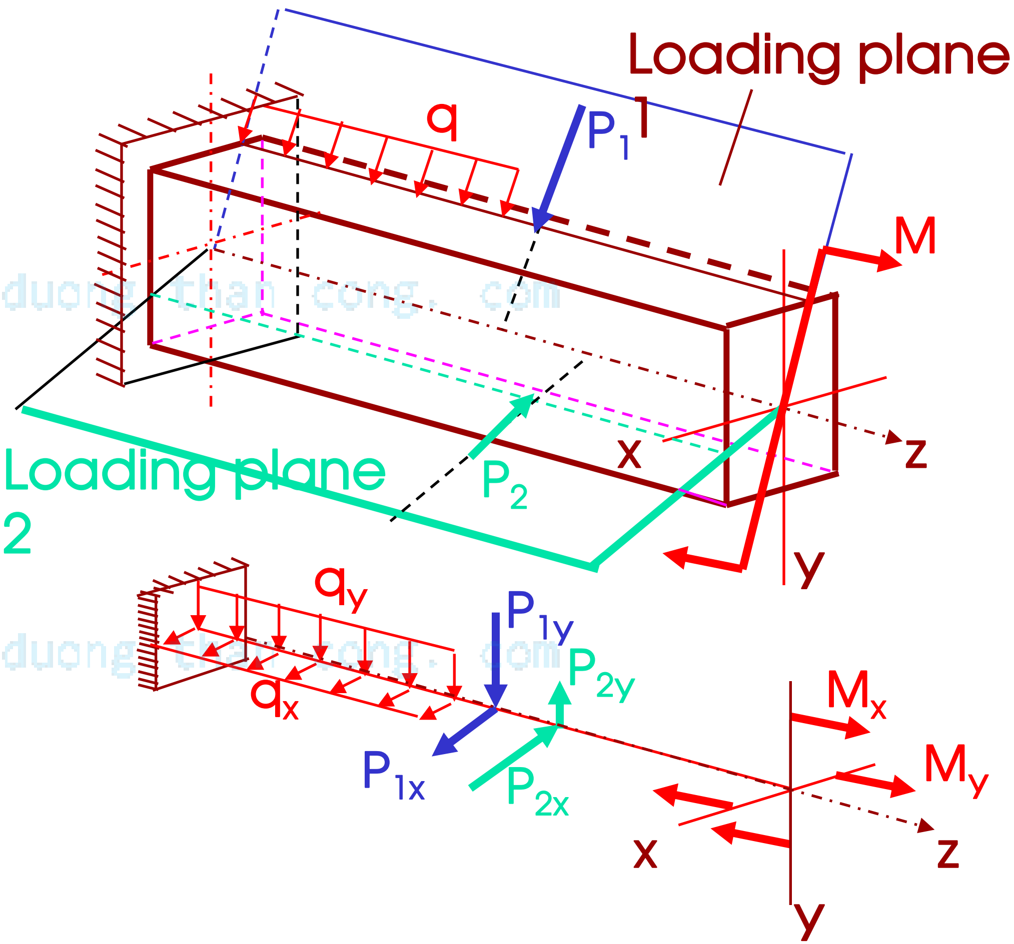
II/ BIAXIAL FLEXURE (UNSYMETRICAL BENDING) (cont.)

- Notations:
  ✓ Bending moments are positive if they have tendency to extend the correspond ing positive fibers

  ✓ Expressing bending moment by a two-arrow vector → the resulting moment M of M_x and M_y lies on an arbitrary plane called loading plane(mặt ph. tải trọng)

II/ BIAXIAL FLEXURE (UNSYMETRICAL BENDING) (cont.)

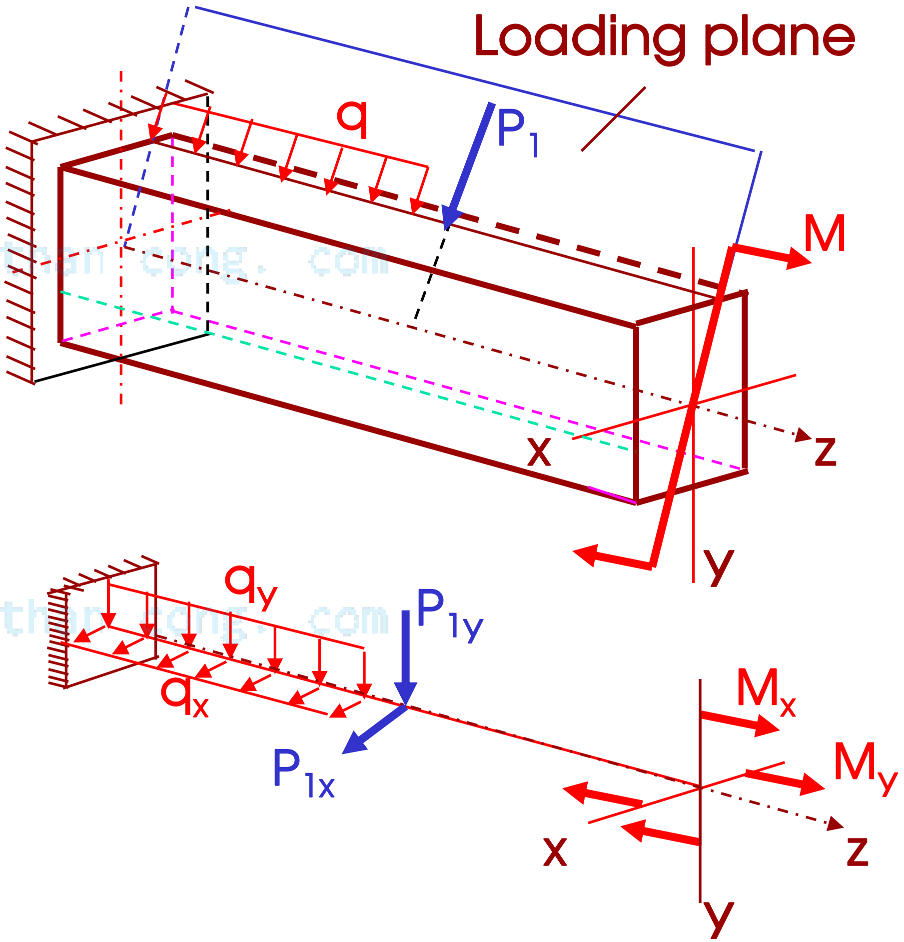
a/ Loadings are in two principal planes



II/ BIAXIAL FLEXURE (UNSYMETRICAL BENDING) (cont.)

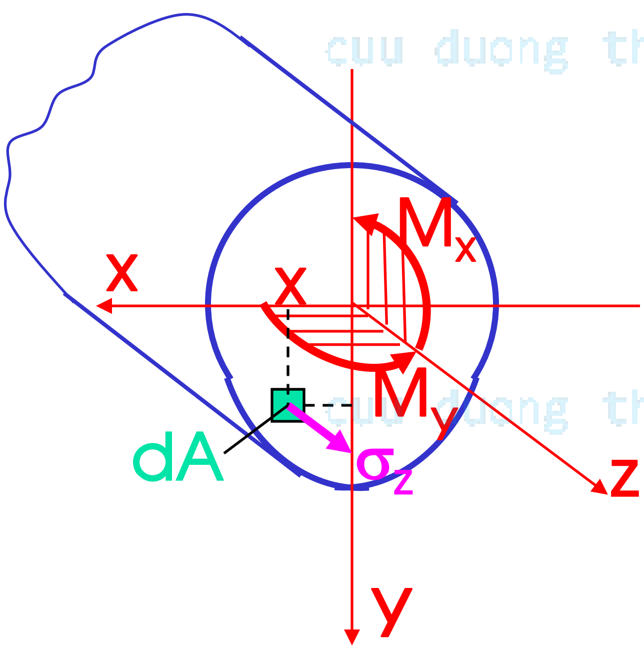
b/ Loadings are in 1 arbitrary plane containing its axis

→ Unsymmetrical Bending



II/ BIAXIAL FLEXURE (UNSYMETRICAL BENDING) (cont.)

- Stress at a point (x,y): Normal Stress, σ_z



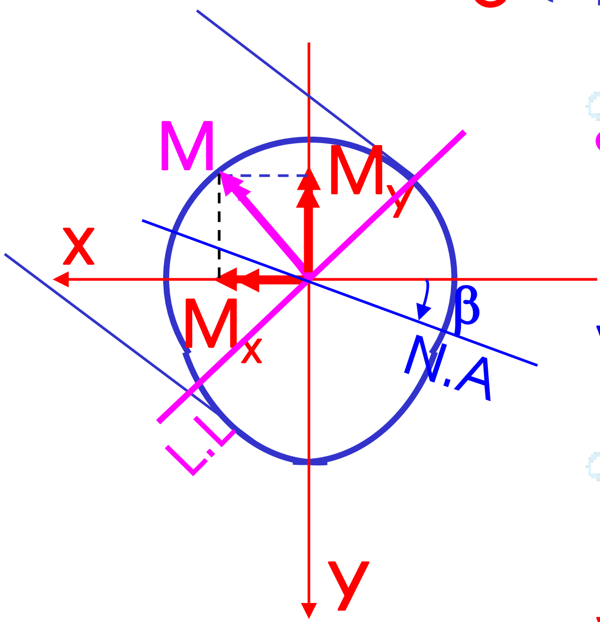
$$\sigma_z = \frac{M_x}{I_x} y + \frac{M_y}{I_y} x$$

Or (hay)

$$\sigma_z = \pm \frac{|M_x|}{I_x} |y| \pm \frac{|M_y|}{I_y} |x|$$

II/ BIAXIAL FLEXURE (UNSYMETRICAL BENDING) (cont.)

- Neutral Axis (Trục trung hòa): set of point (x,y) having (tập hợp các điểm (x,y) có) $\sigma_z (x,y) = 0$



$$\sigma_z = \frac{M_x}{I_x} y + \frac{M_y}{I_y} x = 0 \rightarrow y = ax$$

$$\text{with} \quad a = \tan\beta = -\frac{M_y}{M_x} \cdot \frac{I_x}{I_y} = -\frac{1}{\tan\alpha} \cdot \frac{I_x}{I_y} :$$

slope of N.A.

$$\text{where} \quad \tan\alpha = \frac{M_x}{M_y} : \text{slope of load line}$$

II/ BIAXIAL FLEXURE (UNSYMETRICAL BENDING) (cont.) – Normal Stress

- Properties: $\tan\alpha.\tan\beta = -\dfrac{I_x}{I_y}$

✓ In general, $I_x \neq I_y$: $\tan\alpha.\tan\beta \neq -1$

→ load line (L.L.) is not ⊥ N.A.

✓ If $I_x = I_y$ (square, circular sections,..):

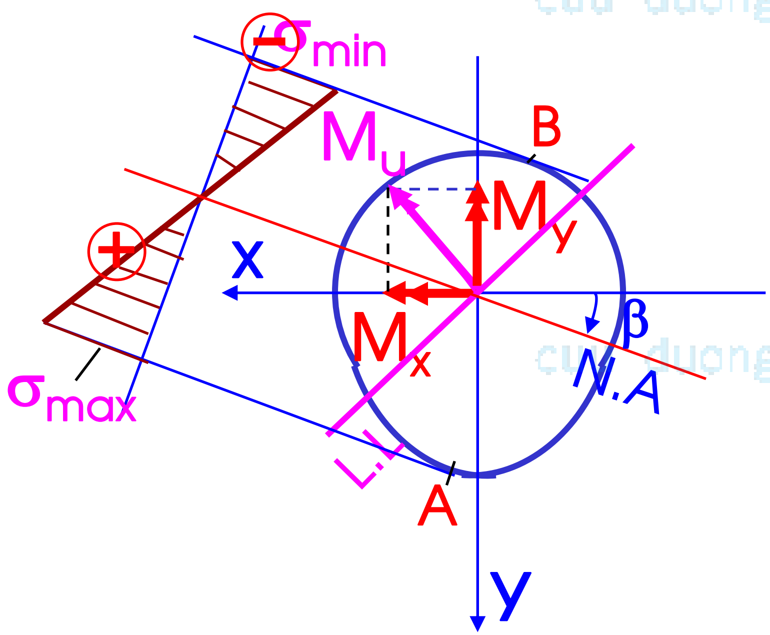
$$\tan\alpha.\tan\beta = -1$$

→ all axes become the principal axes

→ uniplanar bending under $M_u = \sqrt{M_x^2 + M_y^2}$

II/ BIAXIAL FLEXURE (UNSYMETRICAL BENDING) (cont.) – Normal Stress

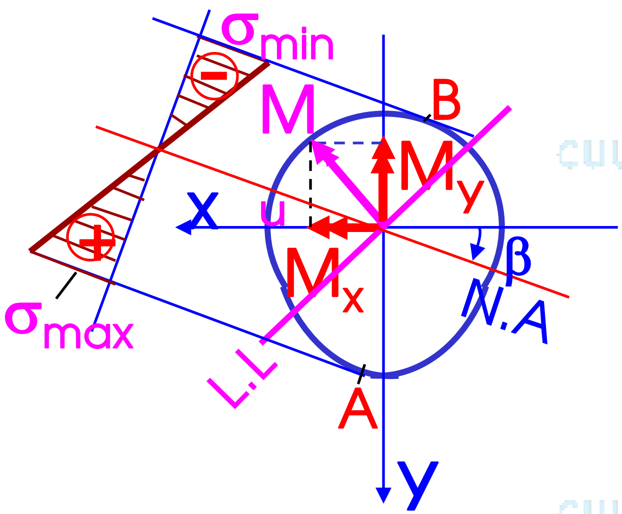
- Normal Stress Distribution
  - ✓ Arbitrary uniaxially symmetric section

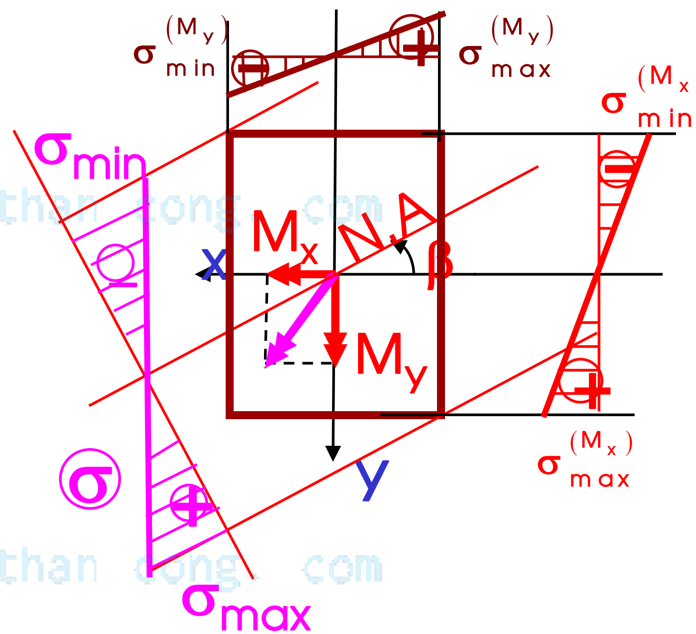


$$\left|\sigma_{max}\right| = \left|\sigma_A\right| = \frac{\left|M_x\right|}{I_x}\left|y_A\right| + \frac{\left|M_y\right|}{I_y}\left|x_A\right|$$

$$\left|\sigma_{min}\right| = \left|\sigma_B\right| = \frac{\left|M_x\right|}{I_x}\left|y_B\right| + \frac{\left|M_y\right|}{I_y}\left|x_B\right|$$

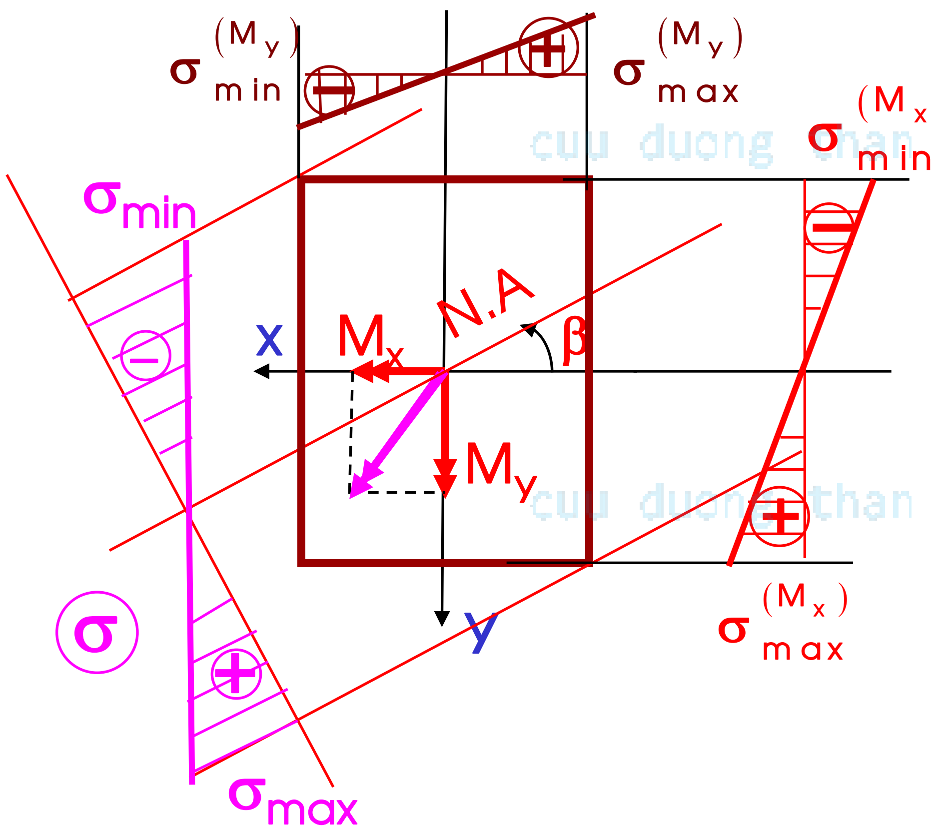
II/ BIAXIAL FLEXURE (UNSYMETRICAL BENDING) (cont.) – Normal Stress





II/ BIAXIAL FLEXURE (UNSYMETRICAL BENDING) (cont.) – Normal Stress

- Biaxially symmetric sections



$$\sigma_{max} = +\frac{|M_x|}{W_x} + \frac{|M_y|}{W_y}$$

$$\sigma_{min} = -\frac{|M_x|}{W_x} - \frac{|M_y|}{W_y}$$

with $W_x = \dfrac{I_x}{y_{max}}$

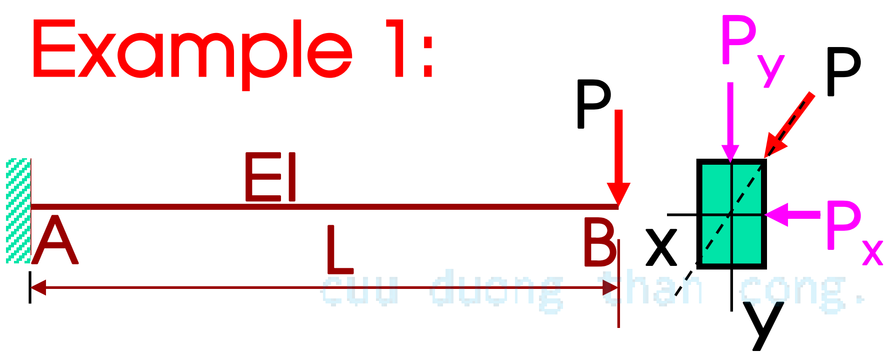
$$W_y = \frac{I_y}{x_{max}}$$

II/ BIAXIAL FLEXURE (UNSYMMETRICAL BENDING) (cont.) – Displacements

- Displacements in Unsymmetric Bending

$$\vec{f} = \vec{f}_x + \vec{f}_y \quad \text{with} \quad f = \sqrt{f_x^2 + f_y^2}$$

- Example 1:



$$P_x = P.\cos\varphi$$

$$P_y = P.\sin\varphi$$

Compute the deflection of point B?

II/ BIAXIAL FLEXURE (UNSYMMETRICAL BENDING) (cont.) – Displacements

- ✓ Deflection of point B only due to P_y

$$f_y = \frac{P_y L^3}{3EI_x} = \frac{(P\sin\varphi)L^3}{3EI_x}$$

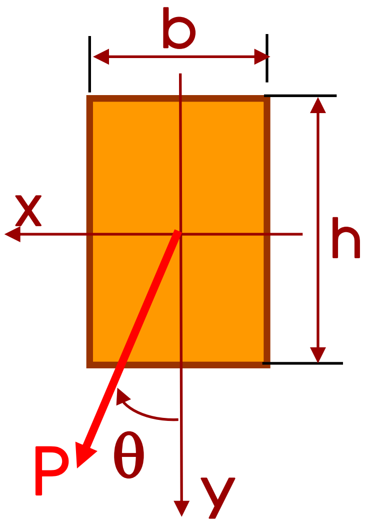
- ✓ Deflection of point B only due to P_x

$$f_x = \frac{P_x L^3}{3EI_y} = \frac{(P\cos\varphi)L^3}{3EI_y}$$

- ✓ Total deflection of the free end B

$$f = \sqrt{f_x^2 + f_y^2}$$

II/ BIAXIAL FLEXURE (UNSYMMETRICAL BENDING) (cont.) –



- Example 2:

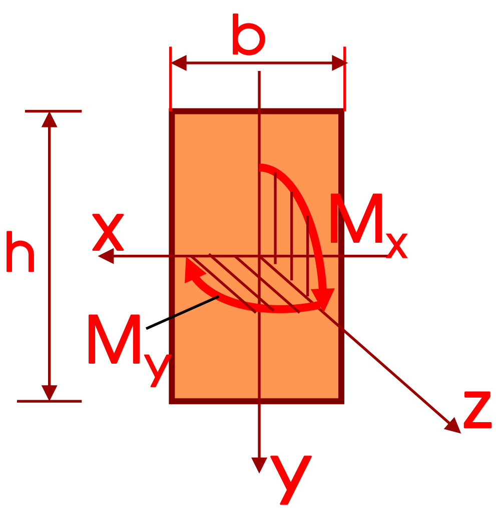
✓A wood of cantiliver beam of rectangular cross section supports a inclined load P at its free end. Calculate the maximum tensile stress σ_{max} due to the load P?

✓ Data for the beam:

b = 75 mm, h = 150 mm, L = 1.4 m, P = 800 N, and $\theta = 30^0$

II/ BIAXIAL FLEXURE (UNSYMMETRICAL BENDING) (cont.) –Example 2

- Solution:
  - ✓ Bending moments at the fixed end section:



$$M_x = -800x(1.4)x\cos 30^0 = -969.95\ N.m$$

$$M_y = -800x(1.4)x\sin 30^0 = -560\ N.m$$

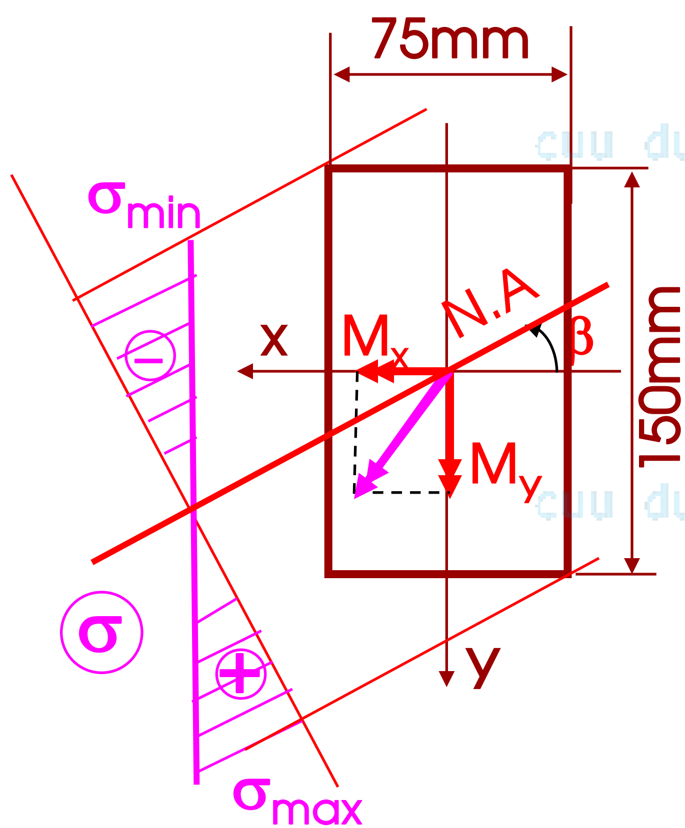
  - ✓ Geometrical properties:

$$W_x = \frac{bh^2}{6} = \frac{(0.075)(0.15)^2}{6} = 2.8x10^{-4}m^3$$

$$W_y = \frac{hb^2}{6} = \frac{(0.15)(0.075)^2}{6} = 1.4x10^{-4}m^3$$

II/ BIAXIAL FLEXURE (UNSYMMETRICAL BENDING) (cont.) – Example 2

✓ Maximum Stress



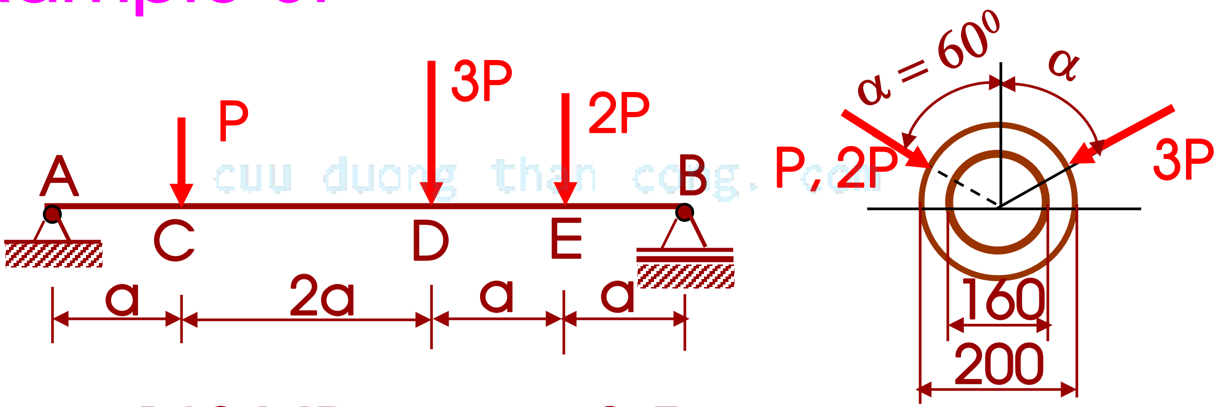
$$\sigma_{max} = \frac{969.95}{2.8\times10^{-4}} + \frac{560}{1.4\times10^{-4}}$$

$$= 7.46\times10^{6}\,Pa = 7.46\,MPa$$

$$\sigma_{min} = -\sigma_{max} = -7.46\,MPa$$

II/ BIAXIAL FLEXURE (UNSYMMETRICAL BENDING) (cont.) – Example 3

- Example 3:

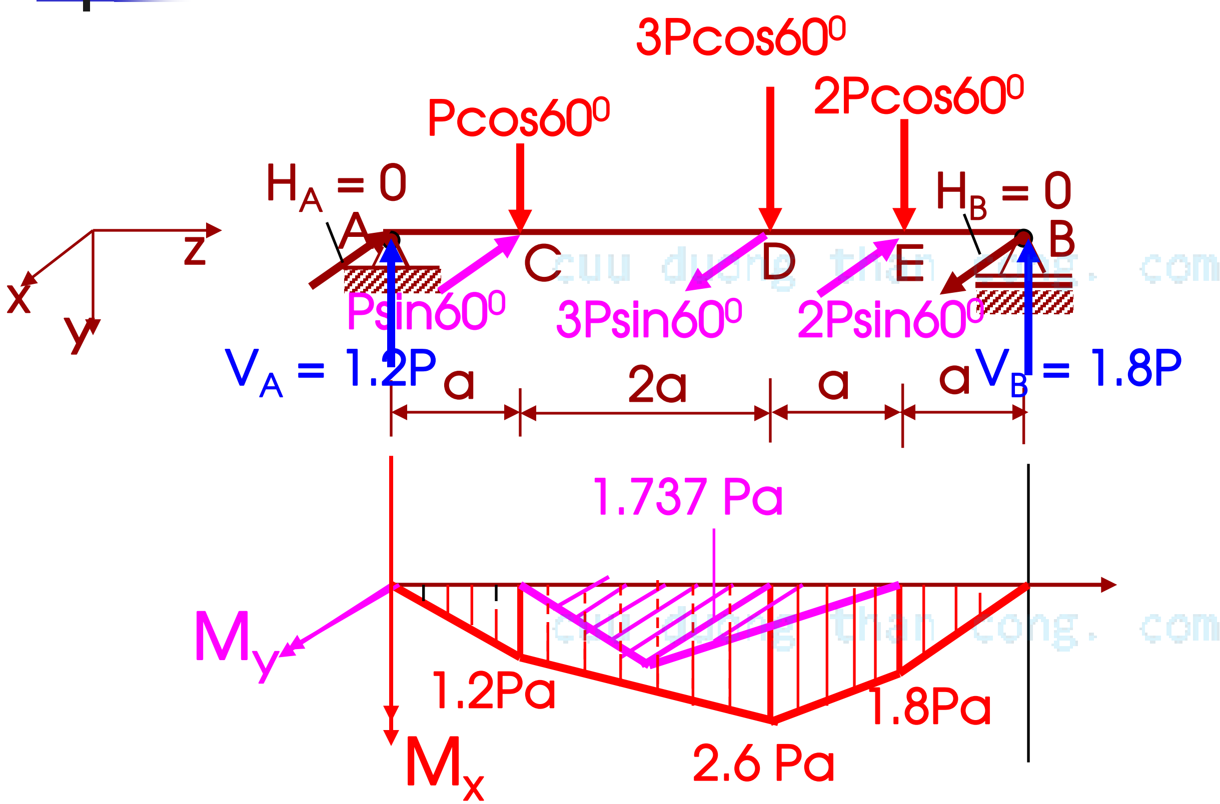


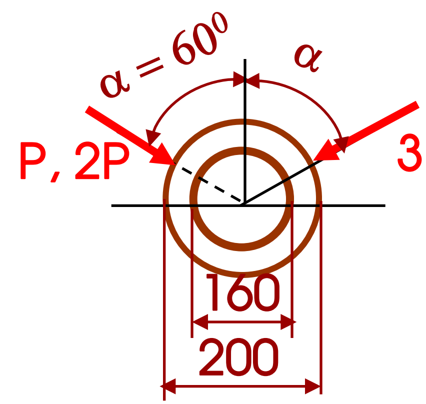
$[\sigma] = 160$ MPa; $a = 0.5$m

a/ Calculate the allowable load value [P]

b/ Determine the N.A. position

II/ BIAXIAL FLEXURE (UNSYMMETRICAL BENDING) (cont.) – Example 3





II/ BIAXIAL FLEXURE (UNSYMMETRICAL BENDING) (cont.) – Example 3

- Solution
  - ✓ Critical section at D:

$$\left.\begin{array}{l} M_x = 2.6Pa, \\ M_y = \sqrt{3}Pa \end{array}\right\} \rightarrow \quad M_u = \sqrt{M_x^2 + M_y^2} = 3.124\ Pa$$

  - ✓ Strength condition:

$$\frac{M_u}{W_u} = \frac{5.655Pa}{\frac{\pi}{32}D^3\left[1-\left(\frac{d}{D}\right)^4\right]} = \frac{5.655\times0.5\times P}{\frac{\pi}{32}\times0.2^3\left[1-\left(\frac{0.16}{0.2}\right)^4\right]} \leq 160MPa$$

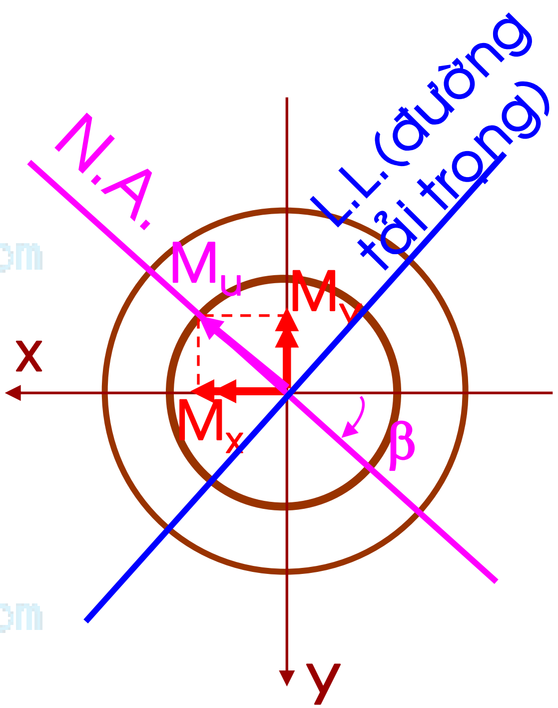
$$\rightarrow\ P \leq 0.0475\ MN = 47.5\ kN$$

II/ BIAXIAL FLEXURE (UNSYMMETRICAL BENDING) (cont.) – Example 3

✓ Neutral Axis

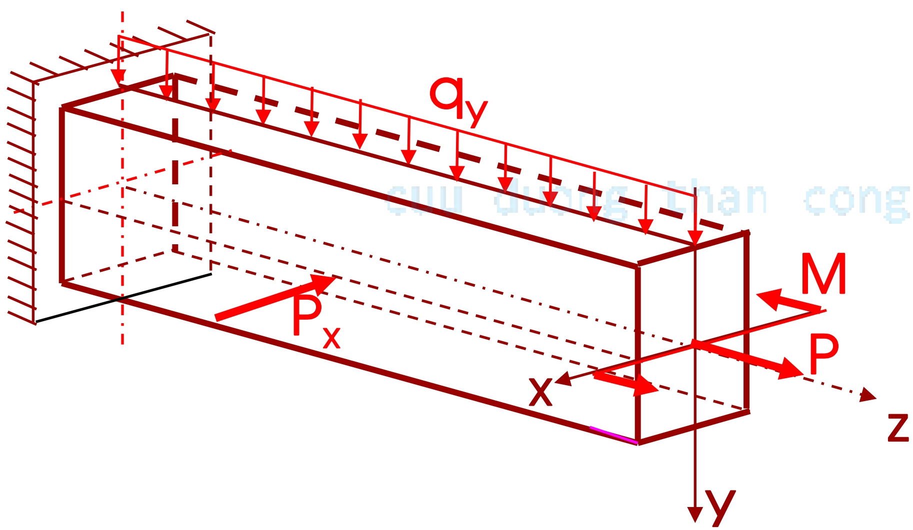
$$\tan\beta = -\frac{M_y}{M_x}\cdot\frac{I_x}{I_y} = -\frac{\sqrt{3}}{2.6} = -0.666$$

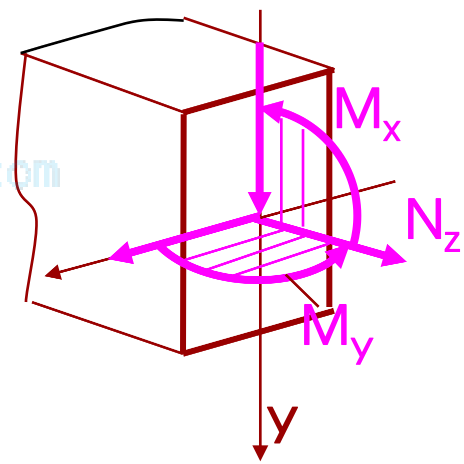
$$\rightarrow \beta = -33{,}6^0$$



III/ COMBINED BENDING AND TENSION OR COMPRESSION (UỐN VÀ KÉO HOẶC NÉN)

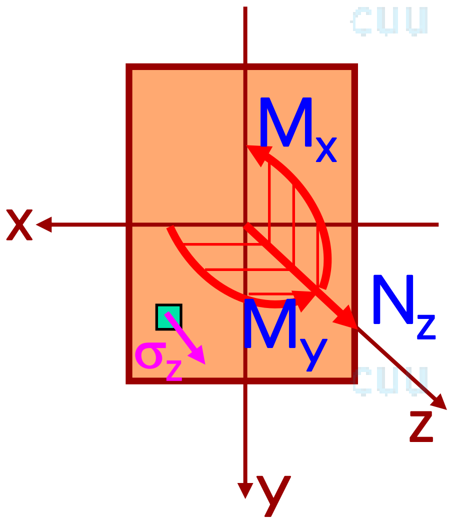
- Internal forces:
  - ✓ Normal force: N_z
  - ✓ Bending moment(s): M_x and/or M_y
  - ✓ Shear forces: Q_y and/or Q_x





III/ COMBINED BENDING AND TENSION OR COMPRESSION (UỐN VÀ KÉO HOẶC NÉN) – Normal stress

- Normal stress due to normal force and bending moments:



$$\sigma_z = \frac{N_z}{A} + \frac{M_x}{I_x} y + \frac{M_y}{I_y} x$$

or

$$\sigma_z = \pm \frac{|N_z|}{A} \pm \frac{|M_x|}{I_x} |y| \pm \frac{|M_y|}{I_y} |x|$$

III/ COMBINED BENDING AND TENSION OR COMPRESSION (UỐN VÀ KÉO HOẶC NÉN) – Normal stress

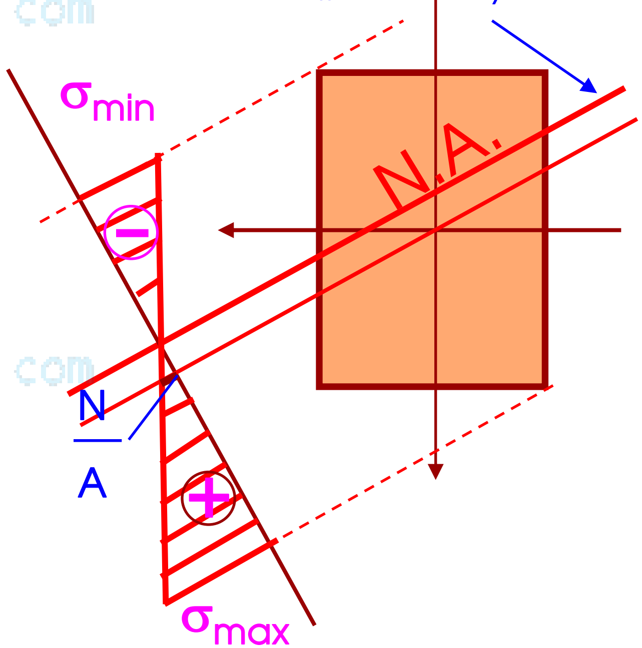
- Neutral Axis – Normal Stress Diagram

✓ Biaxially symmetric sections

$$\frac{N_z}{A} + \frac{M_x}{I_x} y + \frac{M_y}{I_y} = 0$$

$$\sigma_{max} = \pm \frac{|N_z|}{A} + \frac{|M_x|}{W_x} + \frac{|M_y|}{W_y}$$

$$\sigma_{min} = \pm \frac{|N_z|}{A} - \frac{|M_x|}{W_x} - \frac{|M_y|}{W_y}$$

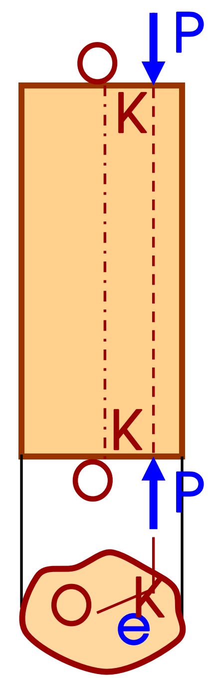


III/ COMBINED BENDING AND TENSION OR COMPRESSION (UỐN VÀ KÉO HOẶC NÉN) – Eccentric tension

- Special case: Eccentric tension (or compression) (kéo hay nén lệch tâm)

✓When the bar is acted upon by two equal and opposite forces P which act along KK parallel to its axis

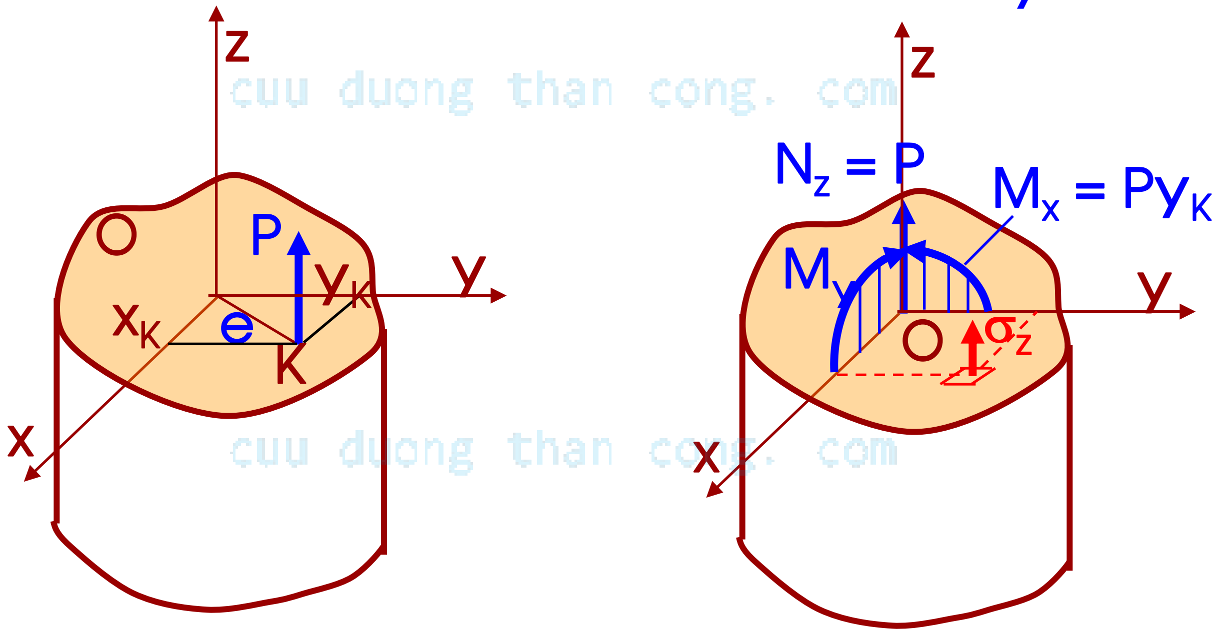
✓ OK = e → eccentricity



III/ COMBINED BENDING AND TENSION OR COMPRESSION (UỐN VÀ KÉO HOẶC NÉN) – Eccentric tension

✓ Internal forces:

$$N_z = P; \;\; M_x = Py_K; \;\; M_y = Px_K$$



III/ COMBINED BENDING AND TENSION OR COMPRESSION (UỐN VÀ KÉO HOẶC NÉN) – Eccentric tension

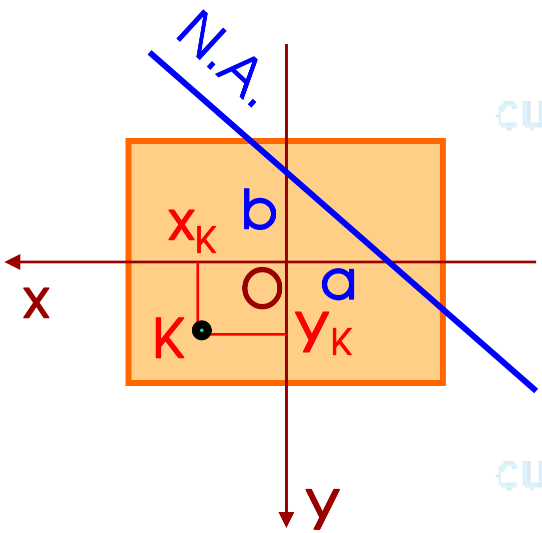
✓ Neutral Axis Equation (P/t đ trung hòa)

$$\sigma_z = \frac{P}{A} + \frac{Py_K}{I_x}y + \frac{Px_K}{I_y}x = \frac{P}{A}\left(1 + \frac{y_K}{r_x^2}y + \frac{x_K}{r_y^2}x\right) = 0$$

$$\text{with} \quad I_x = r_x^2 A; \quad \text{and} \quad I_y = r_y^2 A$$

r_x , r_y – radii of gyration of section about the principal axes (bán kính quán tính)

III/ COMBINED BENDING AND TENSION OR COMPRESSION (UỐN VÀ KÉO HOẶC NÉN) – Eccentric tension



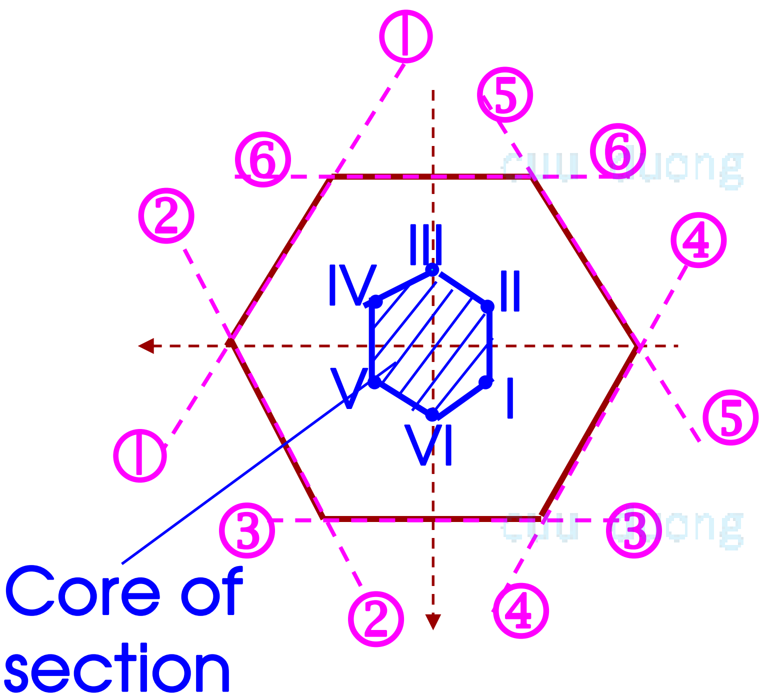
$$\rightarrow \quad 1 + \frac{y_K}{r_x^2} y + \frac{x_K}{r_y^2} x = 0 \qquad \text{or} \qquad \frac{x}{a} + \frac{y}{b} = 0$$

with $\quad a = -\dfrac{r_y^2}{x_K}; \quad b = -\dfrac{r_x^2}{y_K}$

✓ Properties of neutral axis ?

III/ COMBINED BENDING AND TENSION OR COMPRESSION (UỐN VÀ KÉO HOẶC NÉN) – Core of section

- Core of section (Lõi của tiết diện)



Ứng với 1 đường trung hòa i – i (trùng với 1 cạnh của đa giác), tọa độ của điểm đặt lực I:

$$I\begin{cases} x_I = -\dfrac{r_y^2}{a} \\ y_I = -\dfrac{r_y^2}{b} \end{cases};$$

I = 1, số cạnh lồi

III/ COMBINED BENDING AND TENSION OR COMPRESSION (UỐN VÀ KÉO HOẶC NÉN) – Core of section

✓ Core of a rectangular cross-section

- Neutral axis 1-1

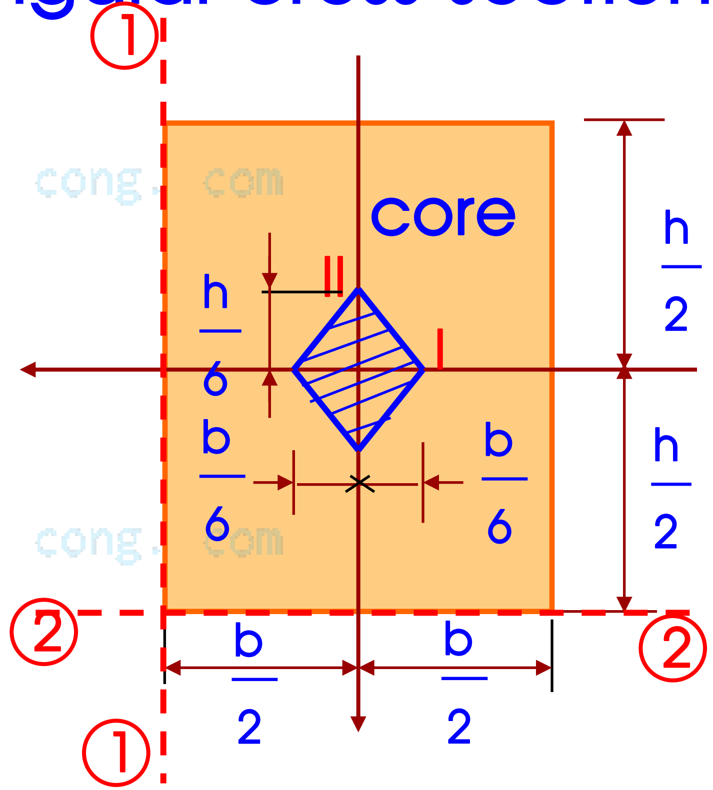
$$a = \frac{b}{2}; \quad b = \infty$$

$$I\left(x_I = -\frac{b}{6}; y_I = 0 \right)$$

- Neutral axis 2-2

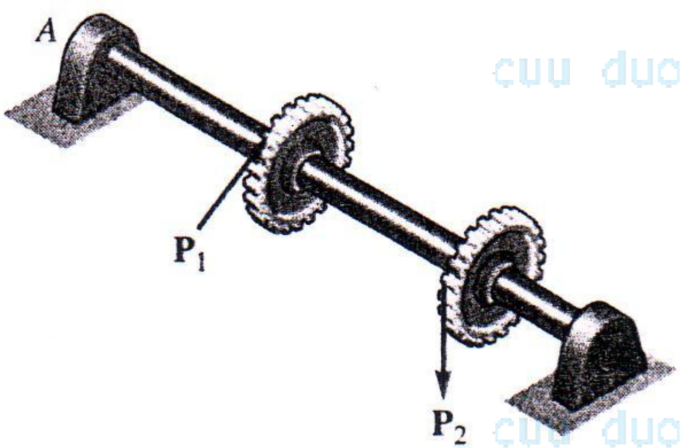
$$a = \infty; \quad b = \frac{h}{2}$$

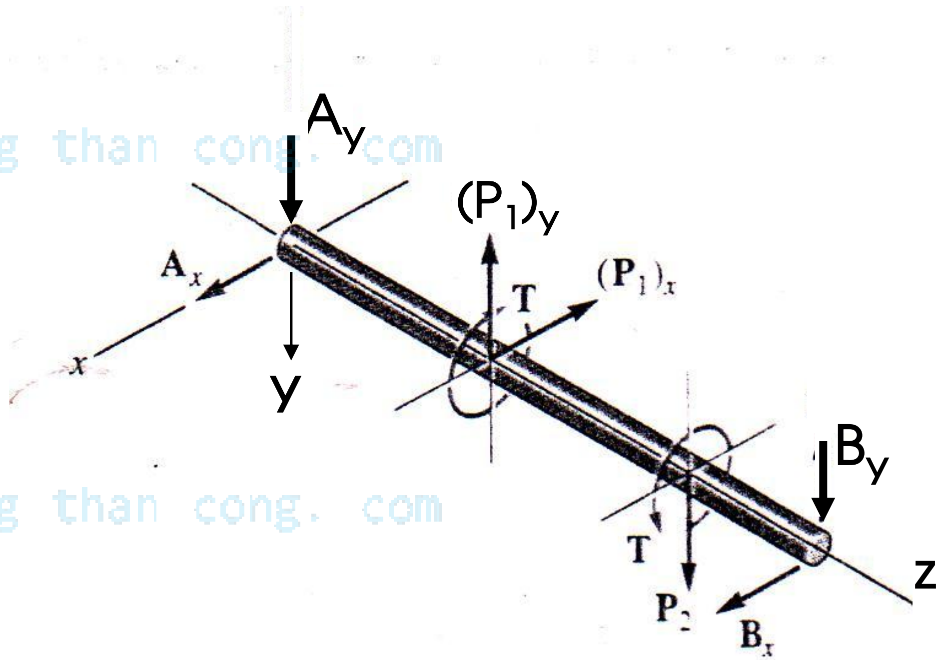
$$II\left(x_{II} = 0; y_{II} = -\frac{h}{6} \right)$$



IV/ COMBINED BENDING AND TORSION (UỐN VÀ XOẮN ĐỒNG THỜI)

- Introduction: Circular Shaft and pulleys



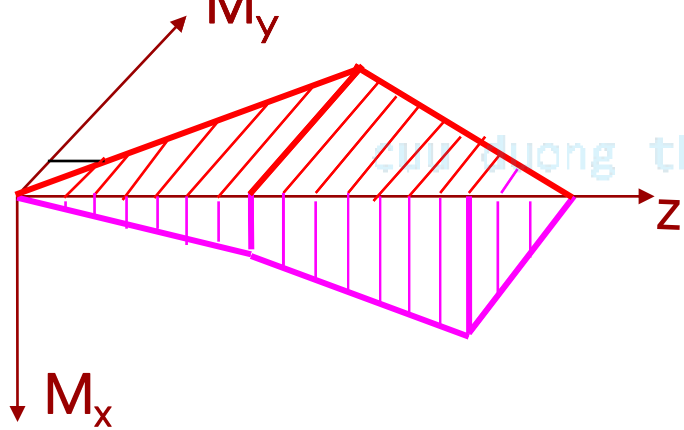


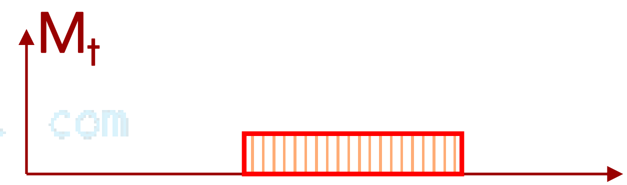
(a) Circular shaft and pulleys

(b) Free body diagram

IV/ COMBINED BENDING AND TORSION (UỐN VÀ XOẮN ĐỒNG THỜI)

Bending moment in x-z plane



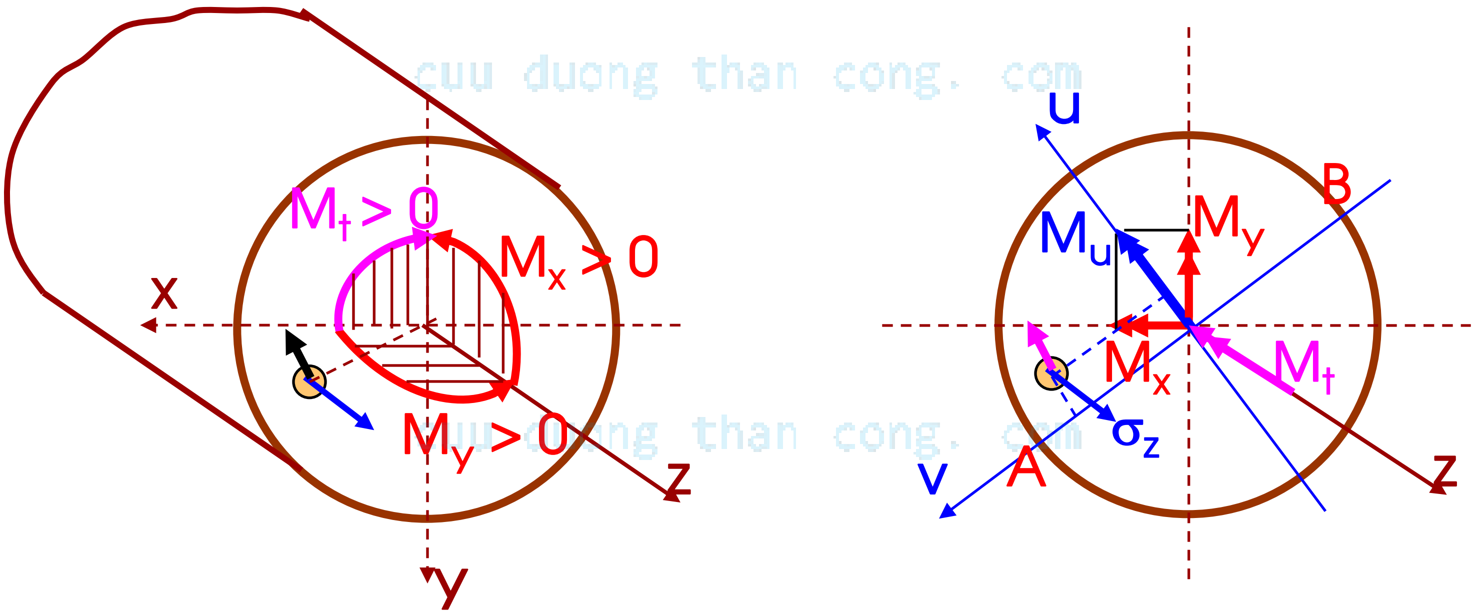


Bending moment in y-z plane

Torsion Moment about z-axis

IV/ COMBINED BENDING AND TORSION (UỐN VÀ XOẮN ĐỒNG THỜI)

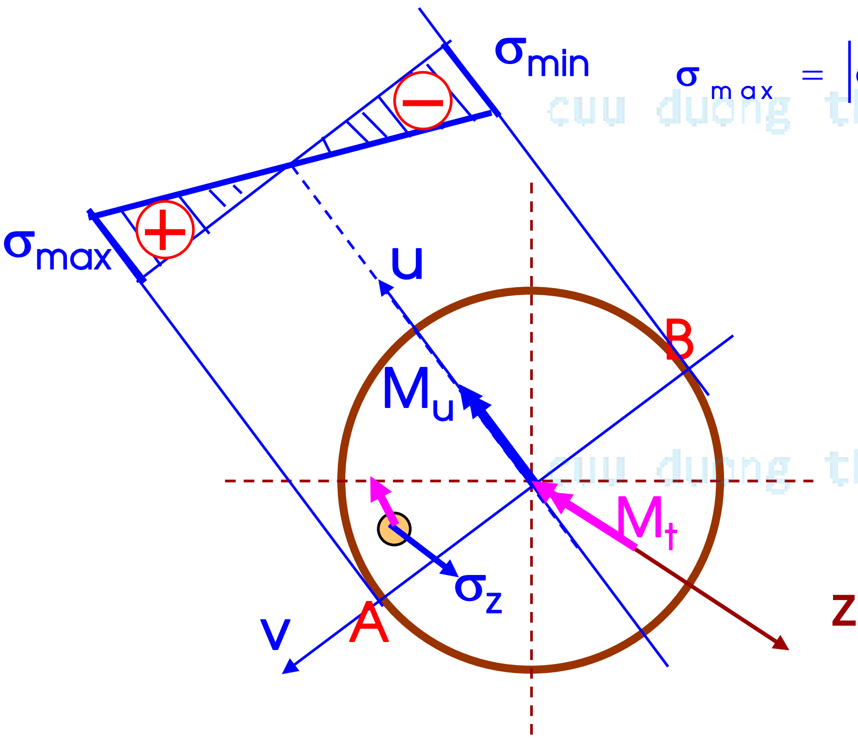
- Moment Components and Sign convention

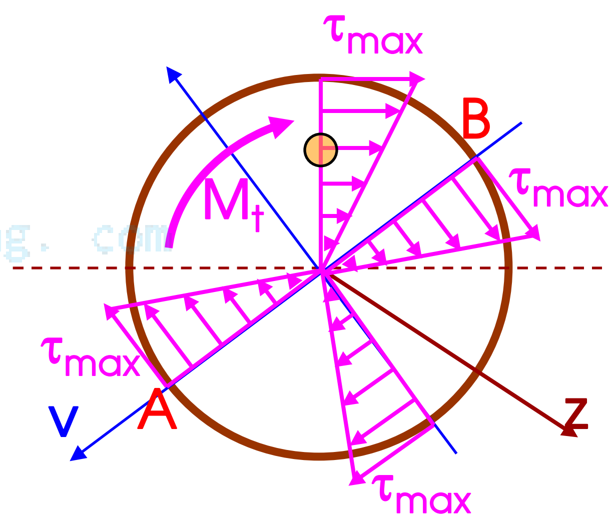


IV/ COMBINED BENDING AND TORSION (UỐN VÀ XOẮN ĐỒNG THỜI)

- Stresses distribution due to moments

$$\sigma_{max} = |\sigma_{min}| = \frac{M_u}{W_u} \qquad \tau_{max} = \frac{M_t}{W_t} = \frac{M_t}{I_t}.\frac{D}{2}$$





IV/ COMBINED BENDING AND TORSION (UỐN VÀ XOẮN ĐỒNG THỜI)

- Strength condition
  - ✓ Critical point: Point A

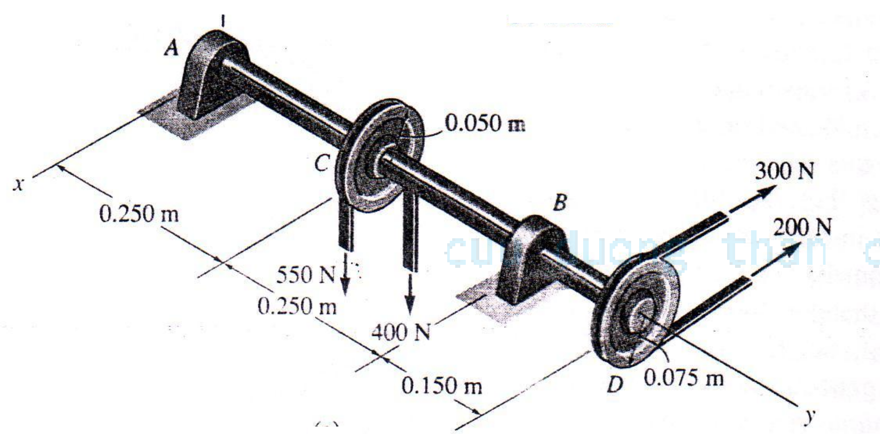
$$\sigma_A = \sigma_{max} = \frac{M_u}{W_u} = \frac{\sqrt{M_x^2 + M_y^2}}{W_u}$$

$$\tau_A = \tau_{max} = \frac{M_t}{W_t} = \frac{M_t}{2W_u}$$

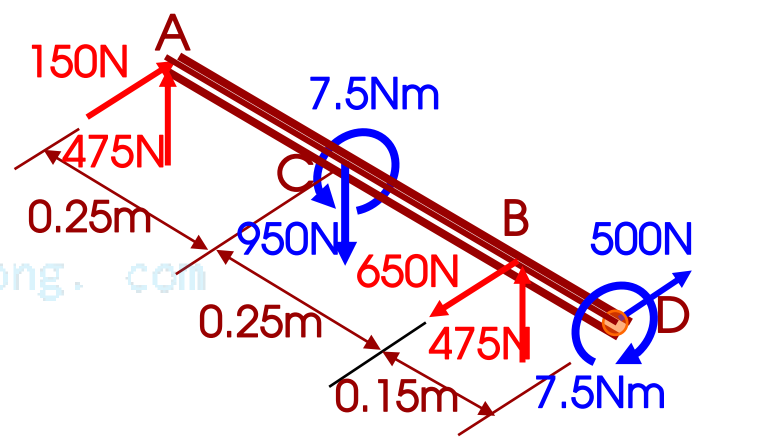
$$\rightarrow \qquad \sigma_{t3} = \sqrt{(\sigma_A)^2 + (\tau_A)^2} \le [\sigma]$$

IV/ COMBINED BENDING AND TORSION (UỐN VÀ XOẮN ĐỒNG THỜI)

- Example: The shaft in fig.a is supported by smooth journal bearings at A and B. Determine the smallest diameter of the shaft using the maximum-shear-stress theory with τ_{allow}= 50MPa



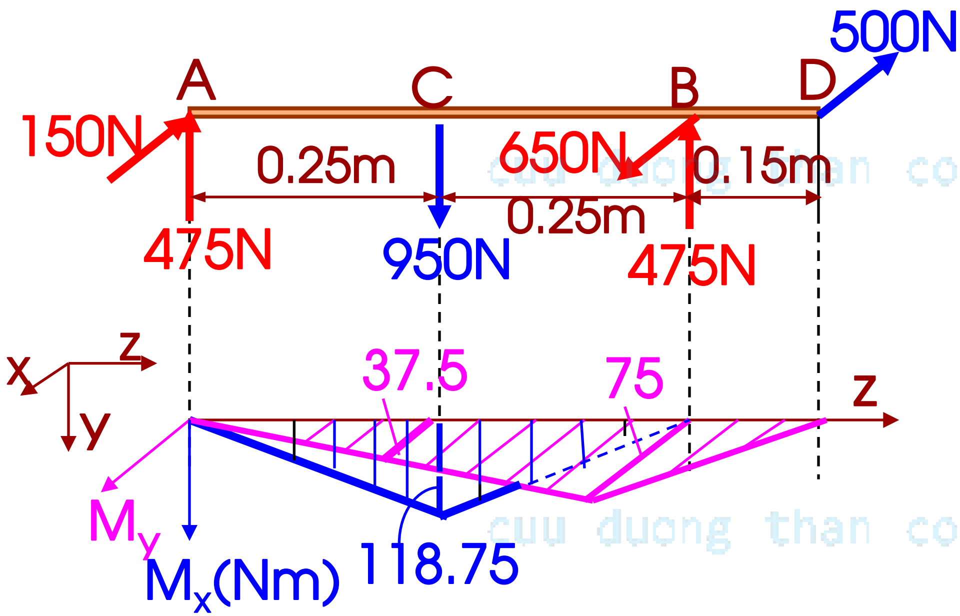
(a) Real scheme

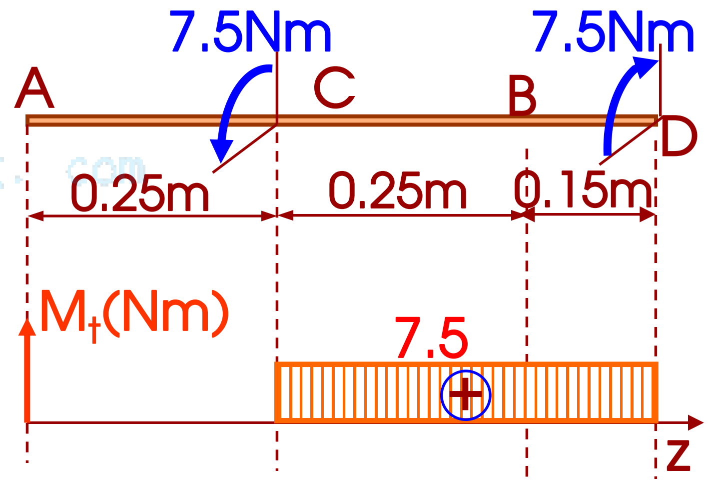


(b)Free-body diagram

IV/ COMBINED BENDING AND TORSION (UỐN VÀ XOẮN ĐỒNG THỜI)

- Solution





Bending moment diagrams

Torsion moment diagram

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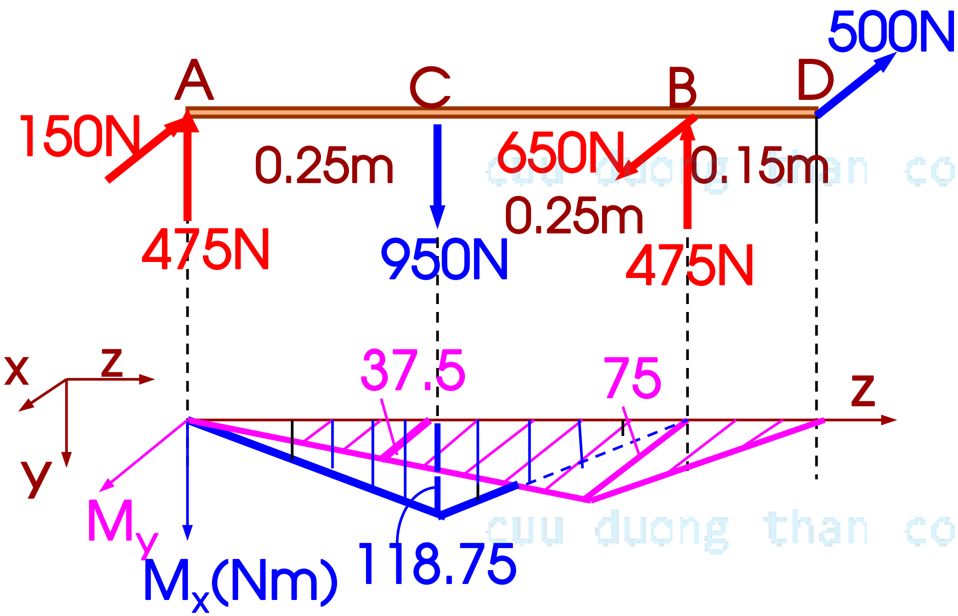
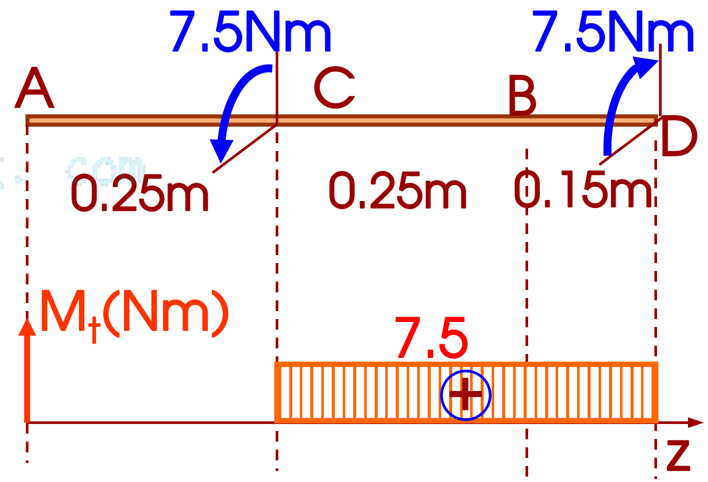




IV/ COMBINED BENDING AND TORSION (UỐN VÀ XOẮN ĐỒNG THỜI)

- **Critical point:** C $\begin{cases} |M_x| = 118.75Nm; |M_y| = 37.5Nm \\ |M_t| = 7.5Nm \end{cases}$

$$M_u = \sqrt{M_x^2 + M_y^2} = \sqrt{(118.75)^2 + (37.5)^2} = 124.53Nm$$

$$\sigma_{t3} = \sqrt{\sigma^2 + 4\tau^2} = \frac{1}{W_u}\sqrt{M_u^2 + M_t^2} \le 2\tau_{allow}$$

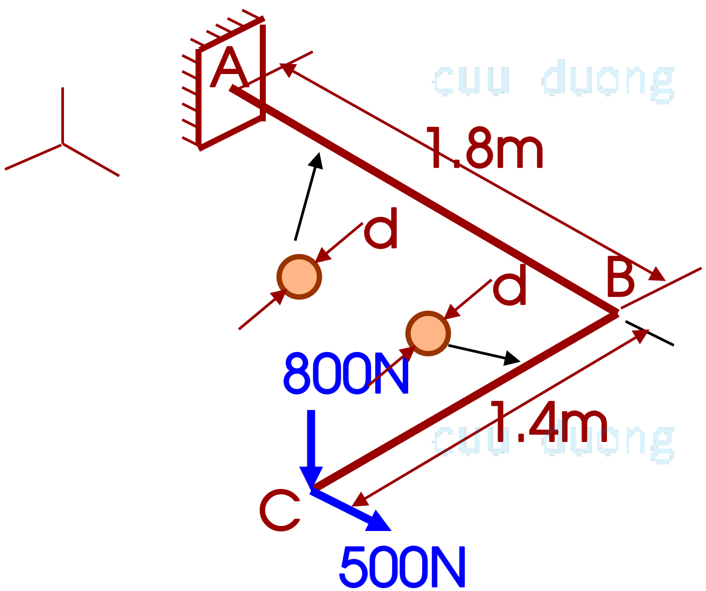
- $$\rightarrow W_u = \frac{\pi}{32}d^3 \ge \frac{\sqrt{M_u^2 + M_t^2}}{2\tau_{allow}} = \frac{\sqrt{(124.53)^2 + (7.5)^2}}{2(50x10^6)} = 124.75x10^{-8}$$

$$d \ge \sqrt[3]{\frac{32x124.75x10^{-8}}{\pi}} = 0.0233m$$

Answer: $d = 23.3mm$

V/ GENERAL COMBINED LOADING

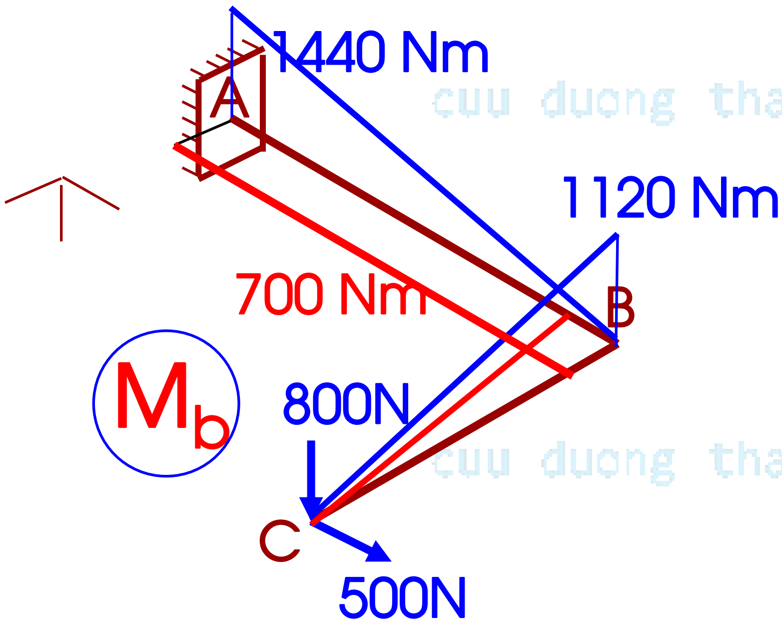
Example

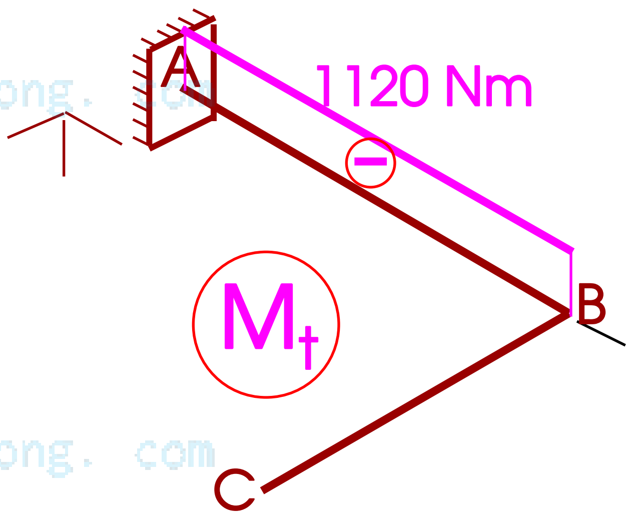


- The solid rod ABC has a radius of d = 50mm. Verify the strength condition using the maximum-shear-stress theory if [σ] = 160 MPa

V/ GENERAL COMBINED LOADING – (continued)

- SOLUTION

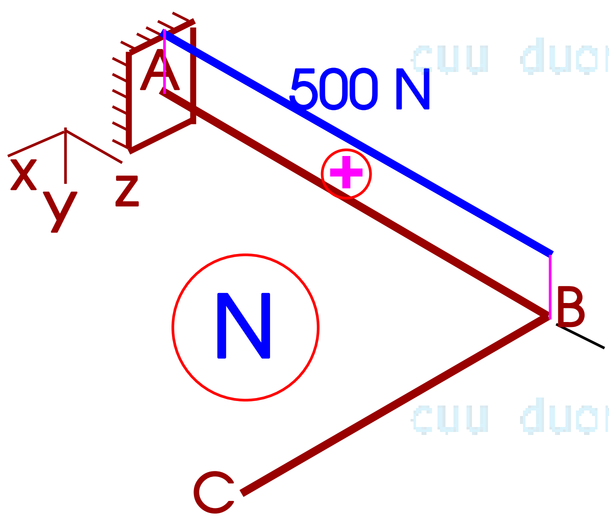




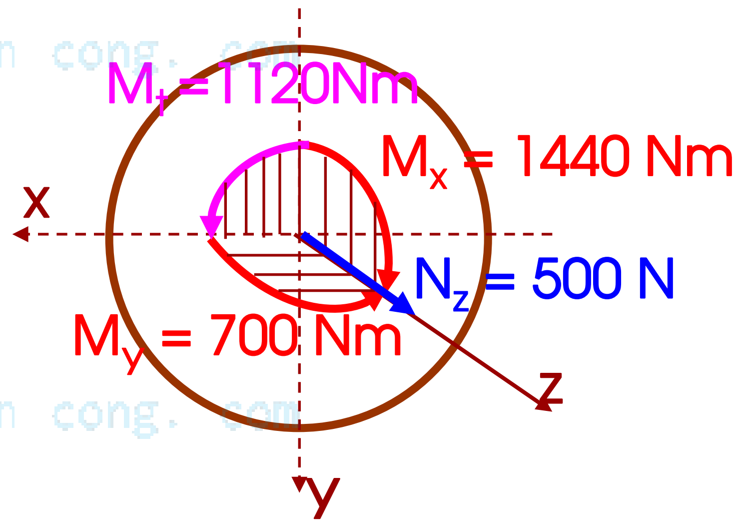
(a) Bending moment diagrams (b) Torsion moment diagram

V/ GENERAL COMBINED LOADING – (continued)

- Internal forces at built-up section A (Nội lực tại tiết diện ngàm A)

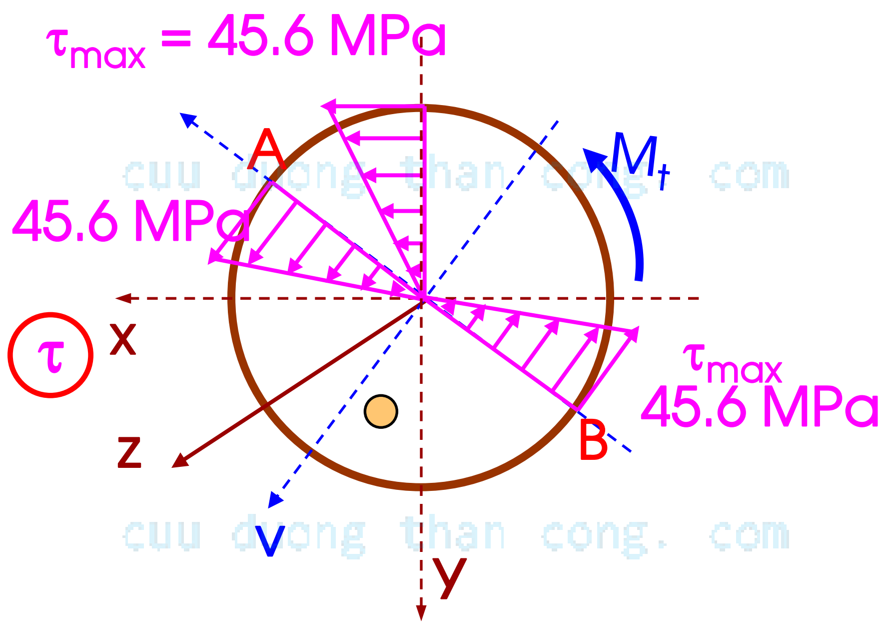


(c) Normal force diagram



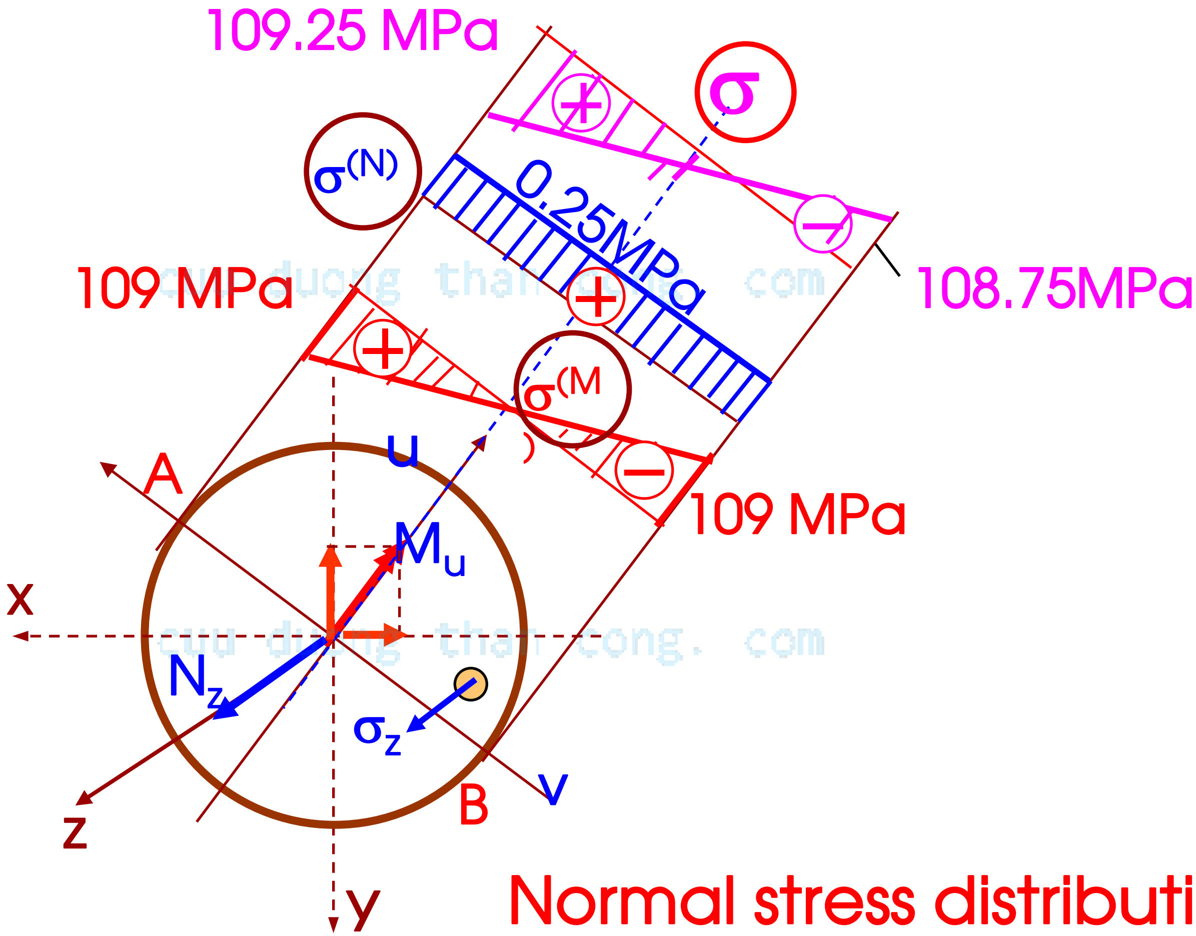
(d) Internal forces at section A

V/ GENERAL COMBINED LOADING – (continued)



Shear Stress Distribution

V/ GENERAL COMBINED LOADING – (continued)



Normal stress distribution

V/ GENERAL COMBINED LOADING – (continued)

✓Point A

$$\sigma^{(N)} = \frac{N}{A} = 0.25 \text{ MPa}$$

$$\sigma_A^{(M)} = \sigma_{max}^{(Mu)} = \frac{M_u}{W_u} = 109 \text{ MPa}$$

$$\sigma_A = \sigma^{(N)} + \sigma_A^{(M)} = 0.25 + 109 = 109.25 \text{ MPa}$$

$$\tau_A = \tau_{max} = \frac{M_t}{W_p} = 45.6 \text{ MPa}$$

$$\sigma_{t3} = \sqrt{(\sigma_A)^2 + 4(\tau_A)^2} = 142.3 \text{ MPa} \le [\sigma] = 160 \text{ MPa}$$