Chapter 9
STABILITY (ỔN ĐỊNH)

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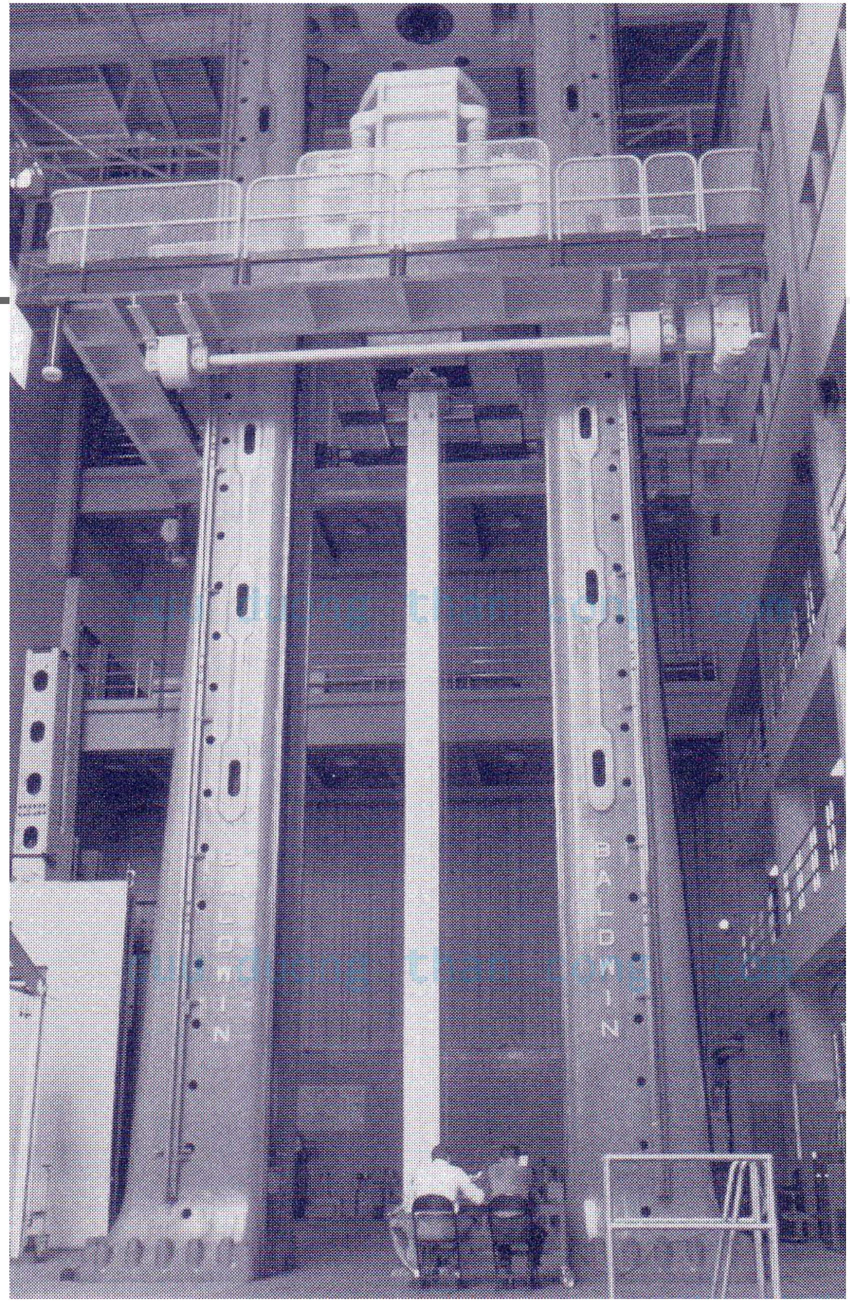

BALDWIN

I/ BUCKLING AND STABILITY

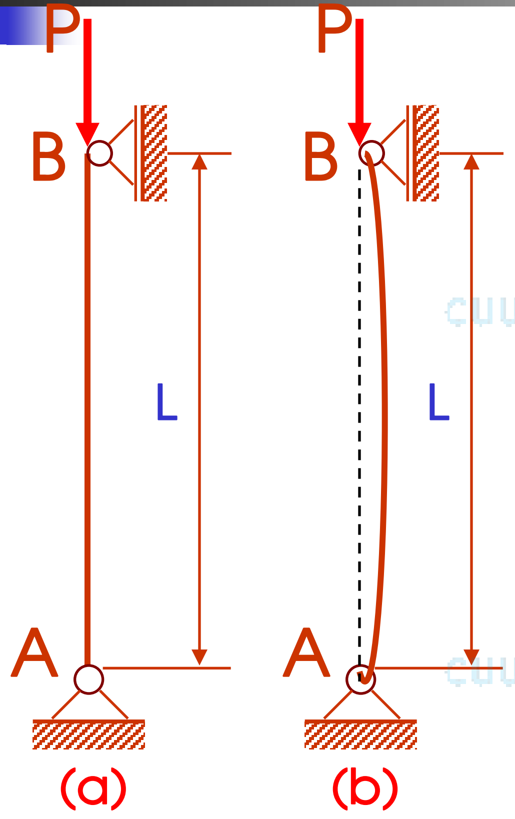


Fig.1 Buckling of a column due to an axial compressive load

✓ Column (Cột) : long, slender structural member loaded axially in compression.

If such a member is slender (mảnh), then instead of failing by direct compression, it may bent and deflect laterally (Fig.1.b)

→ Collapse completely

I/ BUCKLING AND STABILITY (cont.)

- Buckling of a rigid bar supported by a spring

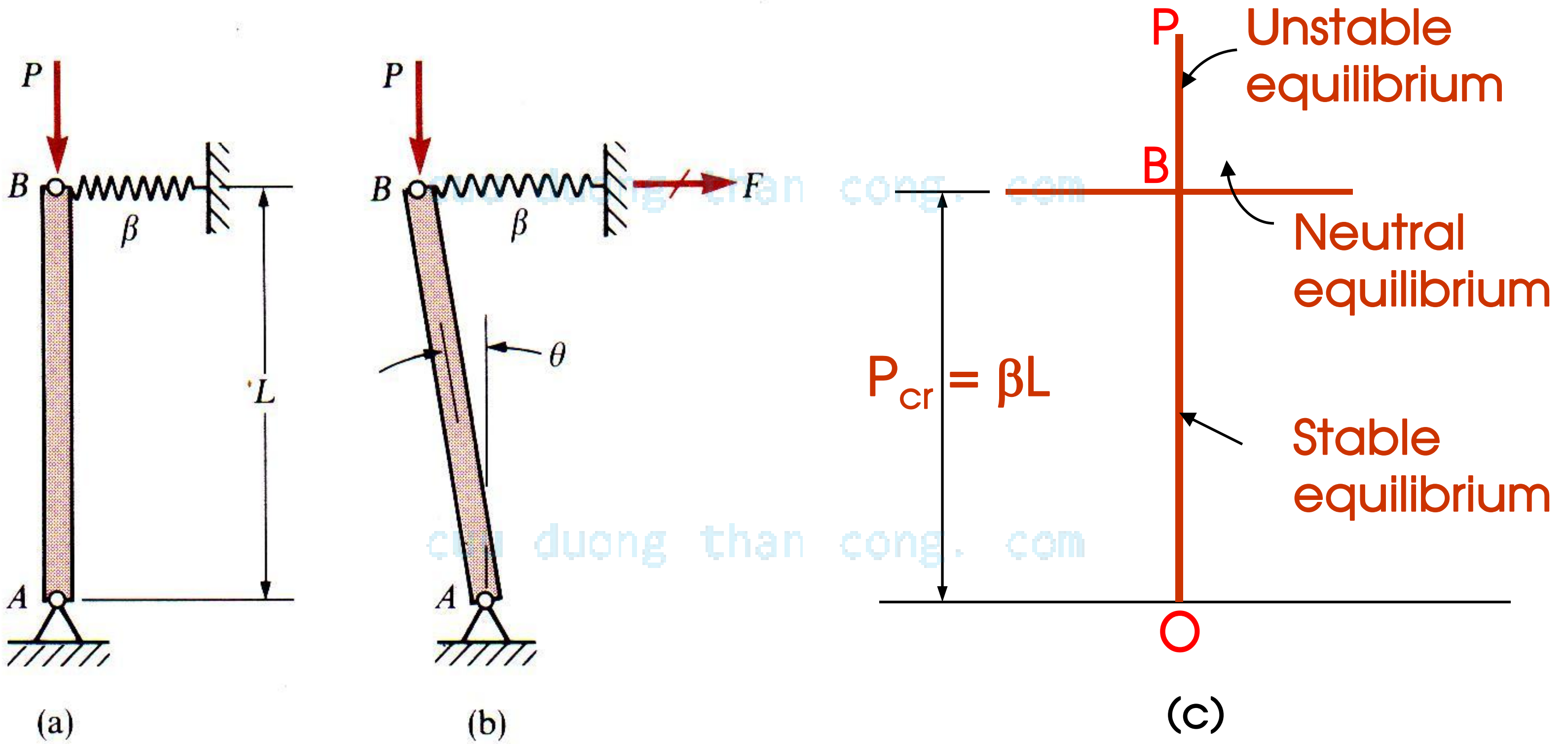


Fig.2. Buckling of a rigid bar supported by a spring

I/ BUCKLING AND STABILITY (cont.)

Phenomenon

✓ Fig. 2a: the bar supports a centrally applied load P → no force in the spring

✓ Fig. 2b: the bar is disturbed by some external force → it rotates a small angle θ about support A

- If P is small, the system is stable and will return to the initial position

- If P is very large, the system is unstable and buckles by undergoing large rotations

I/ BUCKLING AND STABILITY (cont.)

Analyse

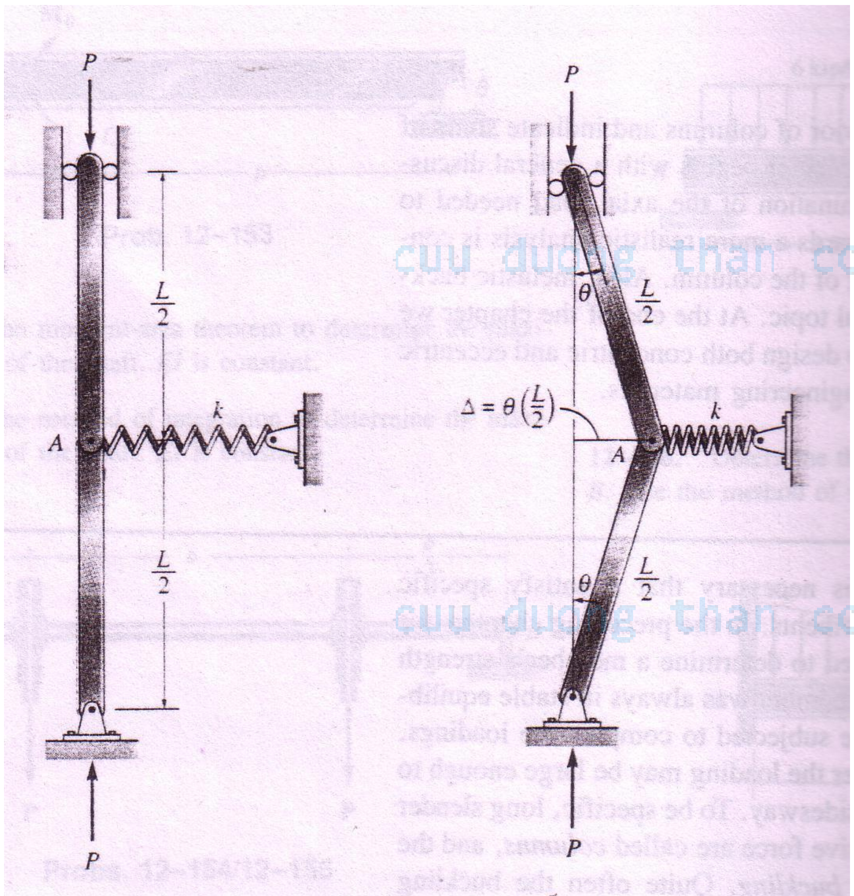
✓ When the bar is rotated θ, the spring is elongated $\theta L \rightarrow$ force $F = \beta\theta L$ in the spring $\rightarrow$ restoring moment $\beta\theta L^2$ (clockwise moment)

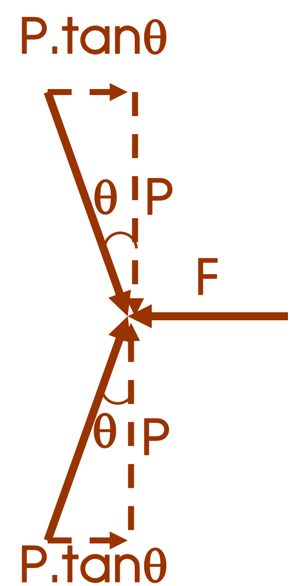
✓ The force P $\rightarrow$ overturning moment $P\theta L$ (counter clockwise moment)

- If $P\theta L < \beta\theta L^2$ or $P < \beta L \rightarrow$ stable system
- If $P\theta L > \beta\theta L^2$ or $P > \beta L \rightarrow$ unstable system

$\rightarrow$ $P_{cr} = \beta L$: critical load (1)

I/ BUCKLING AND STABILITY (cont.)





I/ BUCKLING AND STABILITY (cont.)

- Consider a two-bar mechanism consist-ing of weightless bars that are rigid and pin-connect-ed at their ends
- Fig.a→ P small → the spring is unstretched
- Displacing A a small amount Δ → the spring will produce a restoring force F = kΔ, while the applied load P develops two horizontal components, $P_x = P.\tan\theta$ which tend to push the pin (and the bars) further out of equilibrium.
- So → The restoring spring force, $F = k\theta L/2$

  The disturbing force: $2P_x = 2P\theta$

I/ BUCKLING AND STABILITY (cont.)

- If the restoring force > the disturbing force:

$$\frac{k\theta L}{2} > 2P\theta \Leftrightarrow P < \frac{kL}{4} \qquad \text{stable equilibrium}$$

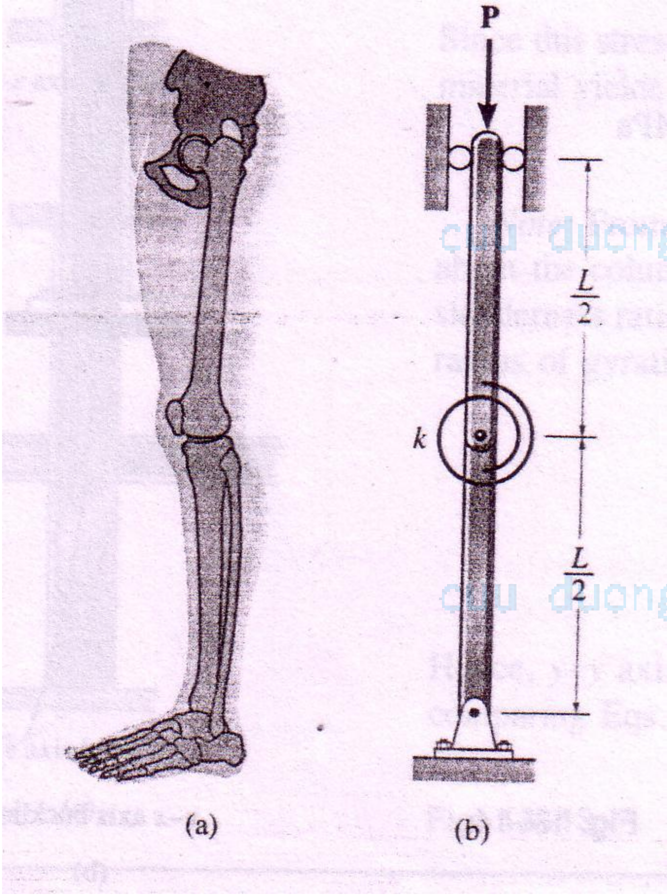
- If the restoring force < the disturbing force:

$$\frac{k\theta L}{2} > 2P\theta \Leftrightarrow P > \frac{kL}{4} \qquad \text{unstable equilibrium}$$

- If the restoring force < the disturbing force:

$$\frac{k\theta L}{2} = 2P\theta \Leftrightarrow P_{cr} = \frac{kL}{4} \qquad \text{neutral equilibrium}$$

I/ BUCKLING AND STABILITY (cont.)



Example: The leg in (a) acts as a column and can be modeled (b) by the two pin-connected members that are attached to a torsional spring having a stiffness k. Determine the critical buckling load. The material can be assumed rigid.

I/ BUCKLING AND STABILITY (cont.)

- Analogy of equilibrium between the systems in Fig.2 and Fig.3

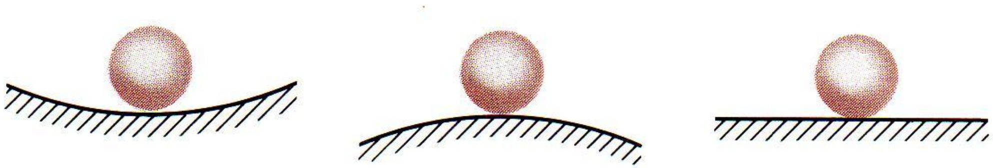
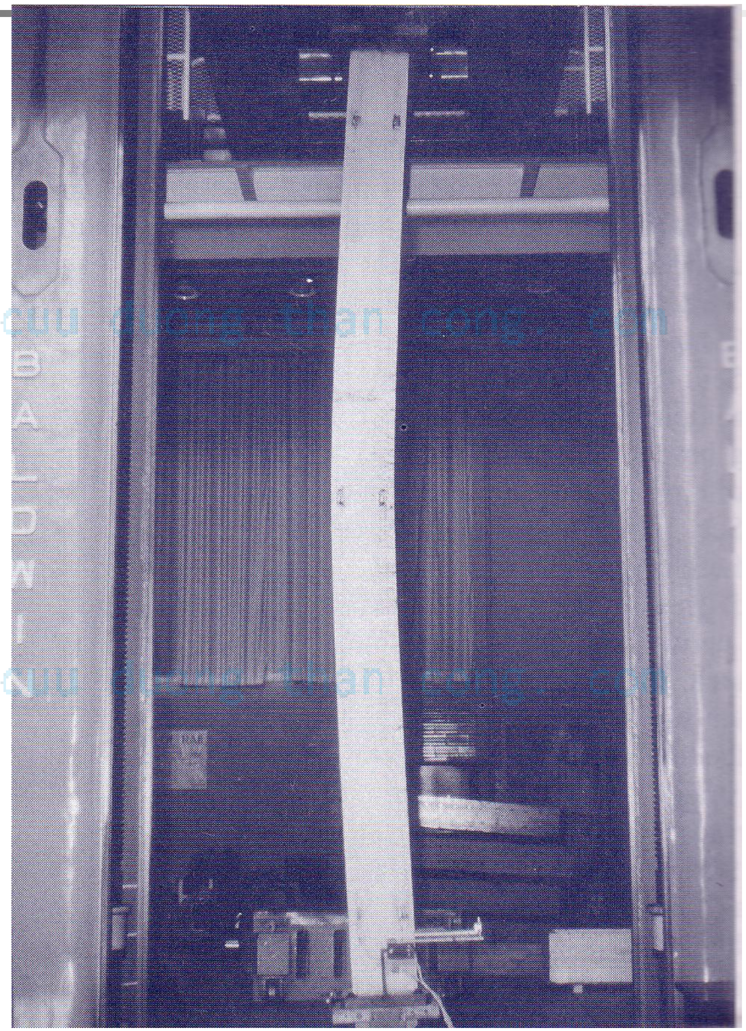


Fig.3 Ball in stable, unstable and neutral equilibrium

Buckling of column



II/ Columns with pinned ends (Euler problem)

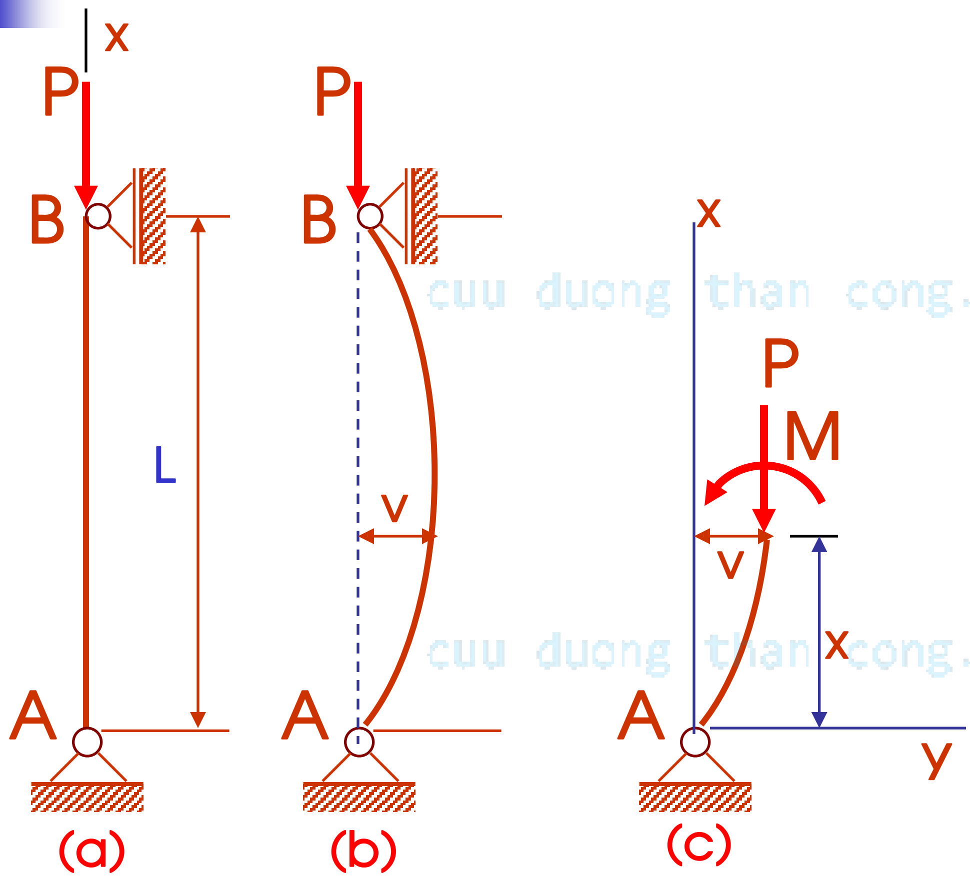


Fig.4 Column with pinned ends:

(a) Ideal column,
(b) bucled shape,
(c) free-body diagram of part of the column

II/ Columns with pinned ends (Euler problem) (cont.)

- Differential equation of deflection curve

$$EIv'' = -M = -Pv \qquad (2)$$

or $$EIv'' + Pv = 0 \qquad (3)$$

Putting: $$k^2 = \frac{P}{EI} \qquad (4)$$

$$\rightarrow \quad v'' + k^2 v = 0 \qquad (5)$$

- General solution:

$$v = C_1 \sin kx + C_2 \cos kx \qquad (6)$$

C_1, C_2 – constants from boundary cond.

II/ Columns with pinned ends (Euler problem) (cont.)

- Boundary conditions:
  - ✓ At A: $v(0) = 0 \rightarrow C_2 = 0$
  - ✓ At B: $v(L) = 0 \rightarrow C_1 \sin kL = 0$
    - $C_1 = 0 \rightarrow v = 0$: bar remains straight
    - $\sin kL = 0 \rightarrow kL = n\pi \quad n = 1,2,3,..$

$$P = \frac{n^2 \pi EI}{L^2} \qquad n = 1, 2, 3, \ldots$$

II/ Columns with pinned ends (Euler problem) (cont.)

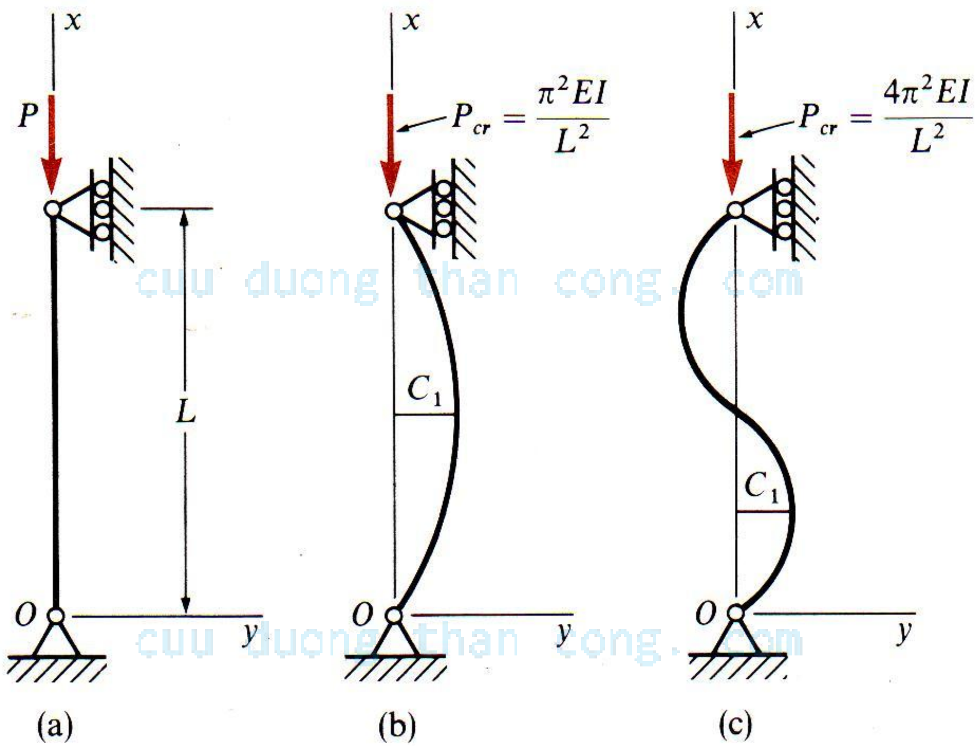
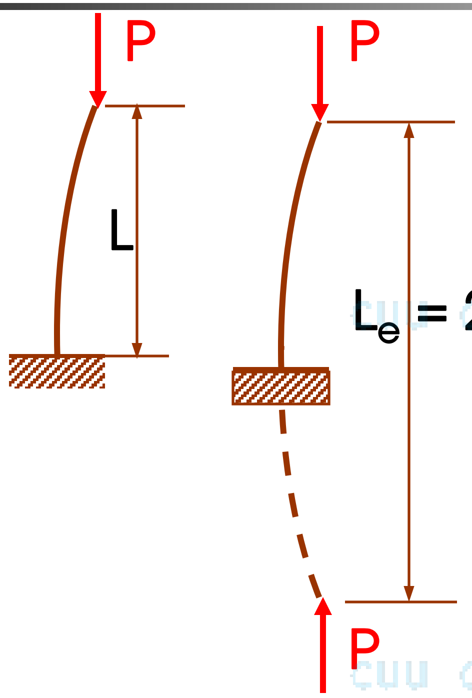


Fig.5 Buckled shapes for an ideal column with pinned ends: (a) initially straight column, (b) buckled shape for n = 1, (c) buckled shape for n = 2

III/ COLUMNS WITH OTHER SUPPORTS – EFFECTIVE LENGTH



- Effective length, L_e: is the length of the equivalent pinned-end column, or the distance between points of inflection in the deflection curve

Fig.6 Effective length L_e for a column fixed at the base and free at the upper end

III/ COLUMNS WITH OTHER SUPPORTS – EFFECTIVE LENGTH

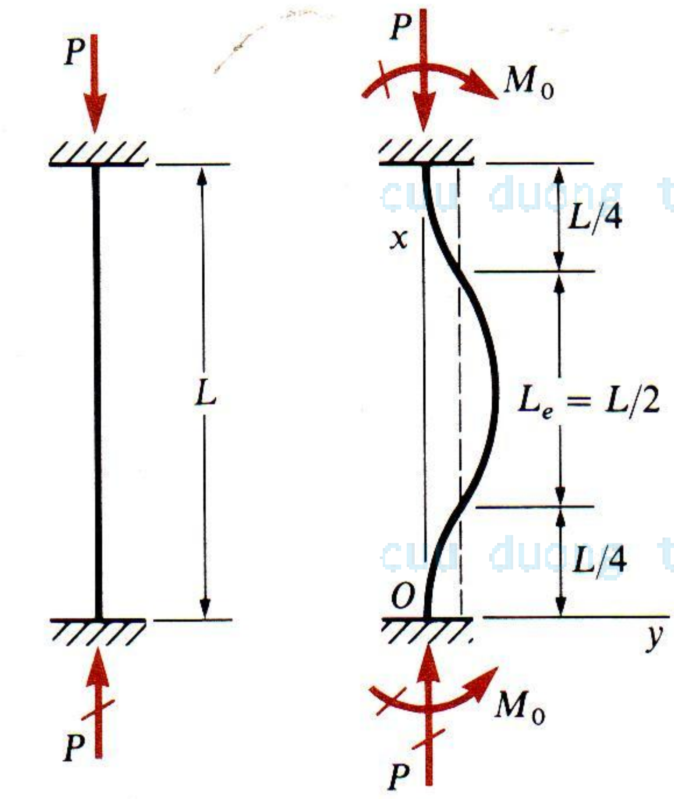


Fig.7 Effective length for a column with both ends fixed against rotation

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 EI}{(0.5L)^2}$$

III/ COLUMNS WITH OTHER SUPPORTS – EFFECTIVE LENGTH

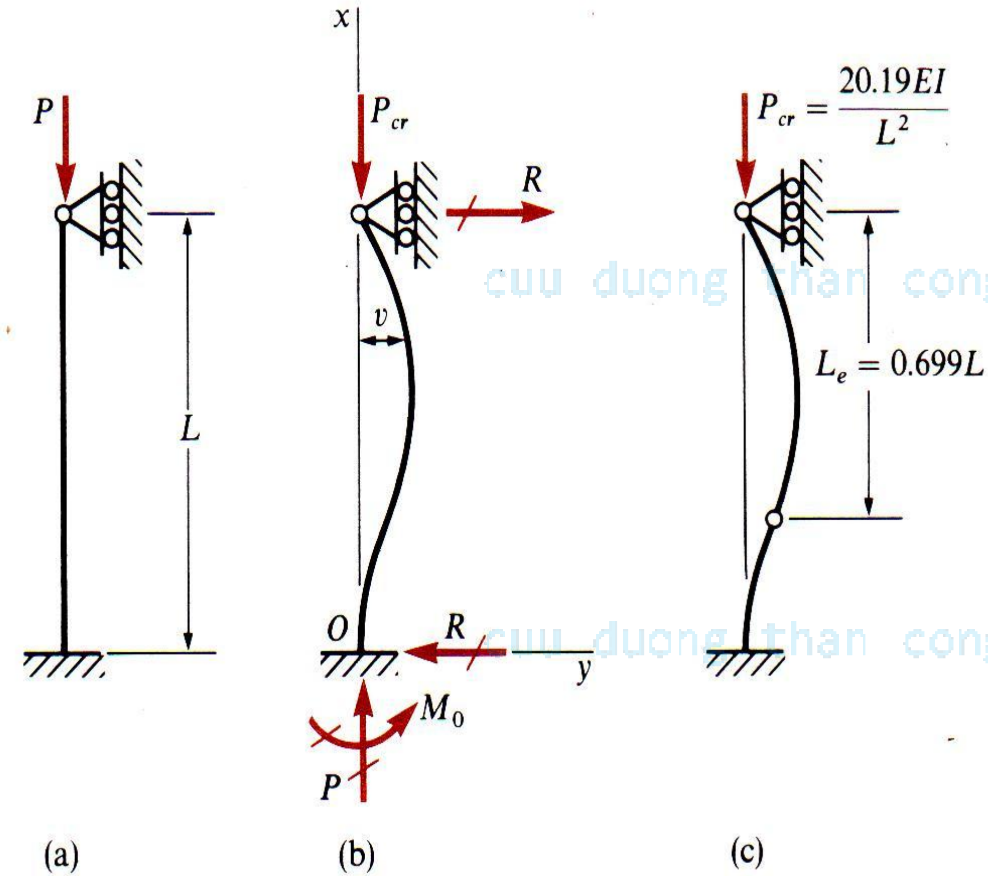
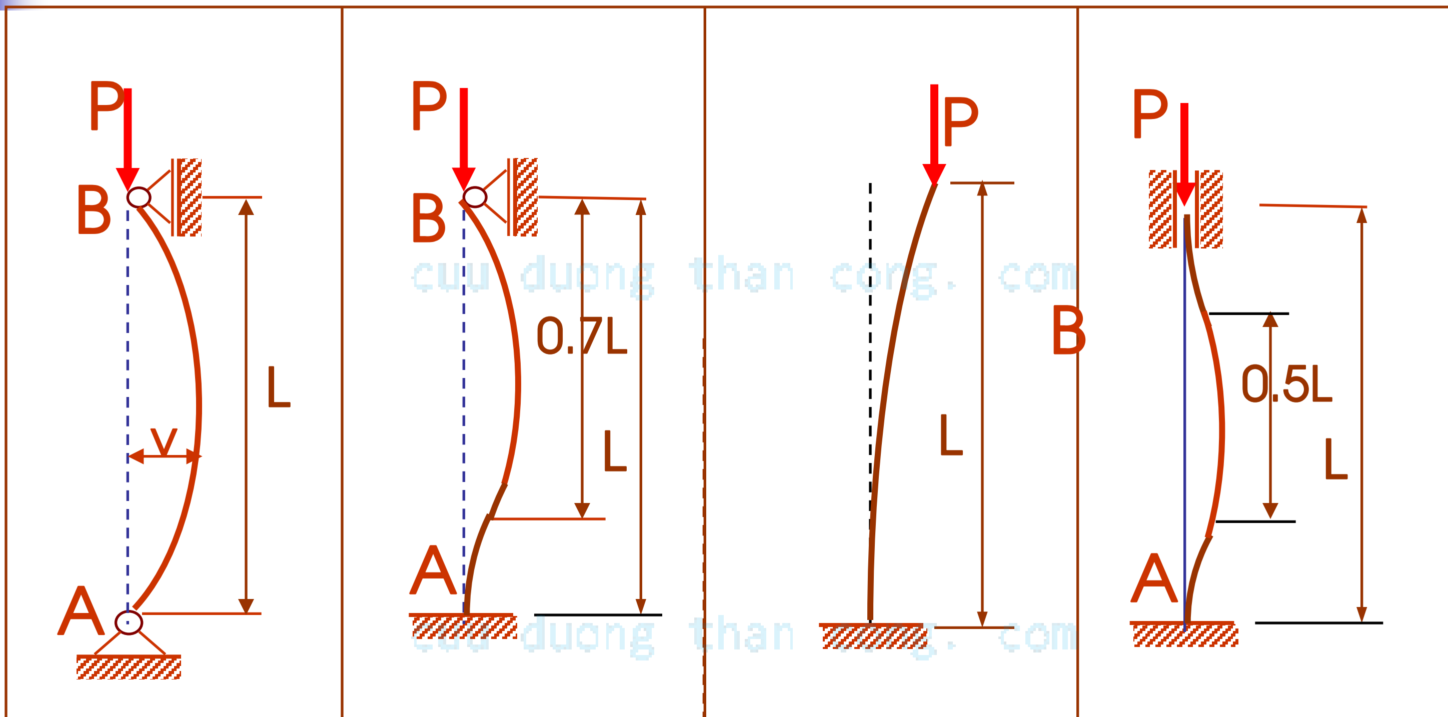


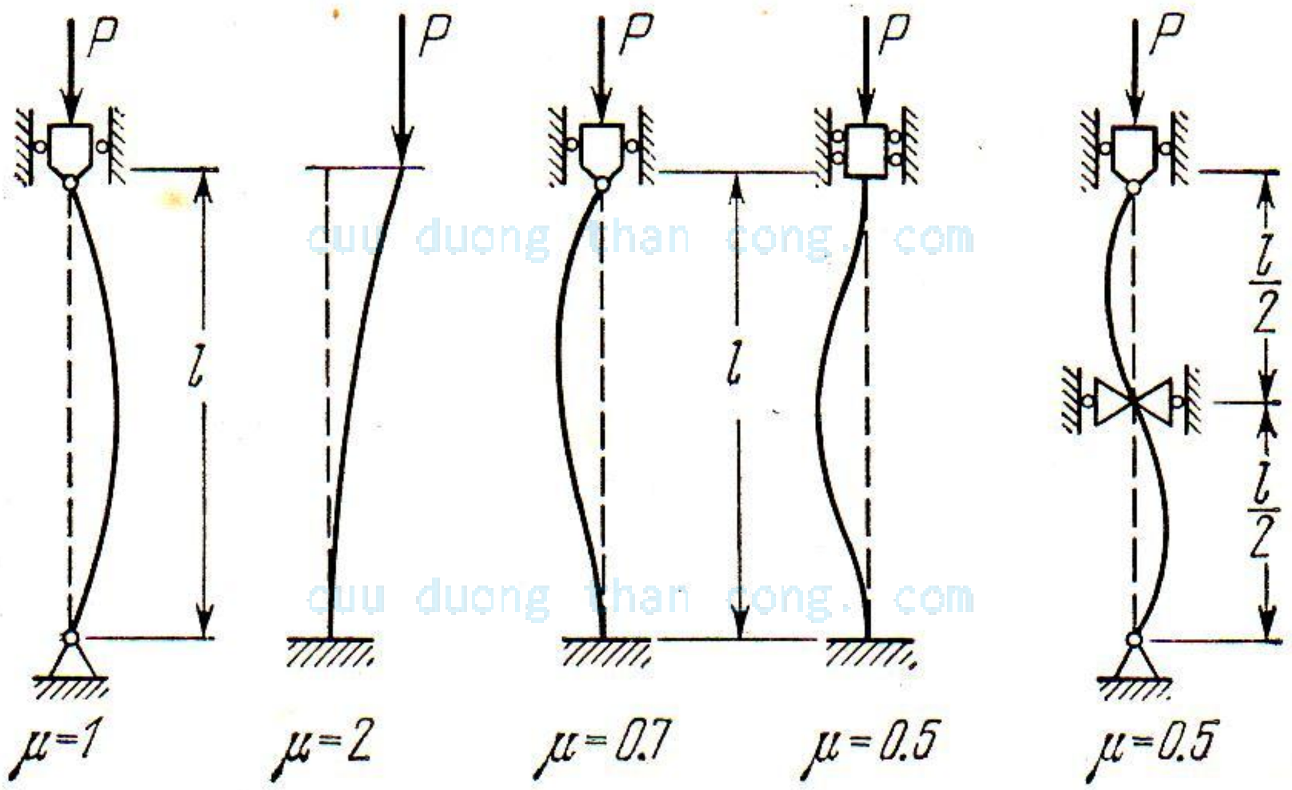
Fig.8 Effective length of column fixed at the base and pinned at the top

III/ COLUMNS WITH OTHER SUPPORTS – EFFECTIVE LENGTH



$\mu = 1$	$\mu = 0.7$	$\mu = 2$	$\mu = 0.5$

III/ COLUMNS WITH OTHER SUPPORTS – EFFECTIVE LENGTH



IV/ LIMIT OF APPLICABILITY OF EULER'S FORMULA

- Euler Critical Stress

$$\sigma_{cr}^{E} = \frac{P_{cr}}{A} = \frac{\pi^2 E I_{min}}{A(\mu L)^2} \quad \text{or} \quad \sigma_{cr}^{E} = \frac{\pi^2 E}{\lambda^2}$$

where $\lambda = \dfrac{\mu L}{i_{min}}$: slenderness (độ mảnh)

with $i_{min} = \sqrt{\dfrac{I_{min}}{A}}$: minimum radius of gyration

IV/ LIMIT OF APPLICABILITY OF EULER'S FORMULA (continued)

- **Limit of applicability of Euler's form.**

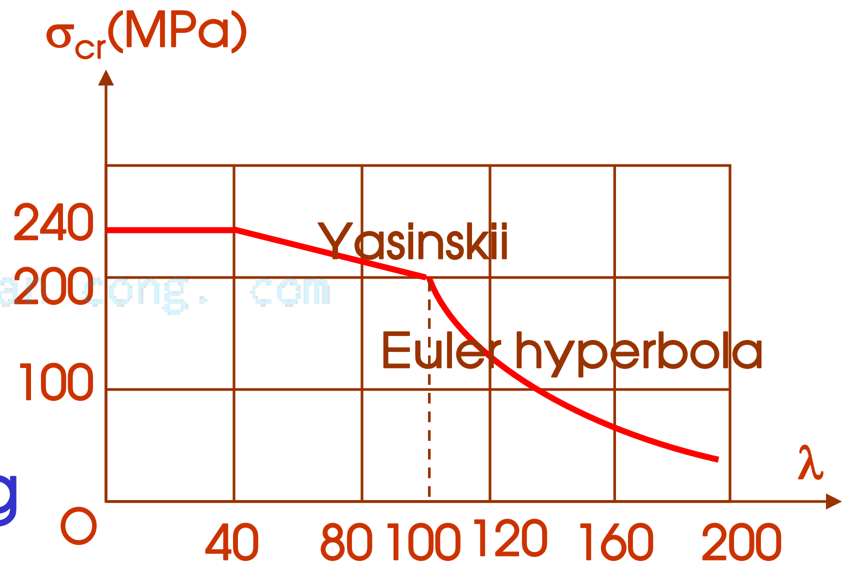
$$\sigma_{cr}^{E} = \frac{\pi^2 E}{\lambda^2} \le \sigma_{pr} \rightarrow \lambda \ge \sqrt{\frac{\pi^2 E}{\sigma_{pr}}} = \lambda_0$$

λ_0 – "limit slenderness" (độ mảnh giới hạn)

- ✓ For steel, σ_{pr} = 200 MPa, E = 200 GPa, λ_0 = 100
- ✓ For cast iron, λ_0 = 85
- ✓ For pine wood, λ_0 = 110,. etc..

V/ INELASTIC BUCKLING – Empirical formula

✓ Empirical formula of Tetmajer-Yasinskii



- $\lambda_1 \leq \lambda \leq \lambda_0$

$$\sigma_{cr} = a - b\lambda$$

a, b – constants depending upon the material

- For mild steel: a = 336 MPa , b = 1.47 MPa
- For timber: a = 29.3 MPa , b = 0.194 MPa

- $\lambda \leq \lambda_1$ $\quad \sigma_{cr} = \sigma_0$

V/ INELASTIC BUCKLING – Tangent Modulus Theory

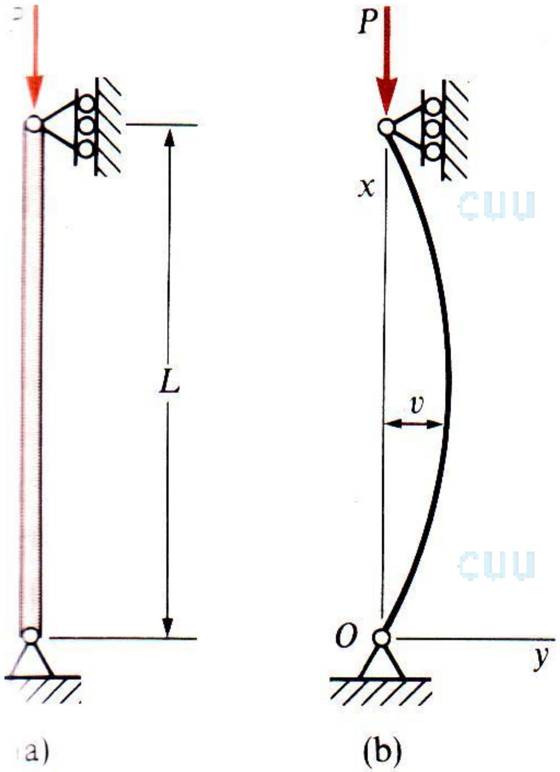


Fig. Ideal column that buckles inelastically

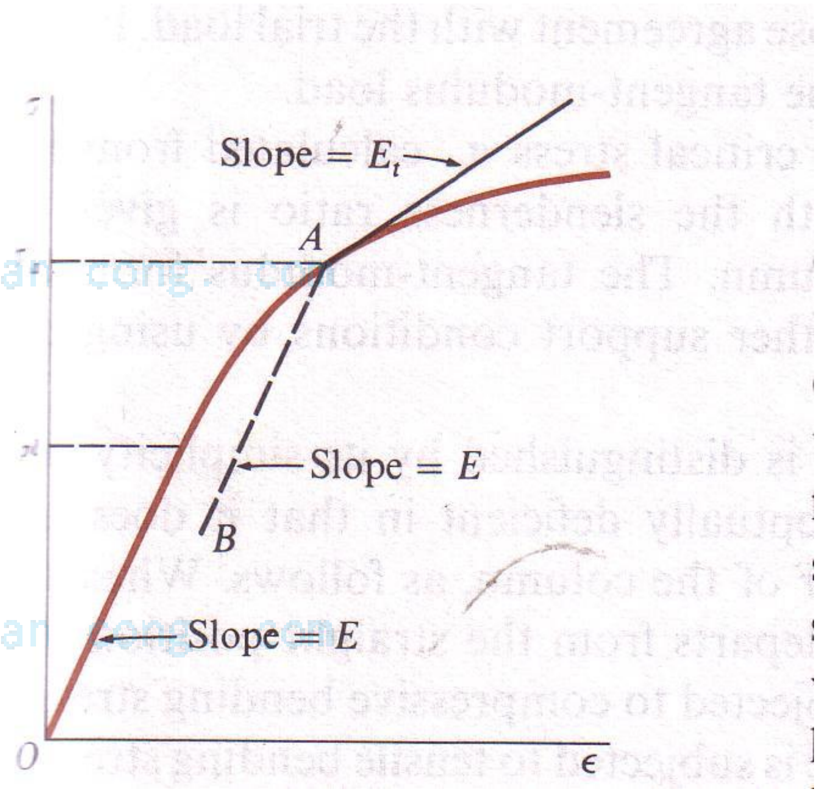


Fig. Stress – strain diagram

V/ INELASTIC BUCKLING – Tangent Modulus Theory (cont.)

- Suppose the stress σ_a in the column is above the proportional limit → point A
- If a small increase in stress ($d\sigma$) → $d\varepsilon$:

  Slope of the tangent → tangent modulus

$$E_t = \frac{d\sigma}{d\varepsilon}$$

- Curvature

$$\kappa = \frac{1}{\rho} = \frac{d^2 v}{dx^2} = -\frac{M}{E_t I}$$

- Tangent-modulus load:

$$P_t = \frac{\pi^2 E_t I}{L^2}$$

VI/ PRACTICAL METHOD FOR STABILITY CHECK OF COMPRESSED BARS

For bars under compression:

- **Strength check:** $\sigma = \dfrac{P}{A} \leq [\sigma]$ with $[\sigma] = \dfrac{\sigma_0}{n}$
- **Stability check:** $\sigma = \dfrac{P}{A} \leq [\sigma_s]$ with $[\sigma_s] = \dfrac{\sigma_{cr}}{k_s}$
- **Combination:** $\varphi = \dfrac{[\sigma_s]}{[\sigma]} = \dfrac{\sigma_{cr}}{\sigma_0} . \dfrac{n}{k_{st}} \leq 1$

$\rightarrow \sigma = \dfrac{P}{A} \leq \varphi[\sigma]$

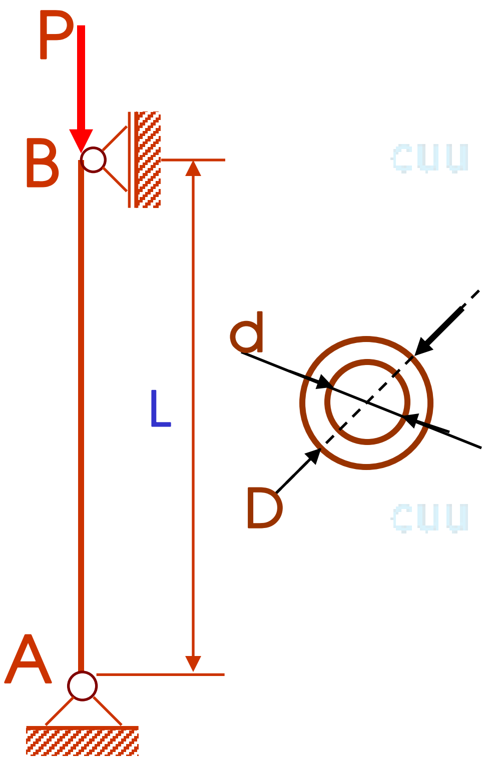
φ – reduction coefficient of the allowable stress

VI/ PRACTICAL METHOD FOR STABILITY CHECK OF COMPRESSED BARS

- Three basic problems:
  - ✓ The stability check problem
  - ✓ Choice of dimensions from stability condition
  - ✓ Choice of load to ensure the stability

VI/ PRACTICAL METHOD FOR STABILITY CHECK OF COMPRESSED BARS

Example:



Find the cross-sectional dimensions of an iron pipe column hinged at both ends and subjected to a force P.

Data: L = 4.8 m, d/D = 0.6

P = 850 kN, [σ] = 120 MPa

VI/ PRACTICAL METHOD FOR STABILITY CHECK OF COMPRESSED BARS

- Solution
  - ✓ Geometrical properties

$$A = \frac{\pi\left(D^2 - d^2\right)}{4} = \frac{\pi D^2}{4}\left[1 - (0.6)^2\right] = 0.503D^2$$

$$i = \sqrt{\frac{I}{A}} = \sqrt{\frac{\pi D^4\left[1 - \eta^4\right]}{64\,\frac{\pi D^2}{4}\left[1 - \eta^2\right]}} = 0.291D$$

VI/ PRACTICAL METHOD FOR STABILITY CHECK OF COMPRESSED BARS

- ✓ Suppose $\varphi_1 = 0.5$

$$A_1 \geq \frac{P}{\varphi_1[\sigma]} = \frac{850000}{0.5x120x10^6} = 0.0142m^2$$

$$\rightarrow D_1 = \sqrt{0.0142/0.503} = 0.168m$$

$$i_1 = 0.291D_1 = 0.291x0.168 = 0.05m$$

$$\lambda_1 = \frac{\mu L}{i_1} = \frac{4.8}{0.049} = 98.17 \rightarrow \varphi_1' = 0.17 \neq \varphi_1 = 0.5$$

VI/ PRACTICAL METHOD FOR STABILITY CHECK OF COMPRESSED BARS

✓ Suppose $\varphi_2 = (0.5 + 0.17)/2 = 0,335$

$$A_2 \geq \frac{P}{\varphi_2[\sigma]} = \frac{850000}{0.335 x 120 x 10^6} = 0.0211 m^2$$

$$\rightarrow D_2 = \sqrt{0.021 / 0.503} = 0.205 m$$

$$i_2 = 0.291 D_2 = 0.291 x 0.205 = 0.06 m$$

$$\lambda_2 = \frac{\mu L}{i_2} = \frac{4.8}{0.06} = 80.45 \rightarrow \varphi_2' = 0.23 \neq \varphi_2 = 0.335$$

VI/ PRACTICAL METHOD FOR STABILITY CHECK OF COMPRESSED BARS

✓ Suppose $\varphi_3 = (0.33 + 0.23)/2 = 0.28$

$$A_3 \geq \frac{P}{\varphi_3[\sigma]} = \frac{850000}{0.28 \times 120 \times 10^6} = 0.025m^2$$

$$\rightarrow D_3 = \sqrt{0.025 / 0.503} = 0.224m$$

$$i_3 = 0.291D_3 = 0.291 \times 0.224 = 0.065m$$

$$\lambda_3 = \frac{\mu L}{i_3} = \frac{4.8}{0.065} = 73.85 \rightarrow \varphi'_3 = 0.31 \neq \varphi_3 = 0.28$$

VI/ PRACTICAL METHOD FOR STABILITY CHECK OF COMPRESSED BARS

- ✓ Suppose $\varphi_4 = (0.31 + 0.28)/2 = 0.30$

$$A_4 \geq \frac{P}{\varphi_4[\sigma]} = \frac{850000}{0.30 \times 120 \times 10^6} = 0.0236 m^2$$

$$\rightarrow D_4 = \sqrt{0.0236 / 0.503} = 0.217 m$$

$$i_4 = 0.291 D_4 = 0.291 \times 0.217 = 0.063 m$$

$$\lambda_4 = \frac{\mu L}{i_4} = \frac{4.8}{0.063} = 76.13 \rightarrow \varphi'_4 = 0.29 \approx \varphi_4 = 0.30$$

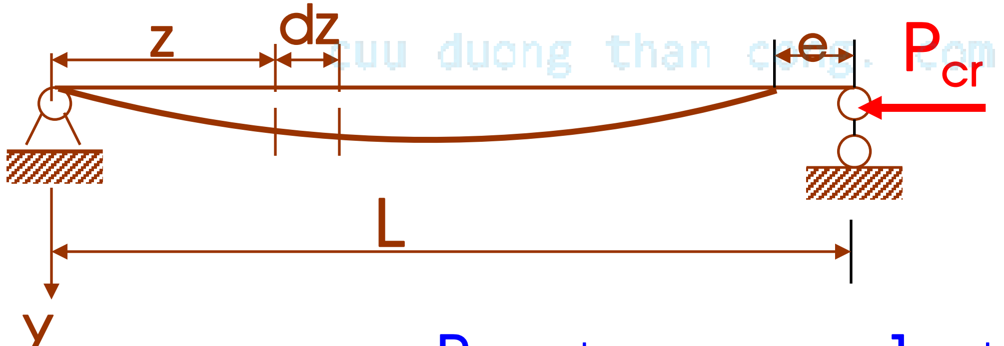
$$\rightarrow A = 0.0236 \text{ m}^2$$

VII/ OPTIMUM SECTION SHAPES & MATERIALS

- Critical load:
  - ✓ Elastic: $$P_{cr} = \frac{\pi^2 EI}{(\mu L)^2}$$
  - ✓ Inelastic: $$P_{cr} = \frac{\pi^2 E_t I}{(\mu L)^2}$$
- To increase P_{cr}
  - • Choose materials with "big value of E"
  - • If $\mu_x = \mu_y \rightarrow$ choose sections $I_x = I_y$: regular polygons, circular,… …
  - • To incease I_x & $I_y \rightarrow$ hollow sections

VIII/ DETERMINE CRITICAL LOAD BY ENERGY METHOD

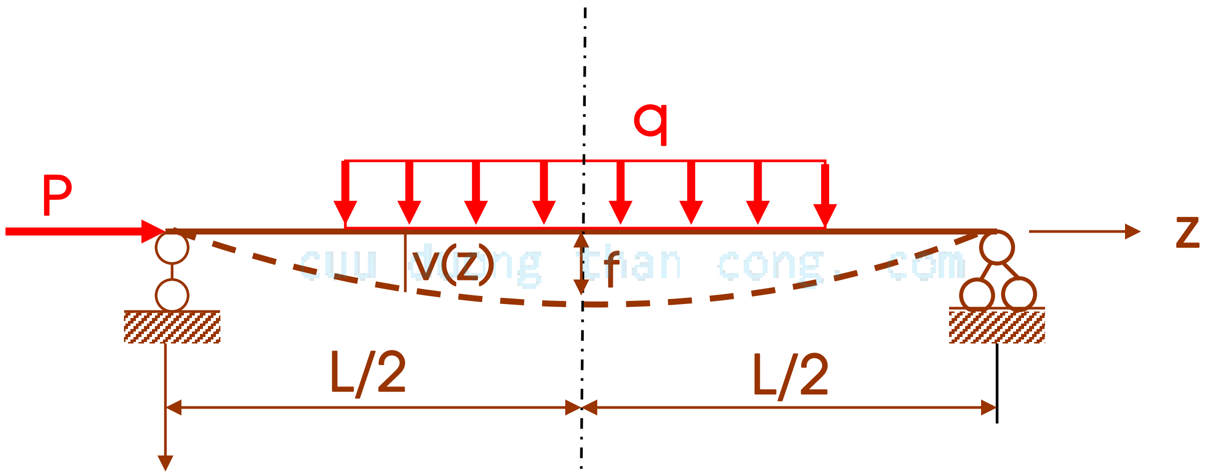
- Application of energy conservative principle to determine critical load



$$W = U \Leftrightarrow \frac{P_{cr}}{2}\int_0^L y'^2 dz = \frac{1}{2}\int_0^L EIy''^2 dz$$

$$\rightarrow \quad P_{cr} = \frac{\int_0^L EIy''^2 dz}{\int_0^L y'^2 dz}$$

IX/ COMPOSED LATERAL AND LONGITUDINAL BENDING (UỐN NGANG – UỐN DỌC ĐỒNG THỜI)



$$M(z) = M^*(z) + P.v(z)$$

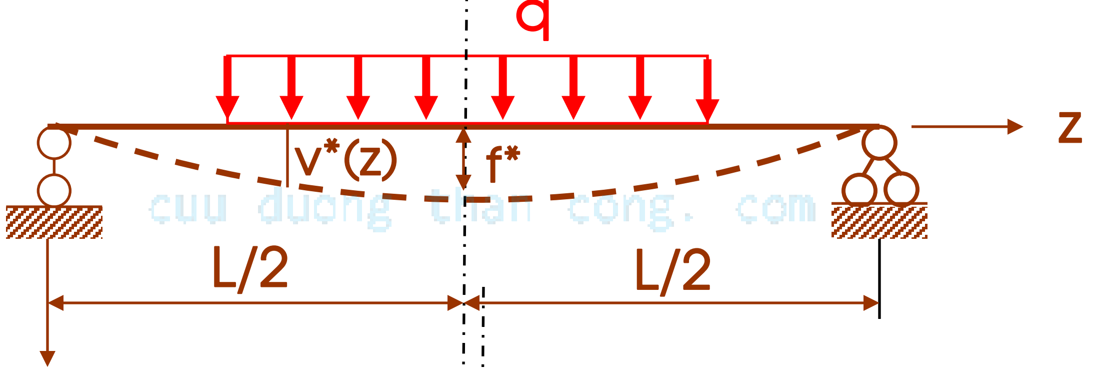
$M^*(z)$ – bending moment due to transversal loads

$P.v(z)$ – bending moment due to longitudinal load P

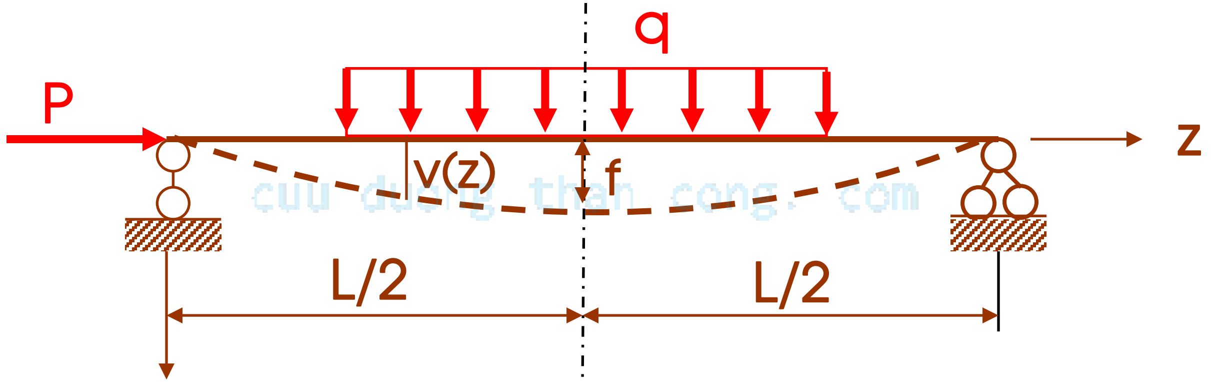
IX/ COMPOSED LATERAL AND LONGITUDINAL BENDING – APPROXIMATIVE METHOD

Consider two cases:

a/



b/



IX/ COMPOSED LATERAL AND LONGITUDINAL BENDING – APPROXIMATIVE METHOD (continued)

- Case 1: Only transversal loads

$$v^*(z) = f^* \sin\frac{\pi z}{L} \rightarrow M^*(z) = -EI\frac{\pi^2}{L^2} f^* \sin\frac{\pi z}{L}$$

- Case 2: Transversal + Longitudinal loads

$$v(z) = f \sin\frac{\pi z}{L} \rightarrow M(z) = -EI\frac{\pi^2}{L^2} f \sin\frac{\pi z}{L}$$

IX/ COMPOSED LATERAL AND LONGITUDINAL BENDING – APPROXIMATIVE METHOD (continued)

$$M = M^* + Pv(z) \rightarrow$$

$$-EIf\sin\frac{\pi z}{L} = -EIf^*\sin\frac{\pi z}{L} + Pf\sin\frac{\pi z}{L}$$

$$v(z) = \frac{v^*(z)}{\left(1 - \frac{P}{P_E}\right)} \quad \text{with} \quad P_E = \frac{\pi^2 EI}{L^2}$$

$$M(z) = \frac{M^*(z)}{\left(1 - \frac{P}{P_E}\right)}; \qquad Q(z) = \frac{Q^*(z)}{\left(1 - \frac{P}{P_E}\right)}$$

IX/ COMPOSED LATERAL AND LONGITUDINAL BENDING – APPROXIMATIVE METHOD (continued)

- Maximum normal stress:

$$\max \sigma = \frac{P}{A} + \frac{M}{W} = \frac{P}{A} + \frac{M^*}{W(1 - P/P_E)}$$

- Load factor – Strength condition

$$\frac{nP}{A} + \frac{nM^*}{W(1 - nP/P_E)} \le \sigma_0$$