CHAPTER FOUR
AXIAL MEMBERS

Learning objectives

1. Understand the theory, its limitations, and its applications for the design and analysis of axial members.
2. Develop the discipline to draw free-body diagrams and approximate deformed shapes in the design and analysis of structures.

The tensile forces supporting the weight of the Mackinaw bridge (Figure 4.1a) act along the longitudinal axis of each cable. The compressive forces raising the weight of the dump on a truck act along the axis of the hydraulic cylinders. The cables and hydraulic cylinders are **axial members**, long straight bodies on which the forces are applied along the longitudinal axis. Connecting rods in an engine, struts in aircraft engine mounts, members of a truss representing a bridge or a building, spokes in bicycle wheels, columns in a building—all are examples of axial members

(a)

(b)

Figure 4.1 Axial members: (a) Cables of Mackinaw bridge. (b) Hydraulic cylinders in a dump truck.

This chapter develops the simplest theory for axial members, following the logic shown in Figure 3.15 but subject to the limitations described in Section 3,13. We can then apply the formulas to statically determinate and indeterminate structures. The two most important tools in our analysis will be free-body diagrams and approximate deformed shapes.

4.1 PRELUDE TO THEORY

As a prelude to theory, we consider two numerical examples solved using the logic discussed in Section 3.2. Their solution will highlight conclusions and observations that will be formalized in the development of the theory in Section 4.2.

EXAMPLE 4.1

Two thin bars are securely attached to a rigid plate, as shown in Figure 4.2. The cross-sectional area of each bar is 20 mm². The force F is to be placed such that the rigid plate moves only horizontally by 0.05 mm without rotating. Determine the force F and its location h for the following two cases: (a) Both bars are made from steel with a modulus of elasticity $E = 200$ GPa. (b) Bar 1 is made of steel ($E = 200$ GPa) and bar 2 is made of aluminum ($E = 70$ GPa).

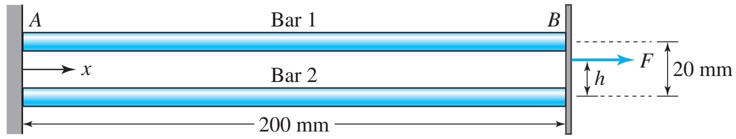


Figure 4.2 Axial bars in Example 4.1.

PLAN

The relative displacement of point B with respect to A is 0.05 mm, from which we can find the axial strain. By multiplying the axial strain by the modulus of elasticity, we can obtain the axial stress. By multiplying the axial stress by the cross-sectional area, we can obtain the internal axial force in each bar. We can draw the free-body diagram of the rigid plate and by equilibrium obtain the force F and its location h.

SOLUTION

1. *Strain calculations:* The displacement of B is $u_B = 0.05$ mm. Point A is built into the wall and hence has zero displacement. The normal strain is the same in both rods:

$$\varepsilon_1 = \varepsilon_2 = \frac{u_B - u_A}{x_B - x_A} = \frac{0.05 \text{ mm}}{200 \text{ mm}} = 250 \ \mu\text{mm/mm} \tag{E1}$$

2. *Stress calculations:* From Hooke's law $\sigma = E\varepsilon$, we can find the normal stress in each bar for the two cases.

Case (a): Because E and ε_1 are the same for both bars, the stress is the same in both bars. We obtain

$$\sigma_1 = \sigma_2 = (200 \times 10^9 \text{ N/m}^2) \times 250 \times 10^{-6} = 50 \times 10^6 \text{ N/m}^2 (\text{T}) \tag{E2}$$

Case (b): Because E is different for the two bars, the stress is different in each bar

$$\sigma_1 = E_1\varepsilon_1 = (200 \times 10^9 \text{ N/m}^2) \times 250 \times 10^{-6} = 50 \times 10^6 \text{ N/m}^2 (\text{T}) \tag{E3}$$

$$\sigma_2 = E_2\varepsilon_2 = 70 \times 10^9 \times 250 \times 10^{-6} = 17.5 \times 10^6 \text{ N/m}^2 (\text{T}) \tag{E4}$$

3. *Internal forces:* Assuming that the normal stress is uniform in each bar, we can find the internal normal force from $\boldsymbol{N} = \sigma A$, where $A = 20 \text{ mm}^2 = 20 \times 10^{-6} \text{ m}^2$.

Case (a): Both bars have the same internal force since stress and cross-sectional area are the same,

$$\boldsymbol{N}_1 = \boldsymbol{N}_2 = (50 \times 10^6 \text{ N/m}^2)(20 \times 10^{-6} \text{ m}^2) = 1000 \text{ N (T)} \tag{E5}$$

Case (b): The equivalent internal force is different for each bar as stresses are different.

$$N_1 = \sigma_1 A_1 = (50 \times 10^6 \text{ N/m}^2)(20 \times 10^{-6} \text{ m}^2) = 1000 \text{ N (T)} \tag{E6}$$

$$\boldsymbol{N}_2 = \sigma_2 A_2 = (17.5 \times 10^6 \text{ N/m}^2)(20 \times 10^{-6} \text{ m}^2) = 350 \text{ N (T)} \tag{E7}$$

4. *External force:* We make an imaginary cut through the bars, show the internal axial forces as tensile, and obtain free-body diagram shown in Figure 4.3. By equilibrium of forces in x direction we obtain

$$F = \boldsymbol{N}_1 + \boldsymbol{N}_2 \tag{E8}$$

By equilibrium of moment point O in Figure 4.3, we obtain

$$\boldsymbol{N}_1(20 - h) - \boldsymbol{N}_2 h = 0 \tag{E9}$$

$$h = \frac{20\boldsymbol{N}_1}{\boldsymbol{N}_1 + \boldsymbol{N}_2} \tag{E10}$$

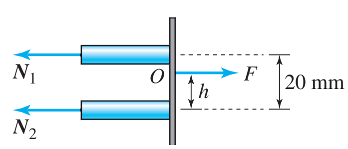


Figure 4.3 Free-body diagram in Example 4.1.

Case (a): Substituting Equation (E5) into Equations (E8) and (E10), we obtain F and h:

$$F = 1000 \text{ N} + 1000 \text{ N} = 2000 \text{ N} \qquad h = \frac{20 \text{ mm} \times 1000 \text{ N}}{(1000 \text{ N} + 1000 \text{ N})} = 10 \text{ mm}$$

ANS. $F = 2000$ N $\qquad h = 10$ mm

Case (b): Substituting Equations (E6) and (E7) into Equations (E8) and (E10), we obtain F and h:

$$F = 1000 \text{ N} + 350 \text{ N} = 1350 \text{ N} \qquad h = \frac{20 \text{ mm} \times 1000 \text{ N}}{(1000 \text{ N} + 350 \text{ N})} = 14.81 \text{ mm}$$

ANS. $F = 1350$ N $h = 14.81$ mm

COMMENTS

1. Both bars, irrespective of the material, were subjected to the same axial strain. This is the fundamental kinematic assumption in the development of the theory for axial members, discussed in Section 4.2.
2. The sum on the right in Equation (E8) can be written $\sum_{i=1}^{n=2} \sigma_i \, \Delta A_i$, where σ_i is the normal stress in the ith bar, ΔA_i is the cross-sectional area of the ith bar, and $n = 2$ reflects that we have two bars in this problem. If we had n bars attached to the rigid plate, then the total axial force would be given by summation over n bars. As we increase the number of bars n to infinity, the cross-sectional area ΔA_i tends to zero (or infinitesimal area dA) as we try to fit an infinite number of bars on the same plate, resulting in a continuous body. The sum then becomes an integral, as discussed in Section 4.1.1.
3. If the external force were located at any point other than that given by the value of h, then the plate would rotate. Thus, for pure axial problems with no bending, a point on the cross section must be found such that the internal moment from the axial stress distribution is zero. To emphasize this, consider the left side of Equation (E9), which can be written as $\sum_{i=1}^{n} y_i \sigma_i \, \Delta A_i$, where y_i is the coordinate of the ith rod's centroid. The summation is an expression of the internal moment that is needed for static equivalency. This internal moment must equal zero if the problem is of pure axial deformation, as discussed in Section 4.1.1.
4. Even though the strains in both bars were the same in both cases, the stresses were different when E changed. Case (a) corresponds to a homogeneous cross section, whereas case (b) is analogous to a laminated bar in which the non-homogeneity affects the stress distribution.

4.1.1 Internal Axial Force

In this section we formalize the key observation made in Example 4.1: the normal stress σ_{xx} can be replaced by an equivalent internal axial force using an integral over the cross-sectional area. Figure 4.4 shows the statically equivalent systems. The axial force on a differential area $\sigma_{xx}\, dA$ can be integrated over the entire cross section to obtain

$$\boxed{\boldsymbol{N} = \int_A \sigma_{xx} \, dA} \tag{4.1}$$

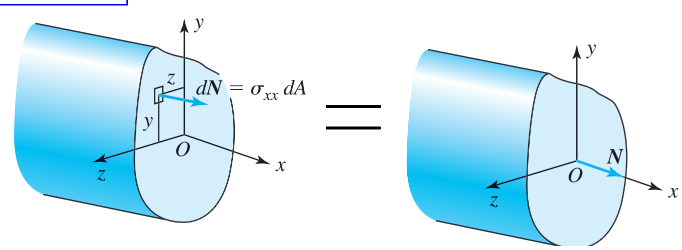


Figure 4.4 Statically equivalent internal axial force.

If the normal stress distribution σ_{xx} is to be replaced by only an axial force at the origin, then the internal moments $\boldsymbol{M}_y$ and $\boldsymbol{M}_z$ must be zero at the origin, and from Figure 4.4 we obtain

$$\int_A y\sigma_{xx} \, dA = 0 \tag{4.2a}$$

$$\int_A z\sigma_{xx} \, dA = 0 \tag{4.2b}$$

Equations (4.1), (4.2a), and (4.2b) are independent of the material models because they represent static equivalency between the normal stress on the cross section and internal axial force. If we were to consider a laminated cross section or nonlinear material, then it would affect the value and distribution of σ_{xx} across the cross section, but Equation (4.1) relating σ_{xx} and $\boldsymbol{N}$ would remain unchanged, and so would the zero moment condition of Equations (4.2a) and (4.2b). Equations (4.2a) and (4.2b) are used to determine the location at which the internal and external forces have to act for pure axial problem without bending, as discussed in Section 4.2.6.

EXAMPLE 4.2

Figure 4.5 shows a homogeneous wooden cross section and a cross section in which the wood is reinforced with steel. The normal strain for both cross sections is uniform, $\varepsilon_{xx} = -200\ \mu$. The moduli of elasticity for steel and wood are $E_{steel} = 30{,}000$ ksi and $E_{wood} = 8000$ ksi. (a) Plot the σ_{xx} distribution for each of the two cross sections shown. (b) Calculate the equivalent internal axial force N for each cross section using Equation (4.1).

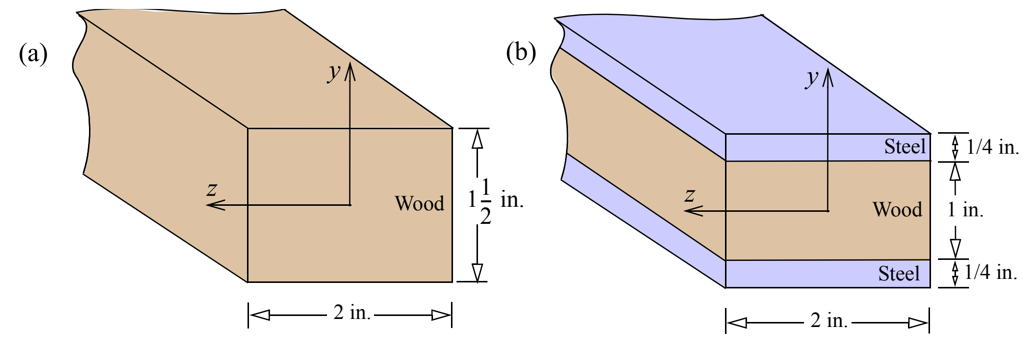


Figure 4.5 Cross sections in Example 4.2. (*a*) Homogeneous. (*b*) Laminated.

PLAN

(a) Using Hooke's law we can find the stress values in each material. Noting that the stress is uniform in each material, we can plot it across the cross section. (b) For the homogeneous cross section we can perform the integration in Equation (4.1) directly. For the nonhomogeneous cross section we can write the integral in Equation (4.1) as the sum of the integrals over steel and wood and then perform the integration to find N.

SOLUTION

(a) From Hooke's law we can write

$$(\sigma_{xx})_{wood} = (8000\text{ ksi})(-200)10^{-6} = -1.6\text{ ksi} \tag{E1}$$

$$(\sigma_{xx})_{steel} = (30000\text{ ksi})(-200)10^{-6} = -6\text{ ksi} \tag{E2}$$

For the homogeneous cross section the stress distribution is as given in Equation (E1), but for the laminated case it switches to Equation (E2), depending on the location of the point where the stress is being evaluated, as shown in Figure 4.6.

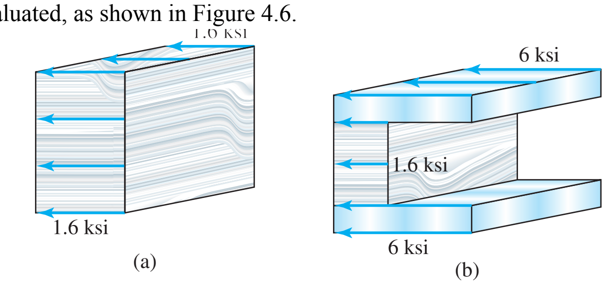


Figure 4.6 Stress distributions in Example 4.2. (*a*) Homogeneous cross section. (*b*) Laminated cross section.

(b) **Homogeneous cross section:** Substituting the stress distribution for the homogeneous cross section in Equation (4.1) and integrating, we obtain the equivalent internal axial force,

$$N = \int_A (\sigma_{xx})_{wood}\, dA = (\sigma_{xx})_{wood} A = (-1.6\text{ ksi})(2\text{ in.})(1.5\text{ in.}) = -4.8\text{ kips} \tag{E3}$$

ANS. $N = 4.8$ kips (C)

Laminated cross section: The stress value changes as we move across the cross section. Let A_{sb} and A_{st} represent the cross-sectional areas of steel at the bottom and the top. Let A_w represent the cross-sectional area of wood. We can write the integral in Equation (4.1) as the sum of three integrals, substitute the stress values of Equations (E1) and (E2), and perform the integration:

$$N = \int_{A_{sb}} \sigma_{xx}\, dA + \int_{A_w} \sigma_{xx}\, dA + \int_{A_{st}} \sigma_{xx}\, dA = \int_{A_{sb}} (\sigma_{xx})_{steel}\, dA + \int_{A_w} (\sigma_{xx})_{wood}\, dA + \int_{A_{st}} (\sigma_{xx})_{steel}\, dA \text{ or} \tag{E4}$$

$$N = (\sigma_{xx})_{steel} A_{sb} + (\sigma_{xx})_{wood} A_w + (\sigma_{xx})_{steel} A_{st} \text{ or} \tag{E5}$$

$$N = (-6\text{ ksi})(2\text{ in.})\left(\frac{1}{4}\text{ in.}\right) + (-1.6\text{ ksi})(1\text{ in.})(2\text{ in.}) + (-6\text{ ksi})(2\text{ in.})\left(\frac{1}{4}\text{ in.}\right) = -9.2\text{ kips} \tag{E6}$$

ANS. $N = 9.2$ kips (C)

COMMENTS

1. Writing the integral in the internal axial force as the sum of integrals over each material, as in Equation (E4), is equivalent to calculating the internal force carried by each material and then summing, as shown in Figure 4.7.

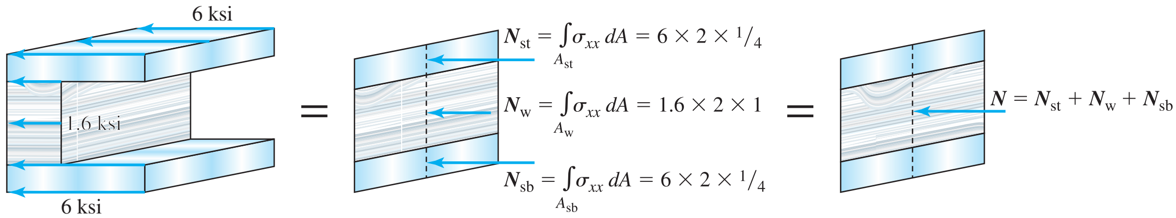


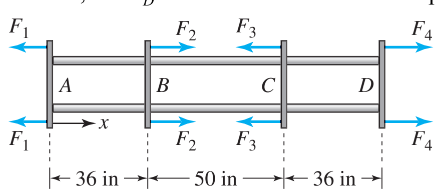
Figure 4.7 Statically equivalent internal force in Example 4.2 for laminated cross section.

2. The cross section is geometrically as well as materially symmetric. Thus we can locate the origin on the line of symmetry. If the lower steel strip is not present, then we will have to determine the location of the equivalent force.
3. The example demonstrates that although the strain is uniform across the cross section, the stress is not. We considered material nonhomogeneity in this example. In a similar manner we can consider other models, such as elastic–perfectly plastic or material models that have nonlinear stress–strain curves.

PROBLEM SET 4.1

4.1 Aluminum bars (E = 30,000 ksi) are welded to rigid plates, as shown in Figure P4.1. All bars have a cross-sectional area of 0.5 in^2. Due to the applied forces the rigid plates at A, B, C, and D are displaced in x direction without rotating by the following amounts: $u_A = -0.0100$ in., u_B = 0.0080 in., $u_C = -0.0045$ in., and $u_D = 0.0075$ in. Determine the applied forces F_1, F_2, F_3, and F_4.

Figure P4.1



4.2 Brass bars between sections A and B, aluminum bars between sections B and C, and steel bars between sections C and D are welded to rigid plates, as shown in Figure P4.2. The rigid plates are displaced in the x direction without rotating by the following amounts: $u_B = -1.8$ mm, $u_C = 0.7$ mm, and $u_D = 3.7$ mm. Determine the external forces F_1, F_2, and F_3 using the properties given in Table P4.2

Figure P4.2

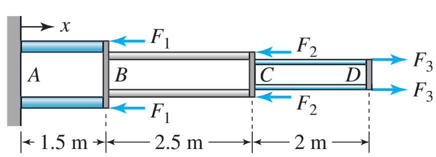
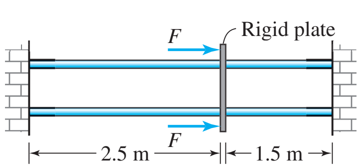


TABLE P4.2

	Brass	Aluminum	Steel
Modulus of elasticity	70 GPa	100 GPa	200 GPa
Diameter	30 mm	25 mm	20 mm

4.3 The ends of four circular steel bars (E = 200 GPa) are welded to a rigid plate, as shown in Figure P4.3. The other ends of the bars are built into walls. Owing to the action of the external force F, the rigid plate moves to the right by 0.1 mm without rotating. If the bars have a diameter of 10 mm, determine the applied force F.

Figure P4.3



4.4 Rigid plates are securely fastened to bars A and B, as shown in Figure P4.4. A gap of 0.02 in. exists between the rigid plates before the forces are applied. After application of the forces the normal strain in bar A was found to be 500 μ. The cross-sectional area and the modulus of elasticity for each bar are as follows: $A_A = 1$ in.^{2}, $E_A = 10{,}000$ ksi, $A_B = 0.5$ in.^{2}, and $E_B = 30{,}000$ ksi. Determine the applied forces F, assuming that the rigid plates do not rotate.

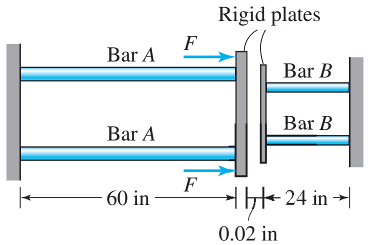


Figure P4.4

4.5 The strain at a cross section shown in Figure P4.5 of an axial rod is assumed to have the uniform value $\varepsilon_{xx} = 200\ \mu$. (a) Plot the stress distribution across the laminated cross section. (b) Determine the equivalent internal axial force N and its location from the bottom of the cross section. Use $E_{alu} = 100$ GPa, $E_{wood} = 10$ GPa, and $E_{steel} = 200$ GPa.

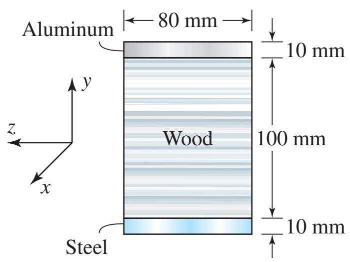


Figure P4.5

4.6 A reinforced concrete bar shown in Figure P4.6 is constructed by embedding 2-in. × 2-in. square iron rods. Assuming a uniform strain $\varepsilon_{xx} = -1500\ \mu$ in the cross section, (a) plot the stress distribution across the cross section; (b) determine the equivalent internal axial force N. Use $E_{iron} = 25{,}000$ ksi and $E_{conc} = 3000$ ksi.

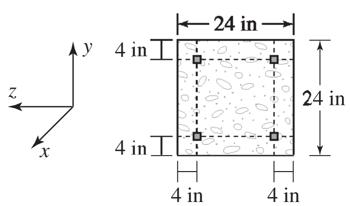


Figure P4.6

4.2 THEORY OF AXIAL MEMBERS

In this section we will follow the procedure in Section 4.1 with variables in place of numbers to develop formulas for axial deformation and stress. The theory will be developed subject to the following limitations:

1. The length of the member is significantly greater than the greatest dimension in the cross section.
2. We are away from the regions of stress concentration.
3. The variation of external loads or changes in the cross-sectional areas is gradual, except in regions of stress concentration.
4. The axial load is applied such that there is no bending.
5. The external forces are not functions of time that is, we have a static problem. (See Problems 4.37, 4.38, and 4.39 for dynamic problems.)

Figure 4.8 shows an externally distributed force per unit length $p(x)$ and external forces F_1 and F_2 acting at each end of an axial bar. The cross-sectional area $A(x)$ can be of any shape and could be a function of x.

> **Sign convention:** The displacement u is considered positive in the positive x direction. The internal axial force N is considered positive in tension negative in compression.

The theory has two objectives:

1. To obtain a formula for the relative displacements $u_2 - u_1$ in terms of the internal axial force N.
2. To obtain a formula for the axial stress σ_{xx} in terms of the internal axial force N.

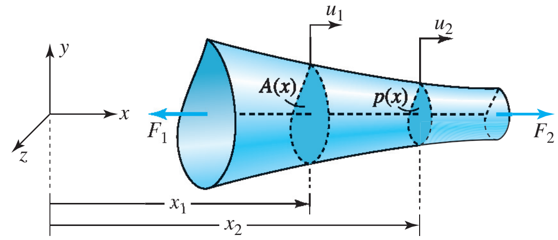


Figure 4.8 Segment of an axial bar.

We will take $\Delta x = x_2 - x_1$ as an infinitesimal distance so that the gradually varying distributed load $p(x)$ and the cross-sectional area $A(x)$ can be treated as constants. We then approximate the deformation across the cross section and apply the logic shown in Figure 4.9. The assumptions identified as we move from each step are also points at which complexities can later be added, as discussed in examples and "Stretch Yourself" problems.

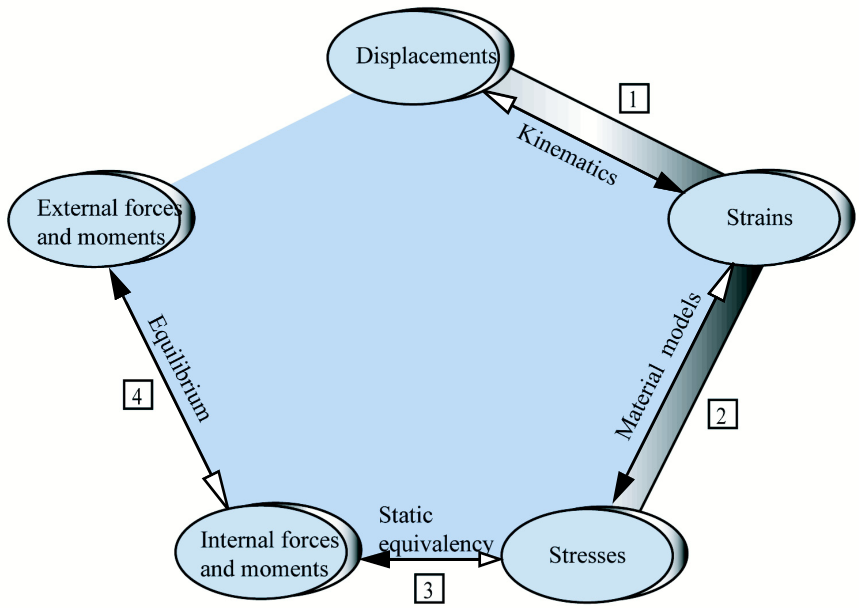


Figure 4.9 Logic in mechanics of materials.

4.2.1 Kinematics

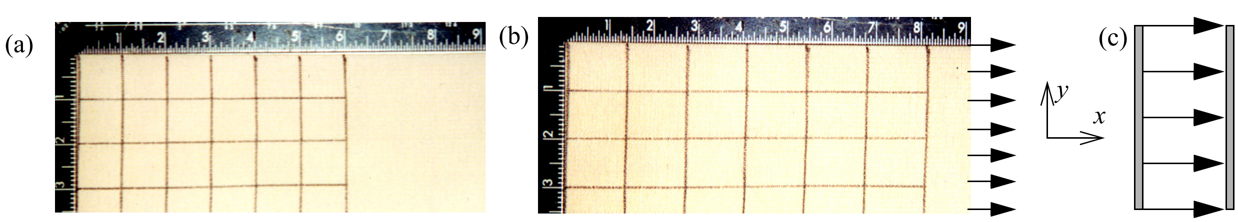


Figure 4.10 Axial deformation: (*a*) original grid; (*b*) deformed grid. (c) u is constant in y direction.

Figure 4.10 shows a grid on an elastic band that is pulled in the axial direction. The vertical lines remain approximately vertical, but the horizontal distance between the vertical lines changes. Thus all points on a vertical line are displaced by equal amounts. If this surface observation is also true in the interior of an axial member, then all points on a cross-section displace by equal amounts, but each cross-section can displace in the x direction by a different amount, leading to Assumption 1.

Assumption 1 Plane sections remain plane and parallel.

Assumption 1 implies that u cannot be a function of y but can be a function of x

$$u = u(x) \tag{4.3}$$

As an alternative perspective, because the cross section is significantly smaller than the length, we can approximate a function such as u by a constant treating it as uniform over a cross section. In Chapter 6, on beam bending, we shall approximate u as a linear function of y.

4.2.2 Strain Distribution

Assumption 2 Strains are small.[1]

If points x_2 and x_1 are close in Figure 4.8, then the strain at any point x can be calculated as

$$\varepsilon_{xx} = \lim_{\Delta x \to 0}\left(\frac{u_2 - u_1}{x_2 - x_1}\right) = \lim_{\Delta x \to 0}\left(\frac{\Delta u}{\Delta x}\right) \quad \text{or}$$

$$\boxed{\varepsilon_{xx} = \frac{du}{dx}(x)} \tag{4.4}$$

Equation (4.4) emphasizes that the axial strain is uniform across the cross section and is only a function of x. In deriving Equation (4.4) we made no statement regarding material behavior. In other words, Equation (4.4) does not depend on the material model if Assumptions 1 and 2 are valid. But clearly if the material or loading is such that Assumptions 1 and 2 are not tenable, then Equation (4.4) will not be valid.

4.2.3 Material Model

Our motivation is to develop a simple theory for axial deformation. Thus we make assumptions regarding material behavior that will permit us to use the simplest material model given by Hooke's law.

Assumption 3 Material is isotropic.

Assumption 4 Material is linearly elastic.[2]

Assumption 5 There are no inelastic strains.[3]

Substituting Equation (4.4) into Hooke's law, that is, $\sigma_{xx} = E\,\varepsilon_{xx}$, we obtain

$$\sigma_{xx} = E\frac{du}{dx} \tag{4.5}$$

Though the strain does not depend on y or z, we cannot say the same for the stress in Equation (4.5) since E could change across the cross section, as in laminated or composite bars.

4.2.4 Formulas for Axial Members

Substituting σ_{xx} from Equation (4.5) into Equation (4.1) and noting that du/dx is a function of x only, whereas the integration is with respect to y and z ($dA = dy\,dz$), we obtain

$$N = \int_A E\frac{du}{dx}\,dA = \frac{du}{dx}\int_A E\,dA \tag{4.6}$$

[1]See Problem 4.40 for large strains.
[2]See Problem 4.36 for nonlinear material behavior.
[3]Inelastic strains could be due to temperature, humidity, plasticity, viscoelasticity, and so on. We shall consider inelastic strains due to temperature in Section 4.5.

Consistent with the motivation for the simplest possible formulas, E should not change across the cross section as implied in Assumption 6. We can take E outside the integral.

Assumption 6 Material is homogeneous across the cross section.

With material homogeneity, we then obtain

$$N = E\frac{\mathrm{d}u}{\mathrm{d}x}\int_A dA = EA\frac{\mathrm{d}u}{\mathrm{d}x} \text{ or}$$

$$\boxed{\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{N}{EA}} \tag{4.7}$$

The higher the value of EA, the smaller will be the deformation for a given value of the internal force. Thus the rigidity of the bar increases with the increase in EA. This implies that an axial bar can be made more rigid by either choosing a stiffer material (a higher value of E) or increasing the cross-sectional area, or both. Example 4.5 brings out the importance of axial rigidity in design. The quantity EA is called **axial rigidity**.

Substituting Equation (4.7) into Equation (4.5), we obtain

$$\boxed{\sigma_{xx} = \frac{N}{A}} \tag{4.8}$$

In Equation (4.8)**,** N and A do not change across the cross section and hence axial stress is uniform across the cross section. We have used Equation (4.8) in Chapters 1 and 3, but this equation is valid only if all the limitations are imposed, and if Assumptions 1 through 6 are valid.

We can integrate Equation (4.7) to obtain the deformation between two points:

$$u_2 - u_1 = \int_{u_1}^{u_2} du = \int_{x_1}^{x_2} \frac{N}{EA}\,dx \tag{4.9}$$

where u_1 and u_2 are the displacements of sections at x_1 and x_2, respectively. To obtain a simple formula we would like to take the three quantities N, E, and A outside the integral, which means these quantities should not change with x. To achieve this simplicity, we make the following assumptions:

Assumption 7 The material is homogeneous between x_1 and x_2. (E is constant)

Assumption 8 The bar is not tapered between x_1 and x_2. (A is constant)

Assumption 9 The external (hence internal) axial force does not change with x between x_1 and x_2. (N is constant)

If Assumptions 7 through 9 are valid, then N, E, and A are constant between x_1 and x_2, and we obtain

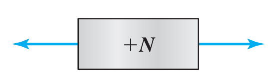
$$\boxed{u_2 - u_1 = \frac{N(x_2 - x_1)}{EA}} \tag{4.10}$$

In Equation (4.10), points x_1 and x_2 must be chosen such that neither N, E, nor A changes between these points.

4.2.5 Sign Convention for Internal Axial Force

The axial stress σ_{xx} was replaced by a statically equivalent internal axial force N. Figure 4.11 shows the sign convention for the positive axial force as tension.

Figure 4.11 Sign convention for positive internal axial force.



N is an internal axial force that has to be determined by making an imaginary cut and drawing a free-body diagram. In what direction should N be drawn on the free-body diagram? There are two possibilities:

1. N is always drawn in tension on the imaginary cut as per our sign convention. The equilibrium equation then gives a positive or a negative value for N. A positive value of σ_{xx} obtained from Equation (4.8) is tensile and a negative value is

compressive. Similarly, the relative deformation obtained from Equation (4.10) is extension for positive values and contraction for negative values. The displacement u will be positive in the positive x direction.

2. ***N*** is drawn on the imaginary cut in a direction to equilibrate the external forces. Since inspection is being used in determining the direction of ***N***, tensile and compressive σ_{xx} and extension or contraction for the relative deformation must also be determined by inspection.

4.2.6 Location of Axial Force on the Cross Section

For pure axial deformation the internal bending moments must be zero. Equations (4.2a) and (4.2b) can then be used to determine the location of the point where the internal axial force and hence the external forces must pass for pure axial problems. Substituting Equation (4.5) into Equations (4.2a) and (4.2b) and noting that du/dx is a function of x only, whereas the integration is with respect to y and z ($dA = dy\,dz$), we obtain

$$\int_A y\sigma_{xx}\,dA = \int_A yE\frac{du}{dx}\,dA = \frac{du}{dx}\int_A yE\,dA = 0 \text{ or}$$

$$\int_A yE\,dA = 0 \tag{4.11a}$$

$$\int_A z\sigma_{xx}\,dA = \int_A zE\frac{du}{dx}\,dA = \frac{du}{dx}\int_A zE\,dA = 0 \text{ or}$$

$$\int_A zE\,dA = 0 \tag{4.11b}$$

Equations (4.11a) and (4.11b) can be used to determine the location of internal axial force for composite materials. If the cross section is homogenous (Assumption 6), then E is constant across the cross section and can be taken out side the integral:

$$\int_A y\,dA = 0 \tag{4.12a}$$

$$\int_A z\,dA = 0 \tag{4.12b}$$

Equations (4.12a) and (4.12b) are satisfied if y and z are measured from the centroid. (See Appendix A.4.) We will have *pure axial deformation if the external and internal forces are colinear and passing through the centroid of a homogenous cross section*. This assumes implicitly that the centroids of all cross sections must lie on a straight line. This eliminates curved but not tapered bars.

4.2.7 Axial Stresses and Strains

In the Cartesian coordinate system all stress components except σ_{xx} are assumed zero. From the generalized Hooke's law for isotropic materials, given by Equations (3.14a) through (3.14c), we obtain the normal strains for axial members:

$$\boxed{\varepsilon_{xx} = \frac{\sigma_{xx}}{E} \qquad \varepsilon_{yy} = -\frac{\nu\sigma_{xx}}{E} = -\nu\varepsilon_{xx} \qquad \varepsilon_{zz} = -\frac{\nu\sigma_{xx}}{E} = -\nu\varepsilon_{xx}} \tag{4.13}$$

where ν is the Poisson's ratio. In Equation (4.13), the normal strains in y and z directions are due to Poisson's effect. Assumption 1, that plane sections remain plane and parallel implies that no right angle would change during deformation, and hence the assumed deformation implies that shear strains in axial members are zero. Alternatively, if shear stresses are zero, then by Hooke's law shear strains are zero.

EXAMPLE 4.3

Solid circular bars of brass (E_{br} = 100 GPa, ν_{br} = 0.34) and aluminum (E_{al} = 70 GPa, ν_{al} = 0.33) having 200 mm diameter are attached to a steel tube (E_{st} = 210 GPa, ν_{st} = 0.3) of the same outer diameter, as shown in Figure 4.12. For the loading shown determine: (a) The movement of the plate at *C* with respect to the plate at *A*. (b) The change in diameter of the brass cylinder. (c) The maximum inner diameter to the nearest millimeter in the steel tube if the factor of safety with respect to failure due to yielding is to be at least 1.2. The yield stress for steel is 250 MPa in tension.

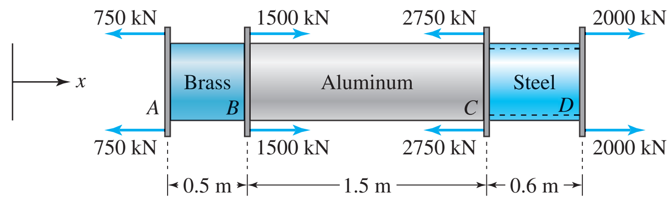


Figure 4.12 Axial member in Example 4.3.

PLAN

(a) We make imaginary cuts in each segment and determine the internal axial forces by equilibrium. Using Equation (4.10) we can find the relative movements of the cross sections at *B* with respect to *A* and at *C* with respect to *B* and add these two relative displacements to obtain the relative movement of the cross section at *C* with respect to the section at *A*. (b) The normal stress σ_{xx} in *AB* can be obtained from Equation (4.8) and the strain ε_{yy} found using Equation (4.13). Multiplying the strain by the diameter we obtain the change in diameter. (c) We can calculate the allowable axial stress in steel from the given failure values and factor of safety. Knowing the internal force in *CD* we can find the cross-sectional area from which we can calculate the internal diameter.

SOLUTION

(a) The cross-sectional areas of segment *AB* and *BC* are

$$A_{AB} = A_{BC} = \frac{\pi}{4}(0.2\text{ m})^2 = 31.41 \times 10^{-3}\text{ m}^2 \tag{E1}$$

We make imaginary cuts in segments *AB*, *BC*, and *CD* and draw the free-body diagrams as shown in Figure 4.13. By equilibrium of forces we obtain the internal axial forces

$$N_{AB} = 1500\text{ kN} \qquad N_{BC} = 1500\text{ kN} - 3000\text{ kN} = -1500\text{ kN} \qquad N_{CD} = \quad = 4000\text{ kN} \tag{E2}$$

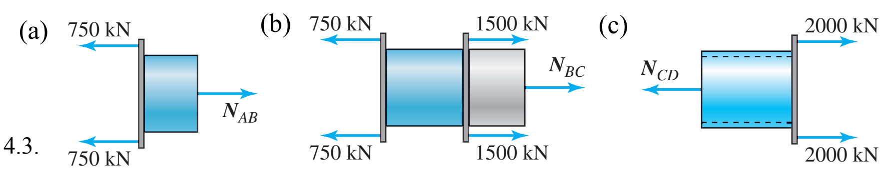


Figure 4.13 Free body diagrams in Example 4.3.

We can find the relative movement of point *B* with respect to point *A*, and *C* with respect to *B* using Equation (4.10):

$$u_B - u_A = \frac{N_{AB}(x_B - x_A)}{E_{AB}A_{AB}} = \frac{(1500 \times 10^3\text{ N})(0.5\text{ m})}{(100 \times 10^9\text{ N/m}^2)(31.41 \times 10^{-3}\text{ m}^2)} = 0.2388 \times 10^{-3}\text{ m} \tag{E3}$$

$$u_C - u_B = \frac{N_{BC}(x_C - x_B)}{E_{BC}A_{BC}} = \frac{(-1500 \times 10^3\text{ N})(1.5\text{ m})}{(70 \times 10^9\text{ N/m}^2)(31.41 \times 10^{-3}\text{ m}^2)} = -1.0233 \times 10^{-3}\text{ m} \tag{E4}$$

Adding Equations (E3) and (E4) we obtain the relative movement of point C with respect to A:

$$u_C - u_A = (u_C - u_B) + (u_B - u_A) = (0.2388\text{ m} - 1.0233\text{ m})10^{-3} = -0.7845 \times 10^{-3}\text{ m} \tag{E5}$$

ANS. $u_C - u_A = 0.7845$ mm contraction

(b) We can find the axial stress σ_{xx} in *AB* using Equation (4.8):

$$\sigma_{xx} = \frac{N_{AB}}{A_{AB}} = \frac{1500 \times 10^3\text{ N}}{(31.41 \times 10^{-3}\text{ m}^2)} = 47.8 \times 10^6\text{ N/m}^2 \tag{E6}$$

Substituting σ_{xx}, E_{br} = 100 GPa, ν_{br} = 0.34 in Equation (4.13), we can find ε_{yy}. Multiplying ε_{yy} by the diameter of 200 mm, we then obtain the change in diameter Δd,

$$\varepsilon_{yy} = -\frac{\nu_{br}\sigma_{xx}}{E_{br}} = -\frac{0.34(47.8 \times 10^6\text{ N/m}^2)}{100 \times 10^9\text{ N/m}^2} = -0.162 \times 10^{-3} = \frac{\Delta d}{200\text{ mm}} \tag{E7}$$

ANS. $\Delta d = -0.032$ mm

(c) The axial stress in segment *CD* is

$$\sigma_{CD} = \frac{N_{CD}}{A_{CD}} = \frac{4000 \times 10^3 \text{ N}}{(\pi/4)[(0.2\text{ m})^2 - D_i^2]} = \frac{16{,}000 \times 10^3}{[\pi(0.2^2 - D_i^2)]} \text{ N/m}^2 \tag{E8}$$

Using the given factor of safety, we determine the value of D_i:

$$K = \frac{\sigma_{\text{yield}}}{\sigma_{CD}} = \frac{250 \times 10^6 \times [\pi(0.2^2 - D_i^2)]}{16{,}000 \times 10^3} = 49.09(0.2^2 - D_i^2) \geq 1.2 \quad \text{or}$$

$$D_i^2 \leq 0.2^2 - 24.45 \times 10^{-3} \qquad \text{or} \qquad D_i \leq 124.7 \times 10^{-3} \text{ m} \tag{E9}$$

To the nearest millimeter, the diameter that satisfies the inequality in Equation (E9) is 124 mm.

ANS. $D_i = 124$ mm

COMMENTS

1. On a free-body diagram some may prefer to show $\boldsymbol{N}$ in a direction that counterbalances the external forces, as shown in Figure 4.14. In such cases the sign convention is not being followed.

We note that $u_B - u_A = 0.2388\ (10^{-3})$ m is extension and $u_C - u_B = 1.0233\ (10^{-3})$ m is contraction. To calculate $u_C - u_A$ we must now manually subtract $u_C - u_B$ from $u_B - u_A$.

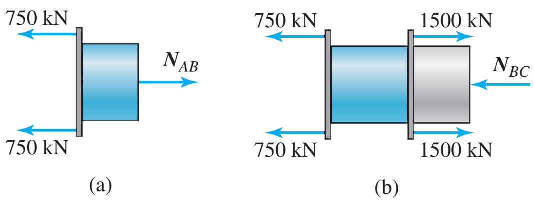


Figure 4.14 Alternative free body diagrams in Example 4.3.

2. An alternative way of calculating of $u_C - u_A$ is

$$u_C - u_A = \int_{x_A}^{x_C} \frac{\boldsymbol{N}}{EA}\,dx = \underbrace{\int_{x_A}^{x_B} \frac{\boldsymbol{N}_{AB}}{E_{AB}A_{AB}}\,dx}_{u_C - u_B} + \underbrace{\int_{x_B}^{x_C} \frac{\boldsymbol{N}_{BC}}{E_{BC}A_{BC}}\,dx}_{u_B - u_A}$$

or, written more compactly,

$$\Delta u = \sum_{i=1}^{n} \frac{\boldsymbol{N}_i \,\Delta x_i}{E_i A_i} \tag{4.14}$$

where n is the number of segments on which the summation is performed, which in our case is 2. Equation (4.14) can be used only if the sign convention for the internal force $\boldsymbol{N}$ is followed.

3. Note that $\boldsymbol{N}_{BC} - \boldsymbol{N}_{AB} = -3000$ kN and the magnitude of the applied external force at the section at B is 3000 kN. Similarly, $\boldsymbol{N}_{CD} - \boldsymbol{N}_{BC}$ = 5500 kN, which is the magnitude of the applied external force at the section at C. In other words, the internal axial force jumps by the value of the external force as one crosses the external force from left to right. We will make use of this observation in the next section, when we develop a graphical technique for finding the internal axial force.

4.2.8 Axial Force Diagram

In Example 4.3 we constructed several free-body diagrams to determine the internal axial force in different segments of the axial member. An axial force diagram is a graphical technique for determining internal axial forces, which avoids the repetition of drawing free-body diagrams.

An **axial force diagram** is a plot of the internal axial force $\boldsymbol{N}$ versus x. To construct an axial force diagram we create a small template to guide us in which direction the internal axial force will jump, as shown in Figure 4.15*a* and Figure 4.15*b*. An **axial template** is a free-body diagram of a small segment of an axial bar created by making an imaginary cut just before and just after the section where the external force is applied.

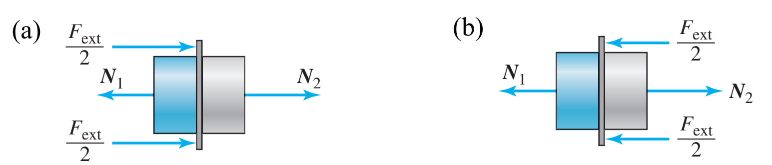


Figure 4.15 Axial bar templates. Template Equations $\boldsymbol{N}_2 = \boldsymbol{N}_1 - F_{\text{ext}}$ $\boldsymbol{N}_2 = \boldsymbol{N}_1 + F_{\text{extt}}$

The external force F_{ext} on the template can be drawn either to the left or to the right. The ends represent the imaginary cut just to the left and just to the right of the applied external force. On these cuts the internal axial forces are drawn in tension. An equilibrium equation—that is, the template equation—is written as shown in Figure 4.15 If the external force on the axial bar is in the direction of the assumed external force on the template, then the value of N_2 is calculated according to the template equation. If the external force on the axial bar is opposite to the direction shown on the template, then N_2 is calculated by changing the sign of F_{ext} in the template equation. Example 4.4 demonstrates the use of templates in constructing axial force diagrams.

EXAMPLE 4.4

Draw the axial force diagram for the axial member shown in Example 4.3 and calculate the movement of the section at *C* with respect to the section at *A*.

PLAN

We can start the process by considering an imaginary extension on the left. In the imaginary extension the internal axial force is zero. Using the template in Figure 4.15*a* to guide us, we can draw the axial force diagram. Using Equation (4.14), we can find the relative displacement of the section at *C* with respect to the section at *A*.

SOLUTION

Let *LA* be an imaginary extension on the left of the shaft, as shown in Figure 4.16*a*. Clearly the internal axial force in the imaginary segment *LA* is zero. As one crosses the section at *A*, the internal force must jump by the applied axial force of 1500 kN. Because the forces at *A* are in the opposite direction to the force F_{ext} shown on the template in Figure 4.15*a*, we must use opposite signs in the template equation. The internal force just after the section at *A* will be +1500 kN. This is the starting value in the internal axial force diagram.

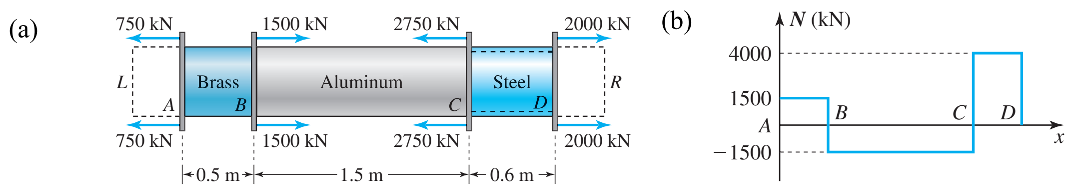


Figure 4.16 (a) Extending the axial bar for an axial force diagram. (b) Axial force diagram.

We approach the section at *B* with an internal force value of +1500 kN. The force at *B* is in the same direction as the force shown on the template in Figure 4.15*a*. Hence we subtract 3000 as per the template equation, to obtain a value of −1500 kN, as shown in Figure 4.16*b*.

We now approach the section at *C* with an internal force value of −1500 kN and note that the forces at *C* are opposite to those on the template in Figure 4.15*a*. Hence we add 5500 to obtain +4000 kN.

The force at *D* is in the same direction as that on the template in Figure 4.15*a*, and after subtracting we obtain a zero value in the imaginary extended bar *DR*. The return to zero value must always occur because the bar is in equilibrium.

From Figure 4.16*b* the internal axial forces in segments *AB* and *BC* are $N_{AB} = 1500$ kN and $N_{BC} = -1500$ kN. The crosssectional areas as calculated in Example 4.3 are $A_{AB} = A_{BC} = 31.41 \times 10^{-3}\ \text{m}^2$ and modulas of elasticity for the two sections are $E_{AB} = 100$ GPa and $E_{BC} = 70$ GPa. Substituting these values into Equation (4.14) we obtain the relative deformation of the section at *C* with respect to the section at *A*,

$$\Delta u = u_C - u_A = \frac{N_{AB}(x_B - x_A)}{E_{AB}A_{AB}} + \frac{N_{BC}(x_C - x_B)}{E_{BC}A_{BC}} \tag{E1}$$

$$u_C - u_A = \frac{(1500 \times 10^3\ \text{N})(0.5\ \text{m})}{(100 \times 10^9\ \text{N/m}^2)(31.41 \times 10^{-3}\ \text{m}^2)} + \frac{(-1500 \times 10^3\ \text{N})(1.5\ \text{m})}{(70 \times 10^9\ \text{N/m}^2)(31.41 \times 10^{-3}\ \text{m}^2)} = -0.7845 \times 10^{-3}\ \text{m or} \tag{E2}$$

ANS. $u_C - u_A = 0.7845$ mm contraction

COMMENT

1. We could have used the template in Figure 4.15*b* to create the axial force diagram.We approach the section at *A* and note that the +1500 kN is in the same direction as that shown on the template of Figure 4.15*b*. As per the template equation we add. Thus our starting value is +1500 kN, as shown in Figure 4.16. As we approach the section at *B*, the internal force N_1 is +1500 kN, and the applied force of 3000 kN is in the opposite direction to the template of Figure 4.15*b*, so we subtract to obtain N_2 as –1500 kN. We approach the section at *C* and note that the applied force is in the same direction as the applied force on the template of Figure 4.15*b*. Hence we add 5500 kN to obtain +4000 kN. The force at section *D* is opposite to that shown on the template of Figure 4.15*b*, so we subtract 4000 to get a zero value in the extended portion *DR*. The example shows that the direction of the external force F_{ext} on the template is immaterial.

EXAMPLE 4.5

A 1-m-long hollow rod is to transmit an axial force of 60 kN. Figure 4.17 shows that the inner diameter of the rod must be 15 mm to fit existing attachments. The elongation of the rod is limited to 2.0 mm. The shaft can be made of titanium alloy or aluminum. The modulus of elasticity E, the allowable normal stress σ_{allow}, and the density γ for the two materials is given in Table 4.1. Determine the minimum outer diameter to the nearest millimeter of the lightest rod that can be used for transmitting the axial force.

TABLE 4.1 **Material properties in Example 4.4**

Material	E (GPa)	σ_{allow} (MPa)	γ (mg/m^3)
Titanium alloy	96	400	4.4
Aluminum	70	200	2.8

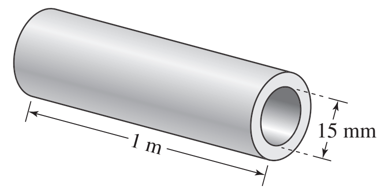


Figure 4.17 Cylindrical rod in Example 4.5.

PLAN

The change in radius affects only the cross-sectional area A and no other quantity in Equations (4.8) and (4.10). For each material we can find the minimum cross-sectional area A needed to satisfy the stiffness and strength requirements. Knowing the minimum A for each material, we can find the minimum outer radius. We can then find the volume and hence the mass of each material and make our decision on the lighter bar.

SOLUTION

We note that for both materials $x_2 - x_1 = 1$ m. From Equations (4.8) and (4.10) we obtain for titanium alloy the following limits on A_{Ti}:

$$(\Delta u)_{Ti} = \frac{(60 \times 10^3 \text{ N})(1 \text{ m})}{(96 \times 10^9 \text{ N/m}^2)A_{Ti}} \le 2 \times 10^{-3} \text{ m} \quad \text{or} \quad A_{Ti} \ge 0.313 \times 10^{-3} \text{ m}^2 \tag{E1}$$

$$(\sigma_{max})_{Ti} = \frac{(60 \times 10^3 \text{ N})}{A_{Ti}} \le 400 \times 10^6 \text{ N/m}^2 \quad \text{or} \quad A_{Ti} \ge 0.150 \times 10^{-3} \text{ m}^2 \tag{E2}$$

Using similar calculations for the aluminum shaft, we obtain the following limits on A_{Al}:

$$(\Delta u)_{Al} = \frac{(60 \times 10^3 \text{ N}) \times 1}{(28 \times 10^9 \text{ N/m}^2)A_{Al}} \le 2 \times 10^{-3} \text{ m} \quad \text{or} \quad A_{Al} \ge 1.071 \times 10^{-3} \text{ m}^2 \tag{E3}$$

$$(\sigma_{max})_{Al} = \frac{(60 \times 10^3 \text{ N})}{A_{Al}} \le 200 \times 10^6 \text{ N/m}^2 \quad \text{or} \quad A_{Al} \ge 0.300 \times 10^{-3} \text{ m}^2 \tag{E4}$$

Thus if $A_{Ti} \ge 0.313 \times 10^{-3}$ m^2, it will meet both conditions in Equations (E1) and (E2). Similarly if $A_{Al} \ge 1.071 \times 10^{-3}$ m^2, it will meet both conditions in Equations (E3) and (E4). The external diameters D_{Ti} and D_{Al} are then

$$A_{Ti} = \frac{\pi}{4}(D_{Ti}^2 - 0.015^2) \ge 0.313 \times 10^{-3} \qquad D_{Ti} \le 24.97 \times 10^{-3} \text{ m} \tag{E5}$$

$$A_{Al} = \frac{\pi}{4}(D_{Al}^2 - 0.015^2) \ge 1.071 \times 10^{-3} \qquad D_{Al} \le 39.86 \times 10^{-3} \text{ m} \tag{E6}$$

Rounding upward to the closest millimeter, we obtain

$$D_{Ti} = 25(10^{-3}) \text{ m} \qquad D_{Al} = 40(10^{-3}) \text{ m} \tag{E7}$$

We can find the mass of each material by taking the product of the material density and the volume of a hollow cylinder,

$$m_{Ti} = \left[(4.4 \times 10^6 \text{ g/m}^3)\left\{\frac{\pi}{4}(0.025^2 - 0.015^2) \text{ m}^2\right\}\right](1 \text{ m}) = 1382 \; g \tag{E8}$$

$$m_{Al} = \left[(2.8 \times 10^6 \text{ g/m}^3)\left\{\frac{\pi}{4}(0.040^2 - 0.015^2) \text{ m}^2\right\}\right](1 \text{ m}) = 3024 \; g \tag{E9}$$

From Equations (E8) and (E9) we see that the titanium alloy shaft is lighter.

ANS. A titanium alloy shaft with an outside diameter of 25 mm should be used.

COMMENTS

1. For both materials the stiffness limitation dictated the calculation of the external diameter, as can be seen from Equations (E1) and (E3).

2. Even though the density of aluminum is lower than that of titanium alloy, the mass of titanium is less. Because of the higher modulus of elasticity of titanium alloy, we can meet the stiffness requirement using less material than with aluminum.
3. The answer may change if cost is a consideration. The cost of titanium per kilogram is significantly higher than that of aluminum. Thus based on material cost we may choose aluminum. However, if the weight affects the running cost, then economic analysis is needed to determine whether the material cost or the running cost is higher.
4. If in Equation (E5) we had 24.05×10^{-3} m on the right-hand side, our answer for D_{Ti} would still be 25 mm because we have to round upward to ensure meeting the greater-than sign requirement in Equation (E5).

EXAMPLE 4.6

A rectangular aluminum bar ($E_{al} = 10{,}000$ ksi, $\nu = 0.25$) of $\frac{3}{4}$-in. thickness consists of a uniform and tapered cross section, as shown in Figure 4.18. The depth in the tapered section varies as $h(x) = (2 - 0.02x)$ in. Determine: (a) The elongation of the bar under the applied loads. (b) The change in dimension in the y direction in section BC.

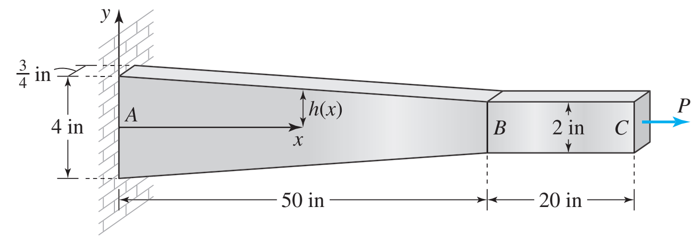


Figure 4.18 Axial member in Example 4.6.

PLAN

(a) We can use Equation (4.10) to find $u_C - u_B$. Noting that cross-sectional area is changing with x in segment AB, we integrate Equation (4.7) to obtain $u_B - u_A$. We add the two relative displacements to obtain $u_C - u_A$ and noting that $u_A = 0$ we obtain the extension as u_C. (b) Once the axial stress in BC is found, the normal strain in the y direction can be found using Equation (4.13). Multiplying by 2 in., the original length in the y direction, we then find the change in depth.

SOLUTION

The cross-sectional areas of AB and BC are

$$A_{BC} = \left(\frac{3}{4}\text{ in.}\right)(2\text{ in.}) = 1.5\text{ in.}^2 \qquad A_{AB} = \left(\frac{3}{4}\text{ in.}\right)(2h\text{ in.}) = 1.5(2 - 0.02x)\text{ in.}^2 \tag{E1}$$

Figure 4.19 Free-body diagrams in Example 4.6.

(a) We can make an imaginary cuts in segment AB and BC, to obtain the free-body diagrams in Figure 4.19. By force equilibrium we obtain the internal forces,

$$N_{AB} = 10\text{ kips} \qquad N_{BC} = 10\text{ kips} \tag{E2}$$

The relative movement of point C with respect to point B is

$$u_C - u_B = \frac{N_{BC}(x_C - x_B)}{E_{BC}A_{BC}} = \frac{(10\text{ kips})(20\text{ in.})}{(10{,}000\text{ ksi})(1.5\text{ in.}^2)} = 13.33 \times 10^{-3}\text{ in.} \tag{E3}$$

Equation (4.7) for segment AB can be written as

$$\left(\frac{du}{dx}\right)_{AB} = \frac{N_{AB}}{E_{AB}A_{AB}} = \frac{10\text{ kips}}{(10{,}000\text{ ksi})[1.5(2 - 0.02x)\text{ in.}^2]} \tag{E4}$$

Integrating Equation (E4), we obtain the relative displacement of B with respect to A:

$$\int_{u_A}^{u_B} du = \left[\int_{x_A=0}^{x_B=50} \frac{10^{-3}}{1.5(2 - 0.02x)}\,dx\right]\text{in. or}$$

$$u_B - u_A = \frac{10^{-3}}{1.5}\frac{1}{(-0.02)}\ln(2 - 0.02x)\Big|_0^{50} = -\frac{10^{-3}}{0.03}[\ln(1) - \ln(2)]\text{ in.} = 23.1 \times 10^{-3}\text{ in.} \tag{E5}$$

We obtain the relative displacement of C with respect to A by adding Equations (E3) and (E5):

$$u_C - u_A = 13.33 \times 10^{-3} + 23.1 \times 10^{-3} = 36.43 \times 10^{-3}\text{ in.} \tag{E6}$$

We note that point A is fixed to the wall, and thus $u_A = 0$.

ANS. $u_C = 0.036$ in. elongation

(b) The axial stress in BC is $\sigma_{AB} = N_{BC}/A_{BC} = 10/1.5 = 6.667$ ksi. From Equation (4.13) the normal strain in y direction can be found,

$$\varepsilon_{yy} = -\frac{\nu_{AB}\sigma_{AB}}{E_{AB}} = -\frac{0.25\times(6.667\text{ ksi})}{(10{,}000\text{ ksi})} = -0.1667\times10^{-3} \tag{E7}$$

The change in dimension in the y direction Δv can be found as

$$\Delta v = \varepsilon_{yy}(2\text{ in.}) = -0.3333\times10^{-3}\text{ in.} \tag{E8}$$

ANS. $\Delta v = 0.3333\times10^{-3}$ in. contraction

COMMENT

1. An alternative approach is to integrate Equation (E4):

$$u(x) = -\frac{10^{-3}}{0.03}\ln(2-0.02x)+c \tag{E9}$$

To find constant of integration c, we note that at $x = 0$ the displacement $u = 0$. Hence, $c = (10^{-3}/0.03)\ln(2)$. Substituting the value, we obtain

$$u(x) = -\frac{10^{-3}}{0.03}\ln\left(\frac{2-0.02x}{2}\right) \tag{E10}$$

Knowing u at all x, we can obtain the extension by substituting $x = 50$ to get the displacement at C.

EXAMPLE 4.7

The radius of a circular truncated cone in Figure 4.20 varies with x as $R(x) = (r/L)(5L - 4x)$. Determine the extension of the truncated cone due to its own weight in terms of E, L, r, and γ, where E and γ are the modulus of elasticity and the specific weight of the material, respectively.

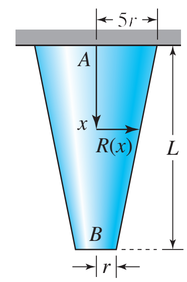


Figure 4.20 Truncated cone in Example 4.7.

PLAN

We make an imaginary cut at location x and take the lower part of the truncated cone as the free-body diagram. In the free-body diagram we can find the volume of the truncated cone as a function of x. Multiplying the volume by the specific weight, we can obtain the weight of the truncated cone and equate it to the internal axial force, thus obtaining the internal force as a function of x. We then integrate Equation (4.7) to obtain the relative displacement of B with respect to A.

SOLUTION

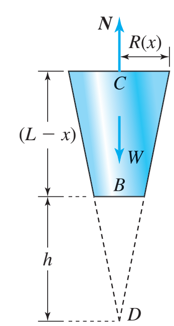


Figure 4.21 Free-body diagram of truncated cone in Example 4.7.

Figure 4.21 shows the free-body diagram after making a cut at some location x. We can find the volume V of the truncated cone by subtracting the volumes of two complete cones between C and D and between B and D. We obtain the location of point D,

$$R(x = L+h) = \frac{r}{L}[5L - 4(L+h)] = 0 \qquad \text{or} \qquad h = L/4 \tag{E1}$$

The volume of the truncated cone is

$$V = \frac{1}{3}\pi R^2\left(L - x + \frac{L}{4}\right) - \frac{1}{3}\pi r^2\frac{L}{4} = \frac{\pi}{12}\left[\frac{r^2}{L^2}(5L-4x)^3 - r^2L\right] \tag{E2}$$

By equilibrium of forces in Figure 4.21 we obtain the internal axial force:

$$N = W = \gamma V = \frac{\gamma\pi r^2}{12L^2}[(5L-4x)^3 - L^3] \tag{E3}$$

The cross-sectional area at location x (point C) is

$$A = \pi R^2 = \pi\frac{r^2}{L^2}(5L-4x)^2 \tag{E4}$$

Equation (4.7) can be written as

$$\frac{du}{dx} = \frac{N}{EA} = \frac{\frac{\gamma\pi r^2}{12L^2}[(5L-4x)^3 - L^3]}{E\pi\frac{r^2}{L^2}(5L-4x)^2} \tag{E5}$$

Integrating Equation (E5) from point A to point B, we obtain the relative movement of point B with respect to point A:

$$\int_{u_A}^{u_B} du = \int_{x_A=0}^{x_B=L} \frac{\gamma}{12E}\left[(5L-4x) - \frac{L^3}{(5L-4x)^2}\right]dx \quad \text{or}$$

$$u_B - u_A = \frac{\gamma}{12E}\left[5Lx - 2x^2 - \frac{L^3}{4(5L-4x)}\right]\Bigg|_0^L = \frac{\gamma L^2}{12E}\left(5 - 2 - \frac{1}{4} + \frac{1}{20}\right) = \frac{7\gamma L^2}{30E} \tag{E6}$$

Point A is built into the wall, hence $u_A = 0$. We obtain the extension of the bar as displacement of point B.

ANS. $u_B = \left(\frac{7\gamma L^2}{30E}\right)$ downward

COMMENTS

1. *Dimension check:* We write $O(\ )$ to represent the dimension of a quantity. F has dimensions of force and L of length. Thus, the modulus of elasticity E, which has dimensions of force per unit area, is represented as $O(F/L^2)$. The dimensional consistency of our answer is then checked as

$$\gamma \to O\left(\frac{F}{L^3}\right) \qquad L \to O(L) \qquad E \to O\left(\frac{F}{L^2}\right) \qquad u \to O(L) \qquad \frac{\gamma L^2}{E} \to O\left(\frac{(F/L^3)L^2}{F/L^2}\right) \to O(L) \to \text{checks}$$

2. An alternative approach to determining the volume of the truncated cone in Figure 4.21 is to find first the volume of the infinitesimal disc shown in Figure 4.22. We then integrate from point C to point B:

$$V = \int_x^L dV = \int_x^L \pi R^2 dx = \int_x^L \pi\frac{r^2}{L^2}(5L-4x)^2\, dx = -\pi\frac{r^2}{L^2}\frac{(5L-4x)^3}{3(-4)}\Bigg|_x^L \tag{E7}$$

3. On substituting the limits we obtain the volume given by (E2), as before:

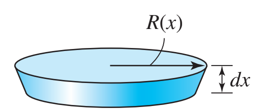


Figure 4.22 Alternative approach to finding volume of truncated cone.

4. The advantage of the approach in comment 2 is that it can be used for any complex function representation of $R(x)$, such as given in Problems 4.27 and 4.28, whereas the approach used in solving the example problem is only valid for a linear representation of $R(x)$.

4.2.9* General Approach to Distributed Axial Forces

Distributed axial forces are usually due to inertial forces, gravitational forces, or frictional forces acting on the surface of the axial bar. The internal axial force N becomes a function of x when an axial bar is subjected to a distributed axial force $p(x)$, as seen in Example 4.7. If $p(x)$ is a simple function, then we can find N as a function of x by drawing a free-body diagram, as we did in Example 4.7. However, if the distributed force $p(x)$ is a complex function, it may be easier to use the alternative described in this section.

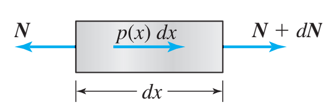


Figure 4.23 Equilibrium of an axial element.

Consider an infinitesimal axial element created by making two imaginary cuts at a distance dx from each other, as shown in Figure 4.23. By equilibrium of forces in the x direction we obtain: $(N + dN) + p(x)dx - N = 0$ or

$$\boxed{\frac{dN}{dx} + p(x) = 0} \tag{4.15}$$

Equation (4.15) assumes that $p(x)$ is positive in the positive x direction. If $p(x)$ is zero in a segment of the axial bar, then the internal force N is a constant in that segment.

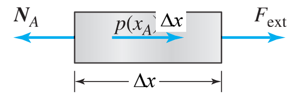


Figure 4.24 Boundary condition on internal axial force.

Equation (4.15) can be integrated to obtain the internal force N. The integration constant can be found by knowing the value of the internal force N at either end of the bar. To obtain the value of N at the end of the shaft (say, point A), a free-body diagram is constructed after making an imaginary cut at an infinitesimal distance Δx from the end as shown in Figure 4.24) and writing the equilibrium equation as

$$\lim_{\Delta x \to 0} [F_{ext} - N_A - p(x_A)\Delta x] = 0 \qquad N_A = F_{ext}$$

This equation shows that the distributed axial force does not affect the boundary condition on the internal axial force. The value of the internal axial force N at the end of an axial bar is equal to the concentrated external axial force applied at the end.

Suppose the weight per unit volume, or, the specific weight of a bar, is γ. By multiplying the specific weight by the cross-sectional area A, we would obtain the weight per unit length. Thus $p(x)$ is equal to γA in magnitude. If x coordinate is chosen in the direction of gravity, then $p(x)$ is positive: $[p(x) = +\gamma A]$. If it is opposite to the direction of gravity, then $p(x)$ is negative:$[p(x) = -\gamma A]$.

EXAMPLE 4.8

Determine the internal force N in Example 4.7 using the approach outlined in Section 4.2.9.

PLAN

The distributed force $p(x)$ per unit length is the product of the specific weight times the area of cross section. We can integrate Equation (4.15) and use the condition that the value of the internal force at the free end is zero to obtain the internal force as a function of x.

SOLUTION

The distributed force $p(x)$ is the weight per unit length and is equal to the specific weight times the area of cross section $A = \pi R^2 = \pi(r^2/L^2)(5L - 4x)^2$:

$$p(x) = \gamma A = \gamma\pi \frac{r^2}{L^2}(5L - 4x)^2 \tag{E1}$$

We note that point B ($x = L$) is on a free surface and hence the internal force at B is zero. We integrate Equation (4.15) from L to x after substituting $p(x)$ from Equation (E1) and obtain N as a function of x,

$$\int_{N_B=0}^{N} dN = -\int_{x_B=L}^{x} p(x)\,dx = -\int_L^x \gamma\left[\pi\frac{r^2}{L^2}(5L-4x)^2\right]dx = -\left(\gamma\pi\frac{r^2}{L^2}\right)\left[\frac{(5L-4x)^3}{-4\times 3}\right]\Bigg|_L^x \tag{E2}$$

ANS. $$N = \frac{\gamma\pi r^2}{12L^2}[(5L-4x)^3 - L^3]$$

COMMENT

1. An alternative approach is to substitute (E1) into Equation (4.15) and integrate to obtain

$$N(x) = \frac{\gamma\pi r^2}{12L^2}(5L-4x)^3 + c_1 \tag{E3}$$

To determine the integration constant, we use the boundary condition that at $N(x = L) = 0$, which yields $c_1 = -(\gamma\pi r^2/12L^2)L^3$. Substituting this value into Equation (E3), we obtain N as before.

Consolidate your knowledge

1. Identity five examples of axial members from your daily life.
2. With the book closed, derive Equation (4.10), listing all the assumptions as you go along.

QUICK TEST 4.1

Time: 20 minutes/Total: 20 points

Answer true or false and justify each answer in one sentence. Grade yourself with the answers given in Appendix E.

1. Axial *strain* is uniform across a nonhomogeneous cross section.

2. Axial *stress* is uniform across a nonhomogeneous cross section.

3. The formula $\sigma_{xx} = N/A$ can be used for finding the stress on a cross section of a tapered axial member.

4. The formula $u_2 - u_1 = N(x_2 - x_1)/EA$ can be used for finding the deformation of a segment of a tapered axial member.

5. The formula $\sigma_{xx} = N/A$ can be used for finding the stress on a cross section of an axial member subjected to distributed forces.

6. The formula $u_2 - u_1 = N(x_2 - x_1)/EA$ can be used for finding the deformation of a segment of an axial member subjected to distributed forces.

7. The equation $N = \int_A \sigma_{xx}\, dA$ *cannot* be used for nonlinear materials.

8. The equation $N = \int_A \sigma_{xx}\, dA$ *can* be used for a nonhomogeneous cross section.

9. External axial forces must be collinear and pass through the centroid of a homogeneous cross section for no bending to occur.

10. Internal axial forces jump by the value of the concentrated external axial force at a section.

PROBLEM SET 4.2

4.7 A crane is lifting a mass of 1000-kg, as shown in Figure P4.7. The weight of the iron ball at *B* is 25 kg. A single cable having a diameter of 25 mm runs between *A* and *B*. Two cables run between *B* and *C*, each having a diameter of 10 mm. Determine the axial stresses in the cables.

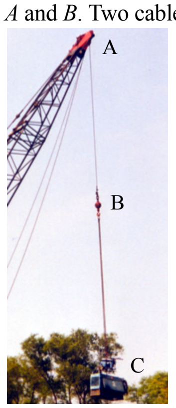


Figure P4.7

4.8 The counterweight in a lift bridge has 12 cables on the left and 12 cables on the right, as shown in Figure P4.8. Each cable has an effective diameter of 0.75 in, a length of 50 ft, a modulus of elasticity of 30,000 ksi, and an ultimate strength of 60 ksi. (a) If the counterweight is 100 kips, determine the factor of safety for the cable. (b) What is the extension of each cable when the bridge is being lifted?

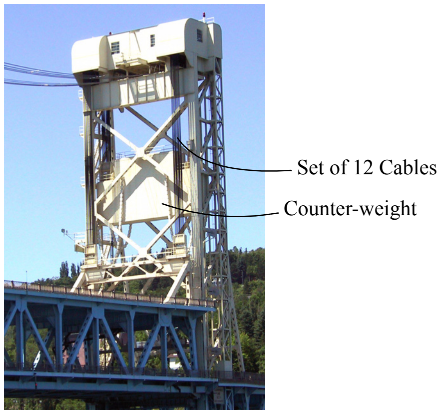


Figure P4.8

4.9 (a) Draw the axial force diagram for the axial member shown in Figure P4.9. (b) Check your results for part *a* by finding the internal forces in segments AB, BC, and CD by making imaginary cuts and drawing free-body diagrams. (c) The axial rigidity of the bar is $EA = 8000$ kips. Determine the movement of the section at *D* with respect to the section at *A*.

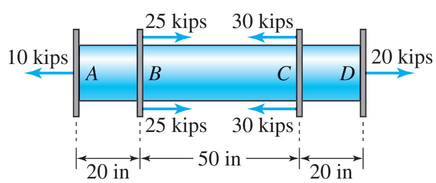


Figure P4.9

4.10 (a) Draw the axial force diagram for the axial member shown in Figure P4.10. (b) Check your results for part *a* by finding the internal forces in segments AB, BC, and CD by making imaginary cuts and drawing free-body diagrams. (c) The axial rigidity of the bar is $EA = 80{,}000$ kN. Determine the movement of the section at *C*.

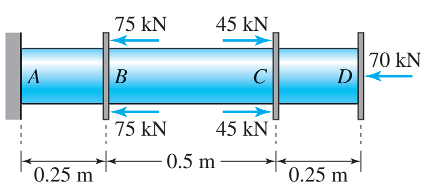


Figure P4.10

4.11 (a) Draw the axial force diagram for the axial member shown in Figure P4.11. (b) Check your results for part *a* by finding the internal forces in segments AB, BC, and CD by making imaginary cuts and drawing free-body diagrams. (c) The axial rigidity of the bar is $EA = 2000$ kips. Determine the movement of the section at *B*.

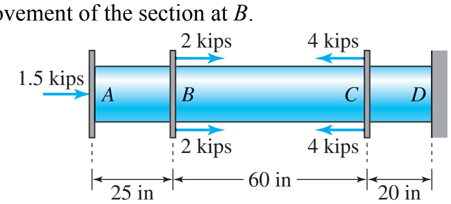


Figure P4.11

4.12 (a) Draw the axial force diagram for the axial member shown in Figure P4.12. (b) Check your results for part *a* by finding the internal forces in segments AB, BC, and CD by making imaginary cuts and drawing free-body diagrams. (c) The axial rigidity of the bar is $EA = 50{,}000$ kN. Determine the movement of the section at *D* with respect to the section at *A*.

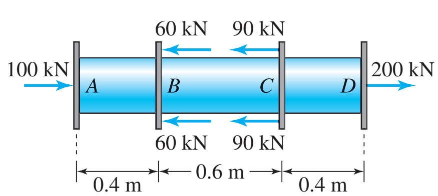


Figure P4.12

4.13 Three segments of 4-in. × 2-in. rectangular wooden bars (E = 1600 ksi) are secured together with rigid plates and subjected to axial forces, as shown in Figure P4.13. Determine: (a) the movement of the rigid plate at D with respect to the plate at A; (b) the maximum axial stress.

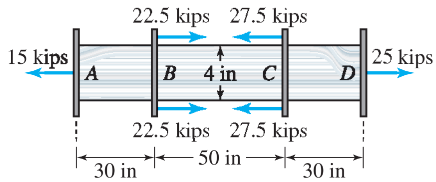


Figure P4.13

4.14 Aluminum bars (E = 30,000 ksi) are welded to rigid plates, as shown in Figure P4.1. All bars have a cross-sectional area of 0.5 in^2. The applied forces are F_1 = 8 kips, F_2 = 12 kips, and F_3= 9 kips. Determine (a) the displacement of the rigid plate at D with respect to the rigid plate at A. (b) the maximum axial stress in the assembly.

4.15 Brass bars between sections A and B, aluminum bars between sections B and C, and steel bars between sections C and D are welded to rigid plates, as shown in Figure P4.2. The properties of the bars are given in Table 4.2 The applied forces are F_1 = 90 kN, F_2 = 40 kN, and F_3= 70 kN. Determine (a) the displacement of the rigid plate at D.(b) the maximum axial stress in the assembly.

4.16 A solid circular steel (E_s = 30,000 ksi) rod BC is securely attached to two hollow steel rods AB and CD as shown. Determine (a) the angle of displacement of section at D with respect to section at A; (b) the maximum axial stress in the axial member.

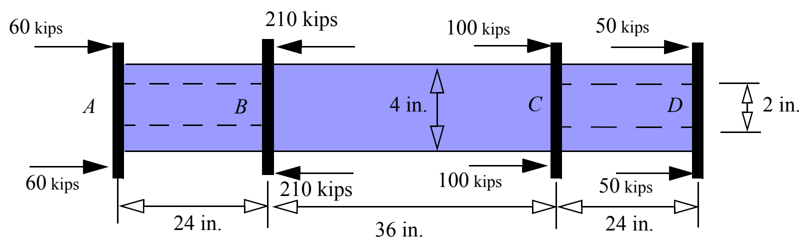


Figure P4.16

4.17 Two circular steel bars (E_s = 30,000 ksi, ν_s = 0.3) of 2-in. diameter are securely connected to an aluminum bar (E_{al} = 10,000 ksi, ν_{al} = 0.33) of 1.5-in. diameter, as shown in Figure P4.17. Determine (a) the displacement of the section at C with respect to the wall; (b) the maximum change in the diameter of the bars.

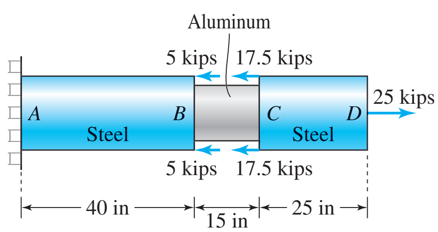


Figure P4.17

4.18 Two cast-iron pipes (E = 100 GPa) are adhesively bonded together, as shown in Figure P4.18. The outer diameters of the two pipes are 50 mm and 70 mm and the wall thickness of each pipe is 10 mm. Determine the displacement of end B with respect to end A.

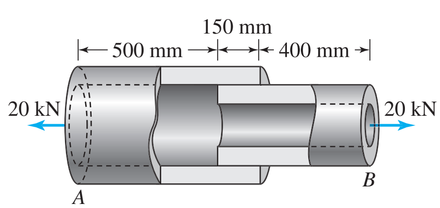


Figure P4.18

Tapered axial members

4.19 The tapered bar shown in Figure P4.19 has a cross-sectional area that varies as $A = K(2L - 0.25\,x)^2$. Determine the elongation of the bar in terms of P, L, E, and K.

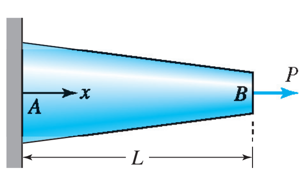


Figure P4.19

4.20 The tapered bar shown in Figure P4.19 has a cross-sectional area that varies as $A = K(4L - 3x)$. Determine the elongation of the bar in terms of *P*, *L*, *E*, and *K*.

4.21 A tapered and an untapered solid circular steel bar (E = 30,000 ksi) are securely fastened to a solid circular aluminum bar (E = 10,000 ksi), as shown in Figure P4.21. The untapered steel bar has a diameter of 2 in. The aluminum bar has a diameter of 1.5 in. The diameter of the tapered bars varies from 1.5 in to 2 in. Determine (a) the displacement of the section at *C* with respect to the section at *A*; (b) the maximum axial stress in the bar.

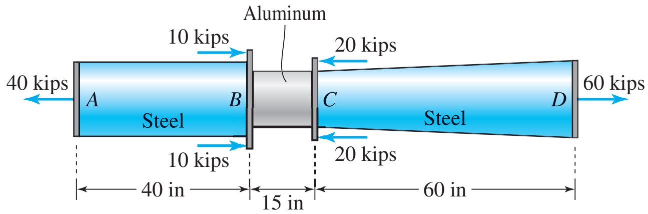


Figure P4.21

Distributed axial force

4.22 The column shown in Figure P4.22 has a length *L*, modulus of elasticity *E*, and specific weight γ. The cross section is a circle of radius *a*. Determine the contraction of each column in terms of *L*, *E*, γ, and *a*.

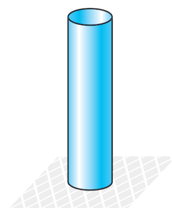


Figure P4.22

4.23 The column shown in Figure P4.23 has a length *L*, modulus of elasticity *E*, and specific weight γ. The cross section is an equilateral triangle of side *a*.Determine the contraction of each column in terms of *L*, *E*, γ, and *a*.

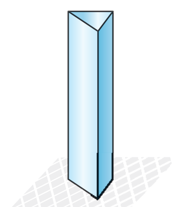


Figure P4.23

4.24 The column shown in Figure P4.24 has a length *L*, modulus of elasticity *E*, and specific weight γ. The cross-sectional area is *A*. Determine the contraction of each column in terms of *L*, *E*, γ, and *A*.

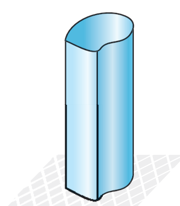


Figure P4.24

4.25 On the truncated cone of Example 4.7 a force $P = \gamma\pi r^2 L/5$ is also applied, as shown in Figure P4.25. Determine the total elongation of the cone due to its weight and the applied force. (*Hint*: Use superposition.)

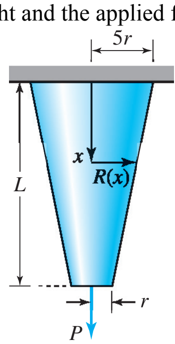


Figure P4.25

4.26 A 20-ft-tall thin, hollow tapered tube of a uniform wall thickness of $\frac{1}{8}$ in. is used for a light pole in a parking lot, as shown in Figure P4.26. The mean diameter at the bottom is 8 in., and at the top it is 2 in. The weight of the lights on top of the pole is 80 lb. The pole is made of aluminum alloy with a specific weight of 0.1 lb/in^3, a modulus of elasticity $E = 11{,}000$ ksi, and a shear modulus of rigidity $G = 4000$ ksi. Determine (a) the maximum axial stress; (b) the contraction of the pole. (*Hint*: Approximate the cross-sectional area of the thin-walled tube by the product of circumference and thickness.)

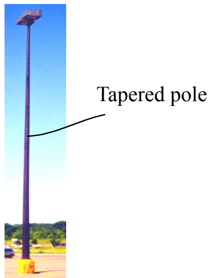


Figure P4.26

4.27 Determine the contraction of a column shown in Figure P4.27 due to its own weight. The specific weight is $\gamma = 0.28$ lb/in.^{3}, the modulus of elasticity is $E = 3600$ ksi, the length is $L = 120$ in., and the radius is $R = \sqrt{240 - x}$, where R and x are in inches.

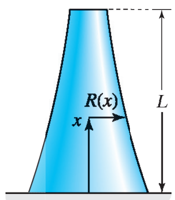


Figure P4.27

4.28 Determine the contraction of a column shown in Figure P4.27 due to its own weight. The specific weight is $\gamma = 24$ kN/m^3 the modulus of elasticity is $E = 25$ GPa, the length is $L = 10$ m and the radius is $R = 0.5e^{-0.07x}$, where R and x are in meters.

4.29 The frictional force per unit length on a cast-iron pipe being pulled from the ground varies as a quadratic function, as shown in Figure P4.29. Determine the force F needed to pull the pipe out of the ground and the elongation of the pipe before the pipe slips, in terms of the modulus of elasticity E, the cross-sectional area A, the length L, and the maximum value of the frictional force $f_{max.}$

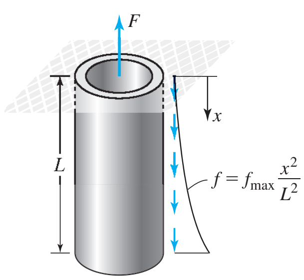


Figure P4.29

Design problems

4.30 The spare wheel in an automobile is stored under the vehicle and raised and lowered by a cable, as shown in Figure P4.30. The wheel has a mass of 25 kg. The ultimate strength of the cable is 300 MPa, and it has an effective modulus of elasticity $E = 180$ GPa. At maximum extension the cable length is 36 cm. (a) For a factor of safety of 4, determine to the nearest millimeter the minimum diameter of the cable if failure due to rupture is to be avoided. (b) What is the maximum extension of the cable for the answer in part (a)?

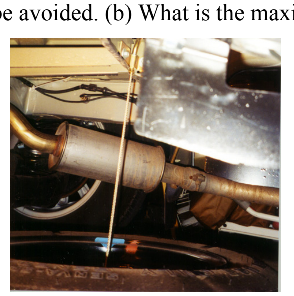


Figure P4.30

4.31 An adhesively bonded joint in wood (E = 1800 ksi) is fabricated as shown in Figure P4.31. If the total elongation of the joint between A and D is to be limited to 0.05 in., determine the maximum axial force F that can be applied.

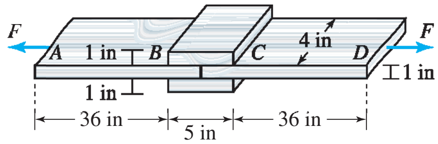


Figure P4.31

4.32 A 5-ft-long hollow rod is to transmit an axial force of 30 kips. The outer diameter of the rod must be 6 in. to fit existing attachments. The relative displacement of the two ends of the shaft is limited to 0.027 in. The axial rod can be made of steel or aluminum. The modulus of elasticity E, the allowable axial stress σ_{allow}, and the specific weight γ are given in Table 4.32. Determine the maximum inner diameter in increments of $\frac{1}{8}$ in. of the lightest rod that can be used for transmitting the axial force and the corresponding weight.

TABLE P4.32 **Material properties**

Material	***E*** **(ksi)**	σ_{allow} **(ksi)**	γ **(lb/in.^{3})**
Steel	30,000	24	0.285
Aluminum	10,000	14	0.100

4.33 A hitch for an automobile is to be designed for pulling a maximum load of 3600 lb. A solid square bar fits into a square tube and is held in place by a pin, as shown in Figure P4.33. The allowable axial stress in the bar is 6 ksi, the allowable shear stress in the pin is 10 ksi, and the allowable axial stress in the steel tube is 12 ksi. To the nearest $\frac{1}{16}$ in., determine the minimum cross-sectional dimensions of the pin, the bar, and the tube. (*Hint:* The pin is in double shear.)

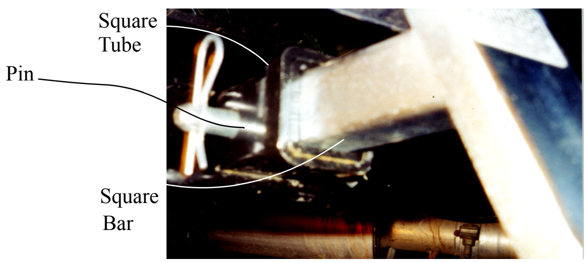


Figure P4.33

Stretch yourself

4.34 An axial rod has a constant axial rigidity EA and is acted upon by a distributed axial force $p(x)$. If at the section at A the internal axial force is zero, show that the relative displacement of the section at B with respect to the displacement of the section at A is given by

$$u_B - u_A = \frac{1}{EA}\left[\int_{x_A}^{x_B} (x - x_B)p(x)\,dx\right] \qquad (4.16)$$

4.35 A composite laminated bar made from n materials is shown in Figure P4.35. E_i and A_i are the modulus of elasticity and cross sectional area of the ith material. (a) If Assumptions from 1 through 5 are valid, show that the stress $(\sigma_{xx})_i$ in the ith material is given Equation (4.17a), where N is the total internal force at a cross section. (b) If Assumptions 7 through 9 are valid, show that relative deformation $u_2 - u_1$ is given by Equation (4.17b). (c) Show that for E_1=E_2=E_3....=E_n=E Equations (4.17a) and (4.17b) give the same results as Equations (4.8) and (4.10).

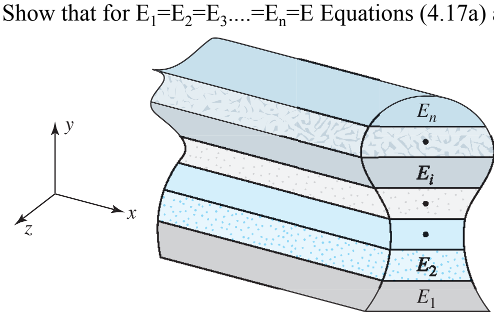


Figure P4.35

$$(\sigma_{xx})_i = \frac{NE_i}{\sum_{j=1}^{n} E_j A_j} \qquad (4.17a)$$

$$u_2 - u_1 = \frac{N(x_2 - x_1)}{\sum_{j=1}^{n} E_j A_j} \qquad (4.17b)$$

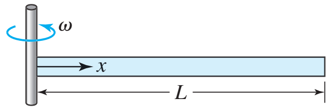
4.36 The stress–strain relationship for a nonlinear material is given by the power law $\sigma = E\varepsilon^n$. If all assumptions except Hooke's law are valid, show that

$$u_2 - u_1 = \left(\frac{N}{EA}\right)^{1/n}(x_2 - x_1) \tag{4.17}$$

and the axial stress σ_{xx} is given by (4.8).

4.37 Determine the elongation of a rotating bar in terms of the rotating speed ω, density γ, length L, modulus of elasticity E, and cross-sectional area A (Figure P4.37). (*Hint:* The body force per unit volume is $\rho\omega^2 x$.)

Figure P4.37



4.38 Consider the dynamic equilibrium of the differential elements shown in Figure P4.38, where N is the internal force, γ is the density, A is the cross-sectional area, and $\partial^2 u/\partial t^2$ is acceleration. By substituting for N from Equation (4.7) into the dynamic equilibrium equation, derive the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \qquad \text{where} \qquad c = \sqrt{\frac{E}{\gamma}} \tag{4.18}$$

The material constant c is the velocity of propagation of sound in the material.

4.39 Show by substitution that the functions $f(x - ct)$ and $g(x + ct)$ satisfy the wave equation, Equation (4.18).

4.40 The strain displacement relationship for large axial strain is given by

$$\varepsilon_{xx} = \frac{du}{dx} + \frac{1}{2}\left(\frac{du}{dx}\right)^2 \tag{4.19}$$

where we recognize that as u is only a function of x, the strain from (4.19) is uniform across the cross section. For a linear, elastic, homogeneous material show that

$$\frac{du}{dx} = \sqrt{1 + \frac{2N}{EA}} - 1 \tag{4.20}$$

The axial stress σ_{xx} is given by (4.8).

Computer problems

4.41 Table P4.41 gives the measured radii at several points along the axis of the solid tapered rod shown in Figure P4.41. The rod is made of aluminum (E = 100 GPa) and has a length of 1.5 m. Determine (a) the elongation of the rod using numerical integration; (b) the maximum axial stress in the rod.

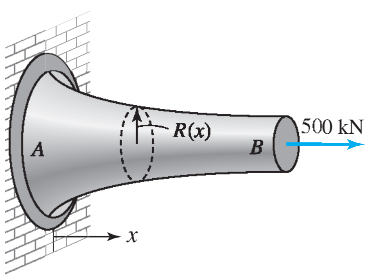


Figure P4.41

TABLE P4.41

x (m)	$R(x)$ (mm)	x (m)	$R(x)$ (mm)
0.0	100.6	0.8	60.1
0.1	92.7	0.9	60.3
0.2	82.6	1.0	59.1
0.3	79.6	1.1	54.0
0.4	75.9	1.2	54.8
0.5	68.8	1.3	54.1
0.6	68.0	1.4	49.4
0.7	65.9	1.5	50.6

4.42 Let the radius of the tapered rod in Problem 4.41 be represented by the equation $R(x) = a + bx$. Using the data in Table P4.41 determine constants a and b by the least-squares method and then find the elongation of the rod by analytical integration.

4.43 Table 4.43 shows the values of the distributed axial force at several points along the axis of the hollow steel rod (E = 30,000 ksi) shown in Figure P4.43. The rod has a length of 36 in., an outside diameter of 1 in., and an inside diameter of 0.875 in. Determine (a) the displacement of end A using numerical integration; (b) the maximum axial stress in the rod.

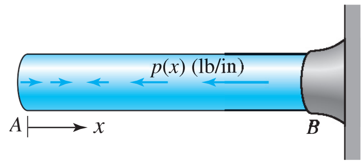


Figure P4.43

TABLE P4.43

x (inches)	$p(x)$ (lb/in.)	x (in.)	$p(x)$ (lb/in.)
0	260	21	−471
3	106	24	−598
6	32	27	−645
9	40	30	−880
12	−142	33	−1035
15	−243	36	−1108
18	−262		

4.44 Let the distributed force $p(x)$ in Problem 4.43 be represented by the equation $p(x) = cx^2 + bx + a$. Using the data in Table P4.43 determine constants a, b, and c by the least-squares method and then find the displacement of the section at A by analytical integration.

4.3 STRUCTURAL ANALYSIS

Structures are usually an assembly of axial bars in different orientations. Equation (4.10) assumes that the bar lies in x direction, and hence in structural analysis the form of Equation (4.21) is preferred over Equation (4.10).

$$\boxed{\delta = \frac{NL}{EA}} \tag{4.21}$$

where $L = x_2 - x_1$ and $\delta = u_2 - u_1$ in Equation (4.10). L represents the original length of the bar and δ represents deformation of the bar in the *original direction* irrespective of the movement of points on the bar. It should also be recognized that L, E, and A are positive. Hence the sign of δ is the same as that of $\boldsymbol{N}$:

- If $\boldsymbol{N}$ is a tensile force, then δ is elongation.
- If $\boldsymbol{N}$ is a compressive force, then δ is contraction.

4.3.1 Statically Indeterminate Structures

Statically indeterminate structures arise when there are more supports than needed to hold a structure in place. These extra supports are included for safety or to increase the stiffness of the structures. Each extra support introduces additional unknown reactions, and hence the total number of unknown reactions exceeds the number of static equilibrium equations. The **degree of static redundancy** is the number of unknown reactions minus the number of equilibrium equations. If the degree of static redundancy is zero, then we have a statically determinate structure and all unknowns can be found from equilibrium equations. If the degree of static redundancy is not zero, then we need additional equations to determine the unknown reactions. These additional equations are the relationships between the deformations of bars. **Compatibility equations** are geometric relationships between the deformations of bars that are derived from the deformed shapes of the structure. The number of compatibility equations needed is always equal to the degree of static redundancy.

Drawing the approximate deformed shape of a structure for obtaining compatibility equations is as important as drawing a free-body diagram for writing equilibrium equations. *The deformations shown in the deformed shape of the structure must be consistent with the direction of forces drawn on the free-body diagram.* Tensile (compressive) force on a bar on free body diagram must correspond to extension (contraction) of the bar shown in deformed shape.

In many structures there are gaps between structural members. These gaps may be by design to permit expansion due to temperature changes, or they may be inadvertent due to improper accounting for manufacturing tolerances. We shall make use of the observation that for a linear system it does not matter how we reach the final equilibrium state. We therefore shall start by assuming that at the final equilibrium state the gap is closed. At the end of analysis we will check if our assumption of gap closure is correct or incorrect and make corrections as needed.

Displacement, strain, stress, and internal force are all related as depicted by the logic shown in Figure 4.9 and incorporated in the formulas developed in Section 4.2. If one of these quantities is found, then the rest could be found for an axial member. Thus theoretically, in structural analysis, any of the four quantities could be treated as an unknown variable. Analysis however, is traditionally conducted using either forces (internal or reaction) or displacements as the unknown variables, as described in the two methods that follow.

4.3.2 Force Method, or Flexibility Method

In this method internal forces or reaction forces are treated as the unknowns. The coefficient L/EA, multiplying the internal unknown force in Equation (4.21), is called the **flexibility coefficient**. If the unknowns are internal forces (rather than reaction forces), as is usually the case in large structures, then the matrix in the simultaneous equations is called the **flexibility matrix**. Reaction forces are often preferred in hand calculations because the number of unknown reactions (degree of static redundancy) is either equal to or less than the total number of unknown internal forces.

4.3.3 Displacement Method, or Stiffness Method

In this method the displacements of points are treated as the unknowns. The minimum number of displacements that are necessary to describe the deformed geometry is called **degree of freedom**. The coefficient multiplying the deformation EA/L is called the **stiffness coefficient**. Using small-strain approximation, the relationship between the displacement of points and the deformation of the bars is found from the deformed shape and substituted in the compatibility equations. Using Equation (4.21) and equilibrium equations, the displacement and the external forces are related. The matrix multiplying the unknown displacements in a set of algebraic equations is called the **stiffness matrix**.

4.3.4 General Procedure for Indeterminate Structure

The procedure outlined can be used for solving statically indeterminate structure problems by either the force method or by the displacement method.

1. If there is a gap, assume it will close at equilibrium.
2. Draw free-body diagrams, noting the tensile and compressive nature of internal forces. Write equilibrium equations relating internal forces to each other.

   or
3. Write equilibrium equations in which the internal forces are written in terms of reaction forces, if the force method is to be used.
4. Draw an exaggerated approximate deformed shape, ensuring that the deformation is consistent with the free body diagrams of step 2. Write compatibility equations relating deformation of the bars to each other.

   or
5. Write compatibility equations in terms of unknown displacements of points on the structure, if displacement method is to be used.
6. Write internal forces in terms of deformations using Equation (4.21).
7. Solve the equations of steps 2, 3, and 4 simultaneously for the unknown forces (for force method) or for the unknown displacements (for displacement method).
8. Check whether the assumption of gap closure in step 1 is correct.

Both the force method and the displacement method are used in Examples 4.10 and 4.11 to demonstrate the similarities and differences in the two methods.

EXAMPLE 4.9

The three bars in Figure 4.25 are made of steel (E = 30,000 ksi) and have cross-sectional areas of 1 in^2. Determine the displacement of point D

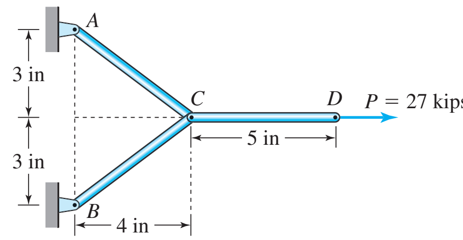


Figure 4.25 Geometry in Example 4.9

PLAN

The displacement of point D with respect to point C can be found using Equation (4.21). The deformation of rod AC or BC can also be found from Equation (4.21) and related to the displacement of point C using small-strain approximation.

SOLUTION

Figure 4.26 shows the free body diagrams. By equilibrium of forces in Figure 4.26*a*, we obtain

$$\boldsymbol{N_{CD}} = 27 \text{ kips} \tag{E1}$$

By equilibrium of forces in Figure 4.26*b*, we obtain

$$\boldsymbol{N_{CA}} = \boldsymbol{N_{CB}} \tag{E2}$$

$$N_{CA}\cos\theta + N_{CB}\cos\theta = 27 \text{ kips} \tag{E3}$$

Substituting for θ from Figure 4.26*c* and solving Equations (E2) and (E3), we obtain

$$2\boldsymbol{N_{CA}}\left(\frac{4}{5}\right) = 27 \text{ kips} \qquad \text{or} \qquad \boldsymbol{N_{CA}} = \boldsymbol{N_{CB}} = 16.875 \text{ kips} \tag{E4}$$

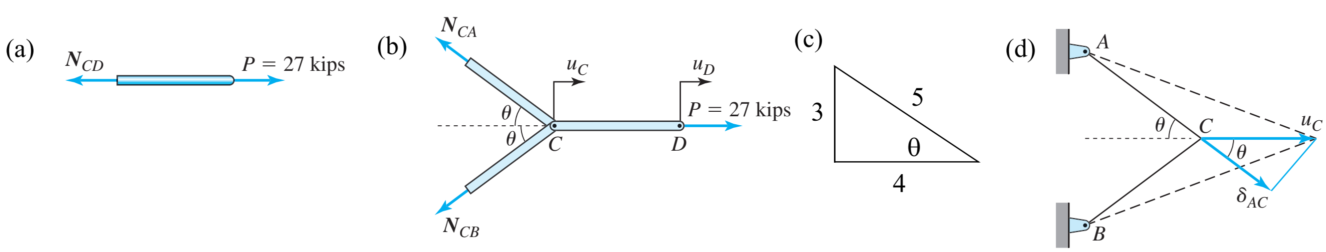


Figure 4.26 Free-body diagrams and deformed geometry in Example 4.9.

From Equation (4.21) we obtain the relative displacement of D with respect to C as shown in Equation (E5) and deformation of bar AC in Equation (E6).

$$\delta_{CD} = u_D - u_C = \frac{N_{CD}L_{CD}}{E_{CD}A_{CD}} = \frac{(27 \text{ kips})(5 \text{ in.})}{(30{,}000 \text{ ksi})(1 \text{ in.}^2)} = 4.5 \times 10^{-3} \text{ in.} \tag{E5}$$

$$\delta_{AC} = \frac{N_{CA}L_{CA}}{E_{CA}A_{CA}} = \frac{(16.875 \text{ kips})(5 \text{ in.})}{(30{,}000 \text{ ksi})(1 \text{ in.}^2)} = 2.8125 \times 10^{-3} \text{ in.} \tag{E6}$$

Figure 4.26*d* shows the exaggerated deformed geometry of the two bars AC and BC. The displacement of point C can be found by

$$u_C = \frac{\delta_{AC}}{\cos\theta} = 3.52 \times 10^{-3} \text{ in.} \tag{E7}$$

Adding Equations (E5) and (E7) we obtain the displacement of point D,

$$u_D = (4.5 + 3.52)10^{-3} \text{ in.}$$

ANS. $u_D = 0.008$ in

COMMENT

1. This was a statically determinate problem as we could find the internal forces in all members by static equilibrium.

EXAMPLE 4.10

An aluminum rod (E_{al} = 70 GPa) is securely fastened to a rigid plate that does not rotate during the application of load P is shown in Figure 4.27. A gap of 0.5 mm exists between the rigid plate and the steel rod (E_{st} = 210 GPa) before the load is applied. The aluminum rod has a diameter of 20 mm and the steel rod has a diameter of 10 mm. Determine (a) the movement of the rigid plate; (b) the axial stress in steel.

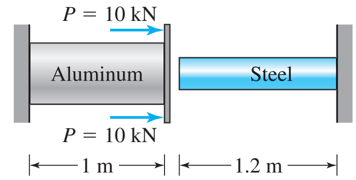


Figure 4.27 Geometry in Example 4.10.

FORCE METHOD: PLAN

We assume that the force P is sufficient to close the gap at equilibrium. The two unknown wall reactions minus one equilibrium equation results in 1 degree of static redundancy. We follow the procedure outlined in Section 4.3.4 to solve the problem.

SOLUTION

Step 1 Assume force P is sufficient to close the gap. If this assumption is correct, then steel will be in compression and aluminum will be in tension.

Step 2 The degree of static redundancy is 1. Thus we use one unknown reaction to formulate our equilibrium equations. We make imaginary cuts at the equilibrium position and obtain the free-body diagrams in Figure 4.28. By equilibrium of forces we can obtain the internal forces in terms of the wall reactions,

$$N_{al} = R_L \qquad N_S = 20(10^3) - R_L \tag{E1}$$

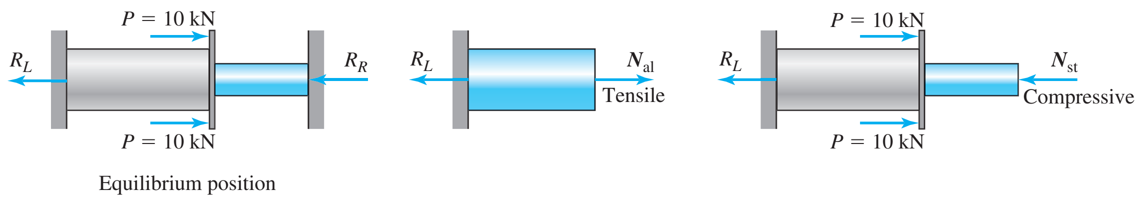


Figure 4.28 Free-body diagrams in Example 4.10.

Step 3 Figure 4.29 shows the exaggerated deformed shape. The deformation of aluminum is extension and steel in contraction, to ensure consistency with the tensile and compressive axial forces shown on the free-body diagrams in Figure 4.28. The compatibility equation can be written

$$\delta_{st} = (\delta_{al} - 0.0005)\ \text{m} \tag{E2}$$

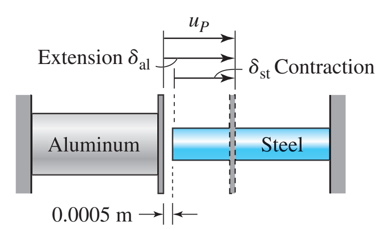


Figure 4.29 Approximate deformed shape in Example 4.10.

Step 4 The radius of the aluminum rod is 0.01 m, and the radius of the steel rod is 0.005 m. We can write the deformation of aluminum and steel in terms of the internal forces,

$$\delta_{al} = \frac{N_{al}L_{al}}{E_{al}A_{al}} = \frac{N_{al}(1\ \text{m})}{(70 \times 10^9\ \text{N/m}^2)[\pi(0.01\ \text{m})^2]} = 0.04547N_{al} \times 10^{-6}\ \text{m} \tag{E3}$$

$$\delta_{st} = \frac{N_{st}L_{st}}{E_{st}A_{st}} = \frac{N_{st}(1.2\ \text{m})}{(210 \times 10^9\ \text{N/m}^2)[\pi(0.005\ \text{m})^2]} = 0.07277N_{st} \times 10^{-6}\ \text{m} \tag{E4}$$

Step 5 Substituting Equation (E1) into Equations (E3) and (E4), we obtain deformation in terms of the unknown reactions,

$$\delta_{al} = 0.04547R_L \times 10^{-6}\ \text{m} \tag{E5}$$

$$\delta_{st} = 0.07277(20 \times 10^3 - R_L)10^{-6}\ \text{m} = (1455.4 - 0.07277R_L)10^{-6}\ \text{m} \tag{E6}$$

Substituting Equations (E5) and (E6) into Equation (E2), we can solve for R_L.

$$1455.4 - 0.07277R_L = 0.04547R_L - 500 \qquad \text{or} \qquad R_L = 16{,}538\ \text{N} \tag{E7}$$

Substituting Equation (E7) into Equations (E1) and (E1) we obtain the internal forces,

$$N_{al} = 16{,}538\ \text{N} \qquad N_{st} = 3462\ \text{N} \tag{E8}$$

Step 6 The positive value of the force in steel confirms that it is compressive and the assumption of the gap being closed is correct.

(a) Substituting Equation (E7) into Equation (E5), we obtain the deformation of aluminum, which is equal to the movement of the rigid plate u_P:

$$u_P = \delta_{al} = (0.04547)\,(16{,}538)\,10^{-6} = 0.752\,(10^{-3})\text{ m}$$

ANS. $u_P = 0.752$ mm

(b) The normal stress in steel can be found from Equation (4.8).

$$\sigma_{st} = \frac{N_{st}}{A_{st}} = \frac{3462\text{ N}}{\pi(0.005\text{ m})^2} = 44.1(10^6)\text{ N/m}^2 \tag{E9}$$

ANS. $\sigma_{st} = 44.1$ MPa (C)

COMMENTS

1. The assumption about gap closure is correct because movement of plate $u_P = 0.752$ mm is greater than the gap.
2. An alternative approach is to use internal forces as the unknowns. We can make a cut on either side of the rigid plate at the equilibrium position and draw the free-body diagram, as shown in Figure 4.30*a*. We can then write the equilibrium equation,

$$N_{st} + N_{al} = 20 \times 10^3\text{ N} \tag{E10}$$

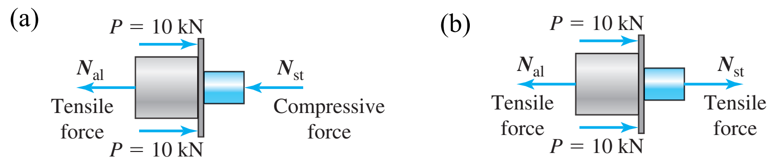


Figure 4.30 (a) Alternative free-body diagram in Example 4.10. (b) Tensile forces in free-body diagram in Example 4.10.

Substituting Equations (E3) and (E4) into (E2), we obtain

$$0.04547N_{al} - 0.07277N_{st} = 500\text{ N} \tag{E11}$$

Equations (E10) and (E11) can be written in matrix form as

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0.04547 & -0.07277 \end{bmatrix}}_{[F]} \underbrace{\begin{Bmatrix} N_{at} \\ N_{st} \end{Bmatrix}}_{\{N\}} = \underbrace{\begin{Bmatrix} 20 \times 10^3 \\ 500 \end{Bmatrix}}_{\{P\}}$$

The matrix $[F]$ is called the *flexibility matrix*.

3. With internal forces as unknowns we had to solve two equations simultaneously, as elaborated in comment 2. With the reaction force as the unknown we had only one unknown, which is the number of degrees of static redundancy. Thus for hand calculations the reaction forces as unknowns are preferred when using the force method. But in computer programs the process of substitution in step 5 is difficult to implement compared to constructing the equilibrium and compatibility equations in terms of internal forces. Thus in computer methods internal forces are treated as unknowns in force methods.
4. Suppose we had started with the direction of the force in steel as tension as shown in Figure 4.30*b*. Then we would get the following equilibrium equation:

$$-N_{st} + N_{al} = 20 \times 10^3\text{ N} \tag{E12}$$

Suppose we incorrectly do not make any changes in Equation (E2) or Equation (E6)—that is, we continue to use the deformation in steel as contraction even though the assumed force is tensile, we then solve Equations (E12) and (E11), we obtain $N_{al} = 34996$ N and $N_{st} = 14996$ N. These answers demonstrate how a simple error in sign produces dramatically different results.

DISPLACEMENT METHOD: PLAN

Let the plate move to the right by the amount u_P and assume that the gap is closed. We follow the procedure outlined in Section 4.3.4 to solve the problem.

SOLUTION

Step 1 Assume the gap is closed.

Step 2 We can substitute (E1) into (E1) to eliminate R_L and obtain the equilibrium equation,

$$N_{st} + N_{al} = 20 \times 10^3\text{ N} \tag{E13}$$

We could also obtain this equation from the free-body diagram shown in Figure 4.30.

Step 3 We draw the exaggerated deformed shape, as shown in Figure 4.29, and obtain the deformation of the bars in terms of the plate displacement u_P as

$$\delta_{al} = u_P \tag{E14}$$

$$\delta_{st} = u_P - 0.0005\text{ m} \tag{E15}$$

Step 4 We can write the internal forces in terms of deformation,

$$N_{al} = \delta_{al}\left(\frac{E_{al}A_{al}}{L_{al}}\right) = 21.99(10^6)\delta_{al}\text{ N} \tag{E16}$$

$$N_{st} = \delta_{st}\left(\frac{E_{st}A_{st}}{L_{st}}\right) = 13.74(10^6)\delta_{st}\ \text{N} \qquad \text{(E17)}$$

Step 5 We can substitute Equations (E14) and (E15) into Equations (E16) and (E17) to obtain the internal forces in terms of u_P.

$$N_{al} = 21.99\,(10^6)\,u_P\,\text{N} \qquad \text{(E18)}$$

$$N_{st} = 13.74\,(u_P - 0.0005)(10^6)\ \text{N} \qquad \text{(E19)}$$

Substituting Equations (E18) and (E19) into (E13) we obtain the displacement u_P.

$$21.99u_P + 13.74(u_P - 0.0005) = 20(10^{-3}) \qquad \text{or} \qquad u_P = 0.752(10^{-3})\ \text{m} \qquad \text{(E20)}$$

ANS. $u_P = 0.752$ mm

Step 6 As $u_P > 0.0005$ m, the assumption of gap closing is correct.

Substituting u_P into Equation (E19), we obtain $N_{st} = 3423.2$ N, which implies that the steel is in compression, as expected. We can now find the axial stress in steel, as before.

COMMENT

1. In the force method as well as in the displacement method the number of unknowns was 1 as the degree of redundancy and the number of degrees of freedom were 1. This is not always the case. In the next example the number of degrees of freedom is less than the degree of redundancy, and hence the displacement method will be easier to implement.

EXAMPLE 4.11

Three steel bars A, B, and C ($E = 200$ GPa) have lengths $L_A = 4$ m, $L_B = 3$ m, and $L_C = 2$ m, as shown in Figure 4.31. All bars have the same cross-sectional area of 500 mm^2. Determine (a) the elongation in bar B; (b) the normal stress in bar C.

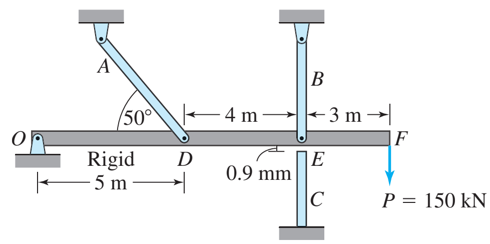


Figure 4.31 Geometry in Example 4.11.

DISPLACEMENT METHOD: PLAN

Assume that the gap is closed. We follow the procedure outlined in Section 4.3.4 to solve the problem.

SOLUTION

Step 1 We assume that the force P is sufficient to close the gap.

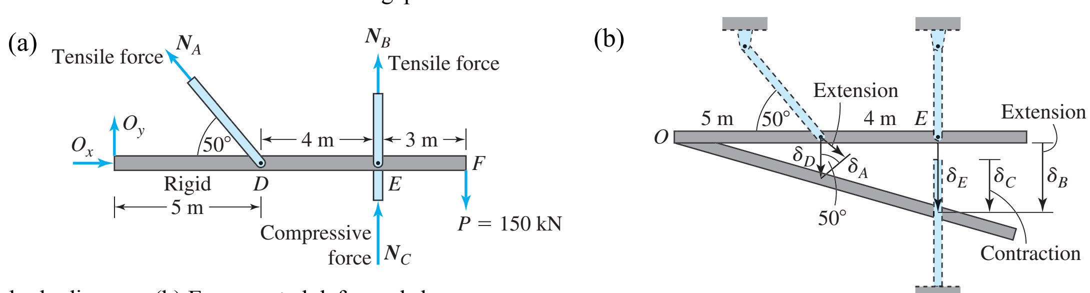


Figure 4.32 (a) Free-body diagram. (b) Exaggerated deformed shape.

Step 2 We draw the free-body diagram of the rigid bar in Figure 4.32*a* with bars A and B in tension and bar C in compression. By equilibrium of moment at point O we obtain the equilibrium equation shown in Equation (E1).

$$N_A \sin 50(5) + N_B(9) + N_C(9) - P(12) = 0 \qquad \text{or} \qquad 3.83N_A + 9N_B + 9N_C = 1800(10^3) \qquad \text{(E1)}$$

Step 3 Figure 4.32*b* shows an exaggerated deformed shape with bars A and B as extension and bar C as contraction to be consistent with the forces drawn in Figure 4.32*a*. Noting that the gap is 0.0009 m, we can write the compatibility equations relating the deformations of bars B and C in terms of the displacement of pin E,

$$\delta_B = \delta_E \qquad \text{(E2)}$$

$$\delta_C = \delta_E - 0.0009\ \text{m} \qquad \text{(E3)}$$

Using similar triangles in Figure 4.32*b* we relate the displacements of point D and E,

$$\frac{\delta_D}{5\text{ m}} = \frac{\delta_E}{9\text{ m}} \tag{E4}$$

Using small-strain approximation we can relate the deformation of bar *A* to the displacement of point *D*,

$$\delta_D = \frac{\delta_A}{\sin 50} \tag{E5}$$

Substituting Equation (E5) into Equation (E4), we obtain

$$\frac{(\delta_A/\sin 50)}{5\text{ m}} = \frac{\delta_E}{9\text{ m}} \qquad \text{or} \qquad \delta_A = 0.4256\delta_E \tag{E6}$$

Step 4 The axial rigidity of all bars is $EA = [200\,(10^9)\text{ N/m}^2]\,[500\,(10^{-6})\text{ m}^2] = 100 \times 10^6\text{ N}$. Using Equation (4.21) we can write,

$$N_A = \frac{100(10^6)}{4}\delta_A\text{ N} = 25(10^6)\delta_A\text{ N} \tag{E7}$$

$$N_B = \frac{100(10^6)}{3}\delta_B = 33.33(10^6)\delta_B\text{ N} \tag{E8}$$

$$N_C = \frac{100(10^6)}{2}\delta_C = 50.00(10^6)\delta_C\text{ N} \tag{E9}$$

Step 5 Substituting Equations (E6), (E2), and (E3) into Equations (E7), (E8), and (E9), we obtain

$$N_A = 25(10)^6(0.4256\delta_E) = 10.64(10^6)\delta_E\text{ N} \tag{E10}$$

$$N_B = 33.33(10^6)\delta_E = 33.33(10^6)\delta_E\text{ N} \tag{E11}$$

$$N_C = 50.00(10^6)(\delta_E - 0.0009) = [50.00(10^6)\delta_E - 45(10^3)]\text{ N} \tag{E12}$$

Substituting Equations (E10), (E11), and (E12) into Equation (E1) we obtain the displacement of pin *E*,

$$3.83(10.64)(10^6)\delta_E + 9(33.33)(10^6)\delta_E + 9[50.00(10^6)\delta_E - 45(10^3)] = 1800(10^3) \qquad \text{or} \qquad \delta_E = 2.788(10^{-3})\text{ m}$$

ANS. $\delta_E = 2.8$ mm

Step 6 The assumption of gap closure is correct as $\delta_E = 2.8$ mm whereas the gap is only 0.9 mm.
From Equation (12) we obtain the internal axial force in bar *C*, from which we obtain the axial stress in bar *C*,

$$N_C = 50.00(10^6)[2.788(10^{-3})] - 45(10^3) = 94.4(10^3)$$

$$\sigma_C = \frac{N_C}{A_C} = \frac{94.4(10^3)\text{ N}}{500(10^{-6})\text{ m}^2} = 188.8 \times 10^6\ N/\text{m}^2 \tag{E13}$$

ANS. $\sigma_C = 189$ MPa (C)

COMMENTS

1. Equation (E4) is a relationship of points on the rigid bar. Equations (E2), (E3), and (E5) relate the motion of points on the rigid bar to the deformation of the rods. This two-step process helps break the complexity into simpler steps.
2. The degree of freedom for this system is 1. In place of δ_E as an unknown, we could have used the displacement of any point on the rigid bar or the rotation angle of the bar, as all of these quantities are related.

FORCE METHOD: PLAN

We assume that the force *P* is sufficient to close the gap. If this assumption is correct, then bar *C* will be in compression. We follow the procedure outlined in Section 4.3.4 to solve the problem.

SOLUTION

Step 1 Assume that the gap closes.
Step 2 Figure 4.32*a* shows the free-body diagram of the rigid bar. By equilibrium we obtain Equation (E1), rewritten here for convenience.

$$3.83N_A + 9N_B + 9N_C = 1800(10^3)\text{ N} \tag{E14}$$

Equation (14) has three unknowns, hence the degree of redundancy is 2. We will need two compatibility equations.
Step 3 We draw the deformed shape, as shown in Figure 4.32*b*, and obtain relationships between points on the rigid bar and the deformation of the bars. Then by eliminating δ_E from Equations (E3) and (E6) and using Equation (E2) we obtain the compatibility equations,

$$\delta_C = \delta_B - 0.0009\text{ m} \tag{E15}$$

$$\delta_A = 0.4256\delta_B \tag{E16}$$

Step 4 We write the deformations of the bars in terms of the internal forces for each member,

$$\delta_A = \frac{N_A L_A}{E_A A_A} = \frac{N_A(4\text{ m})}{100(10^6)\text{ N}} = 0.04\boldsymbol{N}_A(10^{-6})\text{ m} \tag{E17}$$

$$\delta_B = \frac{N_B L_B}{E_B A_B} = \frac{N_B(3\text{ m})}{100(10^6)\text{ N}} = 0.03\boldsymbol{N}_B(10^{-6})\text{ m} \tag{E18}$$

$$\delta_C = \frac{N_C L_C}{E_C A_C} = \frac{N_C(2\text{ m})}{100(10^6)\text{ N}} = 0.02\boldsymbol{N}_C(10^{-6})\text{ m} \tag{E19}$$

Step 5 Substituting Equations (E17), (E18), and (E19) into Equations (E15) and (E16), we obtain

$$0.02\ N_C = 0.03\ N_B - 900\text{ N} \qquad \text{or} \qquad N_C = 1.5N_B - 45(10^3)\text{ N} \tag{E20}$$

$$0.04\ N_A = 0.4256(0.03\ N_B) \qquad \text{or} \qquad N_A = 0.3192\ N_B \tag{E21}$$

Solving Equations (E14), (E20), and (E21), we obtain the internal forces:

$$N_A = 29.67(10^3)\text{ N} \qquad N_B = 92.95(10^3)\text{ N} \qquad N_C = 94.43(10^3)\text{ N} \tag{E22}$$

Step 6 The positive value of $\boldsymbol{N}_C$ confirms it is compressive and the assumption of the gap being closed is valid.

Substituting $\boldsymbol{N}_B$ from Equation (E22) into Equation (E18), we find the deformation of bar *B*, which is the same as the displacement of pin *E*,

$$\delta_E = \delta_B = 0.03[92.95(10^3)](10^{-6}) = 2.788(10^3)\text{ m} \tag{E23}$$

ANS. $\delta_E = 2.8\text{ mm}$

The calculation for the normal stress in bar *C* is as before.

COMMENT

1. Equations (E14), (E20), and (E21) represent three equations in three unknowns. Had we used the reaction forces O_x and O_y in Figure 4.32 as the unknowns, we could have generated two equations in two unknowns, consistent with the fact that this system has a degree of redundancy of 2. But as we saw, the displacement method required only one unknown. Thus our choice of method of solution should be dictated by the number of degrees of freedom and the number of degrees of static redundancy. For fewer degrees of freedom we should use the displacement method; for fewer degrees of static redundancy we should use the force method.

Consolidate your knowledge

1. With the book closed, write a procedure for solving a statically indeterminate problem by the force method.
2. With the book closed, write a procedure for solving a statically indeterminate problem by the displacement method.

PROBLEM SET 4.3

4.45 A rigid bar is hinged at *C* as shown in Figure P4.45. The modulus of elasticity of bar *A* is *E* = 30,000 ksi, the cross-sectional area is *A* = 1.25 in.², and the length is 24 in. Determine the applied force *F* if point *B* moves upward by 0.002 in.

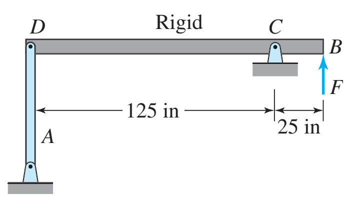


Figure P4.45

4.46 A rigid bar is hinged at *C* as shown in Figure P4.46. The modulus of elasticity of bar *A* is *E* = 30,000 ksi, the cross-sectional area is *A* = 1.25 in.², and the length is 24 in. Determine the applied force *F* if point *B* moves upward by 0.002 in.

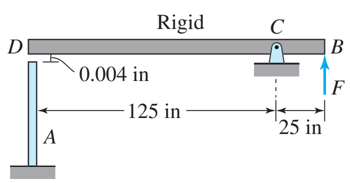


Figure P4.46

4.47 A rigid bar is hinged at C as shown in Figure P4.47. The modulus of elasticity of bar A is $E = 100$ GPa, the cross-sectional area is $A = 15$ mm^2, and the length is 1.2 m. Determine the applied force F if point B moves to the left by 0.75 mm.

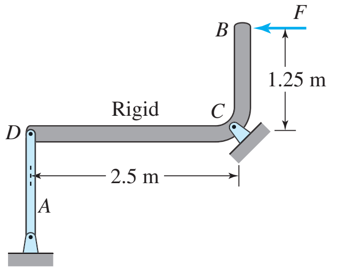


Figure P4.47

4.48 A rigid bar is hinged at C as shown in Figure P4.48. The modulus of elasticity of bar A is $E = 100$ GPa, the cross-sectional area is $A = 15$ mm^2, and the length is 1.2 m. Determine the applied force F if point B moves to the left by 0.75 mm.

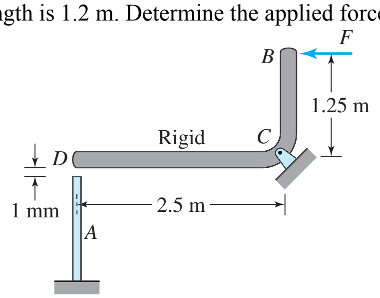


Figure P4.48

4.49 The roller at P in Figure P4.49 slides in the slot due to the force $F = 20$ kN. Member AP has a cross-sectional area $A = 100$ mm^2 and a modulus of elasticity $E = 200$ GPa. Determine the displacement of the roller.

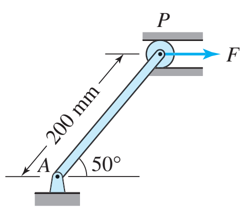


Figure P4.49

4.50 The roller at P in Figure P4.50 slides in the slot due to the force $F = 20$ kN. Member AP has a cross-sectional area $A = 100$ mm^2 and a modulus of elasticity $E = 200$ GPa. Determine the displacement of the roller.

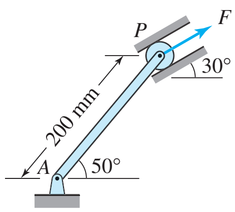
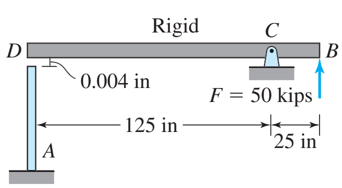


Figure P4.50

4.51 A rigid bar is hinged at C as shown in Figure P4.51. The modulus of elasticity of bar A is $E = 30{,}000$ ksi, the cross-sectional area is $A = 1.25$ in.^{2}, and the length is 24 in. Determine the axial stress in bar A and the displacement of point D on the rigid bar.



FigureP4.51

4.52 A rigid bar is hinged at C as shown in Figure P4.52. The modulus of elasticity of bar A is $E = 30{,}000$ ksi, the cross-sectional area is $A = 1.25$ in.^{2}, and the length is 24 in. Determine the axial stress in bar A and the displacement of point D on the rigid bar.

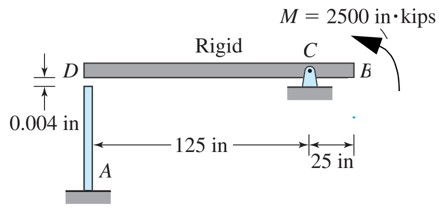
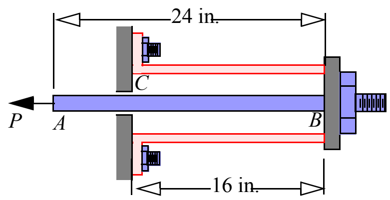


Figure P4.52

4.53 A steel (E = 30,000 ksi, ν = 0.28) rod passes through a copper (E = 15,000 ksi, ν = 0.35) tube as shown in Figure P4.53. The steel rod has a diameter of 1/2 in., and the tube has an inside diameter of 3/4 in. and a thickness of 1/8 in. If the applied load is P = 2.5 kips, determine (a) the movement of point A (b) the change in diameter of the steel rod.



FigureP4.53

4.54 A rigid bar ABC is supported by two aluminum cables (E = 10,000 ksi) with a diameter of 1/2 in. as shown in Figure P4.54. The bar is horizontal before the force is applied. Determine the angle of rotation of the bar from the horizontal when a force P= 5 kips is applied.

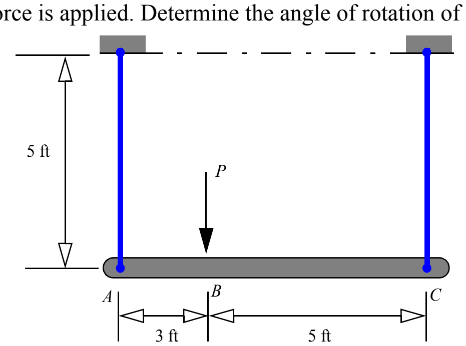


Figure P4.54

4.55 Two rigid beams are supported by four axial steel (E = 210 GPa) rods of diameter 10 mm, as shown in Figure P4.55. Determine the angle of rotation of the bars from the horizontal no load position when a force of P = 5 kN is applied.

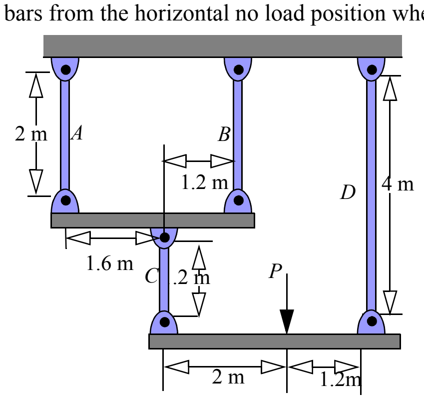


Figure P4.55

4.56 Two rigid beams are supported by four axial steel rods (E = 210 GPa, σ_{yield} = 210 MPa) of diameter 20 mm, as shown in Figure P4.55. For a factor of safety of 1.5, determine the maximum value of force F that can be applied without causing any rod to yield.

4.57 A rigid bar *ABC* is supported by two aluminum cables (E = 10,000 ksi) with a diameter of 1/2 in., as shown in Figure P4.57. Determine the extensions of cables *CE* and *BD* when a force *P*= 5 kips is applied.

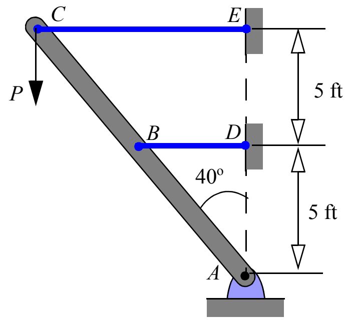


Figure P4.57

4.58 A rigid bar *ABC* is supported by two aluminum cables (E = 10,000 ksi) as shown in Figure P4.57. The yield stress of aluminum is 40 ksi. If the applied force *P* = 10 kips, determine the minimum diameter of cables *CE* and *BD* to the nearest 1/16 in.

4.59 A rigid bar *ABC* is supported by two aluminum cables (E = 10,000 ksi) with a diameter of 1/2 in. as shown in Figure P4.57. The yield stress of aluminum is 40 ksi. Determine the maximum force *P* to the nearest pound that can be applied.

4.60 A force $F = 20$ kN is applied to the roller that slides inside a slot as shown in Figure P4.60. Both bars have a cross-sectional area $A = 100\ \text{mm}^2$ and a modulus of elasticity $E = 200$ GPa. Bars *AP* and *BP* have lengths $L_{AP} = 200$ mm and $L_{BP} = 250$ mm. Determine the displacement of the roller and the axial stress in bar *AP*.

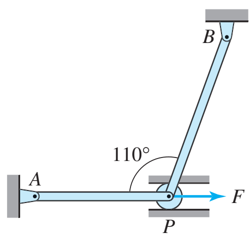


Figure P4.60

4.61 A force $F = 20$ kN is applied to the roller that slides inside a slot as shown in Figure P4.61. Both bars have a cross-sectional area $A = 100\ \text{mm}^2$ and a modulus of elasticity $E = 200$ GPa. Bars *AP* and *BP* have lengths $L_{AP} = 200$ mm and $L_{BP} = 250$ mm. Determine the displacement of the roller and the axial stress in bar *AP*.

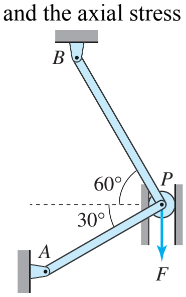


Figure P4.61

4.62 A force $F = 20$ kN is applied to the roller that slides inside a slot as shown in Figure P4.62. Both bars have a cross-sectional area $A = 100\ \text{mm}^2$ and a modulus of elasticity $E = 200$ GPa. Bars *AP* and *BP* have lengths $L_{AP} = 200$ mm and $L_{BP} = 250$ mm. Determine the displacement of the roller and the axial stress in bar *AP*.

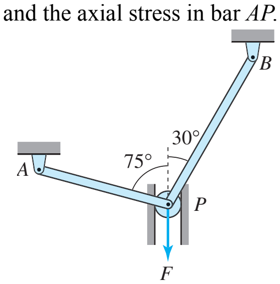


Figure P4.62

4.63 An aluminum (E = 70 GPa, σ_{yield} = 280 MPa, ν = 0.28) wire of diameter 0.5 mm is to hang two flower pots of equal mass as shown in Figure P4.63. (a) Determine the maximum mass of the pots to the nearest gram that can be hung if yielding is to be avoided in all wires. (b) For the maximum mass what is the percentage change in the diameter of the wire *BC*.

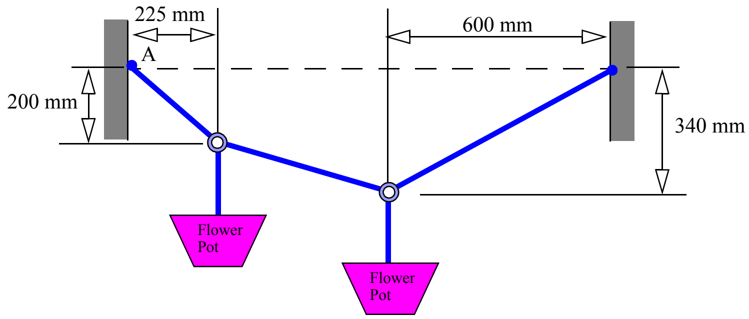
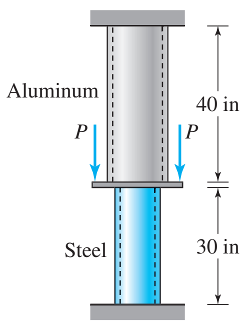


Figure P4.63

4.64 An aluminum (E = 70 GPa, σ_{yield} = 280 MPa, ν = 0.28) wire is to hang two flower pots of equal mass of 5 kg as shown in Figure P4.63. Determine the minimum diameter of the wires to the nearest 1/10 of a millimeter if yielding is to be avoided in all wires.

4.65 An aluminum hollow cylinder (E_{al} = 10,000 ksi, ν_{al} = 0.25) and a steel hollow cylinder (E_{st} = 30,000 ksi, ν_{st} = 0.28) are securely fastened to a rigid plate, as shown in Figure P4.65. Both cylinders are made from $\frac{1}{8}$-in. thickness sheet metal. The outer diameters of the aluminum and steel cylinders are 4 in. and 3 in., respectively. For an applied load of *P* = 20 kips determine (a) the displacement of the rigid plate; (b) the change in diameter of each cylinder.



FigureP4.65

4.66 An aluminum hollow cylinder (E_{al} = 10,000 ksi, ν_{al} = 0.25) and a steel hollow cylinder (E_{st} = 30,000 ksi, ν_{st} = 0.28) are securely fastened to a rigid plate, as shown in Figure P4.65. Both cylinders are made from $\frac{1}{8}$-in. thickness sheet metal. The outer diameters of the aluminum and steel cylinders are 4 in. and 3 in., respectively. The allowable stresses in aluminum and steel are 10 ksi and 25 ksi, respectively. Determine the maximum force *P* that can be applied to the assembly.

4.67 A gap of 0.004 inch exists between the rigid bar and bar *A* before the force *F* is applied as shown in Figure P4.67. The rigid bar is hinged at point *C*. The lengths of bars *A* and *B* are 30 and 50 inches respectively. Both bars have an area of cross-section *A* = 1 in.^{2} and modulus of elasticity *E* = 30,000 ksi. Determine the axial stresses in bars *A* and *B* if *P* = 100 kips.

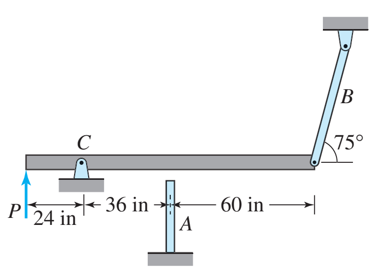


Figure P4.67

4.68 A gap of 0.004 inch exists between the rigid bar and bar *A* before the force *F* is applied as shown in Figure P4.67. The rigid bar is hinged at point *C*. The lengths of bars *A* and *B* are 30 in. and 50 in. respectively. Both bars have an area of cross-section *A* = 1 in.^{2} and modulus of elasticity *E* = 30,000 ksi. If the allowable normal stress in the bars is 20 ksi in tension or compression, determine the maximum force *P* that can be applied.

4.69 In Figure P4.69 a gap exists between the rigid bar and rod A before force F is applied. The rigid bar is hinged at point C. The lengths of bars A and B are 1 m and 1.5 m, and the diameters are 50 mm and 30 mm, respectively. The bars are made of steel with a modulus of elasticity E = 200 GPa and Poisson's ratio ν = 0.28. If F = 75 kN determine (a) the deformation of the two bars; (b) the change in the diameters of the two bars.

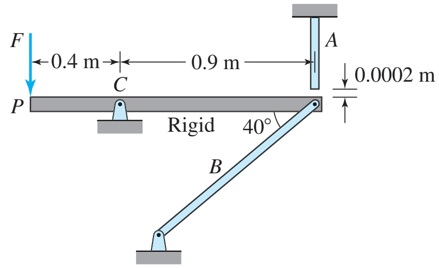


Figure P4.69

4.70 In Figure P4.69 a gap exists between the rigid bar and rod A before force F is applied. The rigid bar is hinged at point C. The lengths of bars A and B are 1 m and 1.5 m, and the diameters are 50 mm and 30 mm, respectively. The bars are made of steel with a modulus of elasticity E = 200 GPa and Poisson's ratio ν = 0.28. If the allowable axial stresses in bars A and B are 110 MPa and 125 MPa, respectively, determine the maximum force F that can be applied.

4.71 A rectangular aluminum bar (E = 10,000 ksi), a steel bar (E = 30,000 ksi), and a brass bar (E = 15,000 ksi) are assembled as shown in Figure P4.71. All bars have the same thickness of 0.5 in. A gap of 0.02 in. exists before the load P is applied to the rigid plate. Assume that the rigid plate does not rotate. If P = 15 kips determine (a) the axial stress in steel; (b) the displacement of the rigid plate with respect to the right wall.

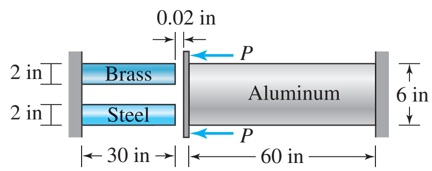


Figure P4.71

4.72 A rectangular aluminum bar (E = 10,000 ksi), a steel bar (E = 30,000 ksi), and a brass bar (E = 15,000 ksi) are assembled as shown in Figure P4.71. All bars have the same thickness of 0.5 in. A gap of 0.02 in. exists before the load P is applied to the rigid plate. Assume that the rigid plate does not rotate. If the allowable axial stresses in brass, steel, and aluminum are 8 ksi, 15 ksi, and 10 ksi, respectively, determine the maximum load P.

4.73 In Figure P4.73 bars A and B have cross-sectional areas of 400 mm^2 and a modulus of elasticity E = 200 GPa. A gap exists between bar A and the rigid bar before the force F is applied. If the applied force F = 10 kN determine: (a) the axial stress in bar B; (b) the deformation of bar A.

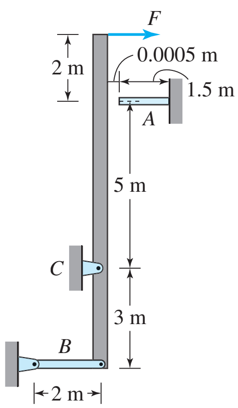


Figure P4.73

4.74 In Figure P4.73 bars A and B have cross-sectional areas of 400 mm^2 and a modulus of elasticity E = 200 GPa. A gap exists between bar A and the rigid bar before the force F is applied. Determine the maximum force F that can be applied if the allowable stress in member B is 120 MPa (C) and the allowable deformation of bar A is 0.25 mm.

4.75 A rectangular steel bar (E = 30,000 ksi, ν = 0.25) of 0.5 in. thickness has a gap of 0.01 in. between the section at D and a rigid wall before the forces are applied as shown in Figure P4.75. Assuming that the applied forces are sufficient to close the gap, determine (a) the movement of rigid plate at C with respect to the left wall; (b) the change in the depth d of segment CD.

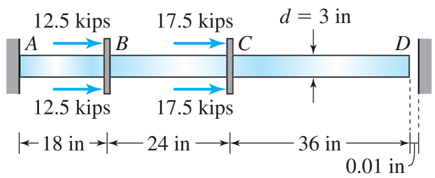


Figure P4.75

4.76 Three plastic members of equal cross sections are shown in Figure P4.76. Member B is smaller than members A by 0.5 mm. A distributed force is applied to the rigid plate, which moves downward without rotating. The moduli of elasticity for members A and B are 1.5 GPa and 2.0 GPa, respectively. Determine the axial stress in each member if the distributed force $W = 20$ MPa.

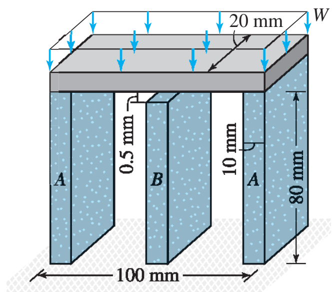


Figure P4.76

4.77 Three plastic members of equal cross sections are shown in Figure P4.76. Member B is smaller than members A by 0.5 mm. A distributed force is applied to the rigid plate, which moves downward without rotating. The moduli of elasticity for members A and B are 1.5 GPa and 2.0 GPa, respectively. Determine the maximum intensity of the distributed force that can be applied to the rigid plate if the allowable stresses in members A and B are 50 MPa and 30 MPa.

4.78 Figure P4.78 shows an aluminum rod (E = 70 GPa, ν = 0.25) inside a steel tube (E = 210 GPa, ν = 0.28). The aluminium rod is slightly longer than the steel tube and has a diameter of 40 mm. The steel tube has an inside diameter of 50 mm and is 10 mm thick. If the applied load $P =$ 200 kN, determine (a) the axial stresses in aluminium rod and steel tube; (b) the change in diameter of aluminium.

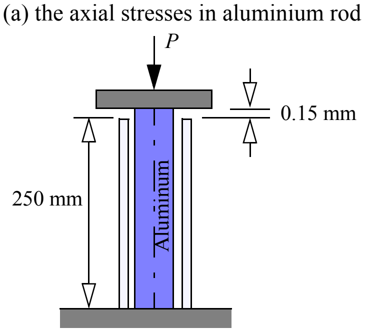


Figure P4.78

4.79 Figure P4.78 shows an aluminium rod (E = 70 GPa, σ_{yield} = 280 MPa) inside a steel tube (E = 210 GPa, σ_{yield} = 210 MPa). The aluminium rod is slightly longer than the steel tube and has a diameter of 40 mm. The steel tube has an inside diameter of 50 mm and is 10 mm thick. What is the maximum force P that can be applied without yielding either material.

4.80 A rigid bar $ABCD$ hinged at one end and is supported by two aluminum cables (E = 10,000 ksi) with a diameter of 1/4 in. as shown in Figure P4.80. The bar is horizontal before the force is applied. Determine the angle of rotation of the bar from the horizontal when a force P= 10kips is applied.

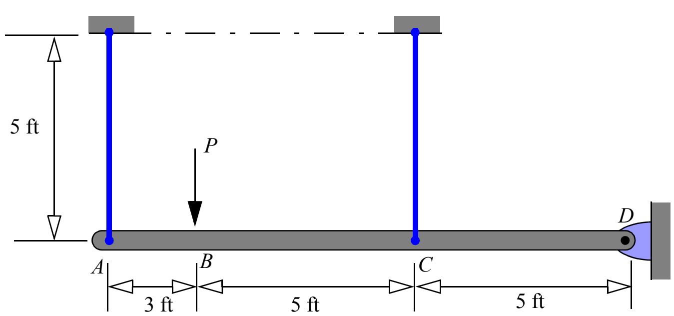


Figure P4.80

4.81 A suspended walkway is modelled as a rigid bar and supported by steel rods (E =30,000 ksi) as shown in Figure P4.81. The rods have a diameter of 2 in., and the nut has a contact area with the bottom of the walkway is 4 in.^{2} The weight of the walk per unit length is w = 725lb/ft. Determine (a) the axial stress in the steel rods; (b) the average bearing stress between the nuts at A and D and the walkway.

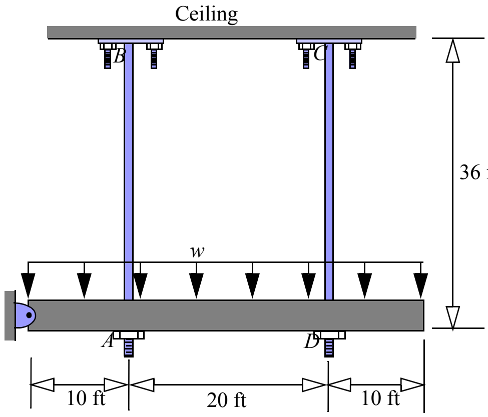


Figure P4.81

4.82 An aluminum circular bar (E_{al} = 70 GPa) and a steel tapered circular bar (E_{st} = 200 GPa) are securely attached to a rigid plate on which axial forces are applied, as shown in Figure P4.82. Determine (a) the displacement of the rigid plate; (b) the maximum axial stress in steel

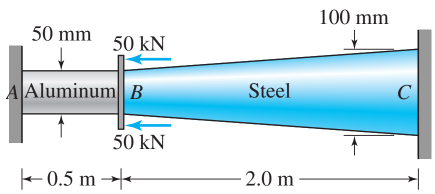


Figure P4.82

Design problems

4.83 A rigid bar hinged at point O has a force P applied to it, as shown in Figure P4.83. Bars A and B are made of steel (E = 30,000 ksi). The cross-sectional areas of bars A and B are A_A = 1 in.^{2} and A_B = 2 in.^{2}. If the allowable deflection at point C is 0.01 in. and the allowable stress in the bars is 25 ksi, determine the maximum force P that can be applied

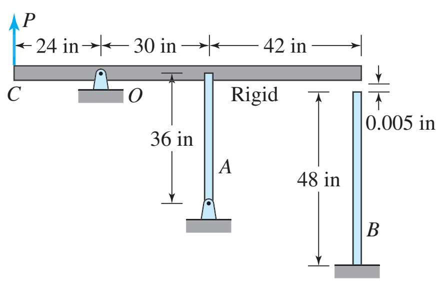


Figure P4.83

4.84 The structure at the base of a crane is modeled by the pin-connected structure shown in Figure P4.84. The allowable axial stresses in members AC and BC are 15 ksi, and the modulus of elasticity is 30,000 ksi. To ensure adequate stiffness at the base, the displacement of pin C in the vertical direction is to be limited to 0.1 in. Determine the minimum cross-sectional areas for members AC and BC.

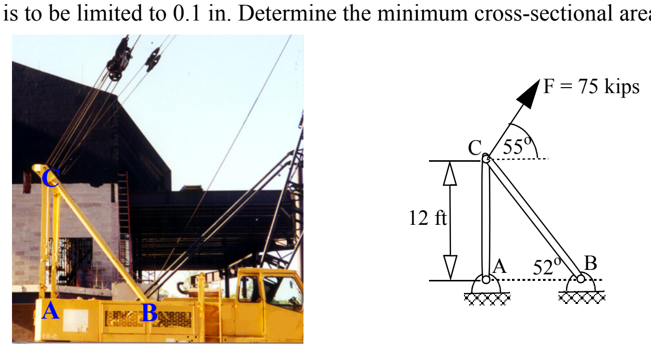


Figure P4.84

4.85 The landing wheel of a plane is modeled as shown in Figure P4.85. The pin at C is in double shear and has an allowable shear stress of 12 ksi. The allowable axial stress for link BC is 30 ksi. Determine the diameter of pin C and the effective cross-sectional area of link BC. (*Note:*

Attachments at *A* and *B* are approximated by pins to simplify analysis. There are two links represented by *BC*, one on either side of the hydraulic cylinder, which we are modeling as a single link with an effective cross-sectional area that is to be determined so that the free-body diagram is two-dimensional.).

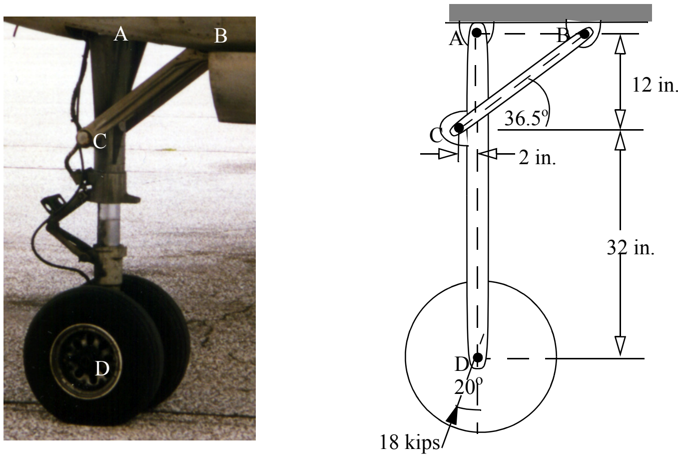


Figure P4.85

Stress concentration

4.86 The allowable shear stress in the stepped axial rod shown in Figure P4.86 is 20 ksi. If $F = 10$ kips, determine the smallest fillet radius that can be used at section *B*. Use the stress concentration graphs given in Section C.4.2.

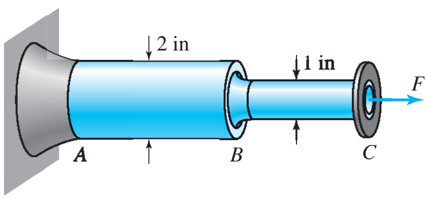


Figure P4.86

4.87 The fillet radius in the stepped circular rod shown in Figure P4.87 is 6 mm. Determine the maximum axial force *F* that can act on the rigid wheel if the allowable axial stress is 120 MPa and the modulus of elasticity is 70 GPa. Use the stress concentration graphs given in Section C.4.2.

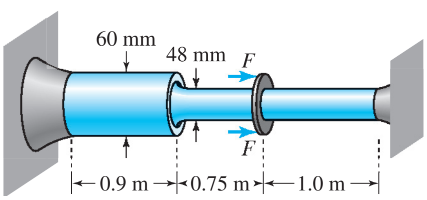


Figure P4.87

Fatigue

4.88 The fillet radius is 0.2 mm in the stepped steel circular rod shown in Figure P4.86. What should be the peak value of the cyclic load *F* to ensure a service life of one-half million cycles? Use the *S–N* curve shown in Figure 3.36

4.89 The aluminum axial rod in Figure P4.87 is subjected to a cyclic load *F*. Determine the peak value of *F* to ensure a service life of one million cycles. Use the *S–N* curves shown in Figure 3.36 and modulus of elasticity $E = 70$ GPa.

MoM in Action: Kansas City Walkway Disaster

On July 17, 1981, nearly 2000 people had gathered to watch a dance competition in the atrium of the Hyatt Regency Hotel in Kansas City, Missouri. At 7:05 P.M. a loud, sharp sound was heard throughout the building. Within minutes, the second- and fourth floor walkways crashed to the ground, killing 114 people and injuring over 200 others. The worst structural failure in the history of United States had taken place. It is a tragic story of multiple design failures – and of failure in professional ethics as well.

Three suspended walkways spanned the hotel atrium, a large open area of approximately 117 ft by 145 ft and 50 ft high. The fourth- floor walkway was directly above the second-floor walkway, while the third-floor walkway (Figure 4.33*a*) was offset 15 ft from the plane of the other two. Each walkway was 120 ft long and 8.6 ft wide, with four 30-ft intervals between support steel rods of diameter 1.25 in. A square box beam was constructed by welding two channel beams, and a hole was drilled through for the supporting axial rods, as shown in Figure 4.33*b*. In the original design (Figure 4.33b), a single continuous steel axial rod passed through the second- and fourth-floor walkways and was attached to the ceiling truss. This design required that the axial rod between the walkways be threaded so that the nuts underneath the box beams could be installed. As designed, the fourth-floor connection had to support only loads from the fourth-floor walkway, while the ceiling truss would support the total load of the second- and fourth-floor walkways together.

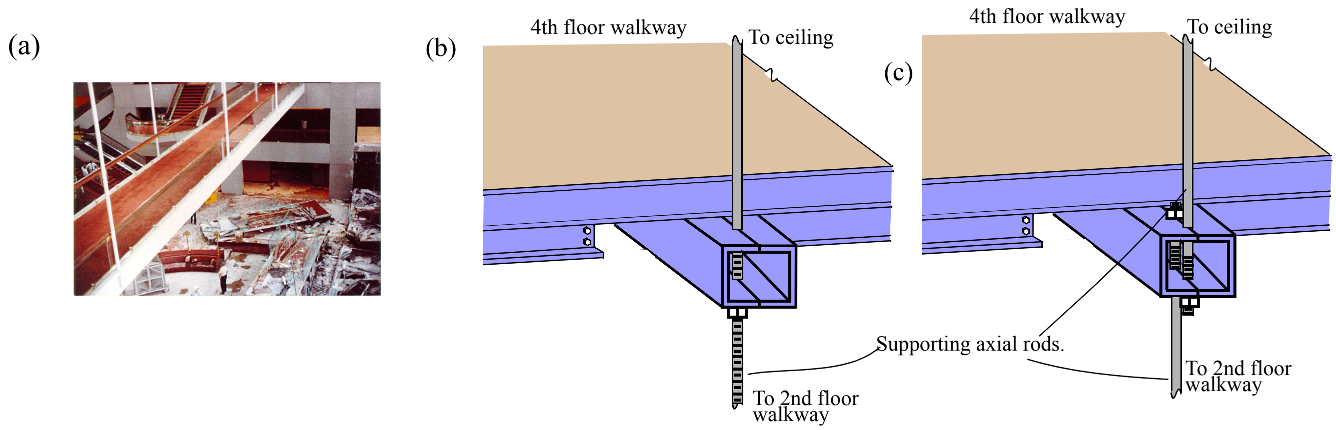


Figure 4.33 Kansas City Hyatt Regency walkways. (a) 3rd floor walkway (Courtesy Dr. Lee Lowery Jr.)(b) Original design connection (c) Fabricated design connection.

The first failure in design was that the box-beam connection could support only 60% of the load specified by the Kansas City building code. Haven's Steel Co. did not want to thread the length of the axial rod between second and fourth floor, so the design was changed to that shown in Figure 4.33*c* in which only the ends of the axial rod were threaded. The change was approved over the phone, with no recalculation of the new design. This transferred the load of the second floor to the box beams of the fourth floor, which thus supported the loads of both walkways at once. This revised design could support only 30% of the load specified by code. Further compounding the design failure, the axial rod passed through the weld, the weakest structural point in the box beam. Close inspection of the connections would have shown the overstressing of the box beam, as was in fact observed in the third-floor walkway (which did not collapse). This inspection was not done.

On that fatal night people stood on the walkways watching the dance competition, with still more spectators on the ground floor. The loud noise preceding the crash was the nut punching though the box beam of the fourth-floor walkway. First one walkway crashed into the other beneath it, and then the two fell to the ground upon the people below.

Engineers Duncan and Gillium were charged with gross negligence, misconduct, and unprofessional conduct, and both their license and their firm's license to practice were revoked in the states of Missouri and Kansas. The tragedy has become a model for the study of engineering design errors and ethics.

4.4* INITIAL STRESS OR STRAIN

Members in a statically indeterminate structure may have an initial stress or strain before the loads are applied. These initial stresses or strains may be intentional or unintentional and can be caused by several factors. A good design must account for these factors by calculating the acceptable levels of prestress.

Nuts on a bolt are usually finger-tightened to hold an assembly in place. At this stage the assembly is usually stress free. The nuts are then given additional turns to pretension the bolts. When a nut is tightened by one full rotation, the distance it moves is called the **pitch**. Alternatively, pitch is the distance between two adjoining peaks on the threads. One reason for pretensioning is to prevent the nuts from becoming loose and falling off. Another reason is to introduce an initial stress that will be opposite in sign to the stress that will be generated by the loads. For example, a cable in a bridge may be pretensioned by tightening the nut and bolt systems to counter the slackening in the cable that may be caused due to wind or seasonal temperature changes.

If during assembly a member is shorter than required, then it will be forced to stretch, thus putting the entire structure into a prestress. Tolerances for the manufacture of members must be prescribed to ensure that the structure is not excessively pre-stressed.

In prestressed concrete, metal bars are initially stretched by applying tensile forces, and then concrete is poured over these bars. After the concrete has set, the applied tensile forces are removed. The initial prestress in the bars is redistributed, putting the concrete in compression. Concrete has good compressive strength but poor tensile strength. After prestressing, the concrete can be used in situations where it may be subjected to tensile stresses.

EXAMPLE 4.12

Bars A and B in the assembly shown in Figure 4.34 are made of steel with a modulus of elasticity $E = 200$ GPa, a cross-sectional area $A = 100\ \text{mm}^2$, and a length $L = 2.5$ m. Bar A is pulled by 3 mm to fill the gap before the force F is applied. (a) Determine the initial axial stress in both bars. (b) If the applied force $F = 10$ kN, determine the total axial stress in both bars.

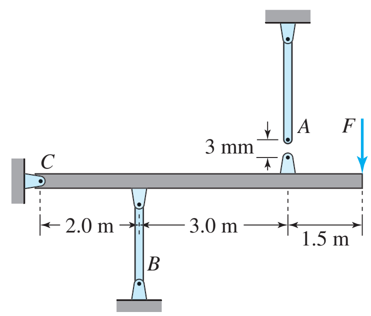


Figure 4.34 Two-bar mechanism in Example 4.12.

PLAN

(a) We can use the force method to solve the problem. After the gap has been closed, the two bars will be in tension. The degree of static redundancy for this problem is 1. We can write one compatibility equation and one equilibrium equation of the moment about C and solve the problem. (b) We can consider calculating the internal forces with just F, assuming the gap has closed and the system is stress free before F is applied. Bar B will be in compression and bar A will be in tension due to the force F. The internal forces in the bars can be found as in part (a). The initial stresses in part (a) can be superposed on the stresses due to solely F, to obtain the total axial stresses.

SOLUTION

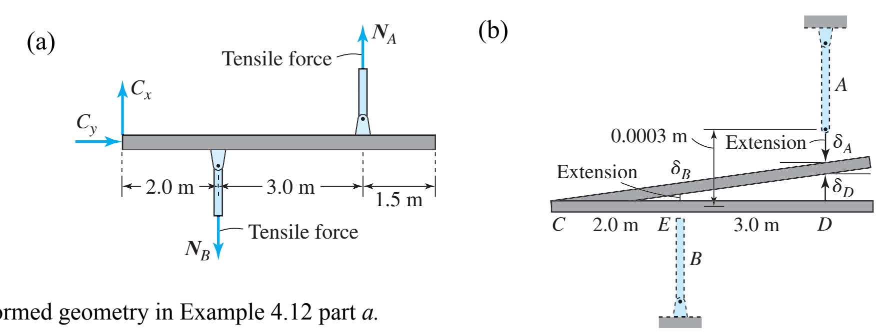


Figure 4.35 (a) Free-body diagram (b) Deformed geometry in Example 4.12 part *a*.

(a) We draw the free-body diagram of the rigid bars with both bars in tension as shown in Figure 4.35a. By moment equilibrium about point C we obtain Equation (E1).

$$N_A(5\text{ m}) = N_B(2\text{ m})\ 1234 \qquad \text{(E1)}$$

Figure 4.35*b* shows the approximate deformed shape. The movement of point *E* is equal to the deformation of bar *B*. The movements of points *E* and *D* on the rigid bar can be related by similar triangles to obtain:

$$\frac{\delta_D}{5\text{ m}} = \frac{\delta_B}{2\text{ m}} \tag{E2}$$

The sum of extension of bar *A* and the movement of point *D* are then equal to the gap:

$$\delta_D + \delta_A = 0.003\text{ m} \tag{E3}$$

From Equations (E2) and (E3) we obtain

$$2.5\delta_B + \delta_A = 0.003\text{ m} \tag{E4}$$

The deformation of bars *A* and *B* can be written as

$$\delta_A = \frac{N_A L_A}{E_A A_A} = \frac{N_A(2.5\text{ m})}{[200(10^9)\text{ N/m}^2][100(10^{-6})\text{ m}^2]} = 0.125N_A(10^{-6})\text{ m} \tag{E5}$$

$$\delta_B = \frac{N_B L_B}{E_B A_B} = \frac{N_B(2.5\text{ m})}{[200(10^9)\text{ N/m}^2][100(10^{-6})\text{ m}^2]} = 0.125N_B(10^{-6})\text{ m} \tag{E6}$$

Substituting Equations (E5) and (E6) into (E4) we obtain

$$2.5(0.125N_B(10^{-6})\text{ m}) + 0.125N_A(10^{-6})\text{ m} = 0.003\text{ m} \qquad \text{or} \qquad 2.5N_B + N_A = 24{,}000 \tag{E7}$$

Solving Equations (E1) and (E7), we obtain the internal forces,

$$N_A = 3310.3\text{ N} \qquad N_B = 8275.9\text{ N} \tag{E8}$$

The stresses in *A* and *B* can now be found:

$$\sigma_A = \frac{N_A}{A_A} = 33.1(10^6)\text{ N/m}^2 \qquad \sigma_B = \frac{N_B}{A_B} = 82.7(10^6)\text{ N/m}^2 \tag{E9}$$

ANS. $\sigma_A = 33.1$ MPa (T) $\sigma_B = 82.7$ MPa (T)

(b) In the calculations that follow, the purpose of the overbars is to distinguish the variables from those in part *a*. We draw the free-body diagram of the rigid bars in Figure 4.36a, with bars *A* in tension and bar *B* in compression, and by moment equilibrium about point *C* obtain

$$F(6.5\text{ m}) - \bar{N}_A(5\text{ m}) - \bar{N}_B(2\text{ m}) = 0 \qquad \text{or} \qquad 5\bar{N}_A + 2\bar{N}_B = 65{,}000\text{ N} \tag{E10}$$

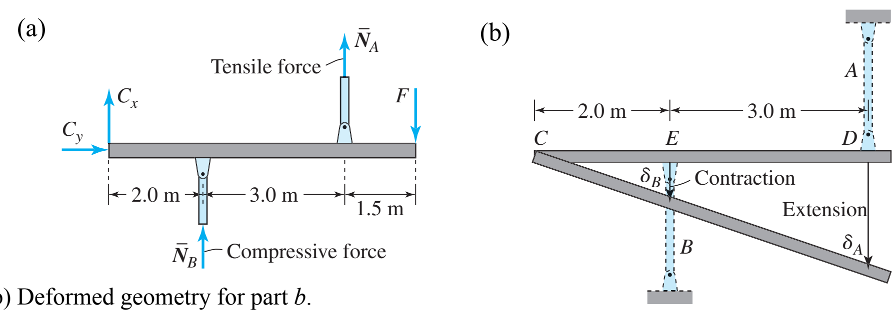


Figure 4.36 (a) Free-body diagram; (b) Deformed geometry for part *b*.

We draw the approximate deformed shape as shown in Figure 4.36b. For this part of the problem the movements of points *D* and *E* are equal to the deformation of the bar. By similar triangles we obtain

$$\frac{\bar{\delta}_A}{5\text{ m}} = \frac{\bar{\delta}_B}{2\text{ m}} \tag{E11}$$

The relation between deformation and internal forces is as before, as shown in Equations (E5) and (E6). Substituting Equations (E5) and (E6) into Equation (E11), we obtain

$$0.125\bar{N}_B(10^{-6}) = 0.4(0.125\bar{N}_A)(10^{-6}) \qquad \text{or} \qquad \bar{N}_B = 0.4\bar{N}_A \tag{E12}$$

Solving Equations (E10) and (E12), we obtain

$$\bar{N}_A = 11.20(10^3)\text{ N} \qquad \bar{N}_B = 4.48(10^3)\text{ N} \tag{E13}$$

The stresses in *A* and *B* are then

$$\bar{\sigma}_A = \frac{\bar{N}_A}{A_A} = 112\text{ MPa (T)} \qquad \bar{\sigma}_B = \frac{\bar{N}_B}{A_B} = 44.8\text{ MPa (C)} \tag{E14}$$

The total axial stress can now be obtained by superposing the stresses in Equations (E9) and (E14).

ANS. $(\sigma_A)_{\text{total}} = 145.1$ MPa (T) $(\sigma_B)_{\text{total}} = 37.9$ MPa (T)

COMMENTS

1. We solved the problem twice, to incorporate the initial stress (strain) due to misfit and then to account for the external load. Since the problem is linear, it should not matter how we reach the final equilibrium position. In the next section we will see that it is possible to solve the problem only once, but it would require an understanding of how initial strain is accounted for in the theory.
2. Consider a slightly different problem. In Figure 4.37, after the nut is finger-tight, it is given an additional quarter turn before the force F is applied. The pitch of the threads is 12 mm. We are required to find the initial axial stress in both bars and the total axial stress. The nut moves by pitch times the number of turns—that is, 3 mm. If we initially ignore the force F and bar B, then the movement of the nut forces the rigid bar to move by the same amount as the gap in Figure 4.34. The mechanisms of introducing the initial strains are different for the problems in Figures 4.34 and 4.37, but the results of the two problems will be identical at equilibrium. The strain due to the tightening of a nut may be hard to visualize, but the analogous problem of strain due to misfit can be visualized and used as an alternative visualization aid.

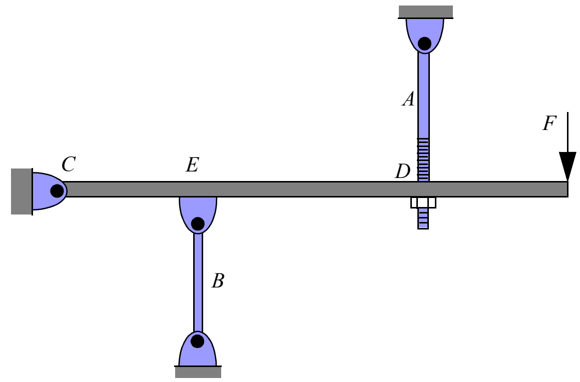


Figure 4.37 Problem similar to Example 4.12.

4.5* TEMPERATURE EFFECTS

Length changes due to temperature variations introduce stresses caused by the constraining effects of other members in a statically indeterminate structure. There are a number of similarities for the purpose of analysis between initial strain and thermal strain. Thus we shall rederive our theory to incorporate initial strain. We once more assume that plane sections remain plane and parallel and we have small strain; that is, Assumptions 1 and 2 are valid. Hence the total strain at any cross section is uniform and only a function of x, as in Equation (4.4). We further assume that the material is isotropic and linearly elastic—that is, Assumptions 3 and 4 are valid. We drop Assumption 5 to account for initial strain ε_0 at a point and write the stress–strain relationship as

$$\varepsilon_{xx} = \frac{du}{dx} = \frac{\sigma_{xx}}{E} + \varepsilon_0 \tag{4.22}$$

Substituting Equation (4.22) into Equation (4.1), assuming that the material is homogeneous and the initial strain ε_0 is uniform across the cross section, we have

$$\boldsymbol{N} = \int_A \left(E\frac{du}{dx} - E\varepsilon_0\right) dA = \frac{du}{dx}\int_A E\ dA - \int_A E\varepsilon_0\ dA = \frac{du}{dx}EA - EA\varepsilon_0 \quad \text{or} \tag{4.23}$$

$$\frac{du}{dx} = \frac{\boldsymbol{N}}{EA} + \varepsilon_0 \tag{4.24}$$

Substituting Equation (4.24) into Equation (4.22), we obtain a familiar relationship:

$$\sigma_{xx} = \frac{\boldsymbol{N}}{A} \tag{4.25}$$

If Assumptions 7 through 9 are valid, and if ε_0 does not change with x, then all quantities on the right-hand side of Equation (4.25) are constant between x_1 and x_2, and by integration we obtain

$$u_2 - u_1 = \frac{\boldsymbol{N}(x_2 - x_1)}{EA} + \varepsilon_0(x_2 - x_1) \tag{4.26}$$

or alternatively,

$$\boxed{\delta = \frac{NL}{EA} + \varepsilon_0 L} \tag{4.27}$$

Equations (4.25) and (4.27) imply that the initial strain affects the deformation but does not affect the stresses. This seemingly paradoxical result has different explanations for the thermal strains and for strains due to misfits or to pretensioning of the bolts.

First we consider the strain ε_0 due to temperature changes. If a body is homogeneous and unconstrained, then no stresses are generated due to temperature changes, as observed in Section 3.9. This observation is equally true for statically determinate structures. The determinate structure simply expands or adjusts to account for the temperature changes. But in an indeterminate structure, the deformation of various members must satisfy the compatibility equations. The compatibility constraints cause the internal forces to be generated, which in turn affects the stresses.

In thermal analysis $\varepsilon_0 = \alpha \Delta T$. An increase in temperature corresponds to extension, whereas a decrease in temperature corresponds to contraction. Equation (4.27) assumes that N is positive in tension, and hence extensions due to ε_0 are positive and contractions are negative. However, if on the free-body diagram N is shown as a compressive force, then δ is shown as contraction in the deformed shape. Consistency requires that contraction due to ε_0 be treated as positive and extension as negative in Equation (4.27). The sign of $\varepsilon_0 L$ due to temperature changes must be consistent with the force N shown on the free-body diagram.

We now consider the issue of initial strains caused by factors discussed in Section 4.4. If we start our analysis with the undeformed geometry even when there is an initial strain or stress, then the implication is that we have imposed a strain that is opposite in sign to the actual initial strain before imposing external loads. To elaborate this issue of sign, we put $\delta = 0$ in Equation (4.27) to correspond to the undeformed state. Also note that N and ε_0 must have opposite signs for the two terms on the right-hand side to combine, yielding a result of zero. But strain and internal forces must have the same sign. For example, if a member is short and has to be pulled to overcome a gap due misfit, then at the undeformed state the bar has been extended and is in tension before external loads are applied. The problem can be corrected only if we think of ε_0 as negative to the actual initial strain. Thus prestrains (stresses) can be analyzed by using ε_0 as negative to the actual initial strain in Equation (4.27).

If we have external forces in addition to the initial strain, then we can solve the problem in two ways. We can find the stresses and the deformation due to initial strain and due to external forces individually, as we did in Section 4.4, and superpose the solution. The advantage of such an approach is that we have a good intuitive feel for the solution process. The disadvantage is that we have to solve the problem twice. Alternatively we could use Equation (4.27) and solve the problem once, but we need to be careful with our signs, and the approach is less intuitive and more mathematical.

EXAMPLE 4.13

Bars A and B in the mechanism shown in Figure 4.38 are made of steel with a modulus of elasticity $E = 200$ GPa, a coefficient of thermal expansion $\alpha = 12\ \mu/^\circ$C, a cross-sectional area $A = 100\ \text{mm}^2$, and a length $L = 2.5$ m. If the applied force $F = 10$ kN and the temperature of bar A is decreased by 100°C, find the total axial stress in both bars.

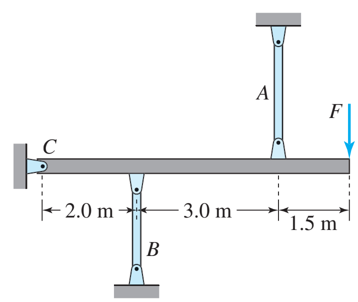


Figure 4.38 Two-bar mechanism in Example 4.13.

PLAN

We can use the force method to solve this problem. The problem has 1 degree of redundancy. We can write one compatibility equation and, using (4.27), get one equation relating the internal forces. By taking the moment about point C in the free-body diagram of the rigid bar, we can obtain the remaining equation and solve the problem.

SOLUTION

The axial rigidity and the thermal strain are

$$EA = [200(10^9)\text{ N/m}^2][100(10^{-6})\text{ m}^2] = 20(10^6)\text{ N} \tag{E1}$$

$$\varepsilon_0 = \alpha\,\Delta T = 12(10^{-6})(-100) = -1200(10^{-6}) \tag{E2}$$

We draw the free-body diagram of the rigid bar with bar A in tension and bar B in compression as shown in Figure 4.39*a*. By moment equilibrium about point C we obtain

$$F(6.5\text{ m}) - \boldsymbol{N}_A(5\text{ m}) - \boldsymbol{N}_B(2\text{ m}) = 0 \qquad \text{or} \qquad 5\boldsymbol{N}_A + 2\boldsymbol{N}_B = 65(10^3)\text{ N} \tag{E3}$$

We draw the approximate deformed shape in Figure 4.39*b*. Noting that the movements of points D and E are equal to the deformation of the bars we obtain from similar triangles

$$\frac{\delta_A}{5\text{ m}} = \frac{\delta_B}{2\text{ m}} \tag{E4}$$

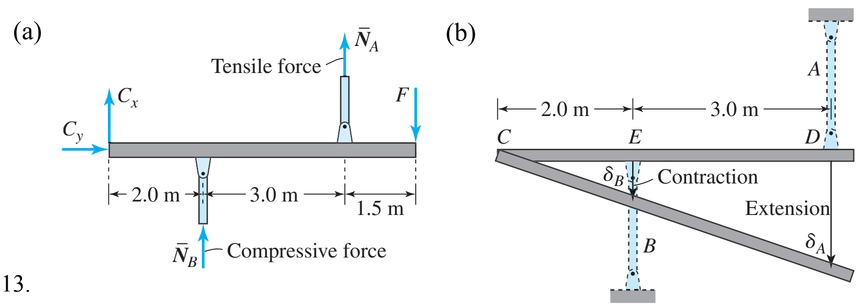


Figure 4.39 Free-body diagram in Example 4.13.

The deformations of bars A and B can be written as

$$\delta_A = \frac{\boldsymbol{N}_A L_A}{E_A A_A} + \varepsilon_0 L_A = \frac{\boldsymbol{N}_A(2.5\text{ m})}{20(10^6)} - 1200(2.5\text{ m})(10^{-6}) = (0.125\boldsymbol{N}_A - 3000)10^{-6}\text{ m} \tag{E5}$$

$$\delta_B = \frac{\boldsymbol{N}_B L_B}{E_B A_B} = \frac{\boldsymbol{N}_B(2.5\text{ m})}{20(10^6)} = 0.125\boldsymbol{N}_B(10^{-6})\text{ m} \tag{E6}$$

Substituting Equations (E5) and (E6) into Equation (E4), we obtain

$$0.125\boldsymbol{N}_B(10^{-6})\text{ m} = 0.4(0.125\boldsymbol{N}_A - 3000)10^{-6}\text{ m} \qquad \text{or} \qquad \boldsymbol{N}_B = 0.4\boldsymbol{N}_A - 9600 \tag{E7}$$

Solving Equations (E3) and (E7), we obtain

$$\boldsymbol{N}_A = 14.51(10^3)\text{ N} \qquad \boldsymbol{N}_B = -3.79(10^3)\text{ N} \tag{E8}$$

Noting that we assumed that bar B is in compression, the sign of $\boldsymbol{N}_B$ in Equation (E8) implies that it is in tension. The stresses in A and B can now be found by dividing the internal forces by the cross-sectional areas.

ANS. $\sigma_A = 145.1\text{ MPa (T)}$ $\qquad \sigma_B = 37.9\text{ MPa (T)}$

COMMENTS

1. In Figure 4.34 the prestrain in member A is $0.0003/2.5 = 1200 \times 10^{-6}$ extension. This means that $\varepsilon_0 = -1200 \times 10^{-6}$. Substituting this value we obtain Equation (E5). Nor will any other equation in this example change for problems represented by Figures 4.34 and 4.37. Thus it is not surprising that the results of this example are identical to those of Example 4.12. But unlike Example 4.12, we solved the problem only once.
2. It would be hard to guess intuitively that bar B will be in tension, because the initial strain is greater than the strain caused by the external force F. But this observation is obvious in the two solutions obtained in Example 4.12.
3. To calculate the initial strain using the method in this example, it is recommended that the problem be formulated initially in terms of the force F. Then to calculate initial strain, substitute $F = 0$. This recommendation avoids some of the confusion that will be caused by a change of the sign of ε_0 in the initial strain calculations.

PROBLEM SET 4.4

Initial strains

4.90 During assembly of a structure, a misfit between bar A and the attachment of the rigid bar was found, as shown in Figure P4.90. If bar A is pulled and attached, determine the initial stress introduced due to the misfit. The modulus of elasticity of the circular bars A and B is E = 10,000 ksi and the diameter is 1 in.

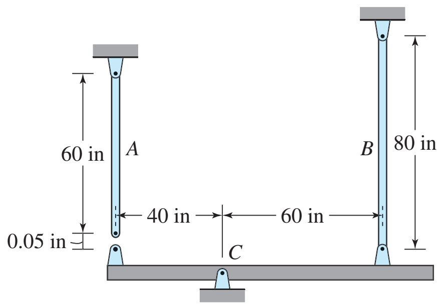


Figure P4.90

4.91 Bar A was manufactured 2 mm shorter than bar B due to an error. The attachment of these bars to the rigid bar would cause a misfit of 2 mm. Calculate the initial stress for each assembly, shown in Figure P4.91. Which of the two assembly configurations would you recommend? Use a modulus of elasticity $E = 70$ GPa and a diameter of 25 mm for the circular bars.

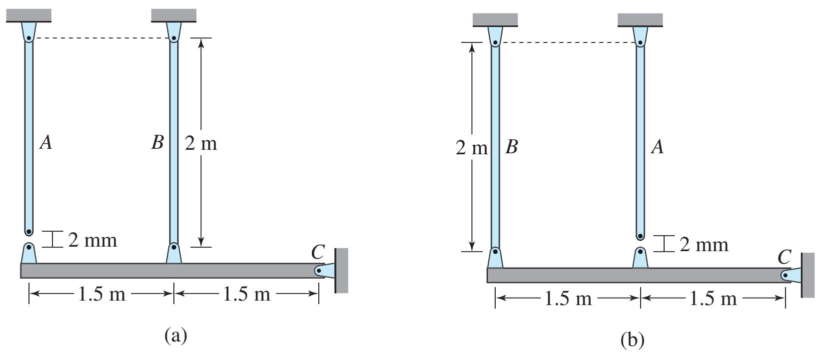


Figure P4.91

4.92 A steel bolt is passed through an aluminum sleeve as shown in Figure P4.92. After assembling the unit by finger-tightening (no deformation) the nut is given a $\frac{1}{4}$ turn. If the pitch of the threads is 3.0 mm, determine the initial axial stress developed in the sleeve and the bolt. The moduli of elasticity for steel and aluminum are $E_{st} = 200$ GPa and $E_{al} = 70$ GPa and the cross-sectional areas are $A_{st} = 500\ \text{mm}^2$ and $A_{al} = 1100\ \text{mm}^2$.

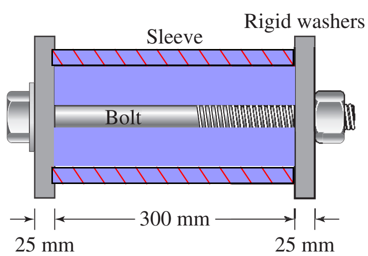


Figure P4.92

4.93 The rigid bar shown in Figure P4.93 is horizontal when the unit is put together by finger-tightening the nut. The pitch of the threads is 0.125 in. The properties of the bars are listed in Table 4.93. Develop a table in steps of quarter turns of the nut that can be used for prescribing the pretension in bar B. The maximum number of quarter turns is limited by the yield stress.

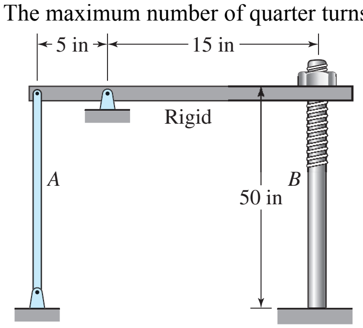


Figure P4.93

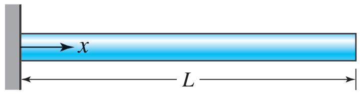
TABLE P4.93 **Material properties**

	Bar A	**Bar B**
Modulus of elasticity	10,000 ksi	30,000 ksi
Yield stress	24 ksi	30 ksi
Cross-sectional area	0.5 in^2	0.75 in^2

Temperature effects

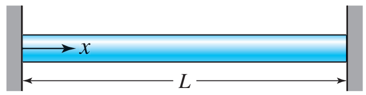
4.94 The temperature for the bar in Figure P4.94 increases as a function of x: $\Delta T = T_L x^2/L^2$. Determine the axial stress and the movement of a point at $x = L/2$ in terms of the length L, the modulus of elasticity E, the cross-sectional area A, the coefficient of thermal expansion α, and the increase in temperature at the end T_L.

Figure P4.94



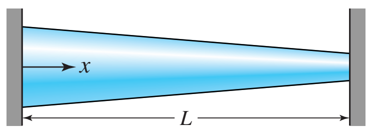
4.95 The temperature for the bar in Figure P4.95 increases as a function of x: $\Delta T = T_L x^2/L^2$. Determine the axial stress and the movement of a point at $x = L/2$ in terms of the length L, the modulus of elasticity E, the cross-sectional area A, the coefficient of thermal expansion α, and the increase in temperature at the end T_L.

Figure P4.95



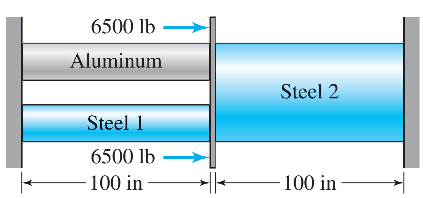
4.96 The tapered bar shown in Figure P4.96 has a cross-sectional area that varies with x as $A = K(L - 0.5x)^2$. If the temperature of the bar increases as $\Delta T = T_L x^2/L^2$, determine the axial stress at midpoint in terms of the length L, the modulus of elasticity E, the cross-sectional area A, the parameter K, the coefficient of thermal expansion α, and the increase in temperature at the end T_L.

Figure P4.96



4.97 Three metallic rods are attached to a rigid plate, as shown in Figure P4.97. The temperature of the rods is lowered by 100°F after the forces are applied. Assuming the rigid plate does not rotate, determine the movement of the rigid plate. The material properties are listed in Table 4.98.

Figure P4.97



4.98 Solve Problem 4.92 assuming that in addition to turning the nut, the temperature of the assembled unit is raised by 40°C. The coefficients of thermal expansion for steel and aluminum are $\alpha_{st} = 12\ \mu/°C$ and $\alpha_{al} = 22.5\ \mu/°C$.

TABLE P4.98 **Material properties**

	Area (in.^{2})	**E (ksi)**	**α (10^{-6}/°F)**
Aluminum	4	10,000	12.5
Steel 1	4	30,000	6.6
Steel 2	12	30,000	6.6

4.99 Determine the axial stress in bar A of Problem 4.93 assuming that the nut is turned 1 full turn and the temperature of bar A is decreased by 80°F. The coefficient of thermal expansion for bar A is $\alpha_{st} = 22.5\ \mu/°F$.

4.6* STRESS APPROXIMATION

Many applications are based on strength design. As was demonstrated in Examples 1.4 and 1.5, we can obtain stress formulas starting with a stress approximation across the cross section and use these in strength design. But how do we deduce a stress behavior across the cross section? In this section we consider the clues that we can use to deduce approximate stress behavior.

In Section 4.7 we will show how to apply these ideas to thin-walled pressure vessels. Section 5.4 on the torsion of thin-walled tubes is another application of the same ideas.

Think of each stress component as a mathematical function to be approximated. The simplest approximation of a function (the stress component) is to assume it to be a constant, as was done in Figure 1.16*a* and *b*. The next level of complexity is to assume a stress component as a linear function, as was done in Figure 1.16*c* and *d*. If we continued this line of thinking, we would next assume a quadratic or higher-order polynomial. The choice of a polynomial for approximating a stress component is dictated by several factors, some of which are discussed in this section.

4.6.1 Free Surface

A segment of a body that has no forces acting on the surface is shown in Figure 4.40. If we consider a point on the surface and draw a stress cube, then the surface with the outward normal in the *z* direction will have no stresses, and we have a situation of plane stress at that point. Because the points on which no forces are acting can be identified by inspection, these points provide us with a clue to making assumptions regarding stress behavior, as will be demonstrated next.

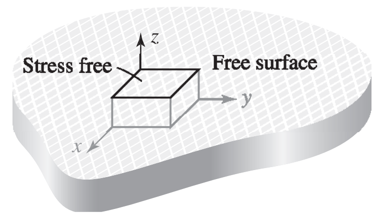


Figure 4.40 Free surface and plane stress.

The drill shown in Figure 4.41 has point *A* located just outside the material that is being drilled. Point A is on a free surface, hence all stress components on this surface, including the shear stress, must go to zero. Point *B* is at the tip of the drill, the point at which the material is being sheared off, that is, at point *B* the shear stress must be equal to the shear strength of the material. Now we have two points of observation. The simplest curve that can be fitted through two points is a straight line. A linear approximation of shear stress, as shown in Figure 4.41, is a better approximation than the uniform behavior we assumed in Example 1.6. It can be confirmed that with linear shear stress behavior, the minimum torque will be 188.5 in. · kips, which is half of what we obtained in Example 1.6. Only experiment can confirm whether the stress approximation in Figure 4.41 is correct. If it is not, then the experimental results would suggest other equations to consider.

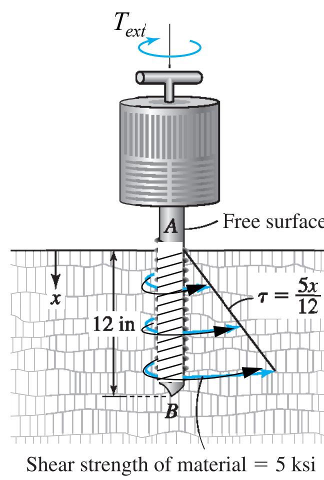


Figure 4.41 Using free surface to guide stress approximation.

4.6.2 Thin Bodies

The smaller the region of approximation, the better is the accuracy of the analytical model. If the dimensions of a cross section are small compared to the length of the body, then assuming a constant or a linear stress distribution across the cross section will introduce small errors in the calculation of internal forces and moments, such as in pins discussed in Section 1.1.2. We now consider another small region of approximation, termed thin bodies. A body is called **thin** if its thickness is an order of magnitude (factor of 10) smaller than the other dimensions.

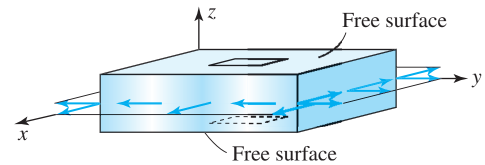


Figure 4.42 Plane stress assumption in thin plates.

Figure 4.42 shows a segment of a plate with loads in the x and y directions. The top and bottom surfaces of the plate are free surfaces, that is, plane stress exists on both surfaces. This does not imply that a point in the middle of the two surfaces is also in a state of plane stress, but if the plate is thin compared to its other dimensions then to simplify analysis, it is reasonable to assume that the entire plate is in plane stress. The other stress components are usually assumed uniform or linear in the thickness direction in thin bodies.

The assumption of plane stress is made in thin bodies even when there are forces acting on one of the surfaces in the thickness direction. The assumption is justified if the maximum stresses in the xy plane turn out to be an order of magnitude greater than the applied distributed load. But the validity of the assumption can be checked only after the stress formula has been developed. Some examples of thin bodies are the skin of an aircraft, the floors and ceilings of buildings, and thin-walled cylindrical or spherical pressure vessels.

4.6.3 Axisymmetric Bodies

A body whose geometry, material properties, and loading are symmetric with respect to an axis is called an **axisymmetric body**. The stress components which are produced cannot depend upon the angular location in an axisymmetric body. In other words, the stress components must also be symmetric with respect to the axis. By using this argument of axisymmetry in thin bodies, we can get good stress approximation, as will be demonstrated by a simple example below and further elaborated in Section 4.7.

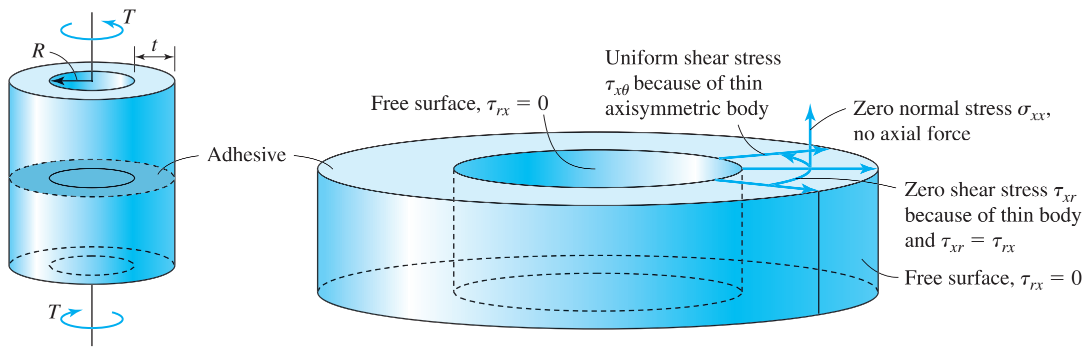


Figure 4.43 Deducing stress behavior in adhesively bonded thin cylinders.

Consider all the stress components acting in the adhesive layer between two thin cylinders subjected to a torque, as shown in Figure 4.43. The shear stress in the radial direction τ_{xr} is assumed to be zero because the symmetric counterpart of this shear stress, τ_{rx}, has to be zero on the inside and outside free surfaces of this thin body. Because the problem is axisymmetric, the normal stress σ_{xx} and the tangential shear stress $\tau_{x\theta}$ cannot depend on the angular coordinate. But a uniform axial stress σ_{xx} would produce an internal axial force. Because no external axial force exists, we approximate the axial stress as zero. Because of thinness, the tangential shear stress $\tau_{x\theta}$ is assumed to be constant in the radial direction. In Example 1.6 we developed the stress formula relating $\tau_{x\theta}$ to the applied torque. In Section 5.4, in a similar manner, we shall deduce the behavior of the shear stress distribution in thin-walled cylindrical bodies of arbitrary cross sections.

4.6.4 Limitations

All analytical models depend on assumptions and are approximations. They are mathematical representations of nature and have errors in their predictions. Whether the approximation is acceptable depends on the accuracy needed and the experimental results. If all we are seeking is an order-of-magnitude value for stresses, then assuming a uniform stress behavior in most cases will give us an adequate answer. But constructing sophisticated models based on stress approximation alone is difficult, if not impossible. Further, an assumed stress distribution may correspond to a material deformation that is physically impossible. For example, the approximation might require holes or corners to form inside the material. Another difficulty is validating the assumption. We need to approximate six independent stress components, which are difficult to visualize and, being internal, cannot be measured directly. These difficulties can be overcome by approximating not the stress but the displacement that can be observed experimentally as discussed in Section 3.2.

We conclude this section with the following observations:

1. A point is in plane stress on a free surface.
2. Some of the stress components must tend to zero as the point approaches the free surface.
3. A state of plane stress may be assumed for thin bodies.
4. Stress components may be approximated as uniform or linear in the thickness direction for thin bodies.
5. A body that has geometry, material properties, and loads that are symmetric about an axis must have stresses that are also symmetric about the axis.

4.7* THIN-WALLED PRESSURE VESSELS

Cylindrical and spherical pressure vessels are used for storage, as shown in Figure 4.44, and for the transportation of fluids and gases. The inherent symmetry and the assumption of thinness make it possible to deduce the behavior of stresses to a first approximation. The argument of symmetry implies that stresses cannot depend on the angular location. By limiting ourselves to thin walls, we can assume uniform radial stresses in the thickness direction. The net effect is that all shear stresses in cylindrical or spherical coordinates are zero, the radial normal stress can be neglected, and the two remaining normal stresses in the radial and circumferential directions are constant. The two unknown stress components can be related to pressure by static equilibrium.

(a)

(b)

Figure 4.44 Gas storage tanks.

The "thin-wall" limitation implies that the ratio of the mean radius R to the wall thickness t is greater than 10. The higher the ratio of R/t, the better is the prediction of our analysis.

4.7.1 Cylindrical Vessels

Figure 4.45 shows a thin cylinder subjected to a pressure of p. The stress element on the right in Figure 4.45 shows the stress components in the cylindrical coordinate system (r, θ, x) on four surfaces. The outer surface of the cylinder is stress free. Hence the shear stresses $\tau_{r\theta}$ and τ_{rx} and the normal stress σ_{rr} are all zero on the outer surface (at A). On the inner surface (at B) there is only a radial force due to pressure p, but there are no tangential forces. Hence on the inner surface the shear stresses $\tau_{r\theta}$ and τ_{rx} are zero. Since the wall is thin, we can assume that the shear stresses $\tau_{r\theta}$ and τ_{rx} are zero across the thickness. The radial normal stress varies from a zero value on the outer surface to a value of the pressure on the inner surface. At the end of our derivation we will justify that the radial stress σ_{rr} can be neglected as it is an order of magnitude less than the other two normal stresses σ_{xx} and $\sigma_{\theta\theta}$. A nonzero value of $\tau_{\theta x}$ will either result in a torque or movement of points the θ direction. Since there is no applied torque, and the movement of a point cannot depend on the angular location because of symmetry, we conclude that the shear stress $\tau_{\theta x}$ is zero.

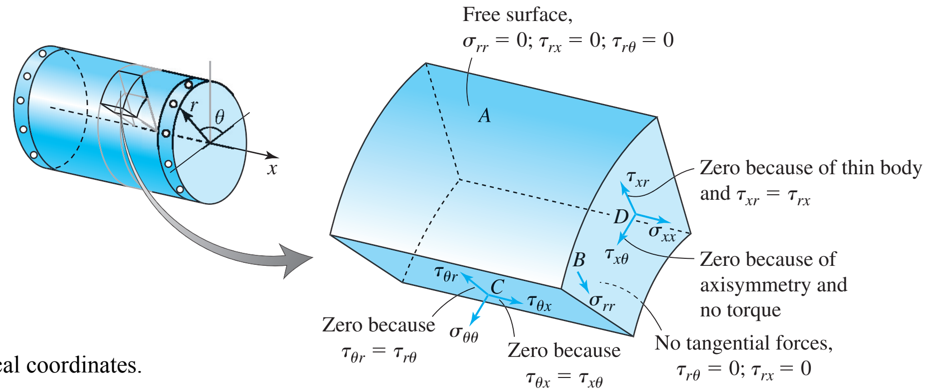


Figure 4.45 Stress element in cylindrical coordinates.

Thus all shear stresses are zero, while the radial normal stress is neglected.The axial stress σ_{xx} and the hoop stress $\sigma_{\theta\theta}$ are assumed uniform across the thickness and across the circumference, as these cannot depend upon angular location. Figure 4.67*a* shows this state of stress. We could start with a differential element and find the internal forces by integrating σ_{xx} and $\sigma_{\theta\theta}$ over appropriate areas. But as these two stresses are uniform across the entire circumference, we can reach the same conclusions by considering two free-body diagrams shown in Figure 4.46*b* and *c*.

By equilibrium of forces on the free-body diagram in Figure 4.46*b,* we obtain $2\sigma_{\theta\theta}(t\ dx) = p(2R)\,dx$, or

$$\boxed{\sigma_{\theta\theta} = \frac{pR}{t}} \tag{4.28}$$

By equilibrium of forces on the free-body diagram in Figure 4.46*c* we obtain $\sigma_{xx}(2\pi R)(t) = p(\pi R^2)$, or

$$\boxed{\sigma_{xx} = \frac{pR}{2t}} \tag{4.29}$$

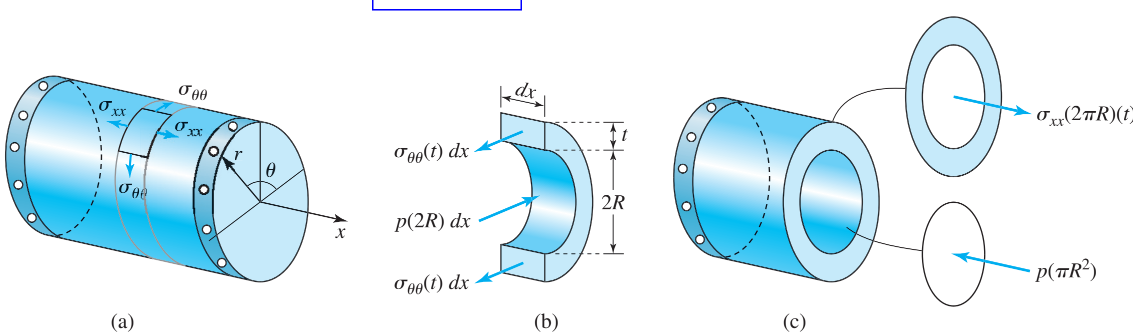


Figure 4.46 Stress analysis in thin cylindrical pressure vessels.

With $R/t > 10$ the stresses σ_{xx} and $\sigma_{\theta\theta}$ are greater than the maximum value of radial stress σ_{rr} (=p) by factors of at least 5 and 10, respectively. This justifies our assumption of neglecting the radial stress in our analysis.

The axial stress σ_{xx} and the hoop stress $\sigma_{\theta\theta}$ are always tensile under internal pressure. The formulas may be used for small applied external pressure but with the following caution. External pressure causes compressive normal stresses that can cause the cylinder to fail due to buckling. The buckling phenomenon is discussed in Chapter 11.

Although the normal stresses are assumed not to vary in the circumferential or thickness direction, our analysis does not preclude variations in the axial direction (x direction). But the variations in the x direction must be gradual. If the variations are very rapid, then our assumption that stresses are uniform across the thickness will not be valid, as can be shown by a more rigorous three-dimensional elasticity analysis.

4.7.2 Spherical Vessels

We use the spherical coordinate system (r, θ, ϕ) for our analysis, as shown in Figure 4.47*a*. Proceeding in a manner similar to the analysis of cylindrical vessels, we deduce the following:

1. All shear stresses are zero:

$$\tau_{r\phi} = \tau_{\phi r} = 0 \qquad \tau_{r\theta} = \tau_{\theta r} = 0 \qquad \tau_{\theta\phi} = \tau_{\phi\theta} = 0 \tag{4.30}$$

2. Normal radial stress σ_{rr} varies from a zero value on the outside to the value of the pressure on the inside. We will once more neglect the radial stress in our analysis and justify it posterior.
3. The normal stresses $\sigma_{\theta\theta}$ and $\sigma_{\phi\phi}$ are equal and are constant over the entire vessel. We set $\sigma_{\theta\theta} = \sigma_{\phi\phi} = \sigma$.

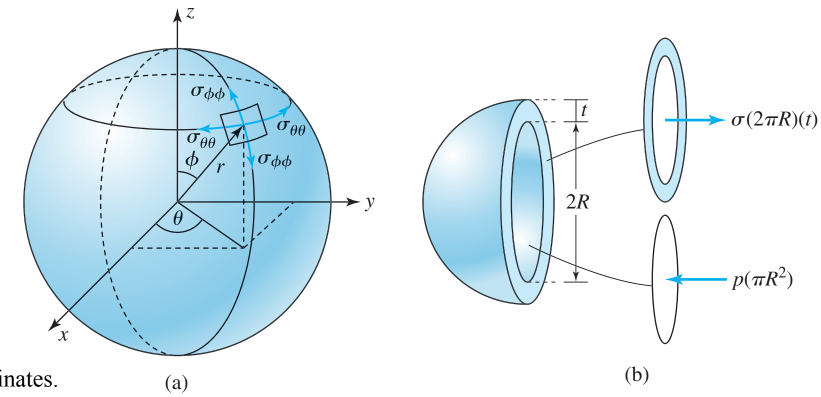


Figure 4.47 Stress analysis in thin spherical coordinates.

As all imaginary cuts through the center are the same, we consider the free-body diagram shown in Figure 4.47*b*. By equilibrium of forces we obtain $\sigma(2\pi R)(t) = p\pi R^2$, or

$$\boxed{\sigma = \frac{pR}{2t}} \tag{4.31}$$

With $R/t > 10$ the normal stress σ is greater than the maximum value of radial stress σ_{rr} $(=p)$ by a factor of at least 5. This justifies our assumption of neglecting the radial stress in our analysis. At each and every point the normal stress in any circumferential direction is the same for thin spherical pressure vessels.

EXAMPLE 4.14

The lid is bolted to the tank in Figure 4.48 along the flanges using 1-in.-diameter bolts. The tank is made from sheet metal that is $\frac{1}{2}$ in. thick and can sustain a maximum hoop stress of 24 ksi in tension. The normal stress in the bolts is to be limited to 60 ksi in tension. A manufacturer can make tanks of diameters from 2 ft to 8 ft in steps of 1 ft. Develop a table that the manufacturer can use to advise customers of the size of tank and the number of bolts per lid needed to hold a desired gas pressure.

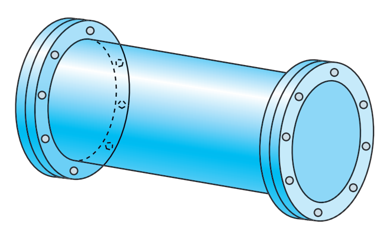


Figure 4.48 Cylindrical tank in Example 4.14.

PLAN

Using Equation (4.28) we can establish a relationship between the pressure p and the radius R (or diameter D) of the tank through the limiting value on hoop stress. We can relate the number of bolts needed by noting that the force due to pressure on the lid is carried equally by the bolts.

SOLUTION

The area of the bolts can be found as shown in (E1).

$$A_{bolt} = \pi(1\text{ in.})^2/4 = (\pi/4)\text{ in.}^2 \qquad \text{(E1)}$$

From Equation (4.28) we obtain (E2).

$$\sigma_{\theta\theta} = \frac{pR}{1/2} \le 24{,}000\text{ psi} \qquad \text{or} \qquad p \le \frac{24,000}{D}\text{ psi} \qquad \text{(E2)}$$

Figure 4.49 shows the free-body diagram of the lid. By equilibrium of forces we obtain (E3).

$$nN_{bolt} = N_{lid} \qquad \text{or} \qquad n\sigma_{bolt}\left(\frac{\pi}{4}\right) = p(\pi R^2) \qquad \text{or} \qquad \sigma_{bolt} \le \frac{4pR^2}{n} \qquad \text{or} \qquad \sigma_{bolt} = \frac{pD^2}{n} \le 60{,}000 \qquad \text{(E3)}$$

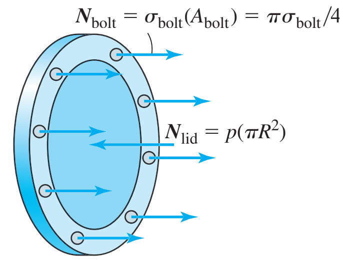


Figure 4.49 Relating forces in bolts and lid in Example 4.14.

Substituting (E3) into (E2) we obtain (E4).

$$\frac{24,000D}{n} \le 60{,}000 \qquad \text{or} \qquad n \ge 0.4D \qquad \text{(E4)}$$

We consider the values of D from 24 in to 96 in. in steps of 12 in and calculate the values of p and n from Equations (E2) and (E4). We report the values of p by rounding downward to the nearest integer that is a factor of 5, and the values of n are reported by rounding upward to the nearest integer, as given in Table 4.2.

TABLE 4.2 **Results of Example 4.14**

Tank Diameter D (ft)	Maximum Pressure p (psi)	Minimum Number of Bolts n
2	1000	10
3	665	15
4	500	20
5	400	24
6	330	30
7	280	34
8	250	39

COMMENT

1. We rounded downwards for p and upwards for n to satisfy the inequalities of Equations (E2) and (E4). Intuitively we know that smaller pressure and more bolts will result is a safer pressure tank.

PROBLEM SET 4.5

Thin-walled pressure vessels

4.100 Fifty rivets of 10-mm diameter are used for attaching caps at each end on a 1000-mm mean diameter cylinder, as shown in Figure P4.100. The wall of the cylinder is 10 mm thick and the gas pressure is 200 kPa. Determine the hoop stress and the axial stress in the cylinder and the shear stress in each rivet.

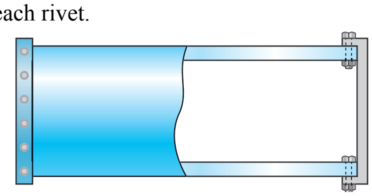


Figure P4.100

4.101 A pressure tank 15 ft long and with a mean diameter of 40 in is to be fabricated from a $\frac{1}{2}$-in.-thick sheet. A 15-ft-long, 8-in.-wide, $\frac{1}{2}$-in.-thick plate is bonded onto the tank to seal the gap, as shown in Figure P4.101. What is the shear stress in the adhesive when the pressure in the tank is 75 psi? Assume uniform shear stress over the entire inner surface of the attaching plate.

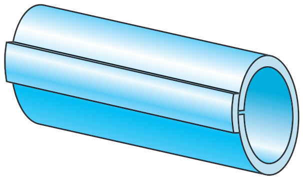


Figure P4.101

Design problems

4.102 A 5-ft mean diameter spherical tank has a wall thickness of $\frac{3}{4}$ in. If the maximum normal stress is not to exceed 10 ksi, determine the maximum permissible pressure.

4.103 In a spherical tank having a 500-mm mean radius and a thickness of 40 mm, a hole of 50-mm diameter is drilled and then plugged using adhesive of 1.2-MPa shear strength to form a safety pressure release mechanism (Figure P4.103). Determine the maximum allowable pressure and the corresponding hoop stress in the tank material.

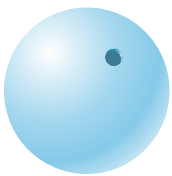


Figure P4.103

4.104 A 20-in. mean diameter pressure cooker is to be designed for a 15-psi pressure (Figure P4.104). The allowable normal stress in the cylindrical pressure cooker is to be limited to 3 ksi. Determine the minimum wall thickness of the pressure cooker. A $\frac{1}{2}$-lb weight on top of the nozzle is used to control the pressure in the cooker. Determine the diameter d of the nozzle.

Figure P4.104

4.105 The cylindrical gas tank shown in Figure P4.105 is made from 8-mm-thick sheet metal and must be designed to sustain a maximum normal stress of 100 MPa. Develop a table of maximum permissible gas pressures and the corresponding mean diameters of the tank in steps of 100 mm between diameter values of 400 mm and 900 mm.

Figure P4.105

4.106 A pressure tank 15 ft long and a mean diameter of 40 in. is to be fabricated from a $\frac{1}{2}$-in.-thick sheet. A 15-ft-long, 8-in.-wide, $\frac{1}{2}$-in.-thick plate is to be used for sealing the gap by using two rows of 90 rivets each. If the shear strength of the rivets is 36 ksi and the normal stress in the tank is to be limited to 20 ksi, determine the maximum pressure and the minimum diameter of the rivets that can be used.

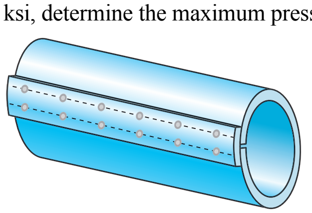


Figure P4.106

4.107 A pressure tank 5 m long and a mean diameter of 1 m is to be fabricated from a 10 mm thick sheet as shown in Figure P4.106. A 5 m-long, 200 mm wide, 10-mm-thick plate is to be used for sealing the gap by using two rows of 100 rivets each. The shear strength of the rivets is 300 MPa and the yield strength of the tank material is 200 MPa. Determine the maximum pressure and the minimum diameter of the rivets to the nearest millimeter that can be used for a factor of safety of 2.

4.8* CONCEPT CONNECTOR

The finite-element method (FEM) is a popular numerical technique for the stress and deformation analysis of planes, ships, automobiles, buildings, bridges, machines, and medical implants, as well as for earthquakes predictions. It is used in both static and dynamic analysis and both linear and nonlinear analysis as well. A whole industry is devoted to developing FEM software, and many commercial packages are already available, including software modules in computer-aided design (CAD), computer-aided manufacturing (CAM), and computer-aided engineering (CAE). This section briefly describes the main ideas behind one version of FEM.

4.8.1 The Finite Element Method

In the *stiffness method* FEM is based on the displacement method, while in the *flexibility method* it is based on the force method. Most commercial FEM software is based on the displacement method.

In the displacement method, the unknowns are the displacements of points called *nodes*, and a set of linear equations represents the force equilibrium at the nodes. For example, the unknowns could be the displacements of pins in a truss, and the linear equations could be the equilibrium equations at each joint written in terms of the displacements. In FEM, however, the equilibrium equations are derived by requiring that the nodal displacements minimize the potential energy of the structure. First equations are created for small, finite elements whose assembly represents the body, and these lead to equations for the entire body. It is assumed that the displacement in an element can be described by a polynomial. Figure 4.50 shows the linear and quadratic displacements in a one-dimensional rod.

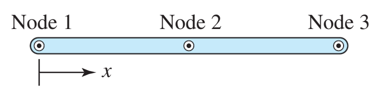
Linear element

$$u(x) = a_0 + a_1 x$$

$$u(x) = u_1\left(\frac{x-x_2}{x_1-x_2}\right) + u_2\left(\frac{x-x_1}{x_2-x_1}\right)$$

$$u(x) = u_1\phi_1(x) + u_2\phi_2(x)$$

Quadratic element



$$u(x) = a_0 + a_1 x + a_2 x^2$$

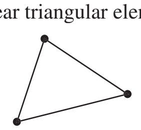
$$u(x) = u_1\left(\frac{x-x_2}{x_1-x_2}\right)\left(\frac{x-x_3}{x_1-x_3}\right) + u_2\left(\frac{x-x_1}{x_2-x_1}\right)\left(\frac{x-x_3}{x_2-x_3}\right) + u_3\left(\frac{x-x_1}{x_3-x_1}\right)\left(\frac{x-x_2}{x_3-x_2}\right)$$

$$u(x) = u_1\phi_1(x) + u_2\phi_2(x) + u_3\phi_3(x)$$

Figure 4.50

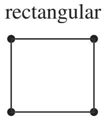
The constants a_i in the polynomials can be found in terms of the nodal displacement values u_i and nodal coordinates x_i as shown in Figure 4.50. The polynomial functions ϕ_i that multiply the nodal displacements are called *interpolation functions*, because we can now interpolate the displacement values from the nodal values. Sometimes the same polynomial functions are also used for representing the shapes of the elements. Then the interpolation functions are also referred to as *shape functions.* When the same polynomials represent the displacement and the shape of an element, then the element is called an *isoparametric* element.

Linear triangular element



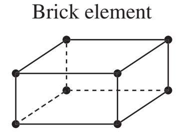
$u(x, y) = a_0 + a_1 x + a_2 y$

Bilinear rectangular element



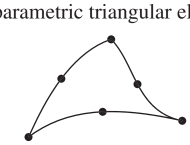
$u(x, y) = a_0 + a_1 x + a_2 y + a_3 xy$

Brick element



$u(x, y, z) = a_0 + a_1 x + a_2 y + a_3 z + a_4 xy + a_5 yz + a_6 xz + a_7 xyz$

Isoparametric triangular element



$u(x, y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2$

Figure 4.51 Examples of elements in finite-element method.

Figure 4.51 shows some popular elements in two and three dimensions. Strains from the displacements can be found by using Equations (2.9a) through (2.9i). The strains are substituted into potential energy, which is then minimized to generate the algebraic equations.

A FEM program consists of three major modules:

1. In first module, called the *pre-processor,* the user: creates the geometry; creates a *mesh* which the discretized geometry of elements; applies loads; and applies the boundary conditions. Figure 4.52 shows a finite-element mesh for a bracket constructed using three-dimensional tetrahedron elements. The bottom of the bracket is welded to another member. The load that is transferred through the bolt must be measured or estimated before a solution can be found. The bottom of the bracket is then modeled as points with zero displacements.
2. In the second module called *solver* the algebraic equations are created and solved. Once the nodal displacements solved are know then stresses are obtained.
3. In the third module called the *post-processor* the results of displacements and stresses are displayed in a variety of forms that are specified by the user.

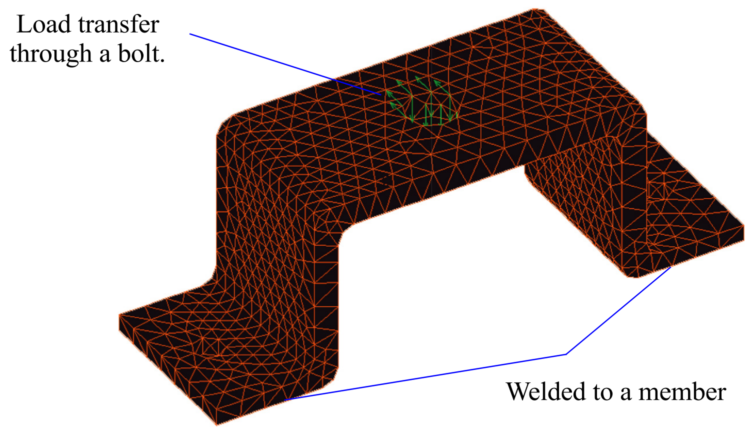


Figure 4.52 Finite-element mesh of bracket. (Courtesy Professor C. R. Vilmann.)

4.9 CHAPTER CONNECTOR

In this chapter we established formulas for deformations and stresses in axial members. We saw that the calculation of stresses and relative deformations requires the calculation of the internal axial force at a section. For statically determinate axial members, the internal axial force can be calculated either (1) by making an imaginary cut and drawing an appropriate free-body diagram or (2) by drawing an axial force diagram.

In statically indeterminate structures there are more unknowns than there are equilibrium equations. Compatibility equations have to be generated from approximate deformed shapes to solve a statically indeterminate problem. In the displacement method the equilibrium and compatibility equations are written either in terms of the deformation of axial members or in terms of the displacements of points on the structure, and the set of equations is solved. In the force method the equilibrium and compatibility equations are written either in terms of internal forces in the axial members or in terms of the reactions at the support of the structure, and the set of equations is again solved.

In Chapter 8, on stress transformation, we shall consider problems in which we first find the axial stress using the stress formula in this chapter and then find stresses on inclined planes, including planes with maximum shear stress. In Chapter 9, on strain transformation, we shall find the axial strain and the strains in the transverse direction due to Poisson's effect. We will then consider strains in different coordinate systems, including coordinate systems in which shear strain is a maximum. In Section 10.1 we shall consider the combined loading problems of axial, torsion, and bending and the design of simple structures that may be determinate or indeterminate.

POINTS AND FORMULAS TO REMEMBER

- Theory is limited to (i) slender members, (ii) regions away from regions of stress concentration, (iii) members in which the variation in cross-sectional areas and external loads is gradual, (iv) members on which axial load is applied such that there is no bending.
- $N = \int_A \sigma_{xx}\, dA$ (4.1) $u = u(x)$ (4.3) Small strain: $\varepsilon_{xx} = \dfrac{du(x)}{dx}$ (4.4)
- where u is the axial displacement, which is positive in the positive x direction, ε_{xx} is the axial strain, σ_{xx} is the axial stress, and N is the internal axial force over cross section A.
- Axial strain ε_{xx} is uniform across the cross section.
- Equations (4.1), (4.3), and (4.4) do not change with material model.
- Formulas below are valid for material that is linear, elastic, isotropic, with no inelastic strains:
- Homogeneous cross-section:
- $\dfrac{du}{dx} = \dfrac{N}{EA}$ (4.7) $\sigma_{xx} = \dfrac{N}{A}$ (4.8) $u_2 - u_1 = \dfrac{N(x_2 - x_1)}{EA}$ (4.10)
- where EA is the axial rigidity of the cross section.
- If N, E, or A change with x, then find deformation by integration of Equation (4.7).
- If N, E, and A do not change between x_1 and x_2, then use Equation (4.10) to find deformation.
- For homogeneous cross sections all external loads must be applied at the centroid of the cross section, and centroids of all cross sections must lie on a straight line.
- Structural analysis: $\delta = \dfrac{NL}{EA}$ (4.21)
- where δ is the deformation in the original direction of the axial bar.
- If N is a tensile force, then δ is elongation. If N is a compressive force, then δ is contraction.
- Degree of static redundancy is the number of unknown reactions minus the number of equilibrium equations.
- If degree of static redundancy is not zero, then we have a statically indeterminate structure.
- Compatibility equations are a geometric relationship between the deformation of bars derived from the deformed shapes of the structure.
- The number of compatibility equations in the analysis of statically indeterminate structures is always equal to the degree of redundancy.
- The direction of forces drawn on the free-body diagram must be consistent with the deformation shown in the deformed shape of the structure.
- The variables necessary to describe the deformed geometry are called degrees of freedom.
- In the displacement method, the displacements of points are treated as unknowns. The number of unknowns is equal to the degrees of freedom.
- In the force method, reaction forces are the unknowns. The number of unknowns is equal to the degrees of redundancy.