Chương 5: XOẮN THUẦN TÚY

5.1. BIẾN DẠNG XOẮN

5.2. MOMENT XOẮN

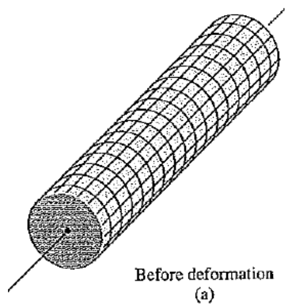
5.3. TRUYỀN ĐỘNG CÔNG SUẤT

5.4. GÓC XOẮN

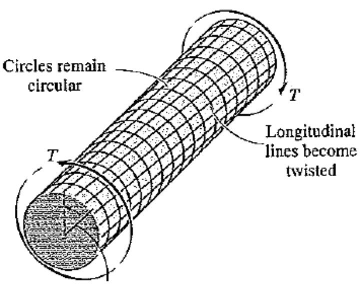
Chương 5: XOẮN THUẦN TÚY

5.1. BIẾN DẠNG XOẮN

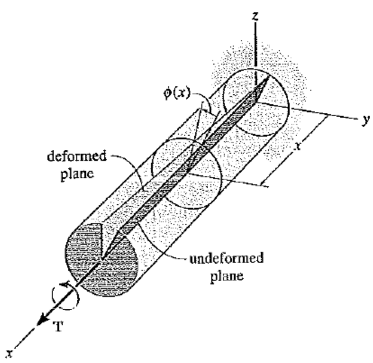
Xét một thanh có mặt cắt ngang hình tròn



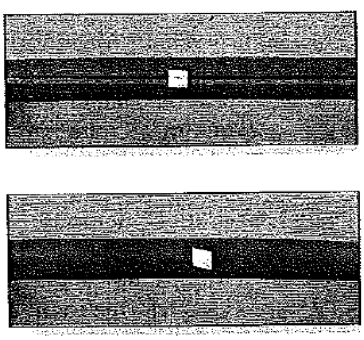
Before deformation
(a)



After deformation
(b)



The angle of twist $\phi(x)$ increases as x increases.

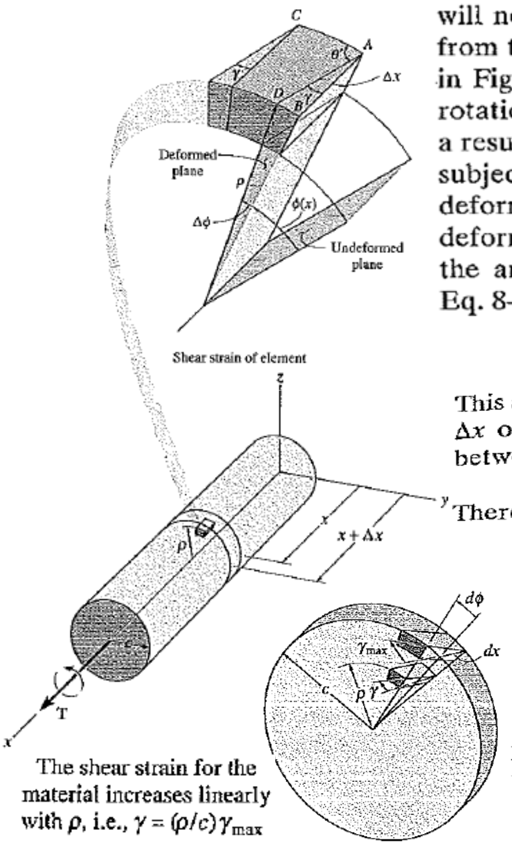


Notice the deformation of the rectangular element when this rubber bar is subjected to a torque.

If the shaft is fixed at one end and a torque is applied to its other end, the shaded plane in Fig. 11–2 will distort into a skewed form as shown. Here a radial line located on the cross section at a distance x from the fixed end of the shaft will rotate through an angle $\phi(x)$. The angle $\phi(x)$, so defined, is called the *angle of twist*. It depends on the position x and will vary along the shaft as shown.

Chương 5: XOẮN THUẦN TÚY

5.1. BIẾN DẠNG XOẮN



The shear strain for the material increases linearly with ρ, i.e., $\gamma = (\rho/c)\gamma_{max}$

In order to understand how this distortion strains the material, we will now isolate a small element located at a radial distance ρ (rho) from the axis of the shaft, Fig. 11–3. Due to the deformation as noted in Fig. 11–2, the front and rear faces of the element will undergo a rotation. The back face by $\phi(x)$, and the front face by $\phi(x) + \Delta\phi$. As a result, the *difference* in these rotations, $\Delta\phi$, causes the element to be subjected to a *shear strain*. To calculate this strain, note that before deformation the angle between the edges AB and AC is 90°; after deformation, however, the edges of the element are AD and AC and the angle between them is θ'. From the definition of shear strain, Eq. 8–13, we have

$$\gamma = \frac{\pi}{2} - \lim_{\substack{C\to A \text{ along } CA \\ B\to A \text{ along } BA}} \theta'$$

This angle, γ, is indicated on the element. It can be related to the length Δx of the element and the difference in the angle of rotation, $\Delta\phi$, between the shaded faces. If we let $\Delta x \to dx$ and $\Delta\phi \to d\phi$, we have

$$BD = \rho\, d\phi = dx\, \gamma$$

Therefore,

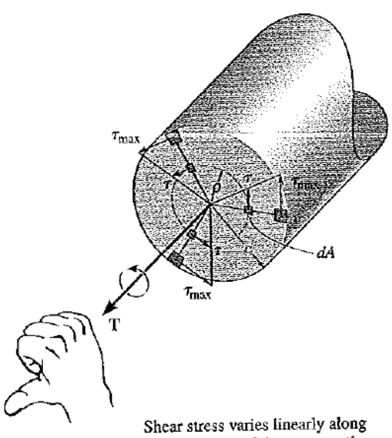
$$\gamma = \rho \frac{d\phi}{dx} \qquad (11\text{–}1)$$

Since dx and $d\phi$ are the *same* for *all elements* located at points on the cross section at x, then $d\phi/dx$ is constant, and Eq. 11–1 states that the magnitude of the shear strain for any of these elements varies only with its radial distance ρ from the axis of the shaft. In other words, the shear strain within the shaft varies linearly along any radial line, from zero at the axis of the shaft to a maximum γ_{max} at its outer boundary, Fig. 11–4. Since $d\phi/dx = \gamma/\rho = \gamma_{max}/c$, then

$$\gamma = \left(\frac{\rho}{c}\right)\gamma_{max} \qquad (11\text{–}2)$$

Chương 5: XOẮN THUẦN TÚY

5.2. MOMENT XOẮN



Shear stress varies linearly along each radial line of the cross section.

Fig. 11–5

If the material is linear-elastic, then Hooke's law applies, $\tau = G\,\gamma$, and consequently a ***linear variation in shear strain***, as noted in the previous section, leads to a corresponding ***linear variation in shear stress*** along any radial line on the cross section. Hence, like the shear-strain variation, for a solid shaft, τ will vary from zero at the shaft's longitudinal axis to a maximum value, τ_{max}, at its outer surface. This variation is shown in Fig. 11–5 on the front faces of a selected number of elements, located at an intermediate radial position ρ and at the outer radius c. Due to the proportionality of triangles, or by using Hooke's law ($\tau = G\,\gamma$) and Eq. 11–2 [$\gamma = (\rho/c)\gamma_{max}$], we can write

$$\tau = \left(\frac{\rho}{c}\right)\tau_{max}$$

. Specifically, each element of area dA, located at ρ, is subjected to a force of $dF = \tau\, dA$. The torque produced by this force is $dT = \rho(\tau\, dA)$. We therefore have for the entire cross section

$$T = \int_A \rho(\tau\, dA) = \int_A \rho\left(\frac{\rho}{c}\right)\tau_{max}\, dA \qquad (11\text{–}4)$$

Since τ_{max}/c is constant,

$$T = \frac{\tau_{max}}{c}\int_A \rho^2\, dA \qquad (11\text{–}5)$$

5.2. MOMENT XOẮN

The integral in this equation depends only on the geometry of the shaft. It represents the ***polar moment of inertia*** of the shaft's cross-sectional area computed about the shaft's longitudinal axis. We will symbolize its value as J, and therefore the above equation can be written in a more compact form, namely,

$$\tau_{max} = \frac{Tc}{J} \qquad (11\text{–}6)$$

where

τ_{max} = the maximum shear stress in the shaft, which occurs at the outer surface

T = the resultant internal torque acting at the cross section. Its value is determined from the method of sections and the equation of moment equilibrium applied about the shaft's longitudinal axis

J = the polar moment of inertia of the cross-sectional area

c = the outer radius of the shaft

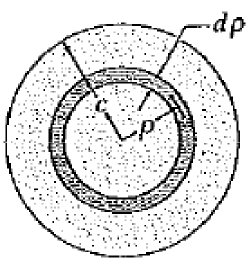
Using Eqs. 11–3 and 11–6, the shear stress at the intermediate distance ρ can be determined from a similar equation:

$$\tau = \frac{T\rho}{J} \qquad (11\text{–}7)$$

5.2. MOMENT XOẮN

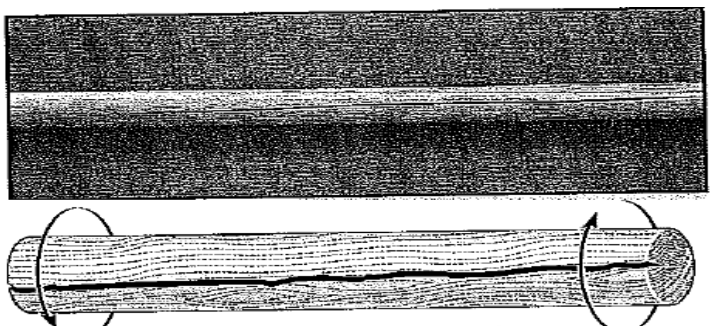
Trường hợp: Trục đặc If the shaft has a solid circular cross section, the polar moment of inertia J can be determined using an area element in the form of a *differential ring* or annulus having a thickness $d\rho$ and circumference $2\pi\rho$, Fig. 11–6. For this ring, $dA = 2\pi\rho\, d\rho$, so

$$J = \int_A \rho^2\, dA = \int_0^c \rho^2 (2\pi\rho\, d\rho) = 2\pi \int_0^c \rho^3\, d\rho = 2\pi\left(\frac{1}{4}\right)\rho^4 \Big|_0^c$$



$$J = \frac{\pi}{2}c^4 \qquad (11\text{–}8)$$

Note that J is a *geometric property* of the circular area and is always positive. Common units used for its measurement are mm^4.

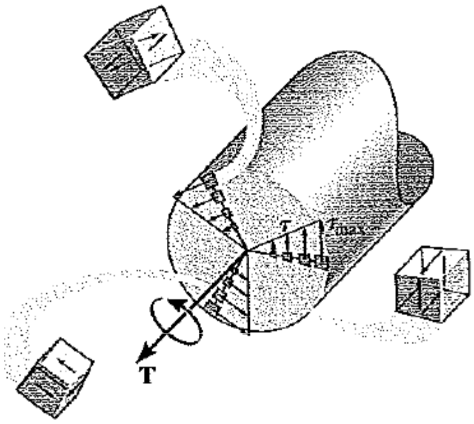


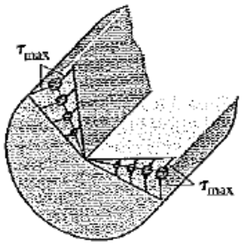
Failure of a wooden shaft due to torsion.

Chương 5: XOẮN THUẦN TÚY

5.2. MOMENT XOẮN

Trường hợp: Trục đặc





Shear stress varies linearly along each radial line of the cross section.
(b)

The shear stress has been shown to vary linearly along each radial line of the cross section of the shaft. However, if a volume element of material on the cross section is isolated, then due to the complementary property of shear, equal shear stresses must also act on four of its adjacent faces as shown in Fig. 11–7*a*. Hence, ***not only does the internal torque T develop a linear distribution of shear stress along each radial line in the plane of the cross-sectional area, but also an associated shear-stress distribution is developed along an axial plane***, Fig. 11–7*b*. It is interesting to note that because of this axial distribution of shear stress, shafts made from wood tend to *split* along the axial plane when subjected to excessive torque, Fig. 11–8. This is because wood is an anisotropic material. Its shear resistance parallel to its grains or fibers, directed along the axis of the shaft, is much less than its resistance perpendicular to the fibers, directed in the plane of the cross section.

Chương 5: XOẮN THUẦN TÚY

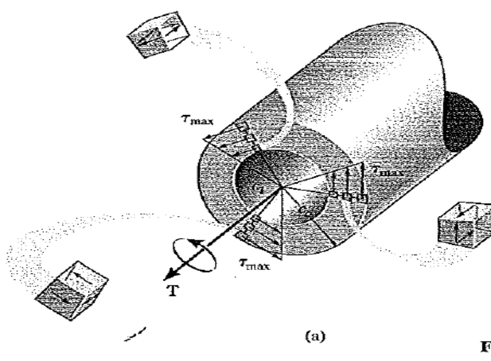
5.2. MOMENT XOẮN

Trường hợp: Trục rỗng

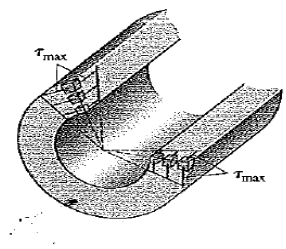
Tubular Shaft. If a shaft has a tubular cross section, with inner radius c_i and outer radius c_o, then from Eq. 11–8 we can determine its polar moment of inertia by subtracting J for a shaft of radius c_i from that determined for a shaft of radius c_o. The result is

$$J = \frac{\pi}{2}(c_o^4 - c_i^4)$$

Like the solid shaft, the shear stress distributed over the tube's cross-sectional area varies linearly along any radial line, Fig. 11–9*a*. Furthermore, the shear stress varies along an axial plane in this same manner, Fig. 11–9*b*. Examples of the shear stress acting on typical volume elements are shown in Fig. 11–9*a*.



(a)



Shear stress varies linearly along each radial line of the cross section.
(b)

Fig. 11–9

5.2. MOMENT XOẮN

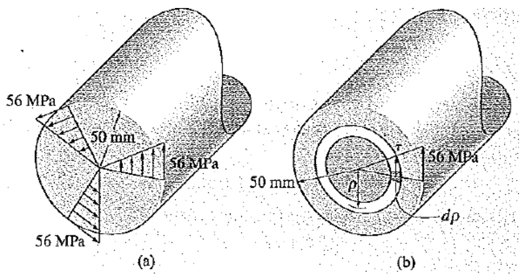
Ứng suất xoắn lớn nhất:

Absolute Maximum Torsional Stress. At any given cross section of the shaft the maximum shear stress occurs at the outer surface. However, if the shaft is subjected to a series of external torques, or the radius (polar moment of inertia) changes, then the maximum torsional stress within the shaft could be different from one section to the next. If the absolute maximum torsional stress is to be determined, then it becomes important to find the location where the ratio Tc/J is a maximum. In this regard, it may be helpful to show the variation of the internal torque T at each section along the axis of the shaft by drawing a ***torque diagram***. Specifically, this diagram is a plot of the internal torque T versus its position x along the shaft's length. As a sign convention, T will be positive if by the right-hand rule the thumb is directed outward from the shaft when the fingers curl in the direction of twist as caused by the torque, Fig. 11–5. Once the internal torque throughout the shaft is determined, the maximum ratio of Tc/J can then be identified.

5.2. MOMENT XOẮN

Ví dụ 01:

The stress distribution in a solid shaft has been plotted along three arbitrary radial lines as shown in Fig. 11–10a. Determine the resultant internal torque at the section.



Solution I

The polar moment of inertia for the cross-sectional area is

$$J = \frac{\pi}{2}(50\text{ mm})^4 = 9.82(10^6)\text{ mm}^4$$

Applying the torsion formula, with $\tau_{max} = 56$ MPa $= 56$ N/mm², Fig. 11–10a, we have

$$\tau_{max} = \frac{Tc}{J}; \qquad 56\text{ N/mm}^2 = \frac{T(50\text{ mm})}{(9.82)(10^6)\text{ mm}^4}$$

$$T = 11.0\text{ kN·m} \qquad \textit{Ans}$$

Solution II

The same result can be obtained by finding the torque produced by the stress distribution about the centroidal axis of the shaft. First we must express $\tau = f(\rho)$. Using proportional triangles, Fig. 11–10b, we have

$$\frac{\tau}{\rho} = \frac{56\text{ N/mm}^2}{50\text{ mm}}$$

$$\tau = 1.12\rho\ \text{N/mm}^2$$

This stress acts on all portions of the differential ring element that has an area $dA = 2\pi\rho\, d\rho$. Since the force created by τ is $dF = \tau\, dA$, the torque is

$$dT = \rho\, dF = \rho(\tau dA) = \rho(1.12\rho)2\pi\rho\, d\rho = 2.24\pi\rho^3\, d\rho$$

For the entire area over which τ acts, we require

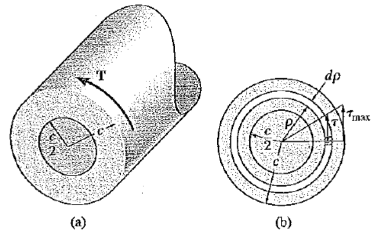
$$T = \int_0^{50} 2.24\pi\rho^3\, d\rho = 2.24\pi\left(\frac{1}{4}\rho^4\right)\Big|_0^{50} = 11.0(10^6)\text{ N·mm}$$

$$= 11.0\text{ kN·m} \qquad \textit{Ans.}$$

5.2. MOMENT XOẮN

Ví dụ 02:

The *solid* shaft of radius c is subjected to a torque **T**, Fig. 11–11a. Determine the fraction of T that is resisted by the material contained within the outer region of the shaft, which has an inner radius of $c/2$ and outer radius c.



The stress in the shaft varies linearly, such that $\tau = (\rho/c)\tau_{max}$, Eq. 11–3. Therefore, the torque dT' on the ring (area) located within the lighter-shaded region, Fig. 11–11b, is

$$dT' = \rho(\tau\, dA) = \rho(\rho/c)\tau_{\max}(2\pi\rho\, d\rho)$$

For the entire lighter-shaded area the torque is

$$T' = \frac{2\pi\tau_{\max}}{c}\int_{c/2}^{c}\rho^3\, d\rho$$

$$= \frac{2\pi\tau_{\max}}{c}\frac{1}{4}\rho^4\Big|_{c/2}^{c}$$

So that

$$T' = \frac{15\pi}{32}\tau_{\max}c^3 \qquad (1)$$

This torque T' can be expressed in terms of the applied torque T by first using the torsion formula to determine the maximum stress in the shaft. We have

$$\tau_{\max} = \frac{Tc}{J} = \frac{Tc}{(\pi/2)c^4}$$

or

$$\tau_{\max} = \frac{2T}{\pi c^3}$$

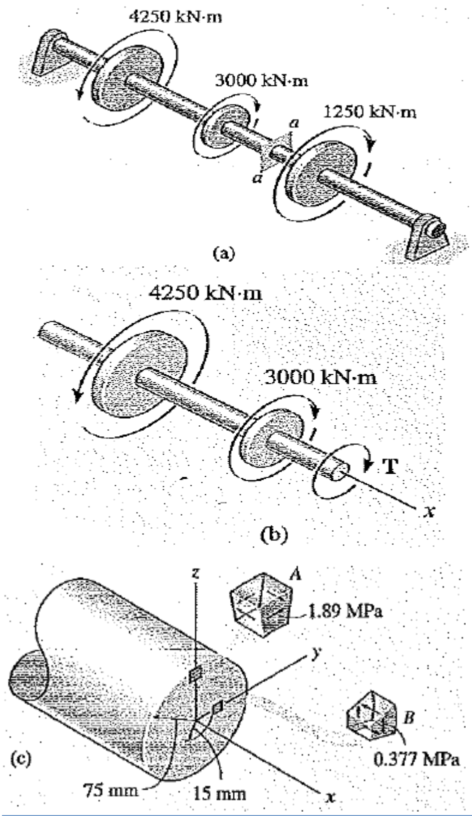
Substituting this into Eq. 1 yields

$$T' = \frac{15}{16}T \qquad \textit{Ans.}$$

Here, approximately 94% of the torque is resisted by the lighter-shaded region, and the remaining 6% of T (or $\frac{1}{16}$) is resisted by the inner "core" of the shaft, $\rho = 0$ to $\rho = c/2$. As a result, the material located at the *outer region* of the shaft is highly effective in resisting torque, which justifies the use of tubular shafts as an efficient means for transmitting torque, and thereby saves material.

5.2. MOMENT XOẮN

Ví dụ 02:



The shaft shown in Fig. 11–12a is supported by two bearings and is subjected to three torques. Determine the shear stress developed at points A and B, located at section $a-a$ of the shaft, Fig. 11–12b.

Internal Torque. The bearing reactions on the shaft are zero, provided the shaft's weight is neglected. Furthermore, the applied torques satisfy moment equilibrium about the shaft's axis.

The internal torque at section $a-a$ will be determined from the free-body diagram of the left segment, Fig. 11–12b. We have

$$\Sigma M_x = 0; \quad 4250\text{ kN·mm} - 3000\text{ kN·mm} - T = 0 \quad T = 1250\text{ kN·mm}$$

Section Property. The polar moment of inertia for the shaft is

$$J = \frac{\pi}{2}(75\text{ mm})^4 = 4.97(10^7)\text{ mm}^4$$

Shear Stress. Since point A is at $\rho = c = 75$ mm,

$$\tau_A = \frac{Tc}{J} = \frac{(1250\text{ kN·mm})(75\text{ mm})}{4.97(10^7)\text{ mm}^4} = 1.89\text{ N/mm}^2 = 1.89\text{ MPa}$$

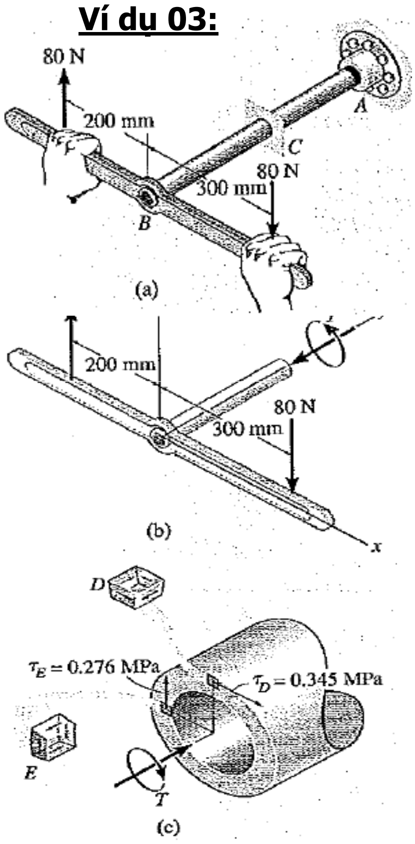
Likewise for point B, at $\rho = 15$ mm, we have

$$\tau_B = \frac{T\rho}{J} = \frac{(1250\text{ kN·mm})(15\text{ mm})}{4.97(10^7)\text{ mm}^4} = 0.377\text{ MPa} \qquad \textit{Ans.}$$

The directions of these stresses on each element at A and B, Fig. 11–12c, are established from the direction of the resultant internal torque **T**, shown in Fig. 11–12b. Note carefully how the shear stress acts on the planes of each of these elements.

5.2. MOMENT XOẮN

Ví dụ 03:



The pipe shown in Fig. 11–13a has an inner diameter of 80 mm and an outer diameter of 100 mm. If its end is tightened against the support at A using a torque wrench at B, determine the shear stress developed in the material at the inner and outer walls along the central portion of the pipe when the 80-N forces are applied to the wrench.

Internal Torque. A section is taken at an intermediate location C along the pipe's axis, Fig. 11–13b. The only unknown at the section is the internal torque **T**. Force equilibrium and moment equilibrium about the x and z axes are satisfied. We require

$$\Sigma M_y = 0; \qquad 80\text{ N}(0.3\text{ m}) + 80\text{ N}(0.2\text{ m}) - T = 0$$
$$T = 40\text{ N}\cdot\text{m}$$

Section Property. The polar moment of inertia for the pipe's cross-sectional area is

$$J = \frac{\pi}{2}[(0.05\text{ m})^4 - (0.04\text{ m})^4] = 5.80(10^{-6})\text{ m}^4$$

Shear Stress. For any point lying on the outside surface of the pipe, $\rho = c_o = 0.05$ m, we have

$$\tau_o = \frac{Tc_o}{J} = \frac{40\text{ N}\cdot\text{m}(0.05\text{ m})}{5.80(10^{-6})\text{ m}^4} = 0.345\text{ MPa} \qquad \textit{Ans.}$$

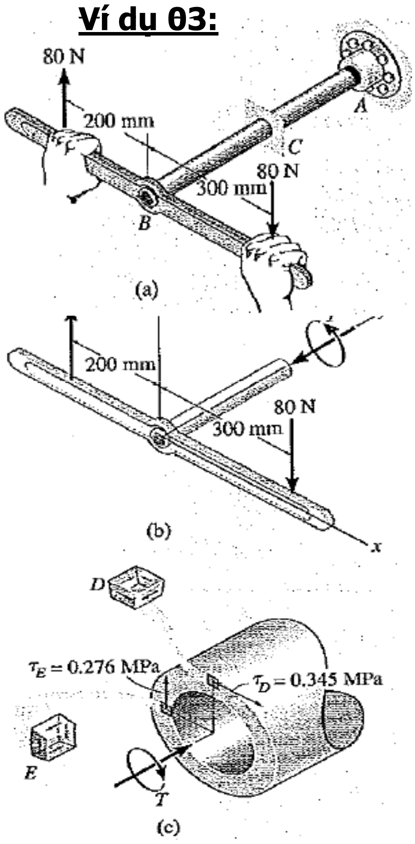
And for any point located on the inside surface, $\rho = c_i = 0.04$ m, so that

$$\tau_i = \frac{Tc_i}{J} = \frac{40\text{ N}\cdot\text{m}(0.04\text{ m})}{5.80(10^{-6})\text{ m}^4} = 0.276\text{ MPa} \qquad \textit{Ans.}$$

Chương 5: XOẮN THUẦN TÚY

5.2. MOMENT XOẮN

Ví dụ 03:



The pipe shown in Fig. 11–13a has an inner diameter of 80 mm and an outer diameter of 100 mm. If its end is tightened against the support at A using a torque wrench at B, determine the shear stress developed in the material at the inner and outer walls along the central portion of the pipe when the 80-N forces are applied to the wrench.

To show how these stresses act at representative points D and E on the cross-sectional area, we will first view the cross section from the front of segment CA of the pipe, Fig. 11–13a. On this section, Fig. 11–13c, the resultant internal torque is equal but opposite to that shown in Fig. 11–13b. The shear stresses at D and E contribute to this torque and therefore act on the shaded faces of the elements in the directions shown. As a consequence, notice how the shear-stress components act on the other three faces. Furthermore, since the top face of D and the inner face of E are in stress-free regions taken from the pipe's outer and inner walls, no shear stress can exist on these faces or on the other corresponding faces of the elements.

5.3. TRUYỀN ĐỘNG CÔNG SUẤT

Shafts and tubes having circular cross sections are often used to transmit power developed by a machine. When used for this purpose, they are subjected to torques that depend on the power generated by the machine and the angular speed of the shaft. ***Power*** is defined as the work performed per unit of time. The work transmitted by a rotating shaft equals the torque applied times the angle of rotation. Therefore, if during an instant of time dt an applied torque $\mathbf{T}$ causes the shaft to rotate $d\theta$, then the instantaneous power is

$$P = \frac{T\,d\theta}{dt}$$

Since the shaft's angular velocity $\omega = d\theta/dt$, we can also express the power as

$$P = T\omega \qquad (11\text{–}10)$$

In the SI system, power is expressed in *watts* when torque is measured in newton-meters (N · m) and ω is in radians per second (rad/s) (1 W = 1 N · m/s).

For machinery, the *frequency* of a shaft's rotation, f, is often reported. This is a measure of the number of revolutions or cycles the shaft makes per second and is expressed in hertz (1 Hz = 1 cycle/s). Since 1 cycle = 2π rad, then $\omega = 2\pi f$, and the above equation for power becomes

$$P = 2\pi f T \qquad (11\text{–}11)$$

5.3. TRUYỀN ĐỘNG CÔNG SUẤT

Thiết kế trục truyền động

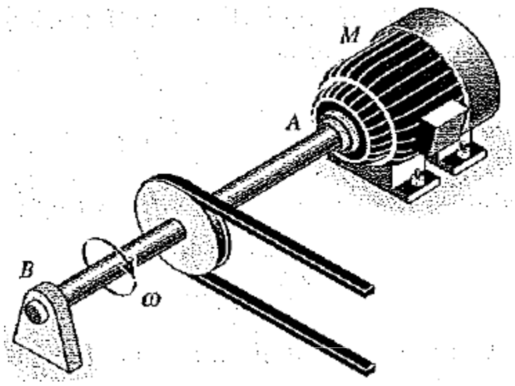
Shaft Design. When the power transmitted by a shaft and its frequency of rotation are known, the torque developed in the shaft can be determined from Eq. 11–11, that is, $T = P/2\pi f$. Knowing T and the allowable shear stress for the material, τ_{allow}, we can determine the size of the shaft's cross section using the torsion formula, provided the material behavior is linear-elastic. Specifically, the design or geometric parameter J/c becomes

$$\frac{J}{c} = \frac{T}{\tau_{allow}} \qquad (11\text{–}12)$$

For a *solid shaft*, $J = (\pi/2)c^4$, and thus, upon substitution, a *unique value* for the shaft's radius c can be determined. If the shaft is *tubular*, so that $J = (\pi/2)(c_o^4 - c_i^4)$, design permits a wide range of possibilities for the solution. This is because an *arbitrary choice* can be made for either c_o or c_i and the other radius can then be determined from Eq. 11–12.

5.3. TRUYỀN ĐỘNG CÔNG SUẤT

Ví dụ 01:



A solid steel shaft AB shown in Fig. 11–14 is to be used to transmit 3750 W from the motor M to which it is attached. If the shaft rotates at $\omega = 175$ rpm and the steel has an allowable shear stress of $\tau_{allow} = 100$ MPa, determine the required diameter of the shaft to the nearest mm.

The torque on the shaft is determined from Eq. 11–10, that is, $P = T\omega$. Expressing P in Newton-meters per second and ω in radians/second, we have

$$P = 3750 \text{ N·m/s}$$

$$\omega = \frac{175 \text{ rev}}{\text{min}}\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 18.33 \text{ rad/s}$$

Thus,

$P = T\omega$; $\quad 3750 \text{ N·m/s} = T(18.33) \text{ rad/s}$

$$T = 204.6 \text{ N·m}$$

Applying Eq. 11–12 yields

$$\frac{J}{c} = \frac{\pi}{2}\frac{c^4}{c} = \frac{T}{\tau_{allow}}$$

$$c = \left(\frac{2T}{\pi\tau_{allow}}\right)^{1/3} = \left(\frac{2(204.6 \text{ N·m})(1000 \text{ mm/m})}{\pi(100 \text{ N/mm}^2)}\right)^{1/3}$$

$$c = 10.92 \text{ mm}$$

Since $2c = 21.84$ mm, select a shaft having a diameter of

$$d = 22 \text{ mm}$$ ***Ans.***

5.3. TRUYỀN ĐỘNG CÔNG SUẤT

Ví dụ 02:

A tubular shaft, having an inner diameter of 30 mm and an outer diameter of 42 mm, is to be used to transmit 90 kW of power. Determine the frequency of rotation of the shaft so that the shear stress will not exceed 50 MPa.

Solution

The maximum torque that can be applied to the shaft is determined from the torsion formula.

$$\tau_{max} = \frac{Tc}{J}$$

$$50(10^6) \text{ N/m}^2 = \frac{T(0.021 \text{ m})}{(\pi/2)[(0.021 \text{ m})^4 - (0.015 \text{ m})^4]}$$

$$T = 538 \text{ N}\cdot\text{m}$$

Applying Eq. 11–11, the frequency of rotation is

$$P = 2\pi f T$$

$$90(10^3) \text{ N}\cdot\text{m/s} = 2\pi f(538 \text{ N}\cdot\text{m})$$

$$f = 26.6 \text{ Hz}$$ ***Ans.***

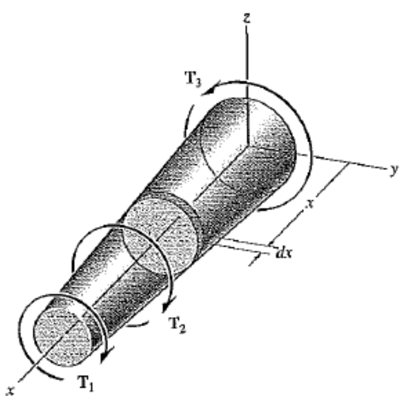
5.4. GÓC XOẮN

In this section we will develop a formula for determining the ***angle of twist*** ϕ (phi) of one end of a shaft with respect to its other end. The shaft is assumed to have a circular cross section that can gradually vary along its length, Fig. 11–15*a*, and the material is assumed to be homogeneous and to behave in a linear-elastic manner when the torque is applied. As in the case of an axially loaded bar, we will neglect the localized deformations that occur at points of application of the torques and where the cross section changes abruptly. By Saint-Venant's principle, these effects occur within small regions of the shaft's length and generally have only a slight effect on the final result.

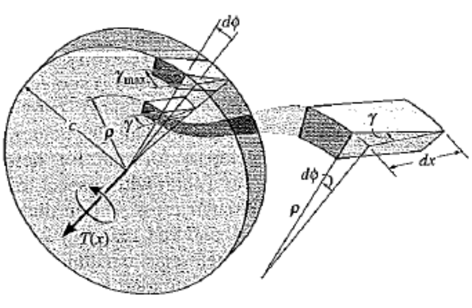
Using the method of sections, a differential disk of thickness dx, located at position x, is isolated from the shaft, Fig. 11–15*b*. The internal resultant torque is represented as $T(x)$, since the external loading may cause it to vary along the axis of the shaft. Due to $T(x)$, the disk will twist, such that the *relative rotation* of one of its faces with respect to the other face is $d\phi$, Fig. 11–15*b*. As a result an element of material located at an arbitrary radius ρ within the disk will undergo a shear strain γ. The values of γ and $d\phi$ are related by Eq. 11–1, namely,

$$d\phi = \gamma \frac{dx}{\rho} \qquad (11\text{–}13)$$

5.4. GÓC XOẮN



(a)



(b)

Using the method of sections, a differential disk of thickness dx, located at position x, is isolated from the shaft, Fig. 11–15*b*. The internal resultant torque is represented as $T(x)$, since the external loading may cause it to vary along the axis of the shaft. Due to $T(x)$, the disk will twist, such that the *relative rotation* of one of its faces with respect to the other face is $d\phi$, Fig. 11–15*b*. As a result an element of material located at an arbitrary radius ρ within the disk will undergo a shear strain γ. The values of γ and $d\phi$ are related by Eq. 11–1, namely,

$$d\phi = \gamma \frac{dx}{\rho} \qquad (11\text{–}13)$$

Since Hooke's law, $\gamma = \tau/G$, applies and the shear stress can be expressed in terms of the applied torque using the torsion formula $\tau = T(x)\rho/J(x)$, then $\gamma = T(x)\rho/J(x)G$. Substituting this into Eq. 11–13, the angle of twist for the disk is

$$d\phi = \frac{T(x)}{J(x)G}dx$$

Integrating over the entire length L of the shaft, we obtain the angle of twist for the entire shaft, namely,

$$\phi = \int_0^L \frac{T(x)dx}{J(x)G} \qquad (11\text{–}14)$$

Here

ϕ = the angle of twist of one end of the shaft with respect to the other end, measured in radians

$T(x)$ = the internal torque at the arbitrary position x, found from the method of sections and the equation of moment equilibrium applied about the shaft's axis

$J(x)$ = the shaft's polar moment of inertia expressed as a function of position x

G = the shear modulus of elasticity for the material

5.4. GÓC XOẮN

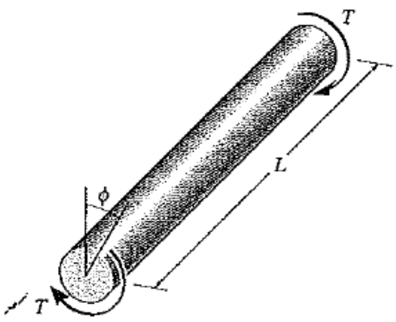


Fig. 11–16

Constant Torque and Cross-Sectional Area. Usually in engineering practice the material is homogeneous so that G is constant. Also, the shaft's cross-sectional area and the applied torque are constant along the length of the shaft, Fig. 11–16. If this is the case, the internal torque $T(x) = T$, the polar moment of inertia $J(x) = J$, and Eq. 11–14 can be integrated, which gives

$$\phi = \frac{TL}{JG} \quad (11\text{–}15)$$

The similarities between the above two equations and those for an axially loaded bar ($\delta = \int P(x)\,dx/A(x)E$ and $\delta = PL/AE$) should be noted.

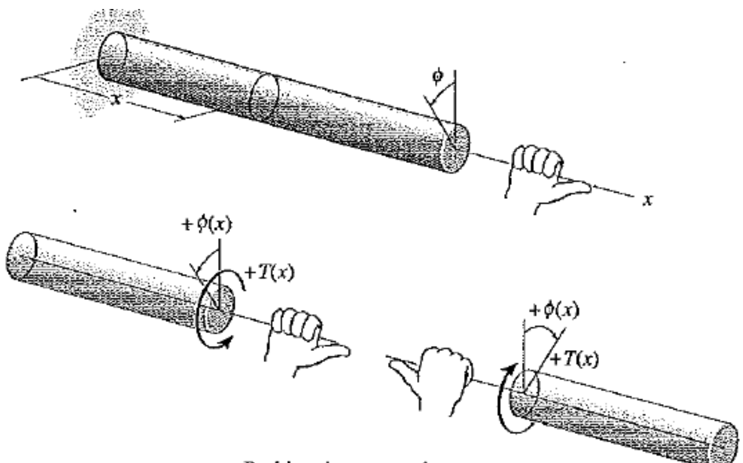
We can use Eq. 11–15 to determine the shear modulus of elasticity G of the material. To do so, a specimen of known length and diameter is placed in a torsion testing machine like the one shown in Fig. 11–17. The applied torque T and angle of twist ϕ are then measured between a gauge length L. Using Eq. 11–15, $G = TL/J\phi$. Usually, to obtain a more reliable value of G, several of these tests are performed and the average value is used.

If the shaft is subjected to several different torques, or the cross-sectional area or shear modulus changes abruptly from one region of the shaft to the next, Eq. 11–15 can be applied to each segment of the shaft where these quantities are all constant. The angle of twist of one end of the shaft with respect to the other is then found from the vector addition of the angles of twist of each segment. For this case,

$$\phi = \sum \frac{TL}{JG} \quad (11\text{–}16)$$

5.4. GÓC XOẮN

Sign Convention. In order to apply the above equation, we must develop a sign convention for the internal torque and the angle of twist of one end of the shaft with respect to the other end. To do this, we will use the right-hand rule, whereby both the torque and angle will be *positive*, provided the *thumb* is directed *outward* from the shaft when the fingers curl to give the tendency for rotation, Fig. 11–18.



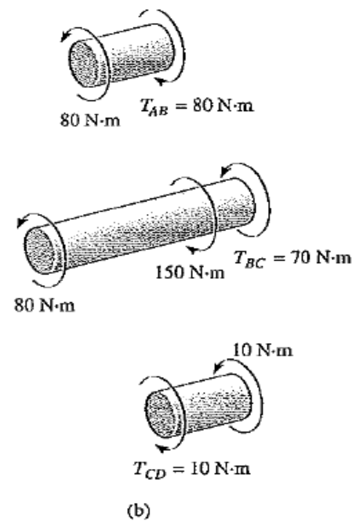
Positive sign convention for T and ϕ

5.4. GÓC XOẮN

To illustrate the use of this sign convention, consider the shaft shown in Fig. 11–19*a*, which is subjected to four torques. The angle of twist of end A with respect to end D is to be determined. For this problem, three segments of the shaft must be considered, since the internal torque changes at B and C. Using the method of sections, the internal torques are found for each segment, Fig. 11–19*b*. By the right-hand rule, with positive torques directed away from the *sectioned end* of the shaft, we have $T_{AB} = +80\ \text{N}\cdot\text{m}$, $T_{BC} = -70\ \text{N}\cdot\text{m}$, and $T_{CD} = -10\ \text{N}\cdot\text{m}$. These results are also shown on the *torque diagram* for the shaft, Fig. 11–19*c*. Applying Eq. 11–16, we have

$$\phi_{A/D} = \frac{(+80\ \text{N}\cdot\text{m})\,L_{AB}}{JG} + \frac{(-70\ \text{N}\cdot\text{m})\,L_{BC}}{JG} + \frac{(-10\ \text{N}\cdot\text{m})\,L_{CD}}{JG}$$

If the other data is substituted and the answer is found as a *positive* quantity, it means that end A will rotate as indicated by the curl of the right-hand fingers when the thumb is directed *away* from the shaft, Fig. 11–19*a*. The double subscript notation is used to indicate this relative angle of twist ($\phi_{A/D}$); however, if the angle of twist is to be determined relative to a *fixed point*, then only a single subscript will be used. For example, if D is located at a fixed support, then the computed angle of twist will be denoted as ϕ_A.



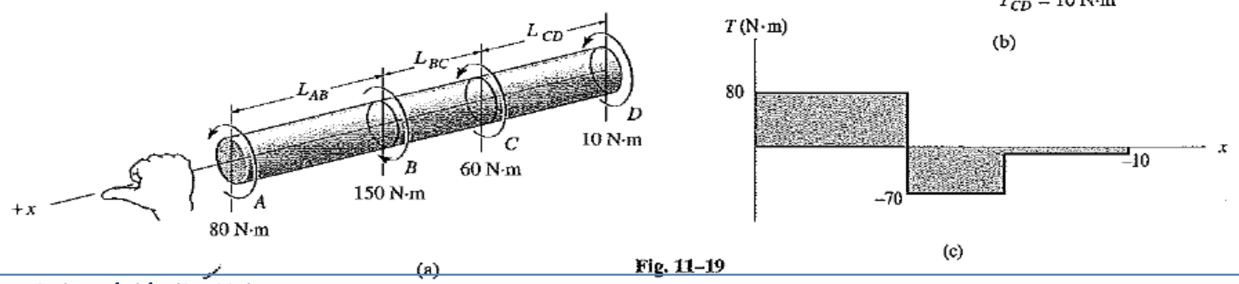
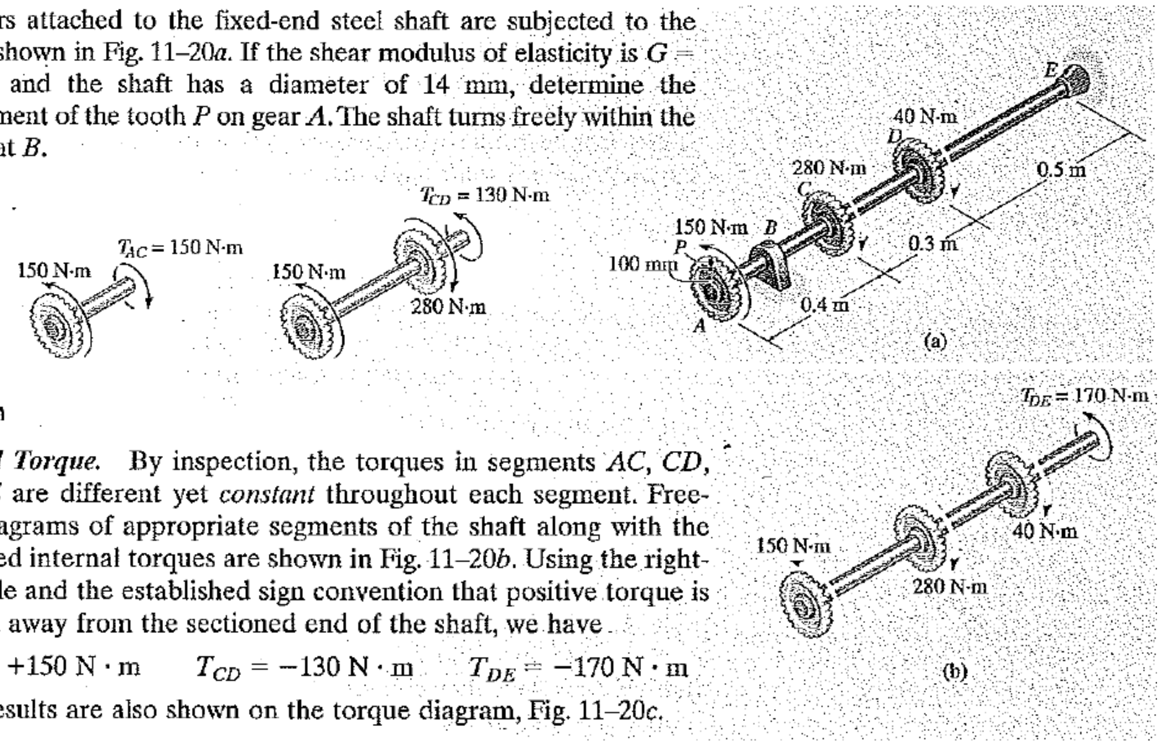


Fig. 11–19

5.4. GÓC XOẮN

Ví dụ 01:

The gears attached to the fixed-end steel shaft are subjected to the torques shown in Fig. 11–20*a*. If the shear modulus of elasticity is G = 80 GPa and the shaft has a diameter of 14 mm, determine the displacement of the tooth P on gear A. The shaft turns freely within the bearing at B.



Solution

Internal Torque. By inspection, the torques in segments AC, CD, and DE are different yet *constant* throughout each segment. Free-body diagrams of appropriate segments of the shaft along with the calculated internal torques are shown in Fig. 11–20*b*. Using the right-hand rule and the established sign convention that positive torque is directed away from the sectioned end of the shaft, we have

$$T_{AC} = +150\ \text{N}\cdot\text{m} \qquad T_{CD} = -130\ \text{N}\cdot\text{m} \qquad T_{DE} = -170\ \text{N}\cdot\text{m}$$

These results are also shown on the torque diagram, Fig. 11–20*c*.

5.4. GÓC XOẮN

Ví dụ 01:

Angle of Twist. The polar moment of inertia for the shaft is

$$J = \frac{\pi}{2}(0.007\ \text{m})^4 = 3.77(10^{-9})\ \text{m}^4$$

Applying Eq. 11–16 to each segment and adding the results algebraically, we have

$$\phi_A = \sum \frac{TL}{JG} = \frac{(+150\ \text{N}\cdot\text{m})(0.4\ \text{m})}{3.77(10^{-9})\ \text{m}^4[80(10^9)\ \text{N/m}^2]} + \frac{(-130\ \text{N}\cdot\text{m})(0.3\ \text{m})}{3.77(10^{-9})\ \text{m}^4[80(10^9)\ \text{N/m}^2)]} + \frac{(-170\ \text{N}\cdot\text{m})(0.5\ \text{m})}{3.77(10^{-9})\ \text{m}^4[80(10^9)\ \text{N/m}^2)]} = -0.212\ \text{rad}$$

Since the answer is negative, by the right-hand rule the thumb is directed *toward* the end E of the shaft, and therefore gear A will rotate as shown in Fig. 11–20*d*.

The displacement of tooth P on gear A is

$$s_P = \phi_A r = (0.212\ \text{rad})(100\ \text{mm}) = 21.2\ \text{mm} \qquad \textit{Ans.}$$

Remember that this analysis is valid only if the shear stress does not exceed the proportional limit of the material.

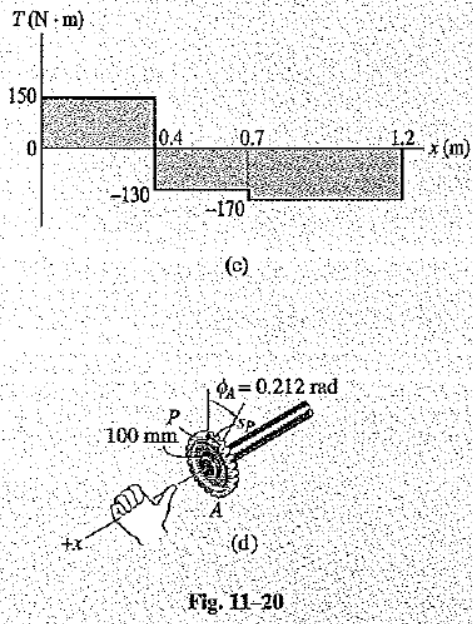
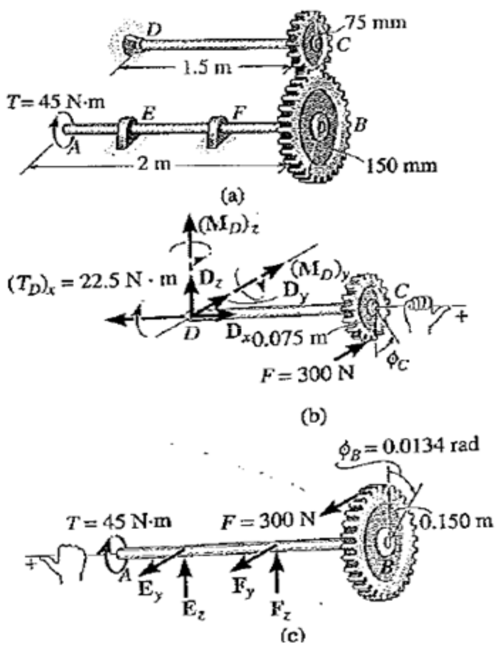


Fig. 11–20

5.4. GÓC XOẮN

Ví dụ 02:



The two solid steel shafts shown in Fig. 11–21*a* are coupled together using the meshed gears. Determine the angle of twist of end A of shaft AB when the torque $T = 45$ N · m is applied. Take $G = 80$ GPa. Shaft AB is free to rotate within bearings E and F, whereas shaft DC is fixed at D. Each shaft has a diameter of 20 mm.

Internal Torque. Free-body diagrams for each shaft are shown in Fig. 11–21*b* and 11–21*c*. Summing moments along the x axis of shaft AB yields the tangential reaction between the gears of $F = 45$ N · m/0.15 m = 300 N. Summing moments about the x axis of shaft DC, this force then creates a torque of $(T_D)_x = 300$ N(0.075 m) = 22.5 N · m on shaft DC.

Angle of Twist. To solve the problem, we will first calculate the rotation of gear C due to the torque of 22.5 N · m in shaft DC, Fig. 11–21*b*. This angle of twist is

$$\phi_C = \frac{TL_{DC}}{JG} = \frac{(+22.5\ \text{N}\cdot\text{m})(1.5\ \text{m})}{(\pi/2)(0.010\ \text{m})^4[80(10^9)\ \text{N/m}^2]} = +0.0269\ \text{rad}$$

Since the gears at the end of the shaft are in mesh, the rotation ϕ_C of gear C causes gear B to rotate ϕ_B, Fig. 11–21*c*, where

$$\phi_B(0.15\ \text{m}) = (0.0269\ \text{rad})(0.075\ \text{m})$$
$$\phi_B = 0.0134\ \text{rad}$$

We will now determine the angle of twist of end A with respect to end B of shaft AB caused by the 45 N · m torque, Fig. 11–21*c*. We have

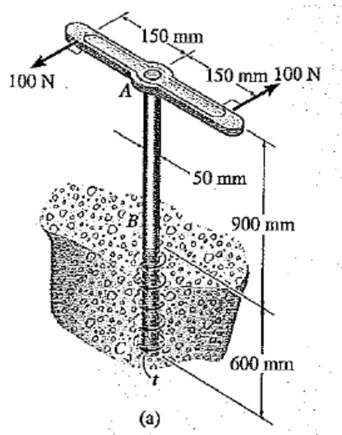
$$\phi_{A/B} = \frac{T_{AB}L_{AB}}{JG} = \frac{(+45\ \text{N}\cdot\text{m})(2\ \text{m})}{(\pi/2)(0.010\ \text{m})^4[80(10^9)\ \text{N/m}^2]} = +0.0716\ \text{rad}$$

The rotation of end A is therefore determined by adding ϕ_B and $\phi_{A/B}$, since both angles are in the *same direction*, Fig. 11–21*c*. We have

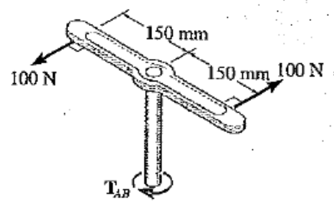
$$\phi_A = \phi_B + \phi_{A/B} = 0.0134\ \text{rad} + 0.0716\ \text{rad} = +0.0850\ \text{rad} \qquad \textit{Ans.}$$

5.4. GÓC XOẮN

Ví dụ 02:



(a)



(b)

The 50-mm-diameter solid cast-iron post shown in Fig. 11–22*a* is buried 600 mm in soil. If a torque is applied to its top using a rigid wrench, determine the maximum shear stress in the post and the angle of twist at its top. Assume that the torque is about to turn the post, and the soil exerts a uniform torsional resistance of t N·mm/mm along its 600 mm buried length. $G = 40(10^3)$ MPa.

Internal Torque. The internal torque in segment AB of the post is constant. From the free-body diagram, Fig. 11–22*b*, we have

$\Sigma M_z = 0; \qquad T_{AB} = 100\text{ N}(300\text{ mm}) = 30(10^3)\text{ N·mm}$

The magnitude of the uniform distribution of torque along the buried segment BC can be determined from equilibrium of the entire post, Fig. 11–22*c*. Here

$$\Sigma M_z = 0; \qquad 100\text{ N}(300\text{ mm}) - t(600\text{ mm}) = 0$$

$$t = 50\text{ N·mm/mm}$$

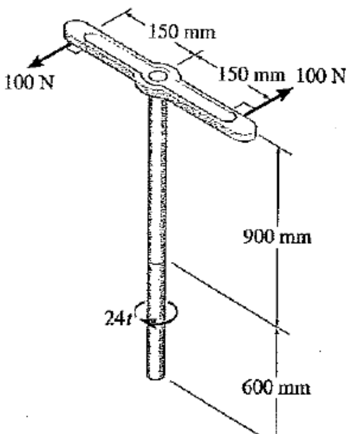
Hence, from a free-body diagram of a section of the post located at the position x within region BC, Fig. 11–22*d*, we have

$$\Sigma M_z = 0; \qquad T_{BC} - 50x = 0$$

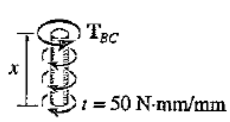
$$T_{BC} = 50x$$

5.4. GÓC XOẮN

Ví dụ 02:



(c)



(d)

Maximum Shear Stress. The largest shear stress occurs in region AB, since the torque is largest there and J is constant for the post. Applying the torsion formula, we have

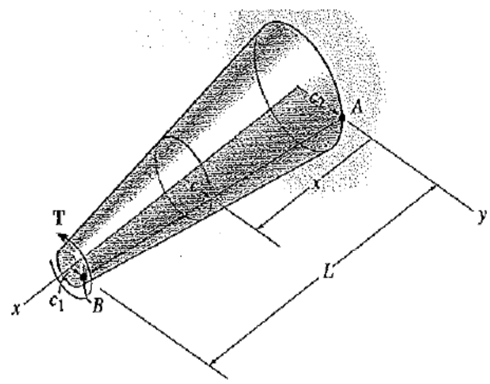
$$\tau_{\max} = \frac{T_{ABC}}{J} = \frac{30 \times 10^3\text{ N·mm }(25\text{ mm})}{(\pi/2)(25\text{ mm})^4} = 1.22\text{ N/mm}^2 \quad \textit{Ans.}$$

Angle of Twist. The angle of twist at the top can be determined relative to the bottom of the post, since it is fixed and yet is about to turn. Both segments AB and BC twist, and so in this case we have

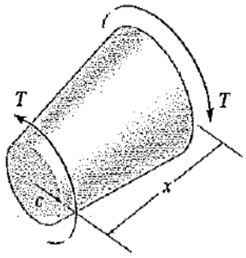
$$\phi_A = \frac{T_{AB}L_{AB}}{JG} + \int_0^{L_{BC}} \frac{T_{BC}\,dx}{JG}$$
$$= \frac{(30(10^3)\text{ N·mm})(900\text{ mm})}{JG} + \int_0^{600} \frac{50x\,dx}{JG}$$
$$= \frac{27(10^6)\text{ N·mm}^2}{JG} + \frac{50[(600)^2/2]\text{ N·mm}^2}{JG}$$
$$= \frac{36(10^6)\text{ N·mm}^2}{(\pi/2)(25\text{ mm})^4 40(10^3)\text{ N/mm}^2} = 0.00147\text{ rad}$$

5.4. GÓC XOẮN

Ví dụ 02:



(a)



The tapered shaft shown in Fig. 11–23*a* is made of a material having a shear modulus *G*. Determine the angle of twist of its end *B* when subjected to the torque.

Internal Torque. By inspection or from the free-body diagram of a section located at the arbitrary position *x*, Fig. 11–23*b*, the internal torque is *T*.

Angle of Twist. Here the polar moment of inertia varies along the shaft's axis and therefore we must express it in terms of the coordinate *x*. The radius *c* of the shaft at *x* can be determined in terms of *x* by proportion of the slope of line *AB* in Fig. 11–23*c*. We have

$$\frac{c_2 - c_1}{L} = \frac{c_2 - c}{x}$$

$$c = c_2 - x\left(\frac{c_2 - c_1}{L}\right)$$

Thus, at *x*,

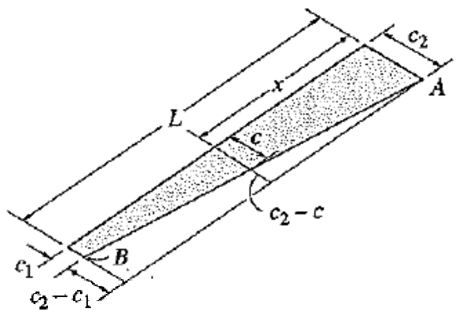
$$J(x) = \frac{\pi}{2}\left[c_2 - x\left(\frac{c_2 - c_1}{L}\right)\right]^4$$

Applying Eq. 11–14, we have

$$\phi = \int_0^L \frac{T\,dx}{\left(\frac{\pi}{2}\right)\left[c_2 - x\left(\frac{c_2 - c_1}{L}\right)\right]^4 G} = \frac{2T}{\pi G}\int_0^L \frac{dx}{\left[c_2 - x\left(\frac{c_2 - c_1}{L}\right)\right]^4}$$

5.4. GÓC XOẮN

Ví dụ 02:



(c)

Performing the integration using an integral table, the result becomes

$$\phi = \left(\frac{2T}{\pi G}\right)\frac{1}{3\left(\frac{c_2 - c_1}{L}\right)\left[c_2 - x\left(\frac{c_2 - c_1}{L}\right)\right]^3}\Bigg|_0^L$$

$$= \frac{2T}{\pi G}\left(\frac{L}{3(c_2 - c_1)}\right)\left(\frac{1}{c_1^3} - \frac{1}{c_2^3}\right)$$

Rearranging terms yields

$$\phi = \frac{2TL}{3\pi G}\left(\frac{c_2^2 + c_1c_2 + c_1^2}{c_1^3c_2^3}\right) \qquad \textit{Ans.}$$

To partially check this result, note that when $c_1 = c_2 = c$, then

$$\phi = \frac{TL}{[(\pi/2)c^4]G} = \frac{TL}{JG}$$

which is Eq. 11–15.