CHAPTER SIX
SYMMETRIC BENDING OF BEAMS

Learning objectives

1. Understand the theory of symmetric bending of beams, its limitations, and its applications for a strength-based design and analysis.
2. Visualize the direction of normal and shear stresses and the surfaces on which they act in the symmetric bending of beams.

On April 29th, 2007 at 3:45 AM, a tanker truck crashed into a pylon on interstate 80 near Oakland, California, spilling 8600 gallons of fuel that ignited. Fortunately no one died. But the heat generated from the ignited fuel, severely reduced the strength and stiffness of the steel beams of the interchange, causing it to collapse under its own weight (Figure 6.1*a*). In this chapter we will study the stresses, hence strength of beams. In Chapter 7 we will discuss deflection, hence stiffness of the beams.

Which structural member can be called a beam? Figure 6.1*b* shows a bookshelf whose length is much greater than its width or thickness, and the weight of the books is perpendicular to its length. Girders, the long horizontal members in bridges and highways transmit the weight of the pavement and traffic to the columns anchored to the ground, and again the weight is perpendicular to the member. Bookshelves and girders can be modeled as **beams**—long structural member on which loads act perpendicular to the longitudinal axis. The mast of a ship, the pole of a sign post, the frame of a car, the bulkheads in an air-craft, and the plank of a seesaw are among countless examples of beams.

The simplest theory for symmetric bending of beams will be developed rigorously, following the logic described in Figure 3.15, but subject to the limitations described in Section 3.13.

(a) (b)

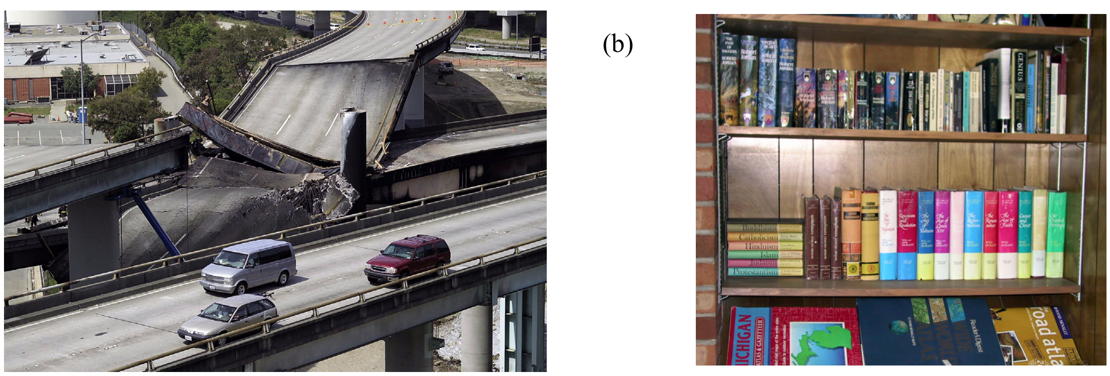


Figure 6.1 (a) I-80 interchange collapse. (b) Beam example.

6.1 PRELUDE TO THEORY

As a prelude to theory, we consider several examples, all solved using the logic discussed in Section 3.2. They highlight observations and conclusions that will be formalized in Section 6.2.

- Example 6.1, discrete bars welded to a rigid plate, illustrates how to calculate the bending normal strains from geometry.
- Example 6.2 shows the similarity of Example 6.1 to the calculation of normal strains for a continuous beam.
- Example 6.3 applies the logic described in Figure 3.15 to beam bending.
- Example 6.4 shows how the choice of a material model alters the calculation of the internal bending moment. As we saw in Chapter 5 for shafts, the material model affects only the stress distribution, leaving all other equations unaffected. Thus, the kinematic equation describing strain distribution is not affected. Neither are the static equivalency equations

between stress and internal moment and the equilibrium equations relating internal forces and moments. Although we shall develop the simplest theory using Hooke's law, most of the equations will apply to complex material models as well.

EXAMPLE 6.1

The left ends of three bars are built into a rigid wall, and the right ends are welded to a rigid plate, as shown in Figure 6.2. The undeformed bars are straight and perpendicular to the wall and the rigid plate. The rigid plate is observed to rotate due to the applied moment by an angle of 3.5° from the vertical plane. If the normal strain in bar 2 is zero, determine the normal strains in bars 1 and 3.

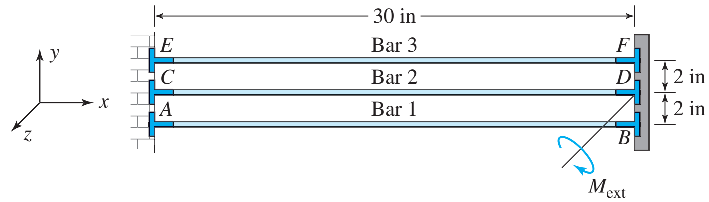


Figure 6.2 Geometry in Example 6.1.

METHOD 1: PLAN

The tangent to a circular arc is perpendicular to the radial line. If the bars are approximated as circular arcs and the wall and the rigid plate are in the radial direction, then the kinematic restriction of bars remaining perpendicular to the wall and plate is satisfied by the deformed shape. We can relate the angle subtended by the arc to the length of arc formed by *CD*, as we did in Example 2.3. From the deformed geometry, the strains of the remaining bars can be found.

SOLUTION

Figure 6.3 shows the deformed bars as circular arcs with the wall and the rigid plate in the radial direction. We know that the length of arc CD_1 is still 30 in., since it does not undergo any strain. We can relate the angle subtended by the arc to the length of arc formed by *CD* and calculate the radius of the arc *R* as

$$\psi = \left(\frac{3.5^\circ}{180^\circ}\right)(3.142 \text{ rad}) = 0.0611 \text{ rad} \qquad CD_1 = R\psi = 30 \text{ in.} \quad \text{or} \quad R = 491.1 \text{ in.} \tag{E1}$$

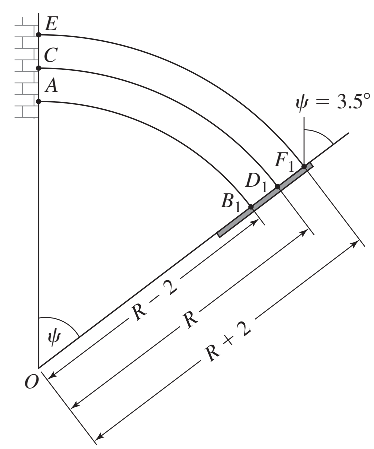


Figure 6.3 Normal strain calculations in Example 6.1.

The arc length AB_1 and EF_1 can be found using Figure 6.3 and the strains in bars 1 and 3 calculated.

$$AB_1 = (R-2)\psi = 29.8778 \text{ in.} \qquad \varepsilon_1 = \frac{AB_1 - AB}{AB} = \frac{-0.1222 \text{ in.}}{30 \text{ in.}} = -0.004073 \text{ in./in.} \tag{E2}$$

ANS. $\varepsilon_1 = -4073\ \mu\text{in./in.}$

$$EF_1 = (R+2)\psi = 30.1222 \text{ in.} \qquad \varepsilon_3 = \frac{EF_1 - EF}{EF} = \frac{0.1222 \text{ in.}}{30 \text{ in.}} = 0.004073 \text{ in./in.} \tag{E3}$$

ANS. $\varepsilon_3 = 4073\ \mu\text{in./in.}$

COMMENT

1. In developing the theory for beam bending, we will view the cross section as a rigid plate that rotates about the *z* axis but stays perpendicular to the longitudinal lines. The longitudinal lines will be analogous to the bars, and bending strains can be calculated as in this example.

METHOD 2: PLAN

We can use small-strain approximation and find the deformation component in the horizontal (original) direction for bars 1 and 3. The normal strains can then be found.

SOLUTION

Figure 6.4 shows the rigid plate in the deformed position. The horizontal displacement of point D is zero as the strain in bar 2 is zero. Points B, D, and F move to B_1, D_1, and F_1 as shown. We can use point D_1 to find the relative displacements of points B and F as shown in Equations (E4) and (E5). We make use of small strain approximation to the sine function by its argument:

$$\Delta u_3 = DF_2 = D_2F_1 = D_1F_1 \sin\psi \approx 2\psi = 0.1222 \text{ in.} \quad \text{(E4)}$$

$$\Delta u_1 = B_2D = D_3D_1 = B_1D_1 \sin\psi \approx 2\psi = 0.1222 \text{ in.} \quad \text{(E5)}$$

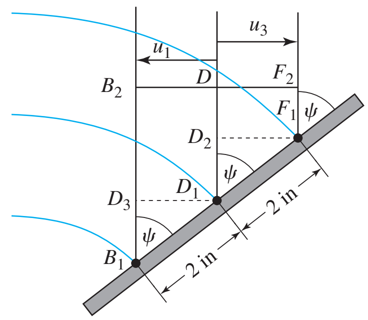


Figure 6.4 Alternate method for normal strain calculations in Example 6.1.

The normal strains in the bars can be found as

$$\varepsilon_1 = \frac{\Delta u_1}{30 \text{ in.}} = \frac{-0.1222 \text{ in.}}{30 \text{ in.}} = -0.004073 \text{ in./in.} \qquad \varepsilon_3 = \frac{\Delta u_3}{30 \text{ in.}} = \frac{0.1222 \text{ in.}}{30 \text{ in.}} = 0.004073 \text{ in./in.} \quad \text{(E6)}$$

ANS. $\varepsilon_1 = -4073\ \mu\text{in./in.} \qquad \varepsilon_3 = 4073\ \mu\text{in./in.}$

COMMENTS

1. Method 1 is intuitive and easier to visualize than method 2. But method 2 is computationally simpler. We will use both methods when we develop the kinematics in beam bending in Section 6.2.
2. Suppose that the normal strain of bar 2 was not zero but $\varepsilon_2 = 800\ \mu$in/in. What would be the normal strains in bars 1 and 3? We could solve this new problem as in this example and obtain $R = 491.5$ in., $\varepsilon_1 = -3272\ \mu$in./in, and $\varepsilon_3 = 4872\ \mu$in./in. Alternatively, we view the assembly was subjected to axial strain before the bending took place. We could then superpose the axial strain and bending strain to obtain $\varepsilon_1 = -4073 + 800 = -3273\ \mu$in./in. and $\varepsilon_3 = 4073 + 800 = 4873\ \mu$in./in. The superposition principle can be used only for linear systems, which is a consequence of small strain approximation, as observed in Chapter 2.

EXAMPLE 6.2

A beam made from hard rubber is built into a rigid wall at the left end and attached to a rigid plate at the right end, as shown in Figure 6.5. After rotation of the rigid plate the strain in line CD at $y = 0$ is zero. Determine the strain in line AB in terms of y and R, where y is the distance of line AB from line CD, and R is the radius of curvature of line CD.

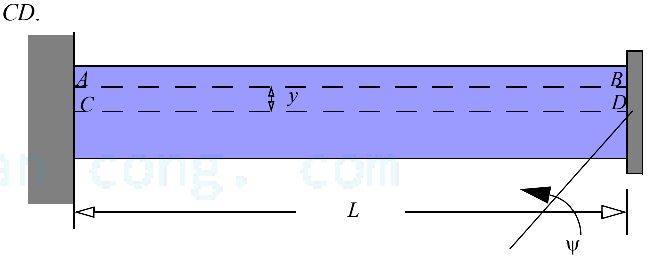


Figure 6.5 Beam geometry in Example 6.2.

PLAN

We visualize the beam as made up of bars, as in Example 6.1, but of infinitesimal thickness. We consider two such bars, AB and CD, and analyze the deformations of these two bars as we did in Example 6.1.

SOLUTION

Because of deformation, point B moves to point B_1 and point D moves to point D_1, as shown in Figure 6.6. We calculate the strain in AB:

$$\varepsilon_{CD} = \frac{CD_1 - CD}{CD} = 0 \quad \text{or} \quad CD_1 = CD = R\psi = L \qquad \psi = \frac{L}{R} \quad \text{(E1)}$$

$$AB_1 = (R-y)\psi = \frac{(R-y)L}{R} \qquad \varepsilon_{AB} = \frac{AB_1 - AB}{AB} = \frac{(R-y)L/R - L}{L} = \frac{L - yL/R - L}{L} \tag{E2}$$

ANS. $\varepsilon_{AB} = \frac{-y}{R}$

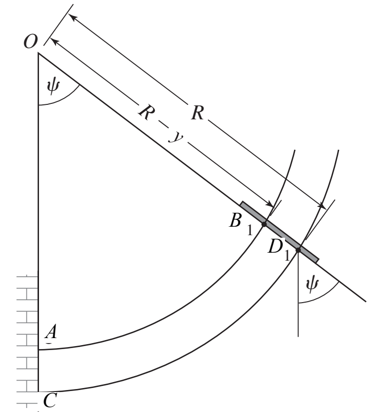


Figure 6.6 Exaggerated deformed geometry in Example 6.2.

COMMENTS

1. In Example 6.1, $R = 491.1$ and $y = +2$ for bar 3, and $y = -2$ for bar 1. On substituting these values into the preceding results, we obtain the results of Example 6.1.
2. Suppose the strain in CD were ε_{CD}. Then the strain in AB can be calculated as in comment 2 of Method 2 in Example 6.1 to obtain $\varepsilon_{AB} = \varepsilon_{CD} - y/R$. The strain ε_{CD} is the axial strain, and the remaining component is the normal strain due to bending.

EXAMPLE 6.3

The modulus of elasticity of the bars in Example 6.1 is 30,000 ksi. Each bar has a cross-sectional area $A = \frac{1}{2}$ in.^{2}. Determine the external moment M_{ext} that caused the strains in the bars in Example 6.1.

PLAN

Using Hooke's law, determine the stresses from the strains calculated in Example 6.1. Replace the stresses by equivalent internal axial forces. Draw the free-body diagram of the rigid plate and determine the moment M_{ext}.

SOLUTION

1. *Strain calculations:* The strains in the three bars as calculated in Example 6.1 are

$$\varepsilon_1 = -4073\ \mu\text{in./in.} \qquad \varepsilon_2 = 0 \qquad \varepsilon_3 = 4073\ \mu\text{in./in.} \tag{E1}$$

2. *Stress calculations:* From Hooke's law we obtain the stresses

$$\sigma_1 = E\varepsilon_1 = (30{,}000\ \text{ksi})(-4073)(10^{-6}) = 122.19\ \text{ksi (C)} \tag{E2}$$

$$\sigma_2 = E\varepsilon_2 = 0 \tag{E3}$$

$$\sigma_3 = E\varepsilon_3 = (30{,}000\ \text{ksi})(4073)(10^{-6}) = 122.19\ \text{ksi (T)} \tag{E4}$$

3. *Internal forces calculations:* The internal normal forces in each bar can be found as

$$N_1 = \sigma_1 A = 61.095\ \text{kips (C)} \qquad N_3 = \sigma_3 A = 61.095\ \text{kips (T)} \tag{E5}$$

4. *External moment calculations:* Figure 6.7 is the free body diagram of the rigid plate. By equilibrium of moment about point O we can find M_z:

$$M_z = N_1(y) + N_3(y) = (61.095\ \text{kips})(2\ \text{in.}) + (61.095\ \text{kips})(2\ \text{in.}) \tag{E6}$$

ANS. $M_z = 244.4\ \text{in.} \cdot \text{kips}$

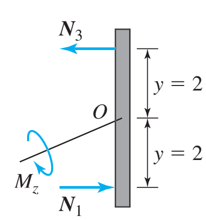


Figure 6.7 Free-body diagram in Example 6.3.

COMMENTS

1. The sum in Equation (E6) can be rewritten $\sum_{i=1}^{2} y\sigma\Delta A_i$, where σ is the normal stress acting at a distance y from the zero strain bar, and ΔA_i is the cross-sectional area of the i^{th} bar. If we had n bars attached to the rigid plate, then the moment would be given by $\sum_{i=1}^{n} y\sigma\Delta A_i$. As we increase the number of bars n to infinity, the cross-sectional area ΔA_i tends to zero, becoming the infinitesimal area dA and the summation is replaced by an integral. In effect, we are fitting an infinite number of bars to the plate, resulting in a continuous body.
2. The total axial force in this example is zero because of symmetry. If this were not the case, then the axial force would be given by the summation $\sum_{i=1}^{n} \sigma\Delta A_i$. As in comment 1, this summation would be replaced by an integral as n tends to infinity, as will be shown in Section 6.1.1.

6.1.1 Internal Bending Moment

In this section we formalize the observation made in Example 6.3: that is, the normal stress σ_{xx} can be replaced by an equivalent bending moment using an integral over the cross-sectional area. Figure 6.8 shows the normal stress distribution σ_{xx} to be replaced by an equivalent internal bending moment $\boldsymbol{M}_z$. Let y represent the coordinate at which the normal stress acts. Static equivalency in Figure 6.8 results in

$$\boldsymbol{M}_z = -\int_A y\sigma_{xx}\, dA \tag{6.1}$$

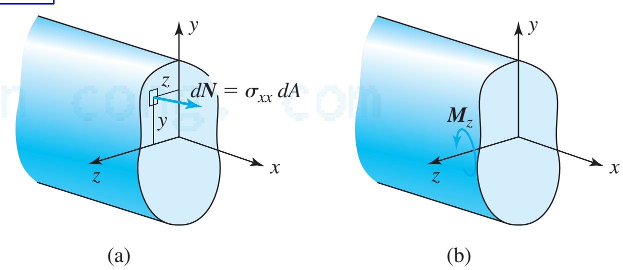


Figure 6.8 Statically equivalent internal moment.

Figure 6.8*a* suggests that for static equivalency there should be an axial force $\boldsymbol{N}$ and a bending moment about the y axis $\boldsymbol{M}_y$. However, the requirement of symmetric bending implies that the normal stress σ_{xx} is symmetric about the axis of symmetry—that is, the y axis. Thus $\boldsymbol{M}_y$ is implicitly zero owing to the limitation of symmetric bending. Our desire to study bending independent of axial loading requires that the stress distribution be such that the internal axial force $\boldsymbol{N}$ should be zero. Thus we must explicitly satisfy the condition

$$\int_A \sigma_{xx}\, dA = 0 \tag{6.2}$$

Equation (6.2) implies that the stress distribution across the cross section must be such that there is no net axial force. That is, the compressive force must equal the tensile force on a cross section in bending. If stress is to change from compression to tension, then there must be a location of zero normal stress in bending. The line on the cross section where the bending normal stress is zero is called **neutral axis**.

Equations (6.1) and (6.2) are independent of the material model. That is because they represent static equivalency between the normal stress on the entire cross section and the internal moment. If we were to consider a composite beam cross section or a nonlinear material model, then the value and distribution of σ_{xx} would change across the cross section yet Equation (6.1) relating σ_{xx} to $\boldsymbol{M}_z$ would remain unchanged. Example 6.4 elaborates on this idea. The origin of the y coordinate is located at the neutral axis irrespective of the material model. Hence, determining the location of the neutral axis is critical in all bending problems. The location of the origin will be discussed in greater detail for a homogeneous, linearly elastic, isotropic material in Section 6.2.4.

EXAMPLE 6.4

Figure 6.9 shows a homogeneous wooden cross section and a cross section in which the wood is reinforced with steel. The normal strain for both cross sections is found to vary as $\varepsilon_{xx} = -200y\ \mu$. The moduli of elasticity for steel and wood are E_{steel} = 30,000 ksi and E_{wood} = 8000 ksi. (a) Write expressions for normal stress σ_{xx} as a function of y, and plot the σ_{xx} distribution for each of the two cross sections shown. (b) Calculate the equivalent internal moment $\boldsymbol{M}_z$ for each cross section.

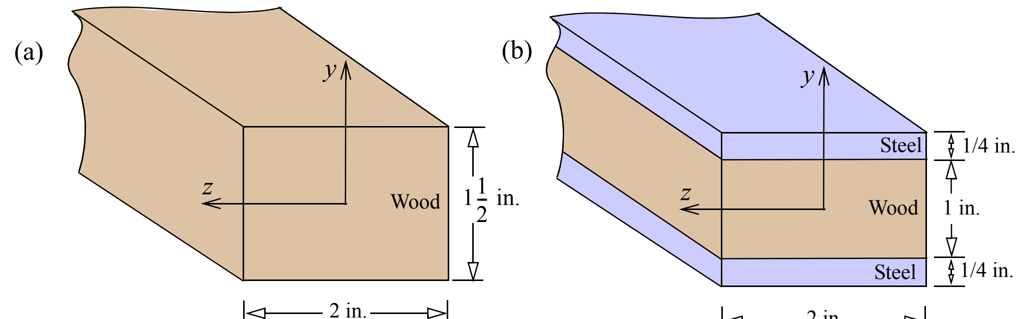


Figure 6.9 Cross sections in Example 6.4. (*a*) Homogeneous. (*b*) Laminated.

PLAN

(a) From the given strain distribution we can find the stress distribution by Hooke's law. We note that the problem is symmetric and stresses in each region will be linear in y. (b) The integral in Equation (6.1) can be written as twice the integral for the top half since the stress distribution is symmetric about the center. After substituting the stress as a function of y in the integral, we can perform the integration to obtain the equivalent internal moment.

SOLUTION

(a) From Hooke's law we can write the stress in each material as

$$(\sigma_{xx})_{wood} = (8000\text{ ksi})(-200y)10^{-6} = -1.6y\text{ ksi} \tag{E1}$$

$$(\sigma_{xx})_{steel} = (30000\text{ ksi})(-200y)10^{-6} = -6y\text{ ksi} \tag{E2}$$

For the homogeneous cross section the stress distribution is given in Equation (E1), but for the laminated case it switches from Equation (E1) to Equation (E2), depending on the value of y. We can write the stress distribution for both cross sections as a function of y.

Homogeneous cross section:

$$\sigma_{xx} = -1.6y\text{ ksi} \qquad -0.75\text{ in.} \le y < 0.75\text{ in.} \tag{E3}$$

Laminated cross section:

$$\sigma_{xx} = \begin{cases} -6y\text{ ksi} & 0.5\text{ in.} < y \le 0.75\text{ in.} \\ -1.6y\text{ ksi} & -0.5\text{ in.} < y < 0.5\text{ in.} \\ -6y\text{ ksi} & -0.75\text{ in.} \le y < -0.5\text{ in.} \end{cases} \tag{E4}$$

Using Equations (E3) and (E4) the strains and stresses can be plotted as a function of y, as shown in Figure 6.10.

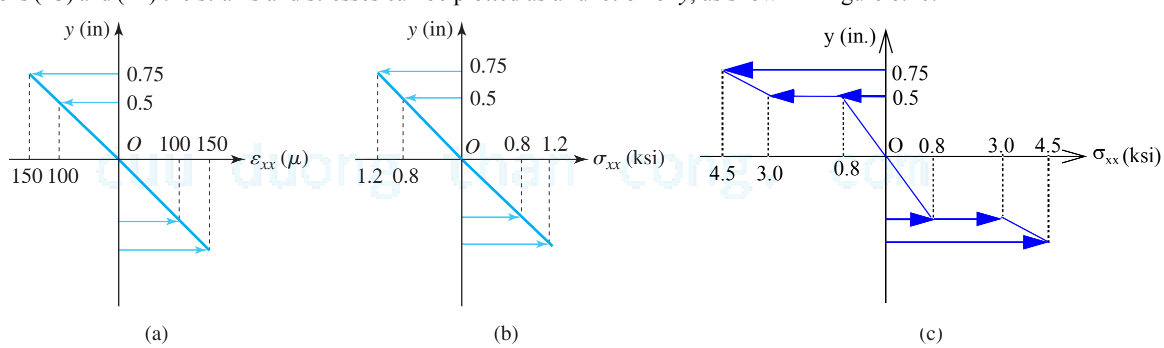


Figure 6.10 Strain and stress distributions in Example 6.4: (*a*) strain distribution; (*b*) stress distribution in homogeneous cross section; (*c*) stress distribution in laminated cross section.

(b) The thickness (dimension in the z direction) is 2 in. Hence we can write $dA = 2dy$. Noting that the stress distribution is symmetric, we can write the integral in Equation (6.1) as

$$M_z = -\int_{-0.75}^{0.75} y\sigma_{xx}(2dy) = -2\left[\int_0^{0.75} y\sigma_{xx}(2dy)\right] \tag{E5}$$

Homogeneous cross section: Substituting Equation (E3) into Equation (E5) and integrating, we obtain the equivalent internal moment.

$$M_z = -2\left[\int_0^{0.75} y(-1.6y\ \text{ksi})(2dy)\right] = 6.4\frac{y^3}{3}\bigg|_0^{0.75} = 6.4\frac{0.75^3}{3} \tag{E6}$$

ANS. $M_z = 0.9\ \text{in.} \cdot \text{kips}$

Laminated cross section: Substituting Equation (E4) into Equation (E5) and integrating, we obtain the equivalent internal moment.

$$M_z = -2\left[\int_0^{0.5} y(-1.6y)(2dy) + \int_{0.5}^{0.75} y(-6y)(2dy)\right] = 4\left(1.6\frac{y^3}{3}\bigg|_0^{0.5} + 6\frac{y^3}{3}\bigg|_{0.5}^{0.75}\right) \tag{E7}$$

ANS. $M_z = 2.64\ \text{in.} \cdot \text{kips}$

COMMENTS

1. As this example demonstrates, although the strain varies linearly across the cross section, the stress may not. In this example we considered material nonhomogeneity. In a similar manner we can consider other models, such as elastic–perfectly plastic model, or material models that have nonlinear stress–strain curves.
2. Figure 6.11 shows the stress distribution on the surface. The symmetry of stresses about the center results in a zero axial force.

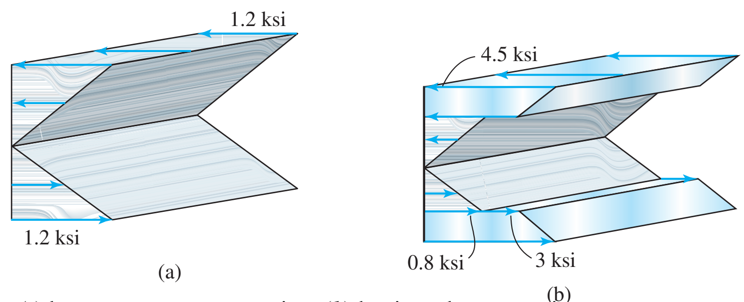


Figure 6.11 Surface stress distributions in Example 6.4 for (*a*) homogeneous cross section; (*b*) laminated cross section.

3. We can obtain the equivalent internal moment for a homogeneous cross section by replacing the triangular load by an equivalent load at the centroid of each triangle. We then find the equivalent moment, as shown in Figure 6.12. This approach is very intuitive. However, as the stress distribution becomes more complex, such as in a laminated cross section, or for more complex cross-sectional shapes, this intuitive approach becomes very tedious. The generalization represented by Equation (6.1) and the resulting formula can then simplify the calculations.

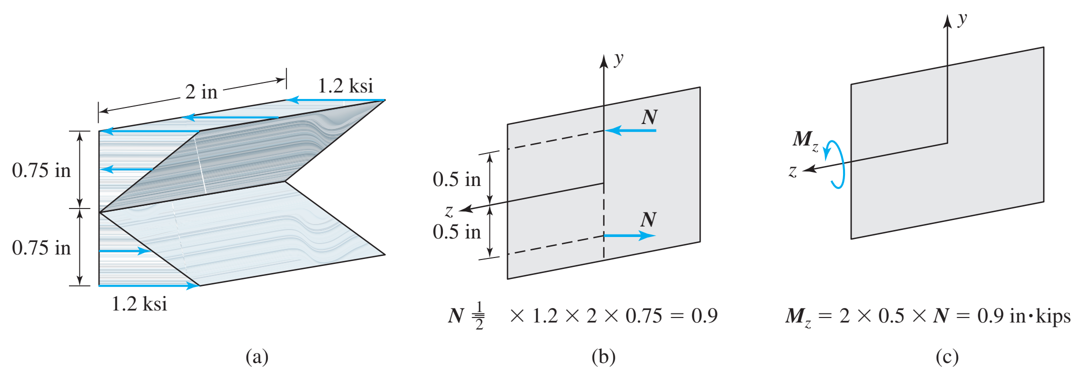


Figure 6.12 Statically equivalent internal moment in Example 6.4.

4. The relationship between the internal moment and the external loads can be established by drawing the appropriate free-body diagram for a particular problem. The relationship between internal and external moments depends on the free-body diagram and is independent of the material homogeneity.

PROBLEM SET 6.1

6.1 The rigid plate that is welded to the two bars in Figure P6.1 is rotated about the z axis, causing the two bars to bend. The normal strains in bars 1 and 2 were found to be $\varepsilon_1 = 2000\ \mu\text{in./in.}$ and $\varepsilon_2 = -1500\ \mu\text{in./in.}$ Determine the angle of rotation ψ.

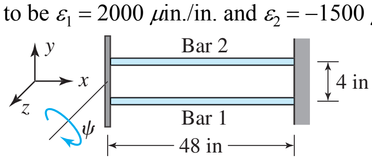


Figure P6.1

6.2 Determine the location h in Figure P6.2 at which a third bar in Problem 6.1 must be placed so that there is no normal strain in the third bar.

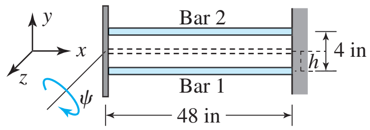


Figure P6.2

6.3 The two rigid plates that are welded to six bars in Figure P6.3 are rotated about the z axis, causing the six bars to bend. The normal strains in bars 2 and 5 were found to be zero. What are the strains in the remaining bars?

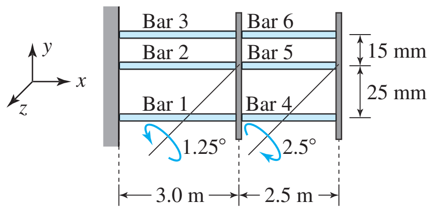


Figure P6.3

6.4 The strains in bars 1 and 3 in Figure P6.4 were found to be $\varepsilon_1 = 800\ \mu$ and $\varepsilon_3 = 500\ \mu$. Determine the strains in the remaining bars.

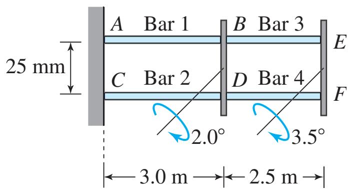


Figure P6.4

6.5 The rigid plate shown in Figure P6.5 was observed to rotate by 2° from the vertical plane due to the action of the external moment M_{ext} and force P, and the normal strain in bar 1 was found to be $\varepsilon_1 = 2000\ \mu$in./in. Both bars have a cross-sectional area $A = \frac{1}{2}$ in.^{2} and a modulus of elasticity $E = 30{,}000$ ksi. Determine the applied moment M_{ext} and force P.

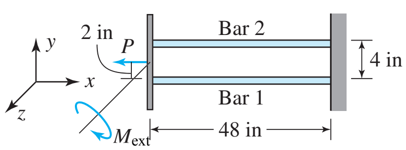


Figure P6.5

6.6 The rigid plate shown in Figure P6.6 was observed to rotate 1.25° from the vertical plane due to the action of the external moment M_{ext} and the force P. All three bars have a cross-sectional area $A = 100\ \text{mm}^2$ and a modulus of elasticity $E = 200$ GPa. If the strain in bar 2 was measured as zero, determine the external moment M_{ext} and the force P.

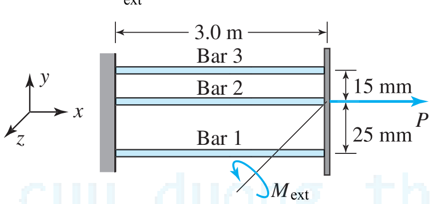


Figure P6.6

6.7 The rigid plates BD and EF in Figure P6.7 were observed to rotate by 2° and 3.5° from the vertical plane in the direction of applied moments. All bars have a cross-sectional area of $A = 125\ \text{mm}^2$. Bars 1 and 3 are made of steel $E_S = 200$ GPa, and bars 2 and 4 are made of aluminum $E_{al} = 70$ GPa. If the strains in bars 1 and 3 were found to be $\varepsilon_1 = 800\ \mu$ and $\varepsilon_3 = 500\ \mu$ determine the applied moment M_1 and M_2 and the forces P_1 and P_2 that act at the center of the rigid plates.

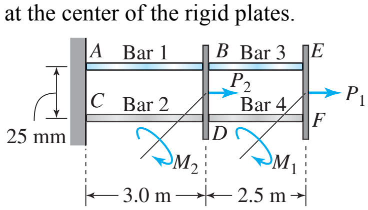


Figure P6.7

6.8 Three wooden beams are glued to form a beam with the cross-section shown in Figure 6.8. The normal strain due to bending about the z axis is ε_{xx} = -0.012y, where y is measured in meters. The modulus of elasticity of wood is 10 GPa. Determine the equivalent internal moment acting at the cross-section. Use t_W =20 mm, h =250 mm, t_F = 20 mm, and d= 125 mm.

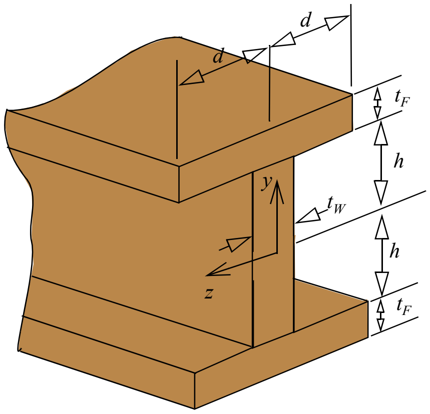


Figure P6.8

6.9 Three wooden beams are glued to form a beam with the cross-section shown in Figure 6.8. The normal strain at the cross due to bending about the z axis is ε_{xx} = -0.015y, where y is measured in meters. The modulus of elasticity of wood is 10 GPa. Determine the equivalent internal moment acting at the cross-section. Use t_W =10 mm, h =50 mm, t_F = 10 mm, and d= 25 mm.

6.10 Three wooden beams are glued to form a beam with the cross-section shown in Figure 6.8. The normal strain at the cross due to bending about the z axis is ε_{xx} = 0.02y, where y is measured in meters. The modulus of elasticity of wood is 10 GPa. Determine the equivalent internal moment acting at the cross-section. Use t_W =15 mm, h =200 mm, t_F = 20 mm, and d= 150 mm.

6.11 Steel strips (E_S = 30,000 ksi) are securely attached to wood (E_W = 2000 ksi) to form a beam with the cross section shown in Figure P6.11. The normal strain at the cross section due to bending about the z axis is $\varepsilon_{xx} = -100y\ \mu$, where y is measured in inches. Determine the equivalent internal moment $\boldsymbol{M}_z$. Use $d = 2$ in., $h_W = 4$ in., and $h_S = \frac{1}{8}$ in.

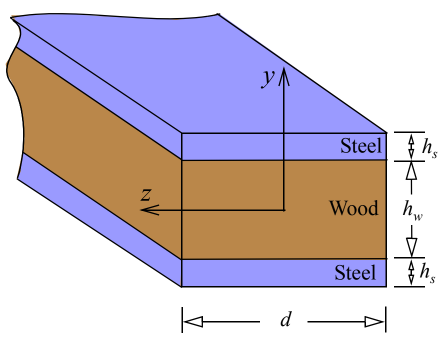


Figure P6.11

6.12 Steel strips (E_S = 30,000 ksi) are securely attached to wood (E_W = 2000 ksi) to form a beam with the cross section shown in Figure P6.11 . The normal strain at the cross section due to bending about the z axis is $\varepsilon_{xx} = -50y\ \mu$, where y is measured in inches. Determine the equivalent internal moment $\boldsymbol{M}_z$. Use $d = 1$ in., $h_W = 6$ in., and $h_S = \frac{1}{4}$ in.

6.13 Steel strips (E_S = 30,000 ksi) are securely attached to wood (E_W = 2000 ksi) to form a beam with the cross section shown in Figure P6.11 . The normal strain at the cross section due to bending about the z axis is $\varepsilon_{xx} = 200y\ \mu$, where y is measured in inches. Determine the equivalent internal moment $\boldsymbol{M}_z$. Use $d = 1$ in., $h_W = 2$ in., and $h_S = \frac{1}{16}$ in.

6.14 Steel strips (E_S = 200 GPa) are securely attached to wood (E_W = *10 GPa)* to form a beam with the cross section shown in Figure P6.14. The normal strain at the cross section due to bending about the *z axis is* $\varepsilon_{xx} = 0.02y$, where *y* is measured in meters. Determine the equivalent internal moment $\boldsymbol{M}_z$ .Use t_W = 15 mm, h_W = 200 mm, t_F = 20 mm, and d_F = 150 mm.

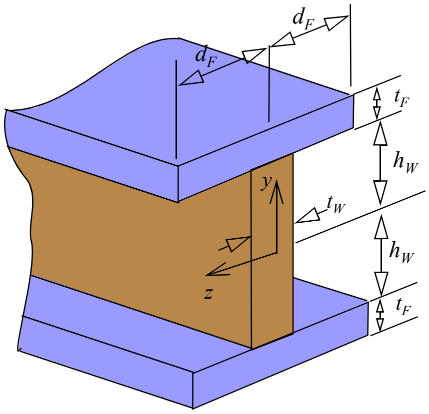


Figure P6.14

Stretch Yourself

6.15 A beam of rectangular cross section shown in Figure 6.15 is made from elastic-perfectly plastic material. If the stress distribution across the cross section is as shown determine the equivalent internal bending moment.

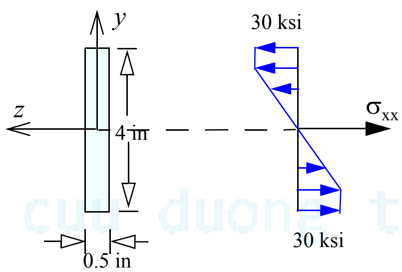


Figure P6.15

6.16 A rectangular beam cross section has the dimensions shown in Figure 6.16. The normal strain due to bending about the *z* axis was found to vary as $\varepsilon_{xx} = -0.01y$, with *y* measured in meters. Determine the equivalent internal moment that produced the given state of strain. The beam is made from elastic-perfectly plastic material that has a yield stress of σ_{yield}= 250 MPa and a modulus of elasticity E = 200 GPa. Assume material that the behaves in a similar manner in tension and compression (see Problem 3.152)

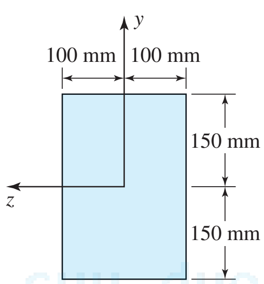


Figure P6.16

6.17 A rectangular beam cross section has the dimensions shown in Figure 6.16. The normal strain due to bending about the *z* axis was found to vary as $\varepsilon_{xx} = -0.01y$, with *y* measured in meters. Determine the equivalent internal moment that would produce the given strain. The beam is made from a bi-linear material that has a yield stress of σ_{yield}= 200 MPa, modulus of elasticity E_1 = 250 GPa, and E_2= 80 GPa. Assume that the material behaves in a similar manner in tension and compression (see Problem 3.153).

6.18 A rectangular beam cross section has the dimensions shown in Figure 6.16. The normal strain due to bending about the *z* axis was found to vary as $\varepsilon_{xx} = -0.01y$, with *y* measured in meters. Determine the equivalent internal moment that would produce the given strain. The beam material has a stress strain relationship given by $\sigma = 952\varepsilon^{0.2}$ MPa . Assume that the material behaves in a similar manner in tension and compression (see Problem 3.154).

6.2 THEORY OF SYMMETRIC BEAM BENDING

In this section we develop formulas for beam deformation and stress. We follow the procedure in Section 6.1 with variables in place of numbers. The theory will be subject to the following limitations:

1. The length of the member is significantly greater than the greatest dimension in the cross section.
2. We are away from the regions of stress concentration;
3. The variation of external loads or changes in the cross-sectional areas are gradual except in regions of stress concentration.
4. The cross section has a plane of symmetry. This limitation separates bending about the z axis from bending about the y axis. (See Problem 6.135 for unsymmetric bending.)
5. The loads are in the plane of symmetry. Load P_1 in Figure 6.13 would bend the beam as well as twist (rotate) the cross section. Load P_2, which lies in the plane of symmetry, will cause only bending. Thus, this limitation decouples the bending problem from the torsion problem[1].
6. The load direction does not change with deformation. This limitation is required to obtain a linear theory and works well as long as the deformations are small.
7. The external loads are not functions of time; that is, we have a static problem. (See Problems 7.50 and 7.51 for dynamic problems.)

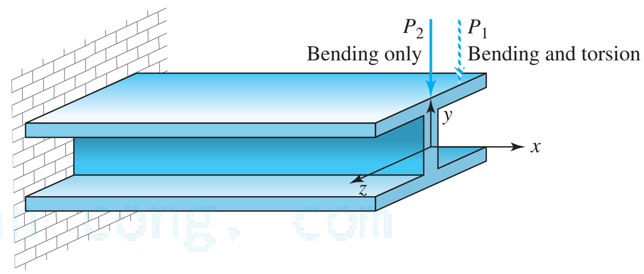


Figure 6.13 Loading in plane of symmetry.

Figure 6.14 shows a segment of a beam with the *x–y* plane as the plane of symmetry. The beam is loaded by transverse forces P_1 and P_2 in the *y* direction, moments M_1 and M_2 about the *z* axis, and a transverse distributed force $p_y(x)$. The distributed force $p_y(x)$ has units of force per unit length and is considered positive in the positive *y* direction. Because of external loads, a line on the beam deflects by v in the *y* direction.

The objectives of the derivation are:

1. To obtain a formula for bending normal stress σ_{xx} and bending shear stress τ_{xy} in terms of the internal moment $\boldsymbol{M_z}$ and the internal shear force $\boldsymbol{V_y}$.
2. To obtain a formula for calculating the beam deflection v(*x*).

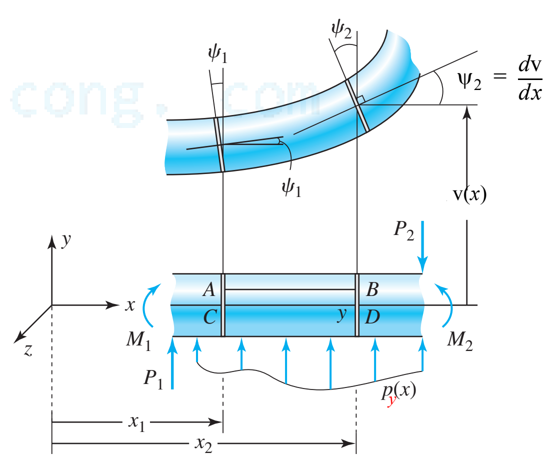


Figure 6.14 Beam segment.

[1]The separation of torsion from bending requires that the load pass through the *shear center*, which always lies on the axis of symmetry.

To account for the gradual variation of $p_y(x)$ and the cross-sectional dimensions, we will take $\Delta x = x_2 - x_1$ as infinitesimal distance in which these quantities can be treated as constants. The logic shown in Figure 6.15 and discussed in Section 3.2 will be used to develop the simplest theory for the bending of beams. Assumptions will be identified as we move from one step to the next. The assumptions identified as we move from each step are also points at which complexities can later be added, as discussed in examples and Stretch Yourself problems.

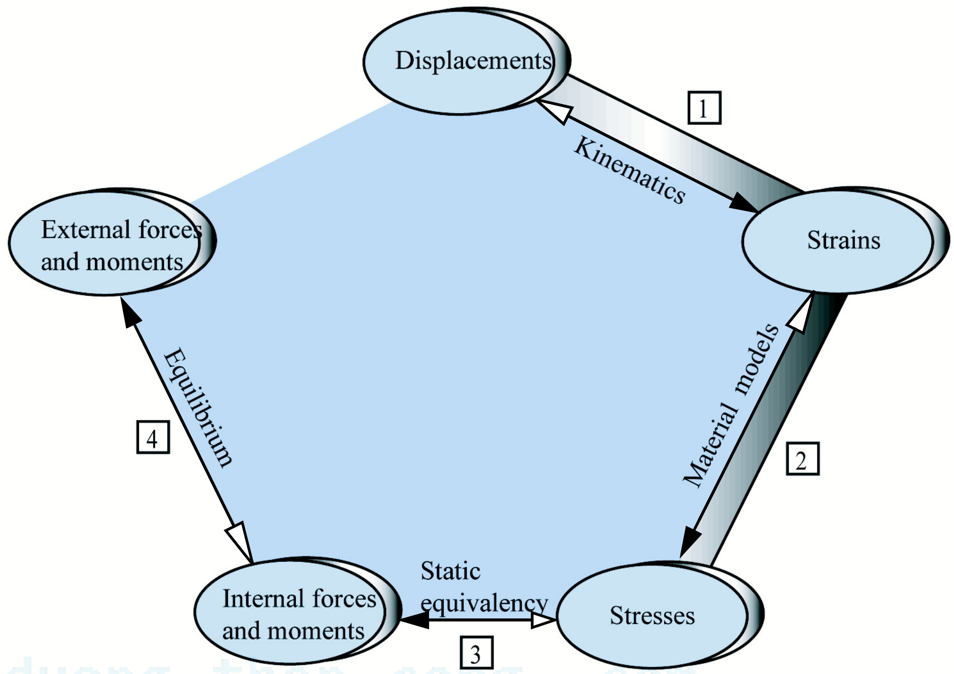


Figure 6.15 Logic in mechanics of materials.

6.2.1 Kinematics

In Example 6.1 we found the normal strains in bars welded to rigid plates rotating about the z axis. Here we state assumptions that will let us simulate the behavior of a cross section like that of the rigid plate. We will consider the experimental evidence justifying our assumptions and the impact of these assumptions on the theory.

Assumption 1: Squashing—that is, dimensional changes in the y direction—is significantly smaller than bending.
Assumption 2: Plane sections before deformation remain planes after deformation.
Assumption 3: Plane sections perpendicular to the beam axis remain *nearly* perpendicular after deformation.

Figure 6.16 shows a rubber beam with a grid on its surface that is bent by hand. Notice that the dimensional changes in the y direction are significantly smaller than those in the x direction, the basis of Assumption 1. The longer the beam, the better is the validity of Assumption 1. Neglecting dimensional changes in the y direction implies that the normal strain in the y direction is small[2] and can be neglected in the kinematic calculations; that is, $\varepsilon_{yy} = \partial v/\partial y \approx 0$. This implies that deflection of the beam v cannot be a function of y:

$$v = v(x) \tag{6.3}$$

Equation (6.3) implies that if we know the curve of one longitudinal line on the beam, then we know how all other longitudinal lines on the beam bend. The curve described by $v(x)$, called the **elastic curve**, will be discussed in detail in the next chapter.

[2]It is accounted for as the *Poisson effect*. However the normal strain in the y direction is not an independent variable and hence is negligible in kinematics.

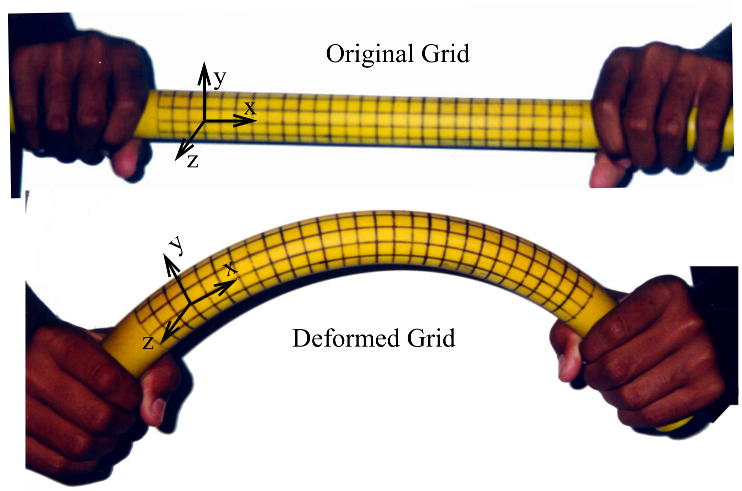


Figure 6.16 Deformation in bending. (Courtesy Professor J. B. Ligon.)

Figure 6.16 shows that lines initially in the y direction continue to remain straight but rotate about the z axis, validating Assumption 2. This implies that the displacement u varies linearly, as shown in Figure 6.17. In other words, the equation for u is

$$u = u_0 - \psi y \tag{6.4}$$

where u_0 is the axial displacement at $y = 0$ and ψ is the slope of the plane. (We accounted for uniform axial displacement u_0 in Chapter 4.) In order to study each problem independently, we will assume $u_0 = 0$. (See Problem 6.133 for $u_0 \neq 0$.)

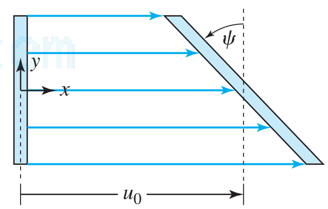


Figure 6.17 Linear variation of axial displacement u.

Figure 6.16 also shows that the right angle between the x and y directions is nearly preserved during bending, validating Assumption 3. This implies that the shear strain γ_{xy} is nearly zero. We cannot use this assumption in building theoretical models of beam bending if shear is important, such as in *sandwich beams* (see comment 3 in Example 6.7) and *Timoshenko beams* (see Problem 7.49). But Assumption 3 helps simplify the theory as it eliminates the variable ψ by imposing the constraint that the angle between the longitudinal direction and the cross section be always 90°. This is accomplished by relating ψ to v as described next.

6.2.2 Strain Distribution

Assumption 4: Strains are small.

The bending curve is defined by v(x). As shown in Figure 6.14, the angle of the tangent to the curve v(x) is equal to the rotation of the cross section when Assumption 3 is valid. For small strains, the tangent of an angle can be replaced by the angle itself, that is, $\tan\psi \approx \psi = dv/dx$. Substituting ψ and $u_0 = 0$ in Equation (6.4), we obtain

$$u = -y\frac{dv}{dx}(x) \tag{6.5}$$

Figure 6.18 shows the exaggerated deformed shape of a segment of the beam. The rotation of the right cross section is taken relative to the left. Thus, the left cross section is viewed as a fixed wall, as in Examples 6.1 and 6.2. We assume that line *CD* representing $y = 0$ has zero bending normal strain. The calculations of Example 6.2 show that the bending normal strain for line *AB* is given by

$$\varepsilon_{xx} = -\frac{y}{R} \tag{6.6a}$$

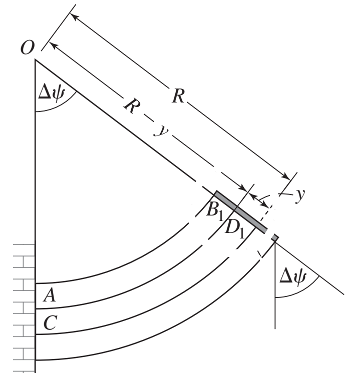


Figure 6.18 Normal strain calculations in symmetric bending.

We can also obtain the equation of bending normal strain by substituting Equation (6.5) into Equation (2.12a) to obtain

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(-y\frac{dv}{dx}(x)\right) \quad \text{or}$$

$$\boxed{\varepsilon_{xx} = -y\frac{d^2v}{dx^2}(x)} \tag{6.6b}$$

Equations 6.6a and 6.6b show that the bending normal strain ε_{xx} varies linearly with y and has a maximum value at either the top or the bottom of the beam. d^2v/dx^2 is the **curvature** of the beam, and its magnitude is equal to $1/R$, where R is the *radius of curvature*.

6.2.3 Material Model

In order to develop a simple theory for the bending of symmetric beams, we shall use the material model given by Hooke's law. We therefore make the following assumptions regarding the material behavior.

Assumption 5: The material is isotropic.
Assumption 6: The material is linearly elastic.[3]
Assumption 7: There are no inelastic strains.[4]

Substituting Equation (6.6b) into Hooke's law $\sigma_{xx} = E\varepsilon_{xx}$, we obtain

$$\sigma_{xx} = -Ey\frac{d^2v}{dx^2} \tag{6.7}$$

Though the strain is a linear function of y, we cannot say the same for stress. The modulus of elasticity E could change across the cross section, as in laminated structures.

6.2.4 Location of Neutral Axis

Equation (6.7) shows that the stress σ_{xx} is a function of y, and its value must be zero at $y = 0$. That is, the origin of y must be at the neutral axis. But where is the neutral axis on the cross section? Section 6.1.1 noted that the distribution of σ_{xx} is such that the total tensile force equals the total compressive force on a cross section, given by Equation (6.2). d^2v/dx^2 is a function of x only, whereas the integration is with respect to y and z ($dA = dy\,dz$). Substituting Equation (6.7) into Equation (6.2), we obtain

[3]See Problems 6.57 and 6.58 for nonlinear material behavior.
[4]Inelastic strains could be due to temperature, humidity, plasticity, viscoelasticity, etc. See Problem 6.134 for including thermal strains.

$$-\int_A Ey\frac{d^2v}{dx^2}(x)\,dA = -\frac{d^2v}{dx^2}(x)\int_A Ey\,dA = 0 \tag{6.8a}$$

The integral in Equation (6.8a) must be zero as shown in Equation (6.8b), because a zero value of d^2v/dx^2 would imply that there is no bending.

$$\int_A yE\,dA = 0 \tag{6.8b}$$

Equation (6.8b) is used for determining the origin (and thus the neutral axis) in composite beams. Consistent with the motivation of developing the simplest possible formulas, we would like to take *E* outside the integral. In other words, *E* should not change across the cross section, as implied in Assumption 8:

Assumption 8: The material is homogeneous across the cross section[5] of a beam.

Equation (6.8b) can be written as

$$E\int_A y\,dA = 0 \tag{6.8c}$$

In Equation (6.8c) either *E* or $\int_A y\,dA$ must be zero. As *E* cannot be zero, we obtain

$$\boxed{\int_A y\,dA = 0} \tag{6.9}$$

Equation (6.9) is satisfied if *y* is measured from the centroid of the cross section. That is, the origin must be at the centroid of the cross section of a linear, elastic, isotropic, and homogeneous material. Equation (6.9) is the same as Equation (4.12a) in axial members. However, in axial problems we required that the internal bending moment that generated Equation (4.12a) be zero. Here it is zero axial force that generates Equation (6.9). Thus by choosing the origin to be the centroid, we decouple the axial problem from the bending problem.

From Equations (6.7) and (6.9) two conclusion follow for cross sections constructed from linear, elastic, isotropic, and homogeneous material:

- The bending normal stress σ_{xx} varies linearly with *y*.
- The bending normal stress σ_{xx} has maximum value at the point farthest from the centroid of the cross section.

The point farthest from the centroid is the top surface or the bottom surface of the beam. Example 6.5 demonstrates the use of our observations.

EXAMPLE 6.5

The maximum bending normal strain on a homogeneous steel (E = 30,000 ksi) cross section shown in Figure 6.19 was found to be ε_{xx} = +1000 μ. Determine the bending normal stress at point *A*.

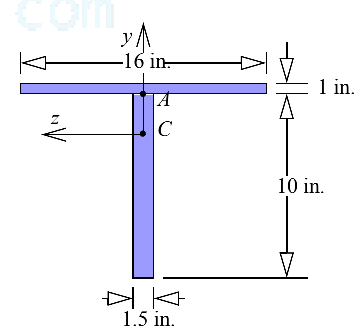


Figure 6.19 T cross section in Example 6.5.

[5]See Problems 6.55 and 6.56 on composite beams for nonhomogeneous cross sections.

PLAN

The centroid C of the cross section can be found where the bending normal stress is zero. The maximum bending normal stress will be at the point farthest from the centroid. Its value can be found from the given strain and Hooke's law. Knowing the normal stress at two points of a linear distribution, we can find the normal stress at point A.

SOLUTION

Figure 6.20a can be used to find the centroid η_c of the cross section.

$$\eta_c = \frac{\sum_i \eta_i A_i}{\sum_i A_i} = \frac{(5\text{ in.})(10\text{ in.})(1.5\text{ in.}) + (10.5\text{ in.})(16\text{ in.})(1\text{ in.})}{(10\text{ in.})(1.5\text{ in.}) + (16\text{ in.})(1\text{ in.})} = 7.84\text{ in.} \tag{E1}$$

The maximum bending normal stress will be at point B, which is farthest from centroid, and its value can be found as

$$\sigma_B = E\varepsilon_{max} = (30{,}000\text{ ksi})(1000)(10^{-6}) = 30\text{ ksi} \tag{E2}$$

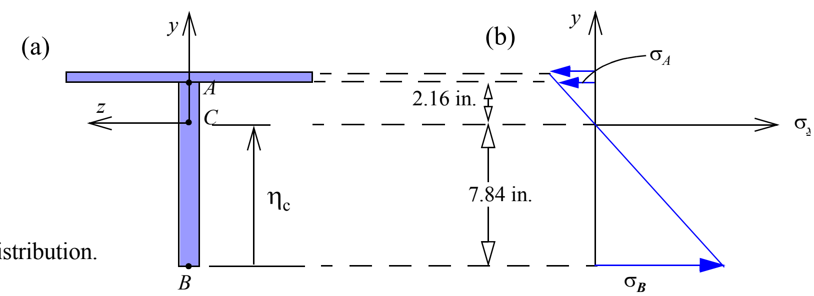


Figure 6.20 (a) Centroid location (b) Linear stress distribution.

The linear distribution of bending normal stress across the cross section can be drawn as shown in Figure 6.20b. By similar triangles we obtain

$$\frac{\sigma_A}{2.16\text{ in.}} = \frac{30\text{ ksi}}{7.84\text{ in.}} \tag{E3}$$

ANS. $\sigma_A = 8.27\text{ ksi (C)}$

COMMENT

1. The stress distribution in Figure 6.20b can be represented as $\sigma_{xx} = -3.82y$ ksi. The equivalent internal moment can be found using Equation (6.1).

6.2.5 Flexure Formulas

Note that d^2v/dx^2 is a function of x only, while integration is with respect to y and z ($dA = dy\,dz$). Substituting σ_{xx} from Equation (6.8b) into Equation (6.1), we therefore obtain

$$\boldsymbol{M}_z = \int_A Ey^2 \frac{d^2v}{dx^2}\,dA = \frac{d^2v}{dx^2}\int_A Ey^2\,dA \tag{6.10}$$

With material homogeneity (Assumption 8), we can take E outside the integral in Equation (6.10) to obtain

$$\boldsymbol{M}_z = E\frac{d^2v}{dx^2}\int_A y^2\,dA \quad \text{or}$$

$$\boxed{\boldsymbol{M}_z = EI_{zz}\frac{d^2v}{dx^2}} \tag{6.11}$$

where $I_{zz} = \int_A y^2\,dA$ is the second area moment of inertia about the z axis passing through the centroid of the cross section. The quantity EI_{zz} is called the **bending rigidity** of a beam cross section. The higher the value of EI_{zz}, the smaller will be the deformation (curvature) of the beam; that is, the beam rigidity increase. A beam can be made more rigid either by choosing a stiffer material (a higher value of E) or by choosing a cross sectional shape that has a large area moment of inertia (see Example 6.7).

Solving for d^2v/dx^2 in Equation (6.11) and substituting into Equation (6.7), we obtain the *bending* stress formula or *flexure* stress formula:

$$\sigma_{xx} = -\frac{M_z y}{I_{zz}} \tag{6.12}$$

The subscript z emphasizes that the bending occurs about the z axis. If bending occurs about the y axis, then y and z in Equation (6.12) are interchanged, as elaborated in Section 10.1 on combined loading.

6.2.6 Sign Conventions for Internal Moment and Shear Force

Equation (6.1) allowed us to replace the normal stress σ_{xx} by a statically equivalent internal bending moment. The normal stress σ_{xx} is positive on two surfaces; hence the equivalent internal bending moment is positive on two surfaces, as shown in Figure 6.21. If we want the formulas to give the correct signs, then we must follow a sign convention for the internal moment when we draw a free body diagram: At the imaginary cut the internal bending moment must be drawn in the positive direction.

Sign Convention: The direction of positive internal moment M_z on a free-body diagram must be such that it puts a point in the positive y direction into compression.

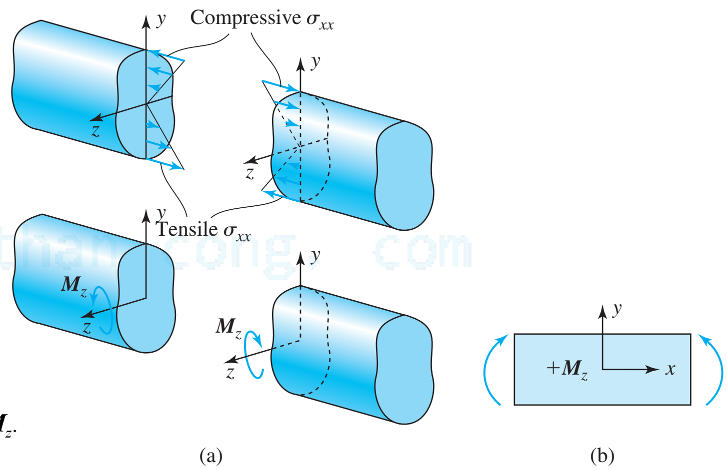


Figure 6.21 Sign convention for internal bending moment M_z.

M_z may be found in either of two possible ways as described next (see also Example 6.8).

1. In one method, on a free-body diagram M_z is always drawn according to the sign convention. The equilibrium equation is then used to get a positive or negative value for M_z. Positive values of stress σ_{xx} from Equation (6.12) are tensile, and negative values of σ_{xx} are compressive.
2. Alternatively, M_z is drawn at the imaginary cut in a direction that equilibrates the external loads. Since inspection is being used in determining the direction of M_z, Equation (6.12) can determine only the magnitude. The tensile and compressive nature of σ_{xx} must be determined by inspection.

Figure 6.22 shows a cantilever beam loaded with a transverse force P. An imaginary cut is made at section AA, and a free-body diagram is drawn. For equilibrium it is clear that we need an internal shear force V_y, which is possible only if there is a nonzero shear stress τ_{xy}. By Hooke's law this implies that the shear strain γ_{xy} cannot be zero. Assumption 3 implied that shear strain was small but not zero. In beam bending, a check on the validity of the analysis is to compare the maximum shear stress τ_{xy} to the maximum normal stress σ_{xx} for the entire beam. If the two stress components are comparable, then the shear strain cannot be neglected in kinematic considerations, and our theory is not valid.

- The maximum normal stress σ_{xx} in the beam should be nearly an order of magnitude greater than the maximum shear stress τ_{xy}.

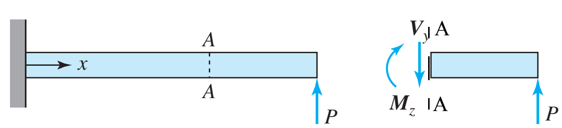


Figure 6.22 Internal forces and moment necessary for equilibrium.

The internal shear force is defined as

$$V_y = \int_A \tau_{xy}\, dA \qquad (6.13)$$

In Section 1.3 we studied the use of subscripts to determine the direction of a stress component, which we can now use to determine the positive direction of τ_{xy}. According to this second sign convention, the equivalent shear force V_y is in the same direction as the shear stress τ_{xy}.

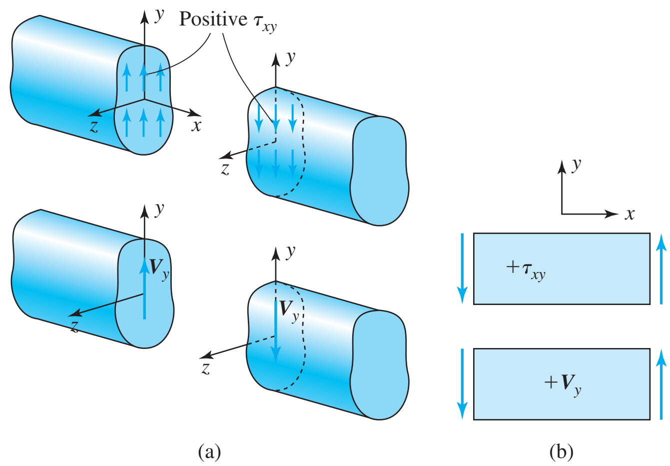


Figure 6.23 Sign convention for internal shear force V_y.

Sign Convention: The direction of positive internal shear force V_y on a free-body diagram is in the direction of the positive shear stress on the surface.[6]

Figure 6.23 shows the positive direction for the internal shear force V_y. The sign conventions for the internal bending moment and the internal shear force are tied to the coordinate system because the sign convention for stresses is tied to the coordinate system. But we are free to choose the directions for our coordinate system. Example 6.6 elaborates this comment further.

EXAMPLE 6.6

Figure 6.24 shows a beam and loading in three different coordinate systems. Determine for the three cases the internal shear force and bending moment at a section 36 in. from the free end using the sign conventions described in Figures 6.21 and 6.23.

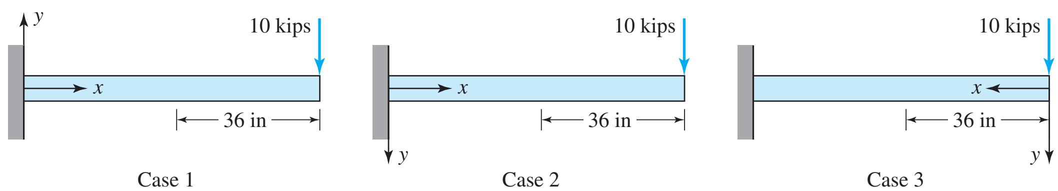


Figure 6.24 Example 6.6 on sign convention.

PLAN

We make an imaginary cut at 36 in. from the free end and take the right-hand part in drawing the free-body diagram. We draw the shear force and bending moment for each of the three cases as per our sign convention. By writing equilibrium equations we obtain the values of the shear force and the bending moment.

SOLUTION

We draw three rectangles and the coordinate axes corresponding to each of the three cases, as shown in Figure 6.25. Point *A* is on the surface that has an outward normal in the positive *x* direction, and hence the force will be in the positive *y* direction to produce a positive shear stress. Point *B* is on the surface that has an outward normal in the negative *x* direction, and hence the force will be in the negative *y* direction to produce a positive shear stress. Point *C* is on the surface where the *y* coordinate is positive. The moment direction is shown to put this surface into compression.

[6]Some mechanics of materials books use an opposite direction for a positive shear force. This is possible because Equation (6.13) is a definition, and a minus sign can be incorporated into the definition. Unfortunately positive shear force and positive shear stress are then opposite in direction, causing problems with intuitive understanding.

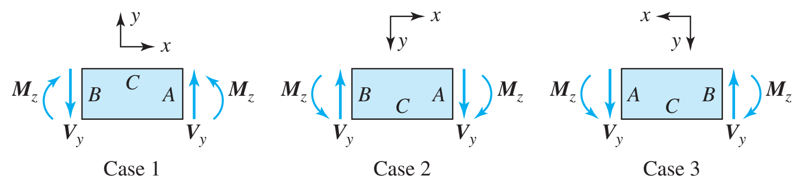


Figure 6.25 Positive shear forces and bending moments in Example 6.6.

Figure 6.26 shows the free body diagram for the three cases with the shear forces and bending moments drawn on the imaginary cut as shown in Figure 6.25. By equilibrium of forces in the *y* direction we obtain the shear force values. By equilibrium of moment about point *O* we obtain the bending moments for each of the three cases as shown in Table 6.1.

TABLE 6.1 **Results for Example 6.6.**

Case 1	*Case 2*	*Case 3*
$V_y = -10$ kips	$V_y = 10$ kips	$V_y = -10$ kips
$M_z = -360$ in.·kips	$M_z = 360$ in.·kips	$M_z = 360$ in.·kips

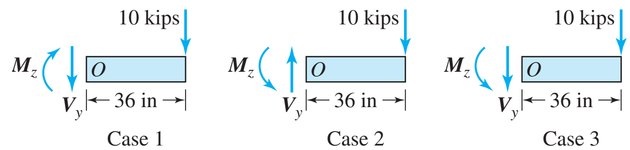


Figure 6.26 Free-body diagrams in Example 6.6.

COMMENTS

1. In Figure 6.26 we drew the shear force and bending moment directions without consideration of the external force of 10 kips. The equilibrium equations then gave us the correct signs. When we substitute these internal quantities, with the proper signs, into the respective stress formulas, we will obtain the correct signs for the stresses.
2. Suppose we draw the shear force and the bending moment in a direction such that it satisfies equilibrium. Then we shall always obtain positive values for the shear force and the bending moment, irrespective of the coordinate system. In such cases the sign for the stresses will have to be determined intuitively, and the stress formulas should be used only for the magnitude. To reap the benefit of both approaches, the internal quantities should be drawn using the sign convention, and the answers should be checked intuitively.
3. All three cases show that the shear force acts upward and the bending moment is counterclockwise, which are the directions for equilibrium.

EXAMPLE 6.7

The two square beam cross sections shown in Figure 6.27 have the same material cross-sectional area *A*. Show that the hollow cross section has a higher area moment of inertia about the *z* axis than the solid cross section.

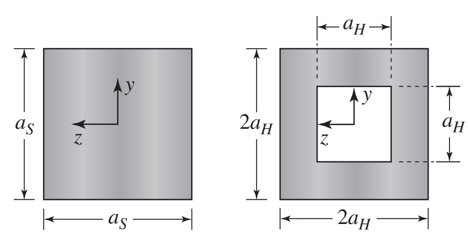


Figure 6.27 Cross sections in Example 6.7.

PLAN

We can find dimensions a_S and a_H in terms of the cross-sectional area *A*. Then we can find the area moments of inertia in terms of *A* and compare.

SOLUTION

The dimensions a_S and a_H in terms of area can be found as

$$A_S = a_S^2 = A \quad \text{or} \quad a_S = \sqrt{A} \quad \text{and} \quad A_H = (2a_H)^2 - a_H^2 = 3a_H^2 = A \quad \text{or} \quad a_H = \sqrt{A/3} \tag{E1}$$

Let I_S and I_H represent the area moments of inertia about the *z* axis for the solid cross section and the hollow cross section, respectively. We can find I_S and I_H in terms of area *A* as

$$I_S = \frac{1}{12}a_S a_S^3 = \frac{1}{12}A^2 \quad \text{and} \quad I_H = \frac{1}{12}(2a_H)(2a_H)^3 - \frac{1}{12}a_H a_H^3 = \frac{15}{12}a_H^4 = \frac{15}{12}\left(\frac{A}{3}\right)^2 = \frac{5}{36}A^2 \tag{E2}$$

Dividing I_H by I_S we obtain

$$\frac{I_H}{I_S} = \frac{5}{3} = 1.677 \tag{E3}$$

ANS. As $I_H > I_S$ the area moment of inertia for the hollow beam is greater than that of the solid beam for the same amount of material.

COMMENTS

1. The hollow cross section has a higher area moment of inertia for the same cross-sectional area. From Equations (6.11) and (6.12) this implies that the hollow cross section will have lower stresses and deformation. Alternatively, a hollow cross section will require less material (and be lighter in weight) giving the same area moment of inertia. This observation plays a major role in the design of beam shapes. Figure 6.28 shows some typical steel beam cross sections used in structures. Notice that in each case material from the region near the centroid is removed. Cross sections so created are thin near the centroid. This thin region near the centroid is called the **web**, while the wide material near the top or bottom is referred to as the **flange**. Section C.6 in Appendix has tables showing the geometric properties of some structural steel members.

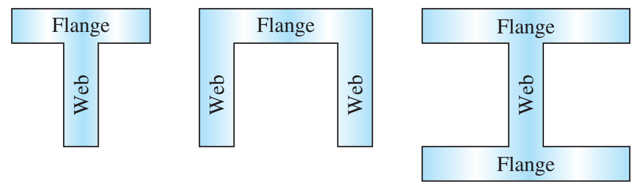


Figure 6.28 Metal beam cross sections.

2. We know that the bending normal stress is zero at the centroid and maximum at the top or bottom surfaces. We take material near the centroid, where it is not severely stressed, and move it to the top or bottom surface, where stress is maximum. In this way, we use material where it does the most good in terms of carrying load. This phenomenological explanation is an alternative explanation for the design of the cross sections shown in Figure 6.28. It is also the motivation in design of **sandwich beams**, in which two stiff panels are separated by softer and lighter core material. Sandwich beams are common in the design of lightweight structures such as aircrafts and boats.
3. Wooden beams are usually rectangular as machining costs do not offset the saving in weight.

EXAMPLE 6.8

An S180 × 30 steel beam is loaded and supported as shown in Figure 6.29. Determine: (a) The bending normal stress at a point *A* that is 20 mm above the bottom of the beam. (b) The maximum compressive bending normal stress in a section 0.5 m from the left end.

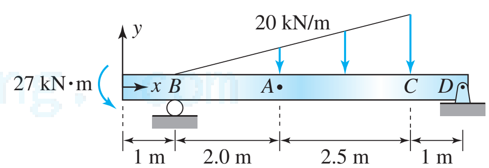


Figure 6.29 Beam in Example 6.8.

PLAN

From Section C.6 we can find the cross section, the centroid, and the moment of inertia. Using free body diagram for the entire beam, we can find the reaction force at B. Making an imaginary cut through *A* and drawing the free body diagram, we can determine the internal moment. Using Equation (6.12) we determine the bending normal stress at point *A* and the maximum bending normal stress in the section.

SOLUTION

From Section C.6 we obtain the cross section of S180 × 30 shown in Figure 6.30*a* and the area moment of inertia:

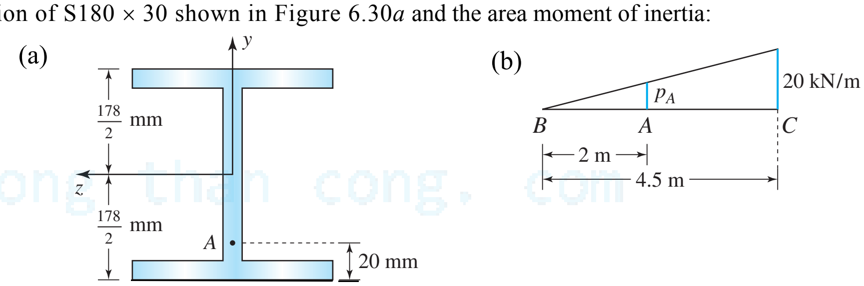


Figure 6.30 (a) S180 x 30 cross section in Example 6.8. (b) Intensity of distributed force at point *A* in Example 6.8

$$I_{zz} = 17.65(10^6)\ \text{mm}^4 \tag{E1}$$

The coordinates of point *A* can be found from Figure 6.30*a*, as shown in Equation (E2). The maximum bending normal stress will occur at the top or at the bottom of the cross section.The *y* coordinates are

$$y_A = -\left(\frac{178\ \text{mm}}{2} - 20\ \text{mm}\right) = -69\ \text{mm} \qquad y_{max} = \pm\frac{178\ \text{mm}}{2} = \pm 89\ \text{mm} \tag{E2}$$

We draw the free-body diagram of the entire beam with distributed load replaced by a statically equivalent load placed at the centroid of the load as shown in Figure 6.31a. By equilibrium of moment about point *D* we obtain R_B

$$R_B\ (5.5\ \text{m}) - (27\ \text{kN}\cdot\text{m}) - (45\ \text{kN})\ (2.5\ \text{m}) = 0 \qquad \text{or} \qquad R_B = 25.36\ \text{kN} \tag{E3}$$

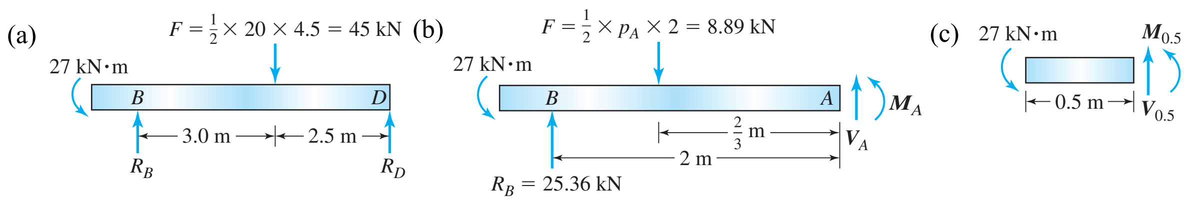


Figure 6.31 Free-body diagrams in Example 6.8 for (a) entire beam (b) calculation of M_A (c) calculation of $M_{0.5}$.

Figure 6.30*b* shows the variation of distributed load. The intensity of the distributed load acting on the beam at point *A* can be found from similar triangles,

$$\frac{p_A}{2\text{ m}} = \frac{20\text{ kN/m}}{4.5\text{ m}} \quad \text{or} \quad p_A = 8.89\text{ kN/m} \tag{E4}$$

We make an imaginary cut through point *A* in Figure 6.29 and draw the internal bending moment and the shear force using our sign convention. We also replace that portion of the distributed load acting at left of *A* by an equivalent force to obtain the free-body diagram shown in Figure 6.31b. By equilibrium of moment at point *A* we obtain the internal moment.

$$M_A + (27\text{ kN}\cdot\text{m}) - (25.36\text{ kN})(2\text{ m}) + (8.89\text{ kN})\left(\frac{2}{3}\text{ m}\right) = 0 \quad \text{or} \quad M_A = 17.8\text{ kN}\cdot\text{m} \tag{E5}$$

(a) Using Equations (6.12) we obtain the bending normal stress at point *A*.

$$\sigma_A = -\frac{M_A y_A}{I_{zz}} = -\frac{[17.8(10^3)\text{ N}\cdot\text{m}][-69(10^{-3})\text{ m}]}{17.65(10^{-6})\text{ m}^4} = 69.6(10^6)\text{ N/m}^2 \tag{E6}$$

ANS. $\sigma_A = 69.6$ MPa (T)

(b) We make an imaginary cut at 0.5 m from the left, draw the internal bending moment and the shear force using our sign convention to obtain the free body diagram shown in Figure 6.30c. By equilibrium of moment we obtain

$$M_{0.5} = -27\text{ kN}\cdot\text{m} \tag{E7}$$

The maximum compressive bending normal stress will be at the bottom of the beam, where $y = -88.9\ (10^{-3})$ m. Its value can be calculated as

$$\sigma_{0.5} = -\frac{-[27(10^3)\text{ N}\cdot\text{m}][-89(10^{-3})\text{ m}]}{17.65(10^{-6})\text{ m}^4} \tag{E8}$$

ANS. $\sigma_{0.5} = 136.1$ MPa (C)

COMMENT

1. For an *intuitive check* on the answer, we can draw an approximate deformed shape of the beam, as shown in Figure 6.32. We start by drawing the approximate shape of the bottom surface (or the top surface). At the left end the beam deflects downward owing to the applied moment. At the support point *B* the deflection must be zero. Since the slope of the beam must be continuous (otherwise a corner will be formed), the beam has to deflect upward as one crosses *B*. Now the externally distributed load pushes the beam downward. Eventually the beam will deflect downward, and finally it must have zero deflection at the support point *D*. The top surface is drawn parallel the bottom surface.

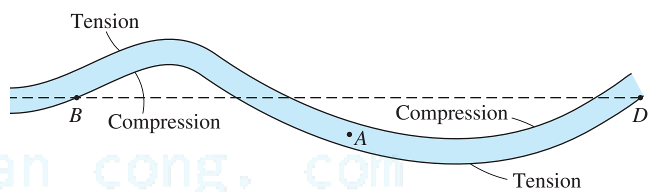


Figure 6.32 Approximate deformed shape of beam in Example 6.8.

2. By inspection of Figure 6.32 we see that point *A* is in the region where the bottom surface is in tension and the top surface in compression. If point *A* were closer to the inflection point, then we would have greater difficulty in assessing the situation. This once more emphasizes that intuitive checks are valuable but their conclusions must be viewed with caution.

Consolidate your knowledge

1. Identity five examples of beams from your daily life.
2. With the book closed, derive Equations (6.11) and (6.12), listing all the assumptions as you go along.

MoM in Action: Suspension Bridges

The Golden Gate Bridge (Figure 6.33*a*) opened May 27, 1937, spanning the opening of San Francisco Bay. More than 100,000 vehicles cross it every day, and more than 9 million visitors come to see it each year. The first bridge to span the Tacoma Narrows, between the Olympic peninsula and the Washington State mainland, opened just three years later, on July 1, 1940. It quickly acquired the name *Galloping Gertie* (Figure 6.33*b*) for its vertical undulations and twisting of the bridge deck in even moderate winds. Four months later, on November 7, it fell. The two suspension bridges, one famous, the other infamous, are a story of pushing design limits to cut cost.

Bridges today frequently have spans of up to 7000 ft and high clearances, for large ships to pass through. But suspension bridges are as old as the vine and rope bridges (Figure 6.33*c*) used across the world to ford rivers and canyons. Simply walking on rope bridges can cause them to sway, which can be fun for a child on a playground but can make a traveler very uncomfortable crossing a deep canyon. In India in the 4th century C.E., cables were introduced – first of plaited bamboo and later iron chains – to increase rigidity and decrease swaying. But the modern form, in which a roadway is suspended by cables, came about in the early nineteenth century in England, France, and America to bridge navigable streams. Still, early bridges were susceptible to stability and strength failures from wind, snow, and droves of cattle. John Augustus Roebling solved the problem, first in bridging Niagara Falls Gorge and again with his masterpiece—the Brooklyn Bridge, completed in 1883. Roebling increased rigidity and strength by adding on either side, a truss underneath the roadway.

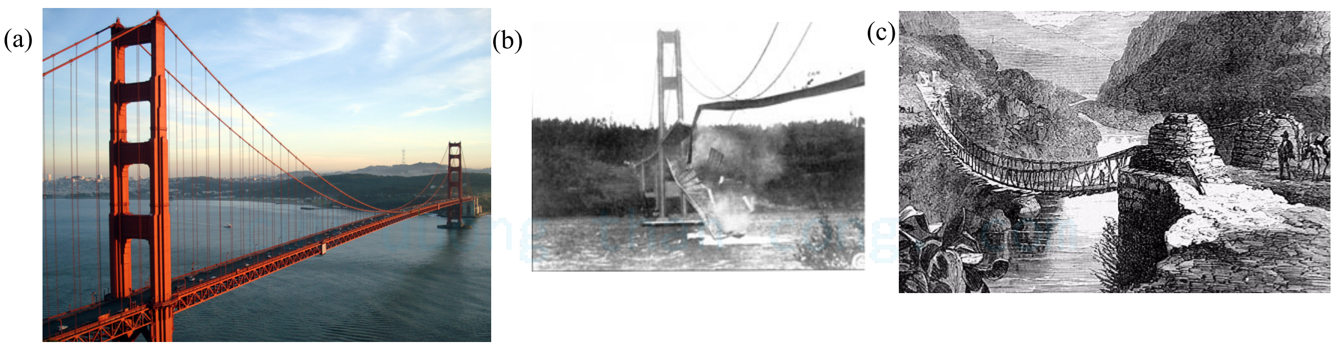


Figure 6.33 Suspension bridges: (a) Golden Gate (Courtesy Mr. Rich Niewiroski Jr.); (b) Galloping Gertie collapse; (c) Inca's rope bridge

Clearly engineers have long been aware of the impact of wind and traffic loads on the strength and motion of suspension bridges. Galloping Gertie was strong enough to withstanding bending stresses from winds of 120 mph. However, the cost of public works is always a serious consideration, and in case of Galloping Gertie it led to design decisions with disastrous consequences. The six-lane Golden Gate Bridge is 90 feet wide, has a bridge-deck depth of 25 feet and a center-span length to width ratio of 47:1. Galloping Gertie's two lanes were only 27 feet wide, a bridge-deck depth of only 8 feet, and center-span length to width ratio of 72:1. Thus, the bending rigidity (*EI)* and torsional rigidity (*GJ*) per unit length of Galloping Gertie were significantly less than the Golden Gate bridge. To further save on construction costs, the roadway was supported by solid I-beam girders, which unlike the open lattice of Golden Gate did not allow wind to pass through it but rather over and under it— that is, the roadway behaved like a wing of a plane. The bridge collapsed in a wind of 42 mph, and torsional and bending rigidity played a critical role.

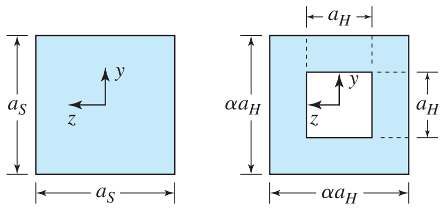
There are two kinds of aerodynamic forces: *lift*, which makes planes rise into air, and *drag*, a dissipative force that helps bring the plane back to the ground. Drag and lift forces depend strongly on the wind direction relative to the structure. If the structure twists, then the relative angle of the wind changes. The structure's rigidity resists further deformation due to changes in torsional and bending loads. However, when winds reach the *flutter speed*, torsional and bending deformation couple, with forces and deformations feeding each other till the structure breaks. This aerodynamic instability, known as *flutter*, was not understood in bridge design in 1940.

Today, wind-tunnel tests of bridge design are mandatory. A Tacoma Narrows Bridge with higher bending and torsional rigidity and an open lattice roadway support was built in 1950. Suspension bridges are as popular as ever. The Pearl Bridge built in 1998, linking Kobe, Japan, with Awaji-shima island has the world's longest center span at 6532 ft. Its mass dampers swing to counter earthquakes and wind. Galloping Gertie, however, will be remembered for the lesson it taught in design decisions that are penny wise but pound foolish.

PROBLEM SET 6.2

Second area moments of inertia

6.19 A solid and a hollow square beam have the same cross-sectional area A, as shown in Figure P6.19. Show that the ratio of the second area moment of inertia for the hollow beam I_H to that of the solid beam I_S is given by the equation below.



$$\frac{I_H}{I_S} = \frac{\alpha^2 + 1}{\alpha^2 - 1}$$

Figure P6.19

6.20 Figure P6.20a shows four separate wooden strips that bend independently about the neutral axis passing through the centroid of each strip. Figure 6.15b shows the four strips glued together and bending as a unit about the centroid of the glued cross section. (a) Show that $I_G = 16I_S$, where I_G is the area moment of inertia for the glued cross section and I_S is the total area moment of inertia of the four separate beams. (b) Also show that $\sigma_G = \sigma_S/4$, where σ_G and σ_S are the maximum bending normal stresses at any cross section for the glued and separate beams, respectively.

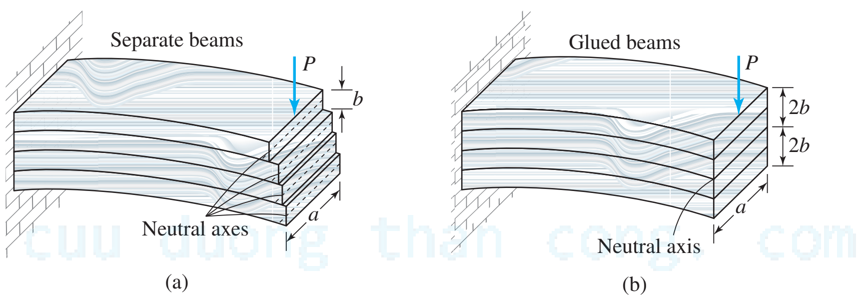


Figure P6.20

6.21 The cross sections of the beams shown in Figure P6.21 is constructed from thin sheet metal of thickness t. Assume that the thickness $t \ll a$. Determine the second area moments of inertia about an axis passing through the centroid in terms of a and t.

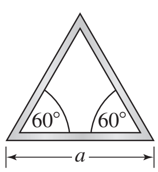


Figure P6.21

6.22 The cross sections of the beams shown in Figure P6.22 is constructed from thin sheet metal of thickness t. Assume that the thickness $t \ll a$. Determine the second area moments of inertia about an axis passing through the centroid in terms of a and t.

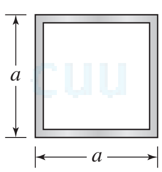


Figure P6.22

6.23 The cross sections of the beams shown in Figure P6.23 is constructed from thin sheet metal of thickness t. Assume that the thickness $t \ll a$. Determine the second area moments of inertia about an axis passing through the centroid in terms of a and t.

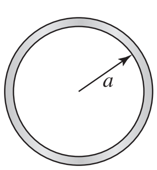


Figure P6.23

6.24 The same amount of material is used for constructing the cross sections shown in Figures P6.21, P6.22, and P6.23. Let the maximum bending normal stresses be σ_T, σ_S, and σ_C for the triangular, square, and circular cross sections, respectively. For the same moment-carrying capability determine the proportional ratio of the maximum bending normal stresses; that is, σ_T: σ_S: σ_C. What is the proportional ratio of the section moduli?

Normal stress and strain variations across a cross section

6.25 Due to bending about the z axis the normal strain at point A on the cross section shown in Figures P6.25 is $\varepsilon_{xx} = 200\ \mu$. The modulus of elasticity of the beam material is E = 8000 ksi. Determine the maximum tensile and compressive normal stress on the cross-section.

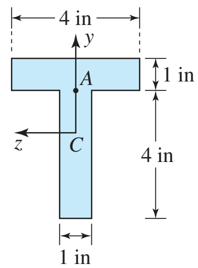


Figure P6.25

6.26 Due to bending about the z axis the maximum bending normal stress on the cross section shown in Figures P6.26 was found to be 40 ksi (C). The modulus of elasticity of the beam material is E = 30,000 ksi. Determine (a) the bending normal strain at point A. (b) the maximum bending tensile stress.

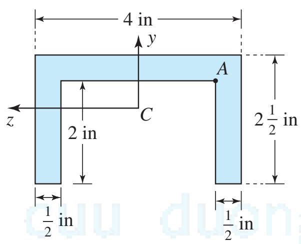


Figure P6.26

6.27 A composite beam cross section is shown in Figure 6.27. The bending normal strain at point A due to bending about the z axis was found to be $\varepsilon_{xx} = -200\ \mu$. The modulus of elasticity of the two materials are E_1 = 200 GPa, E_2 = 70 GPa. Determine the maximum bending stress in each of the two materials.

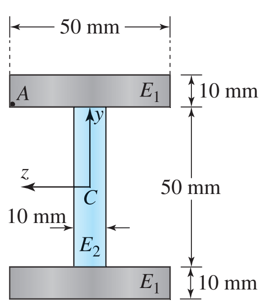


Figure P6.27

6.28 A composite beam cross section is shown in Figure 6.28. The bending normal strain at point A due to bending about the z axis was found to be $\varepsilon_{xx} = 300\ \mu$. The modulus of elasticity of the two materials are E_1 = 30,000 ksi, E_2 = 20,000 ksi. Determine the maximum bending stress in each of the two materials.

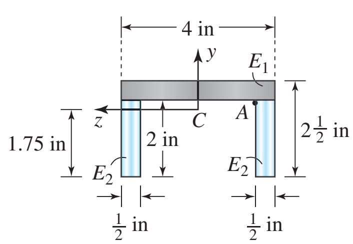


Figure P6.28

6.29 The internal moment due to bending about the *z* axis, at a beam cross section shown in Figures P6.29 is M_z = 20 in.·kips. Determine the bending normal stresses at points *A*, *B*, and *D*.

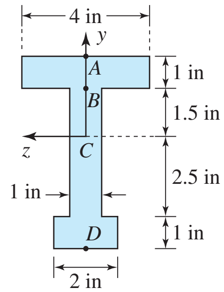


Figure P6.29

6.30 The internal moment due to bending about the *z* axis, at a beam cross section shown in Figures P6.30 is M_z = 10 kN·m. Determine the bending normal stresses at points *A*, *B*, and *D*.

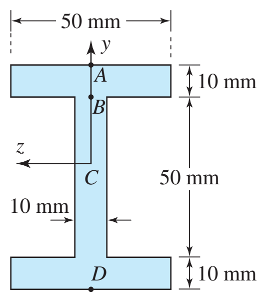


Figure P6.30

6.31 The internal moment due to bending about the *z* axis, at a beam cross section shown in Figures P6.31 is M_z = –12 kN·m. Determine the bending normal stresses at points *A*, *B*, and *D*.

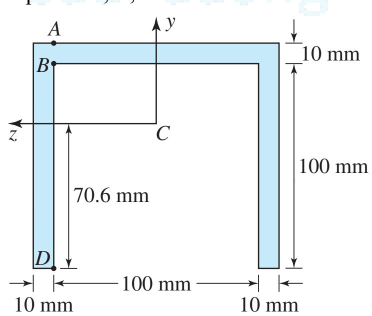


Figure P6.31

Sign convention

6.32 A beam and loading in three different coordinate systems is shown in Figures P6.32. Determine the internal shear force and bending moment at the section containing point *A* for the three cases shown using the sign convention described in Section 6.2.6.

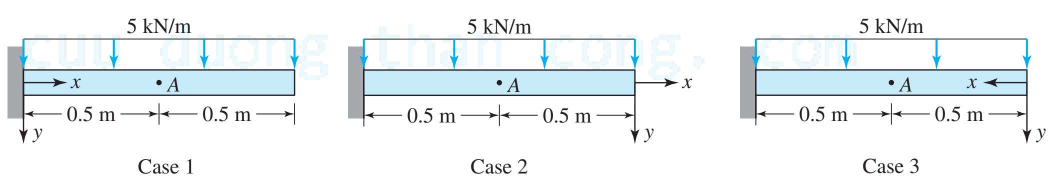


Figure P6.32

6.33 A beam and loading in three different coordinate systems is shown in Figures P6.33. Determine the internal shear force and bending moment at the section containing point *A* for the three cases shown using the sign convention described in Section 6.2.6.

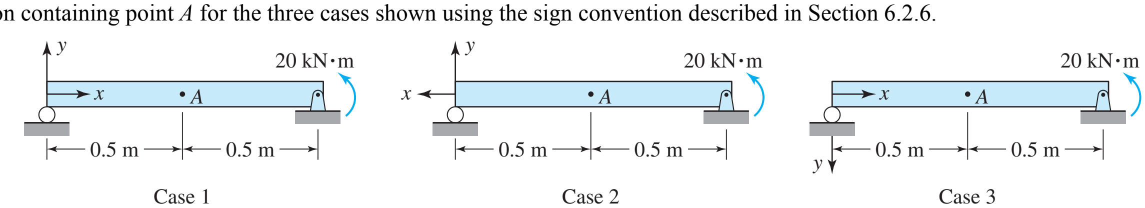


Figure P6.33

6.34 A beam and loading in three different coordinate systems is shown in Figures P6.34. Determine the internal shear force and bending moment at the section containing point A for the three cases shown using the sign convention described in Section 6.2.6.

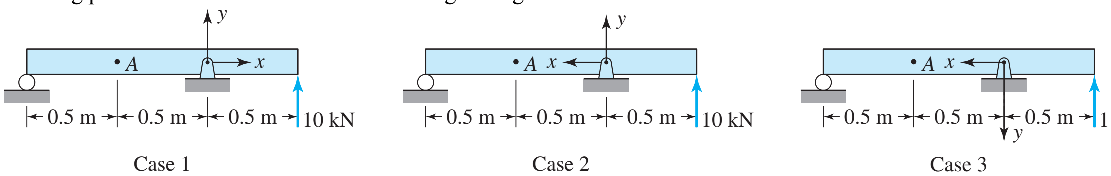


Figure P6.34

Sign of stress by inspection

6.35 Draw an approximate deformed shape of the beam for the beam and loading shown in Figure P6.35. By inspection determine whether the bending normal stress is tensile or compressive at points A and B.

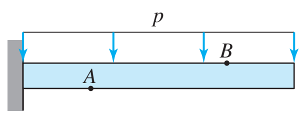


Figure P6.35

6.36 Draw an approximate deformed shape of the beam for the beam and loading shown in Figure P6.36. By inspection determine whether the bending normal stress is tensile or compressive at points A and B.

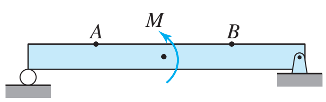


Figure P6.36

6.37 Draw an approximate deformed shape of the beam for the beam and loading shown in Figure P6.37. By inspection determine whether the bending normal stress is tensile or compressive at points A and B.

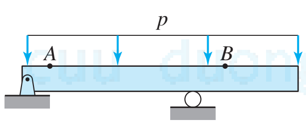


Figure P6.37

6.38 Draw an approximate deformed shape of the beam for the beam and loading shown in Figure P6.38. By inspection determine whether the bending normal stress is tensile or compressive at points A and B.

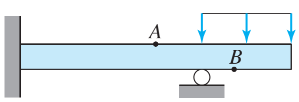


Figure P6.38

6.39 Draw an approximate deformed shape of the beam for the beam and loading shown in Figure P6.39. By inspection determine whether the bending normal stress is tensile or compressive at points A and B.

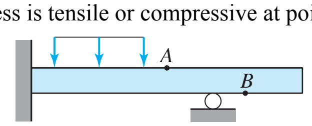


Figure P6.39

6.40 Draw an approximate deformed shape of the beam for the beam and loading shown in Figure P6.40. By inspection determine whether the bending normal stress is tensile or compressive at points A and B.

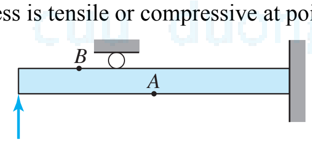


Figure P6.40

Bending normal stress and strain calculations

6.41 A W150 × 24 steel beam is simply supported over a length of 4 m and supports a distributed load of 2 kN/m. At the midsection of the beam, determine (a) the bending normal stress at a point 40 mm above the bottom surface; (b) the maximum bending normal stress.

6.42 A W10 × 30 steel beam is simply supported over a length of 10 ft and supports a distributed load of 1.5 kips/ft. At the midsection of the beam, determine (a) the bending normal stress at a point 3 in below the top surface; (b) the maximum bending normal stress.

6.43 An S12 × 35 steel cantilever beam has a length of 20 ft. At the free end a force of 3 kips acts downward. At the section near the built-in end, determine (a) the bending normal stress at a point 2 in above the bottom surface; (b) the maximum bending normal stress.

6.44 An S250 × 52 steel cantilever beam has a length of 5 m. At the free end a force of 15 kN acts downward. At the section near the built-in end, determine (a) the bending normal stress at a point 30 mm below the top surface; (b) the maximum bending normal stress.

6.45 Determine the bending normal stress at point *A* and the maximum bending normal stress in the section containing point *A* for the beam and loading shown in Figure P6.45.

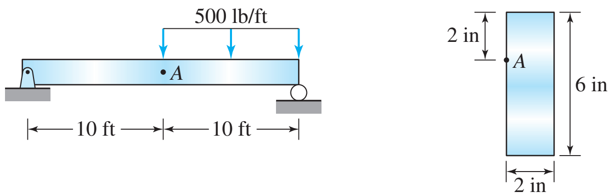


Figure P6.45

6.46 Determine the bending normal stress at point *A* and the maximum bending normal stress in the section containing point *A* for the beam and loading shown in Figure P6.46.

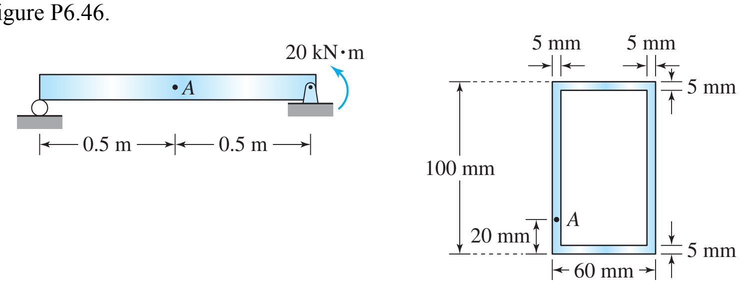


Figure P6.46

6.47 Determine the bending normal stress at point *A* and the maximum bending normal stress in the section containing point *A* for the beam and loading shown in Figure P6.47.

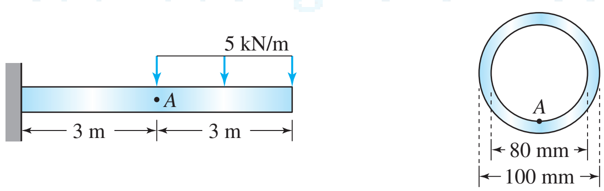


Figure P6.47

6.48 A simply supported beam with its cross section is shown in Figure P6.48. The intensity of distributed load reaches a maximum value of 5 kN/m. Determine the bending normal stress at point *A* and the maximum bending normal stress in the section containing point *A*.

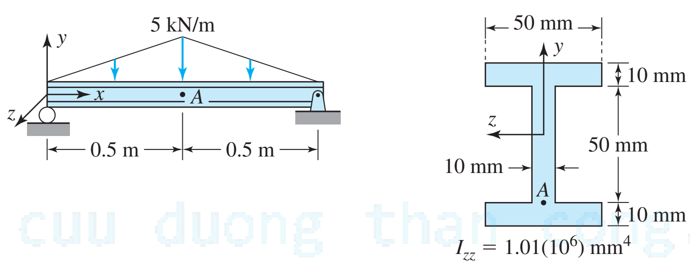


Figure P6.48

6.49 A cantilever beam with cross section is shown in Figure P6.49. The distributed load reaches its maximum intensity of 300 lb/in. Determine the bending normal stress at point *A* and the maximum bending normal stress in the section containing point *A* for the beam and loading

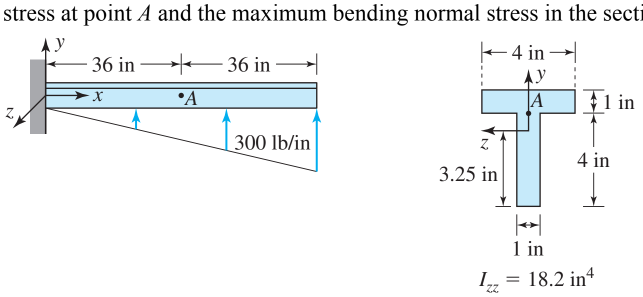


Figure P6.49

6.50 Determine the bending normal stress at point A and the maximum bending normal stress in the section containing point A for the beam and loading shown in Figure P6.50.

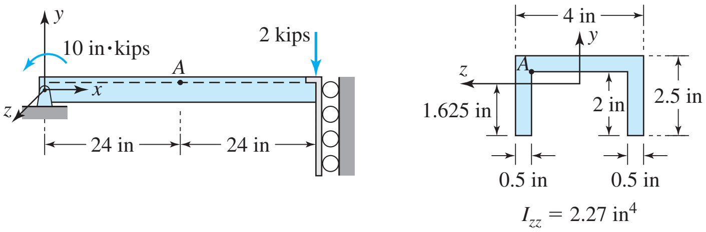


Figure P6.50

6.51 A wooden rectangular beam (E = 10 GPa), its loading, and its cross section are as shown in Figure P6.51. If the distributed force w = 5 kN/m, determine the normal strain ε_{xx} at point A on the bottom of the beam.

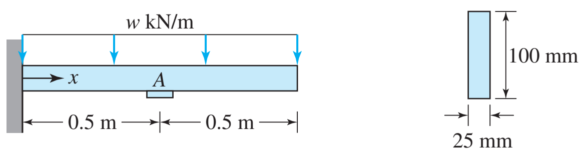


Figure P6.51

6.52 A wooden rectangular beam (E = 10 GPa), its loading, and its cross section are as shown in Figure P6.51. The normal strain at point A was measured as $\varepsilon_{xx} = -600\ \mu$. Determine the distributed force w that is acting on the beam.

6.53 A wooden beam (E = 8000 ksi), its loading, and its cross section are as shown in Figure P6.53. If the applied load P = 6 kips, determine the normal strain ε_{xx} at point A.

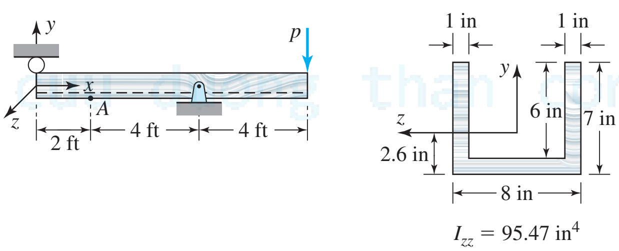


Figure P6.53

6.54 A wooden beam (E = 8000 ksi), its loading, and its cross section are as shown in Figure P6.53. The normal strain at point A was measured as $\varepsilon_{xx} = -250\ \mu$. Determine the load P.

Stretch Yourself

6.55 A composite beam made from n materials is shown in Figure 6.55. If Assumptions 1 through 7 are valid, show that the location of neutral axis η_c is given by Equation (6.14), where η_j, E_j, and A_j are location of the centroid, the modulus of elasticity, and cross sectional area of the j^{th} material.

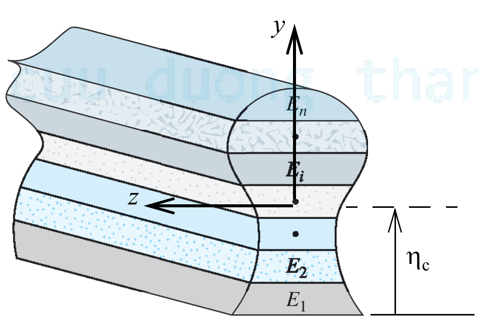


Figure P6.55

$$\eta_c = \frac{\sum_{j=1}^{n} \eta_j E_j A_j}{\sum_{j=1}^{n} E_j A_j} \qquad (6.14)$$

6.56 A composite beam made from n materials is shown in Figure 6.55. If Assumptions 1 through 7 are valid, show that the moment curvature relationship and the equation for bending normal stress $(\sigma_{xx})_i$ in the i[th] material are as given by

$$M_z = \frac{d^2 v}{dx^2}\left[\sum_{j=1}^{n} E_j (I_{zz})_j\right] \qquad (6.15)$$

$$(\sigma_{xx})_i = -E_i y \frac{M_z}{\left[\sum_{j=1}^{n} E_j (I_{zz})_j\right]} \qquad (6.16)$$

where E_j and $(I_{zz})_j$ are the modulus of elasticity and cross sectional area, and second area moment of inertia of the j[th] material. Show that if $E_1=E_2=...=E_n=E$ then Equations (6.15) and (6.16) reduce to Equations (6.11) and (6.12).

6.57 The stress–strain curve in tension for a material is given by $\sigma = K\varepsilon^{0.5}$. For the rectangular cross section shown in Figure P6.57, show that the bending normal stress is given by the equations below.

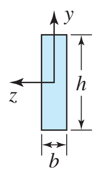


Figure P6.57

$$\sigma_{xx} = \begin{cases} \dfrac{-5\sqrt{2}}{bh^2}\left(\dfrac{y}{h}\right)^{0.5} M_z & y > 0 \\ \dfrac{5\sqrt{2}}{bh^2}\left(-\dfrac{y}{h}\right)^{0.5} M_z, & y < 0 \end{cases}$$

6.58 The hollow square beam shown in Figure P6.58 is made from a material that has a stress–strain relation given by $\sigma = K\varepsilon^{0.4}$. Assume the same behavior in tension and in compression. In terms of K, L, a, and M_{ext} determine the bending normal strain and stress at point A.

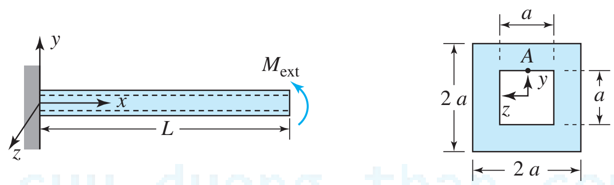


Figure P6.58

6.3 SHEAR AND MOMENT BY EQUILIBRIUM

Equilibrium equations at a point on the beam are differential equations relating the distributed force p_y, the shear force V_y, and the bending moment M_z. The differential equations can be integrated analytically or graphically to obtain V_y and M_z as a function of x. These in turn can be used to determine the maximum values of V_y and M_z, and hence the maximum values of the bending normal stress from Equation (6.12) and the maximum bending shear stress, as discussed in Section 6.6. M_z as a function of x is also needed when integrating Equation (6.11) to find the deflection of the beam, as we will discuss in Chapter 7.

Consider a differential element Δx of the beam shown at left in Figure 6.34. Recall that a positive distributed force p_y acts in the positive y direction, as shown in Figure 6.14. Internal shear forces and the internal moment change as one moves across the element, as shown in Figure 6.34. By replacing the distributed force by an equivalent force, we obtain the diagram on the right of Figure 6.34.

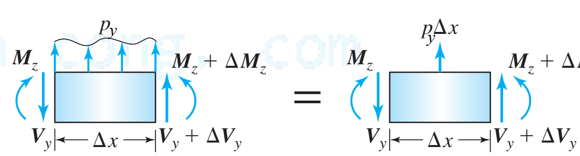


Figure 6.34 Differential beam element.

By equilibrium of forces in the y direction, we obtain

$$-V_y + (V_y + \Delta V_y) + p_y \Delta x = 0 \qquad \text{or} \qquad \frac{\Delta V_y}{\Delta x} = -p_y$$

As $\Delta x \to 0$, we obtain

$$\boxed{\frac{dV_y}{dx} = -p_y} \qquad (6.17)$$

By equilibrium of moment in the z direction about an axis passing through the right side, we obtain

$$-\boldsymbol{M}_z + (\boldsymbol{M}_z + \Delta\boldsymbol{M}_z) + \boldsymbol{V}_y\Delta x + (p_y\Delta x)\frac{\Delta x}{2} = 0 \quad \text{or} \quad \frac{\Delta \boldsymbol{M}_z}{\Delta x} + \frac{p\Delta x}{2} = -\boldsymbol{V}_y$$

As $\Delta x \to 0$, we obtain

$$\boxed{\frac{d\boldsymbol{M}_z}{dx} = -\boldsymbol{V}_y} \tag{6.18}$$

Equations (6.17) and (6.18) are differential equilibrium equations that are applicable at every point on the beam, except where $\boldsymbol{V}_y$ and $\boldsymbol{M}_z$ are discontinuous. In Example 6.10 we shall see that $\boldsymbol{V}_y$ and $\boldsymbol{M}_z$ are discontinuous at the points where concentrated (point) external forces or moments are applied. We shall consider two methods for finding $\boldsymbol{V}_y$ and $\boldsymbol{M}_z$ as a function of x:

1. We can integrate Equation (6.17) to obtain $\boldsymbol{V}_y$ and then integrate Equation (6.18) to obtain $\boldsymbol{M}_z$. The integration constants can be found from the values of $\boldsymbol{V}_y$ and $\boldsymbol{M}_z$ at the end of the beam, as illustrated in Example 6.9.
2. Alternatively, we can make an imaginary cut at some location defined by the variable x and draw the free-body diagram. We then determine $\boldsymbol{V}_y$ and $\boldsymbol{M}_z$ in terms of x by writing equilibrium equations. We can check our results by substituting the expressions of $\boldsymbol{V}_y$ and $\boldsymbol{M}_z$ in Equations (6.17) and (6.18), respectively.

The first approach, by integration, is a general approach. This is particularly useful if p_y is represented by a complicated function. But for uniform and linear variations of p_y the free-body diagram method is simpler. Example 6.9 compares the two methods, and Example 6.10 elaborates the use of the free-body diagram approach further.

EXAMPLE 6.9

Figure 6.35 shows two models of wind pressure on a light pole. Find $\boldsymbol{V}_y$ and $\boldsymbol{M}_z$ as a function of x for the two distributions shown. Neglect the weights of the light and the pole.

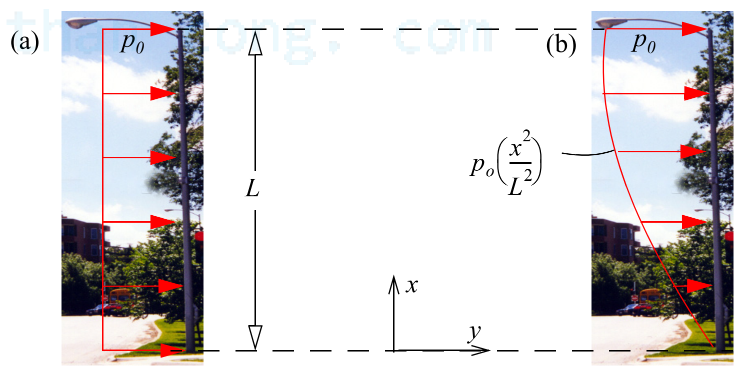


Figure 6.35 Light pole in Example 6.9. (*a*) Uniform distribution. (*b*) Quadratic distribution.

PLAN

For uniform distribution we can find $\boldsymbol{V}_y$ and $\boldsymbol{M}_z$ as a function of x by making an imaginary cut at a distance x from the bottom and drawing the free-body diagram of the top part. For the quadratic distribution we can first integrate Equation (6.17) to find $\boldsymbol{V}_y$ and then integrate Equation (6.18) to find $\boldsymbol{M}_z$. To find the integration constants, we can construct a free-body diagram of infinitesimal length at the top ($x = L$) and obtain the boundary conditions on $\boldsymbol{V}_y$ and $\boldsymbol{M}_z$. Using boundary conditions and integrated expressions, we can obtain $\boldsymbol{V}_y$ and $\boldsymbol{M}_z$ as a function of x for the quadratic distribution.

SOLUTION

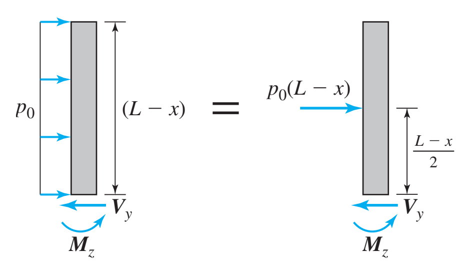


Figure 6.36 Shear force and bending moment by free-body diagram.

Uniform distribution: We can make an imaginary cut at location x and draw the free-body diagram of the top part, as shown in Figure 6.36.We then replace the distributed load by an equivalent force and write the equilibrium equations.

$$V_y = p_0(L-x) \qquad M_z = p_0(L-x)\left(\frac{L-x}{2}\right) = \frac{p_0}{2}(x^2 - 2xL + L^2) \tag{E1}$$

Check: Differentiating Equation (E1), we obtain

$$\frac{dV_y}{dx} = -p_0 = -p_y \qquad \frac{dM_z}{dx} = -p_0(L-x) = -V_y \tag{E2}$$

Equation (E2) shows that the equilibrium Equations (6.17) and (6.18) are satisfied.

Quadratic distribution: Substituting $p_y = p_0(x^2/L^2)$ into Equation (6.17) and integrating, we obtain

$$V_y = -\left(\frac{p_0}{3L^2}\right)x^3 + C_1 \tag{E3}$$

Substituting Equation (E3) into Equation (6.18) and integrating, we obtain

$$M_z = \frac{p_0}{3L^2}\left(\frac{x^4}{4}\right) - C_1 x + C_2 \tag{E4}$$

We make an imaginary cut at a distance Δx from the top and draw the free-body diagram shown in Figure 6.37.By equilibrium of forces in the y direction and equilibrium of moment about point O and letting Δx tend to zero we obtain the boundary conditions:

$$\lim_{\Delta x \to 0}[V_y(x=L) + p\Delta x] = 0 \qquad \text{or} \qquad V_y(x=L) = 0 \tag{E5}$$

$$\lim_{\Delta x \to 0}\left[M_z(x=L) + \frac{p\Delta x^2}{2}\right] = 0 \qquad \text{or} \qquad M_z(x=L) = 0 \tag{E6}$$

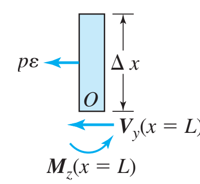


Figure 6.37 Boundary conditions on shear force and bending moments.

Substituting $x = L$ into Equation (E3) and using the condition Equation (E5), we obtain

$$-\left(\frac{p_0}{3L^2}\right)L^3 + C_1 = 0 \qquad \text{or} \qquad C_1 = \frac{p_0 L}{3} \tag{E7}$$

Substituting Equation (E7) into Equation (E3), we obtain the shear force,

$$V_y = -\frac{p_0}{3L^2}x^3 + \frac{p_0 L}{3} \tag{E8}$$

ANS. $V_y = \dfrac{p_0}{3L^2}(L^3 - x^3)$

Substituting $x = L$ into Equation (E4), and using Equations (E6) and (E7), we obtain

$$\frac{p_0}{3L^2}\left(\frac{L^4}{4}\right) - \frac{p_0 L}{3}(L) + C_2 = 0 \qquad \text{or} \qquad C_2 = \frac{p_0 L^2}{4} \tag{E9}$$

From Equation (E4) we obtain the moment

$$M_z = \frac{p_0 x^4}{12L^2} - \frac{p_0}{3}xL + \frac{p_0 L^2}{4} \tag{E10}$$

ANS. $M_z = \dfrac{p_0}{12L^2}(x^4 - 4xL^3 + 3L^4)$

COMMENTS

1. Suppose that for the uniform distribution we integrate Equation (6.17) after substituting $p_y = p_0$. We would obtain $V_y = -p_0 x + C_3$. On substituting this into Equation (6.18) and integrating, we would obtain $M_z = p_0(x^2/2) - C_3 x + C_4$. Substituting $x = L$ in the expressions of V_y and M_z and equating the results to zero, we obtain $C_3 = p_0 L$ and $C_4 = p_0 L^2/2$. Substituting these in the expressions of V_y and M_z, we obtain Equation (E1).
2. The free-body diagram approach is simpler than the integration approach for uniform distribution for two reasons. First, we did not have to perform any integration to obtain the equivalent load $p_0 L$ or to determine its location when we constructed the free-body diagram in Figure 6.36. Second, we do not have to impose zero boundary conditions on the shear force and bending moments at $x = L$, because these conditions are implicitly included in the free-body diagram in Figure 6.36.

3. The free-body diagram approach would present difficulties for the quadratic distribution, as we would need to find the equivalent load and its location. Both involve the same integrals as obtained from Equations (6.17) and (6.18). Thus for simple distributions the free-body diagram approach is preferred, whereas the integration approach is better for more complex loading.

EXAMPLE 6.10

(a) Write the equations for the internal shear force V_y and the internal bending moments M_z as a function of x for the entire beam shown in Figure 6.38. (b) Determine the values of V_y and M_z just before and after point B.

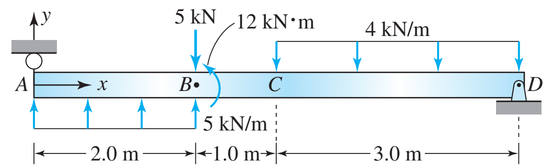


Figure 6.38 Beam in Example 6.10.

PLAN

By considering the free-body diagram of the entire beam we can determine the reactions at supports A and D. (a) The loading changes at points B and C. Thus shear force and bending moment will be represented by different functions in AB, BC, and CD. We draw free body diagrams after making imaginary cuts in AB, BC, and CD and determine shear force and bending moment by equilibrium. We can use Equations (6.17) and (6.18) to check our answers. (b) By substituting $x = 2$ m in the expressions for V_y and M_z in segment AB we can find the values just before B, and by substituting $x = 2$ in segment BC we find the values just after B.

SOLUTION

(a) We replace the distributed loads by equivalent forces and draw the free-body diagram of the entire beam as shown in Figure 6.39.By equilibrium of moment about point D and equilibrium of forces in the y direction we obtain the reaction forces.

$$R_A(6\text{ m}) - (10\text{ kN})(5\text{ m}) + (12\text{ kN}\cdot\text{m}) + (5\text{ kN})(4\text{ m}) + (12\text{ kN})(1.5\text{ m}) = 0 \quad \text{or} \quad R_A = 0 \tag{E1}$$

$$-R_A + 10\text{ kN} - 5\text{ kN} - 12\text{ kN} + R_D = 0 \quad \text{or} \quad R_D = 7\text{ kN} \tag{E2}$$

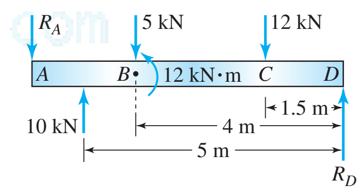


Figure 6.39 Free-body diagram of entire beam in Example 6.10.

Segment AB, $0 \le x < 2$: We make an imaginary cut at some location x in segment AB. We take the left part of the cut and draw the free-body diagram after replacing the distributed force over the distance x by a statically equivalent force, as shown in Figure 6.40a. We write the equilibrium equations to obtain V_y and M_z as a function of x.

$$V_y + 5x = 0 \quad \text{or} \quad V_y = -5x\text{ kN} \tag{E3}$$

$$M_z - 5x\left(\frac{x}{2}\right) = 0 \quad \text{or} \quad M_z = \frac{5}{2}x^2\text{ kN·m} \tag{E4}$$

ANS. $V_y = -5x\text{ kN} \quad M_z = \frac{5}{2}x^2\text{ kN·m}$

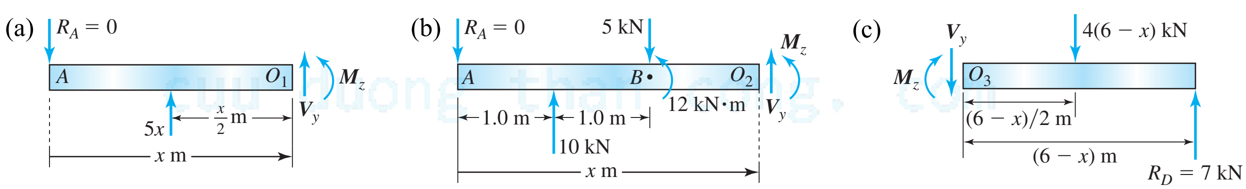


Figure 6.40 Free body diagrams in Example 6.10 after imaginary cut in (a) AB (b) BC (c) CD.

Check: Differentiating the shear force and bending moment, we obtain

$$\frac{dV_y}{dx} = -5 = -p_y, \qquad \frac{dM_z}{dx} = 5x = -V_y \tag{E5}$$

Equation (E5) shows that Equations (6.17) and (6.18) are satisfied.

Segment BC, $2 < x < 3$: We make an imaginary cut at some location x in segment BC. We take the left part of the cut and draw the free-body diagram after replacing the distributed force by a statically equivalent force, as shown in Figure 6.40b. We write the equilibrium equations to obtain V_y and M_z as a function of x.

$$V_y + 10\text{ kN} - 5\text{ kN} = 0 \quad \text{or} \quad V_y = -5\text{ kN} \tag{E6}$$

$$M_z - (10\text{ kN})(x-1) + (12\text{ kN}\cdot\text{m}) + (5\text{ kN})(x-2) = 0 \quad \text{or} \quad M_z = (5x - 12)\text{ kN}\cdot\text{m} \tag{E7}$$

ANS. $V_y = -5\text{ kN}$ $M_z = (5x-12)\text{ kN}\cdot\text{m}$

Check: Differentiating shear force and bending moment, we obtain

$$\frac{dV_y}{dx} = 0 = -p_y \qquad \frac{dM_z}{dx} = 5 = -V_y \tag{E8}$$

Equation (E8) shows that Equations (6.17) and (6.18) are satisfied.

Segment CD, $3 < x < 6$: We make an imaginary cut at some location x in segment *CD*. We take the right part of the cut and note that left part is x m long and the right part hence is $6 - x$ m long. We draw the free-body diagram after replacing the distributed force by a statically equivalent force, as shown in Figure 6.40c.We write the equilibrium equations to obtain V_y and M_z as a function of x,

$$V_y + (4)(6-x)\text{ kN} - (7\text{ kN}) = 0 \quad \text{or} \quad V_y = (4x - 17)\text{ kN} \tag{E9}$$

$$M_z + [4(6-x)\text{ kN}]\left(\frac{6-x}{2}\right) - (7\text{ kN})(6-x) = 0 \quad \text{or} \quad M_z = (-2x^2 + 17x - 30)\text{ kN}\cdot\text{m} \tag{E10}$$

ANS. $V_y = (4x-17)\text{ kN}$ $M_z = (-2x^2 + 17x - 30)\text{ kN}\cdot\text{m}$

Check: Differentiating shear force and bending moment, we obtain

$$\frac{dV_y}{dx} = 4 = -p_y, \qquad \frac{dM_z}{dx} = -4x + 17 = -V_y \tag{E11}$$

Equation (E11) shows that Equations (6.17) and (6.18) are satisfied.

(b) Substituting $x = 2$ m into Equations (E3) and (E6) we obtain the values of V_y and M_z just before point *B*,

ANS. $V_y(2^-) = -10\text{ kN}$, $M_z(2^-) = +10\text{ kN}\cdot\text{m}$

where the superscripts – refer to just before $x = 2$ m. Substituting $x = 2$ m into Equations (E4) and (E7) we obtain the values of V_y and M_z just after point *B*,

ANS. $V_y(2^+) = -5\text{ kN}$, $M_z(2^+) = -2\text{ kN}\cdot\text{m}$

where the superscripts + refer to just after $x = 2$m.

COMMENTS

1. In Figures 6.40a and 6.40b the left part after the imaginary cut was taken and the distance from *A* was labeled x. In Figure 6.40c the right part of the imaginary cut was taken, and the distance from the right end was labeled $(6 - x)$. These free-body diagrams emphasize that x defines the location of the imaginary cut, irrespective of the part used in drawing the free-body diagram. Furthermore, the distance (coordinate) x is always measured from the same point in all free-body diagrams, which in this problem is point *A*.
2. We note that $V_y(2^+) - V_y(2^-) = 5$ kN, which is the magnitude of the applied external force at point *B*. Similarly, $M_z(2^+) - M_z(2^-) = -12$ kN·m, which is the magnitude of the applied external moment at point *B*. This emphasizes that the external point force causes a jump in internal shear force, and the external point moment causes a jump in the internal bending moment. We will make use of these observations in the next section in plotting the shear force—bending moment diagrams.
3. We can obtain V_y and M_z in each segment by integrating Equations (6.17) and (6.18). Observe that the shear force and bending moment jump by the value of applied force and moment, respectively causing additional difficulties in determining integration constants. Thus, the free-body approach is easier then the method of integration in this case.
4. For beam deflection, Section 7.4* introduces a method based on the integration approach, that eliminates drawing free-body diagrams for each segment to account for jumps in the loading. But that method requires an additional concept—*discontinuity functions* (also called *singularity functions*).

6.4 SHEAR AND MOMENT DIAGRAMS

Shear and moment diagrams are plots of internal shear force and internal bending moment as a function of x. By looking at these plots, we can immediately see the maximum values of the shear force and the bending moment, as well as the location of these maximum values. One way of making these plots is to determine the shear force and bending moment as a function of x, as in Section 6.3, and plot the results. However, for simple loadings there exists an easier alternative. We first discuss how the distributed forces are accounted, then how to account for the point forces and moments.

6.4.1 Distributed Force

The graphical technique described in this section is based on the interpretation of an integral as the area under a curve. The minus signs[7] in Equations (6.17) and (6.18) lead to positive areas being subtracted and negative areas being added. To over-come this problem of flip-flop of sign in the graphical procedure, we introduce $V = -V_y$. Let V_1 and V_2 be the values of V at x_1 and x_2, respectively. Let M_1 and M_2 be the of values of M_z at x_1 and x_2, respectively. Equations (6.17) and (6.18) can be written in terms of V as $dV/dx = p_y$ and $dM_z/dx = V$. Integration then yields

$$V_2 = V_1 + \int_{x_1}^{x_2} p_y \, dx \qquad (6.19)$$

$$M_2 = M_1 + \int_{x_1}^{x_2} V \, dx \qquad (6.20)$$

The key idea is to recognize that the values of the integrals in Equations (6.19) and (6.20) are the areas under the load curve p_y and the curve defining V, respectively. If we know V_1 and M_1, then by adding or subtracting the areas under the respective curves, we can find V_2 and M_2. We then move to point 2, where we now know the shear force and bending moment, and consider it as point 1 for the next segment of the beam. Moving in this bootstrap manner, we go across the beam accounting for the distributed forces.

Shear force curve

Recall that p_y is positive in the positive y direction. Thus in Figure 6.41a and b, $p_y = +w$, and from Equation (6.19) we obtain $V_2 = V_1 + w(x_2 - x_1)$. Similarly, in Figure 6.46c and d, $p_y = -w$, and from Equation (6.19) we obtain $V_2 = V_1 - w(x_2 - x_1)$. The term $w(x_2 - x_1)$ is the area of the rectangle and represents the magnitude of the integral in Equation (6.19).The line joining the values of V_1 and V_2 is a straight line because the integral of a constant function will result in a linear function.

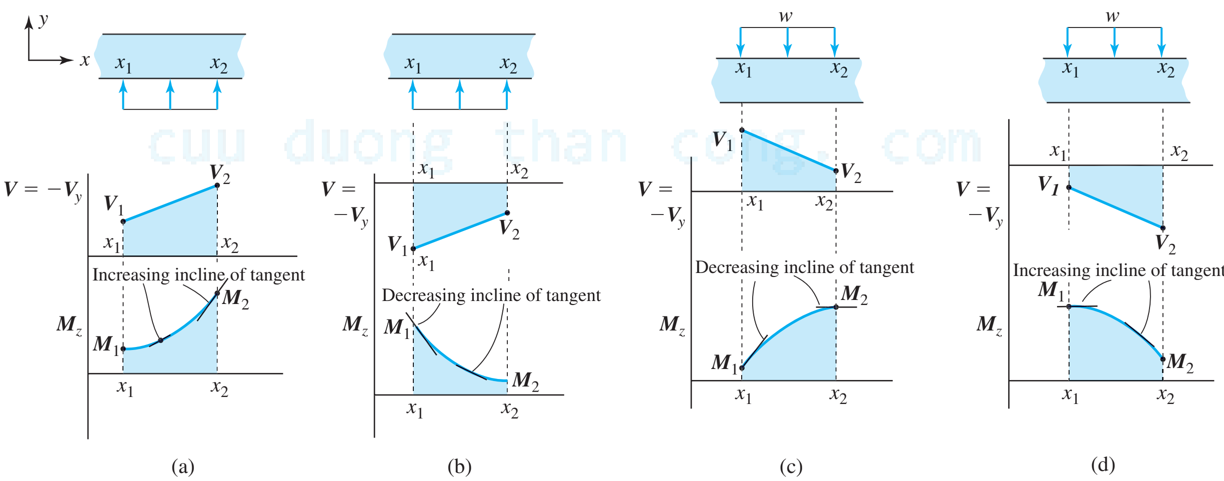


Figure 6.41 Shear and moment diagrams for uniformly distributed load.

Bending moment curve

The integral in Equation (6.20) represents the area under the curve defining V, that is, the areas of the trapezoids shown by the shaded regions in Figure 6.41. In Figure 6.41a and c, V is positive and we add the area to M_1 to get M_2. In Figure 6.41b and d, V is negative and we subtract the area from M_1 to get M_2. As V is linear between x_1 and x_2, the integral in Equation (6.20) will generate a quadratic function. But what would be the curvature of the moment curve, concave or convex? To answer this question, we note that the derivative of the moment curve—that is, the slope of the tangent—is equal to the value on the shear force diagram. To avoid some ambiguities associated with the sign[8] of a slope, we consider the inclination of the tangent to the moment curve, $|dM_z/dx| = |V|$. If the magnitude of V is increasing, the inclination of the tangent to the moment curve must increase, as shown in Figure 6.41a and d. If the magnitude of V is decreasing, the inclination of the tangent to the moment curve must decrease, as shown in Figure 6.41b and c.

[7]This is a consequence of trying to stay mathematically consistent while keeping the directions of shear force and shear stress the same. See footnote 6.
[8]We avoid statements such as "increasing negative slope," which could mean more negative or less negative. "Decreasing negative slope." is similarly ambiguous.

An alternative approach to getting the curvature of the moment curve is to note that if we substitute Equation (6.18) into Equation (6.17), we obtain $d^2\boldsymbol{M}_z/dx^2 = p_y$. If p_y is positive, then the curvature of the moment curve is positive, and hence the curve is concave, as shown in Figure 6.41*a* and *b*. If p_y is negative, then the curvature of the moment curve is negative, and the curve is convex, as shown in Figure 6.41*c* and *d*.

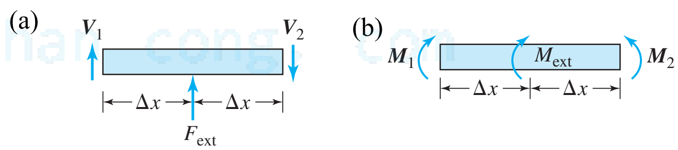
We call our conclusions the *curvature rule* for quadratic $\boldsymbol{M}_z$ curves:

> The curvature of the $\boldsymbol{M}_z$ curve must be such that the incline of the tangent to the $\boldsymbol{M}_z$ curve must increase (or decrease) as the magnitude of $\boldsymbol{V}$ increases (or decreases).
>
> or (6.21)
>
> The curvature of the moment curve is concave if p_y is positive, and convex if p_y is negative.

6.4.2 Point Force and Moments

It was noted in Comment 2 of Example 6.10 that the values of the internal shear force and the bending moment jump as one crosses an applied point force and moment, respectively. In Section 4.2.8 on axial force diagrams and in Section 5.2.6 on torque diagrams we used a template to give us the correct direction of the jump. We use the same idea here.

A template is a small segment (Δx tends to zero in Figure 6.42) of a beam on which the external moment M_{ext} and an external force F_{ext} are drawn. The directions of F_{ext} and M_{ext} are arbitrary. The ends at $+\Delta x$ and $-\Delta x$ from the applied external force and moment represent the imaginary cut just to the left and just to the right of the applied external forces and moments. On these cuts the internal shear force and the internal bending moment are drawn. Equilibrium equations are written for this $2\Delta x$ segment of the beam to obtain the template equations.



Template Equations $\boldsymbol{V}_2 = \boldsymbol{V}_1 + F_{ext}$ $\boldsymbol{M}_2 = \boldsymbol{M}_1 + M_{ext}$

Figure 6.42 Beam templates and equations for (*a*) Shear force (*b*) Moment.

Shear force template

Notice that the internal forces $\boldsymbol{V_1}$ and $\boldsymbol{V_2}$ are drawn opposite to the direction of positive internal shear forces, as per the definition $\boldsymbol{V} = -\boldsymbol{V}_y$, which is an additional artifact of the procedure to remember. To avoid this, we note that the sign of F_{ext} is the same as the direction in which $\boldsymbol{V_2}$ will move relative to $\boldsymbol{V_1}$. In the future we will not draw the shear force template but use the following observation:

- $\boldsymbol{V}$ will jump in the direction of the external point force.

Moment template

On the moment template, the internal moments are drawn according to our sign convention, discussed in Section 6.2.6. Unlike the observation about the jump in $\boldsymbol{V}$, there is no single observation that is valid for all coordinate systems. Thus the moment template must be drawn and the corresponding template equation used as follows.

If the external moment on the beam is in the direction of the assumed moment M_{ext} on the template, then the value of $\boldsymbol{M_2}$ is calculated according to the template equation. If the external moment on the beam is opposite to the direction of M_{ext} on the template, then $\boldsymbol{M_2}$ is calculated by changing the sign of M_{ext} in the template equation.

6.4.3 Construction of Shear and Moment Diagrams

Figure 6.43 is used to elaborate the procedure for constructing shear and moment diagrams as we outline it next.

Step 1 Determine the reaction forces and moments.

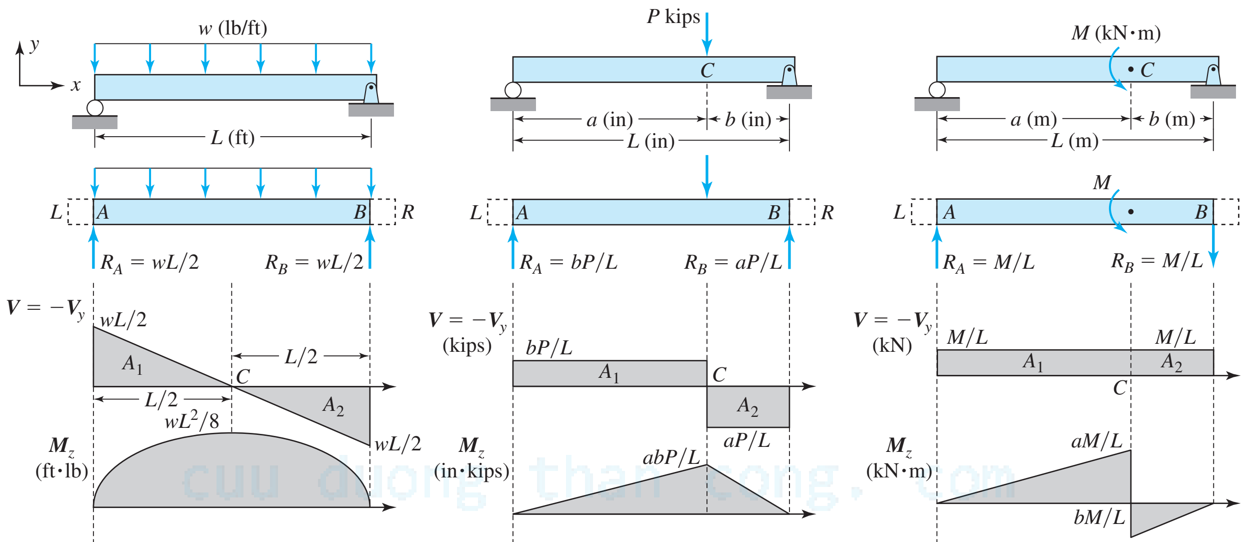
The free-body diagram for the entire beam is drawn, and the reaction forces and moments are calculated at the supports at *A* and *B*, as shown in Figure 6.43.

Step 2 Draw and label the vertical axes for V and M_z along with the units to be used.

We show $V = -V_y$ on the axis to remind ourselves that the positive and negative values read from the plots are for V, whereas the formula that will be developed in Section 6.6 for the bending shear stress will be in terms of V_y.

Step 3 Draw the beam with all forces and moments. At each change of loading draw a vertical line.

The vertical lines define the segments of the beam between two points x_1 and x_2 where the values of shear force and moment will be calculated. The vertical lines also represent points where V and M_z values may jump, such as at point *C* in cases 2 and 3 in Figure 6.43.



$A_1 = \frac{1}{2}wL/2(L/2) = \frac{wL^2}{8}$ $A_1 = \frac{bFP}{L}a = \frac{abFP}{L}$ $A_1 = \frac{M}{L}a = \frac{aM}{L}$

$A_2 = \frac{1}{2}wL/2(L/2) = \frac{wL^2}{8}$ $A_2 = \frac{aFP}{L}b = \frac{abFP}{L}$ $A_2 = \frac{M}{L}b = \frac{bM}{L}$

Case 1 Case 2 Case 3

Figure 6.43 Construction of shear and moment diagrams.

Step 4 Consider imaginary extensions on the left and right ends of the beam. V and M_z are zero in these imaginary extensions.

In the imaginary left extension, *LA* at the beams shown in Figure 6.43, V_1 and M_1 are zero, and we can start our process at this segment. Point *A* (the start of the beam) can now be treated like any other point on the beam at which there is a point force and/or point moment. At the right imaginary extension *BR*, the values of the shear force and bending moment must return to zero, providing a check on our solution procedure.

Shear force diagram

Step 5 If there is a point force, then increase the value of V in the direction of the point force.

Just before point *A* in Figure 6.43, $V_1 = 0$ as we are in the imaginary extension. As we cross point *A*, the value of V jumps upward (positive) by the value of the reaction force R_A, which is in the upward direction.

At point *C* in case 2 we jump in the direction of *P*, which is pointed downward; that is, we subtract *P* from the value of V_1. In other words, $V_2 = bP/L - P = (b - L)P/L = -aP/L$ just after point *C*, as shown.

The reaction force R_B is upward in cases 1 and 2, so we add the value of R_B to V_1. In case 3 R_B is downward, so we subtract the value of R_B from V_1. As expected in all cases, we return to a zero value for force V in the imaginary extension *BR*.

Step 6 Compute the area under the curve of the distributed load. Add the area to the value of V_1 if p_y is positive, and subtract it if p_y is negative, to obtain the value of V_2.

In case 1 the area under the distributed force is wL and p_y is negative. Therefore we subtract wL from the value of V just after A $(+wL/2)$ to get the value of V just before B $(-wL/2)$.

Step 7 Repeat Steps 5 and 6 until the imaginary extension at the right of the beam is reached. If the value of V is not zero in the imaginary extension, then check Steps 5 and 6 for each segment of the beam.

For the three simple cases considered in Figure 6.43, this step is not required.

Step 8 Draw additional vertical lines at any point where the value V is zero. Determine the location of these points by using geometry.

The points where V is zero represent the location of the maximum or minimum values of the bending moment because $dM_z/dx = 0$ at these points. In case 1 $V = 0$ at point C. The location can be found by using similar triangles.

Step 9 Calculate the areas under the V curve and between two adjacent vertical lines.

Areas A_1 and A_2 can be found and recorded as shown in Figure 6.48.

Moment diagram

Step 10 If there is a point moment, then use the moment template and the template equation to determine the direction of the jump.

In case 3 there is a point moment at point C. Comparing the direction of the moment at C to that in the template in Figure 6.42, we conclude that $M_{ext} = -M$. Just before C, $M_1 = aM/L$. As per the template equation, $M_2 = aM/L - M = (a - L)M/L = -bM/L$, which is the value just after C.

In all three cases there is no point moment at A, hence our starting value is zero. If there were a point moment at A, we would use the moment template and the template equation to determine the starting value as we move from the imaginary segment to just right of A.

Step 11 To move from the right of one vertical line to the left of the next vertical line, add the areas under the V curve if V is positive, and subtract the areas if V is negative. Draw the curve according to the curvature rule in Equation (6.21).

In all three cases the area A_1 is positive and we add the value of the area to the value of the moment at point A to obtain the moment just before C. In cases 1 and 2 the area A_2 is negative. Hence we subtract the value of A_2 from the moment value just after C to get a zero value just before B. In case 3 A_2 is positive and we add the value of A_2 to the moment value just after C.

In cases 2 and 3 the V curve is constant in each segment, and hence the M_z curve is linear in each segment. In case 1 the V curve is linear. Hence the M_z curve is quadratic and we need to determine the curvature of the curve. The inclination of the tangent to the M_z curve at A is nonzero, and it decreases to zero at C. Thus the inclination of the tangent decreases as the magnitude of V decreases. Similarly, as we move from C to B, the inclination of the tangent increases from zero to a nonzero value, consistent with the increasing magnitude of V.

Alternatively, in case 1 $p_y = -w$, and hence the curvature of the moment curve is convex.

Step 12 Repeat Steps 10 and 11 until you reach the imaginary extension on the right of the beam. If the value of M_z is not zero in the imaginary extension, then check Steps 10 and 11 for each segment of the beam.

This procedure is applied and elaborated in Examples 6.11 and 6.12.

6.5 STRENGTH BEAM DESIGN

This section addresses two issues. The first relates to choosing a standard, commonly manufactured beam cross section that will be cheapest to use. The second issue relates to determining the maximum tensile or compressive bending normal stress.

6.5.1 Section Modulus

In the design of steel beams, the tensile and compressive strength are usually assumed to be equal. We calculate the magnitude of the maximum bending normal stress using Equation (6.12), which can be written as $\sigma_{max} = M_{max} y_{max} / I_{zz}$, where M_{max} is the magnitude of the maximum internal bending moment, and y_{max} the distance of the point farthest from the neutral axis. The moment of inertia I_{zz} and y_{max} depend on the geometry of the cross section. Thus, Equation (6.12) requires two variables to determine the best geometric shape in a particular design. A variable called **section modulus** S, simplifies the equation for the maximum bending normal stress

$$S = \frac{I_{zz}}{y_{max}} \qquad \sigma_{max} = \frac{M_{max}}{S} \tag{6.22}$$

Section C.6 give the section modulus S for steel beams of standard shapes and Example 6.12 shows its use in design.

6.5.2 Maximum Tensile and Compressive Bending Normal Stresses

Section 3.1 observed that a brittle material usually ruptures when the maximum tensile normal stress exceeds the ultimate tensile stress of the material. Cracks in materials propagate due to tensile stress. Adhesively bonded material debonds from tensile normal stress called *peel stress*. Thus a structure designed for maximum normal stress may fail when the maximum tensile stress is less in magnitude than the maximum compressive stress. Similarly, failure may occur when the maximum compressive normal stress is less than the maximum tensile normal stress. This may happen because *buckling*, which is discussed in Chapter 11. Proper beam design must take into account failure due to tensile or compressive normal stresses.

To account for these stresses, it may be necessary to determine two stress values—the maximum tensile and compressive bending normal stress. These may differ when the top and the bottom of the beam are at different distances from the neutral axis of the cross section. Since both M_z and y affect the sign of the bending normal stress in Equation (6.12), stresses must be checked at four points:

- On the top and bottom surfaces on the cross-section location where M_z is a maximum positive value.
- On the top and bottom surfaces on the cross-section location where M_z is a maximum negative value.

Example 6.11 elaborates this issue.

EXAMPLE 6.11

Figure 6.44 shows a loaded beam and cross section. (a) Draw the shear force and bending moment diagrams for the beam, and determine the maximum shear force and bending moment. (b) Determine the maximum tensile and compressive bending normal stress in the beam.

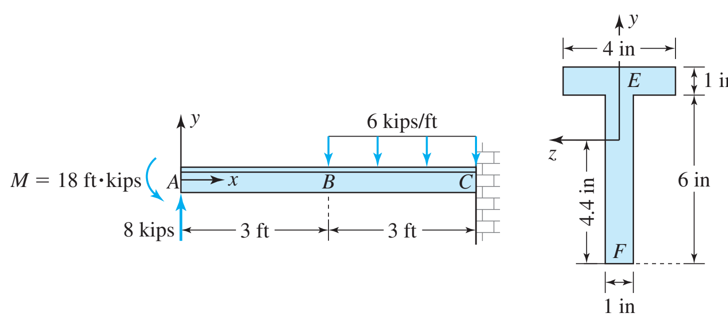


Figure 6.44 Beam and loading in Example 6.11.

$I_{zz} = 47.7\ \text{in}^4$

PLAN

(a) We can determine the reaction force and moment at wall C and follow the procedure for drawing shear and moment diagrams described in Section 6.4.3. (b) We can find σ_{xx} from Equation (6.12) at points E and F at those cross sections where the M_z value is maximum positive and maximum negative. From these four values we can find the maximum tensile and compressive bending normal stresses.

SOLUTION

(a) We draw the shear force and bending moment diagram as per the procedure outlined in Section 6.4.3.

Step 1: From the free-body diagram shown in Figure 6.45 we can determine the value of the reaction force R_w by equilibrium of forces in the y direction. By equilibrium of moment about point C, we can determine the reaction moment M_w. The values of these reactions are

$$R_w = 10\ \text{kips} \tag{E1}$$

$$M_w = 3\ \text{ft·kips} \tag{E2}$$

Step 2: We draw and label the axes for V and M_z and record the units.

Step 3: The beam is shown in Figure 6.45 with all forces and moments acting on it. Vertical lines at points A, B, and C are drawn as shown.

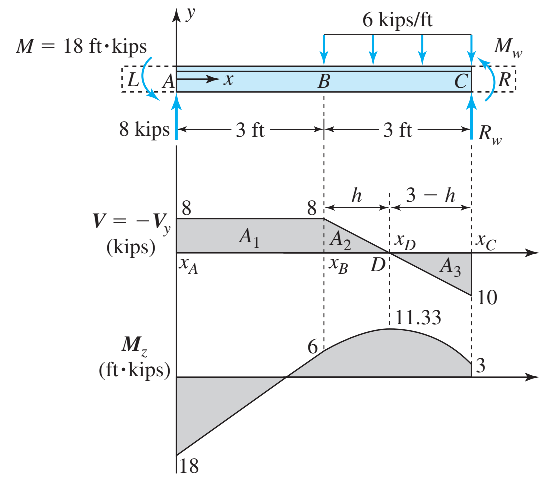


Figure 6.45 Shear and moment diagrams in Example 6.11.

Step 4: We draw imaginary extensions LA and CR to the beam.

Shear force diagram in Figure 6.45

Steps 5, 6, 7: In segment LA the shear force is zero, and hence $V_1 = 0$. The 8-kips force at A is upward, so we jump to a value of $V_2 = +8$ kips just to the right of point A. $p_y = 0$ in segment AB, and hence the value of V remains at 8 kips just before B. Since there is no point force at B, there is no jump in V at B.

In segment BC the area under the distributed load is 18 kips. As p_y is negative, we subtract the area from 8 kips to get a value of –10 kips just before C. We join it by a straight line as p_y is uniform in BC.

The reaction force R_w is upward, so we add the value to −10 kips to get a zero value just after C, confirming the correctness of our solution.

Step 8: At point D, where $V = 0$, we draw another vertical line. To find the location of point D, we use the two similar triangles on either side of point D to get the value of h in Equation (E3).

$$\frac{8\text{ kips}}{h} = \frac{10\text{ kips}}{3\text{ ft} - h} \qquad \text{or} \qquad h = 1.333\text{ ft} \tag{E3}$$

Step 9: We calculate areas A_1, A_2, and A_3 as

$$A_1 = (8)(3) = 24 \qquad A_2 = \frac{1}{2}(8)(h) = 5.33 \qquad A_3 = \frac{1}{2}10(3-h) = 8.33 \tag{E4}$$

Moment diagram in Figure 6.45

Steps 10, 11, 12: In segment LA the bending moment is zero, and hence $M_1 = 0$. Comparing the 18-ft·kips couple at point A with M_{ext} in the moment template in Figure 6.47, we obtain $M_{ext} = –18$ ft·kips. Hence from the template equation $M_2 = –18$ ft·kips just to the right of point A.

The area A_1 is positive, so we add its value to −18 ft·kips to obtain $M_2 = +6$ ft·kips just before B. As V was constant in AB, we join the moments at points A and B by a straight line, as shown.

The area A_2 is positive, so we add its value to +6 ft·kips to obtain $M_2 = +11.33$ ft·kips just before D. As V is linear between B and D, the integral will result in a quadratic function. The magnitude of the shear force is decreasing. Hence the incline of the tangent to the moment curve must decrease as we move from point B toward point D, resulting in the convex curve shown between B and D. Alternatively, since p_y is negative between B and D, the curve is convex.

The area A_3 is negative, so we subtract its value from 11.33 ft·kips to obtain +3 ft·kips just before C. As V is linear between D and C, the integral will result in a quadratic function. Since the magnitude of the shear force is increasing. Hence the incline of the tangent to the moment curve must increase as we move from point D toward C, resulting in the convex curve shown between D and C.

Comparing the moment M_w at C with M_{ext} in the template in Figure 6.42, we obtain $M_{ext} = –M_w = –3$ ft·kips. Hence from the template equation $M_2 = 0$ just to the right of point C. That is, in the imaginary segment CR the moment is zero as expected, confirming the correctness of our construction.

From Figure 6.45 we see that the maximum values of V and M_z are –10 kips and –18 ft·kips, respectively. Recollect that, $V = –V_y$. This gives us the maximum values of the shear force and the bending moment.

ANS. $(V_y)_{max} = 10$ kips $\qquad (M_z)_{max} = –18$ ft·kips

(b) The maximum positive moment occurs at D ($M_D = +11.33$ ft·kips = 136 in·kips) and the maximum negative moment occurs at A ($M_A = –18$ ft·kips = –216 in.·kips). We can evaluate the bending normal stress at points E ($y_E = +2.6$ in) and F ($y_F = –4.4$ in.) on the cross sections at A and D using Equation (6.12):

- On cross section A, at point E, the bending normal stress is

$$\sigma_{AE} = -\frac{(-216\ \text{in.}\cdot\text{kips})(2.6\ \text{in.})}{47.7\ \text{in.}^4} = +11.8\ \text{ksi} \tag{E5}$$

- On cross section *A*, at point *F*, the bending normal stress is

$$\sigma_{AF} = -\frac{(-216\ \text{in.}\cdot\text{kips})(-4.4\ \text{in.})}{47.7\ \text{in.}^4} = -19.9\ \text{ksi} \tag{E6}$$

- On cross section *D*, at point *E*, the bending normal stress is

$$\sigma_{DE} = -\frac{(136\ \text{in.}\cdot\text{kips})(2.6\ \text{in.})}{47.7\ \text{in.}^4} = -7.4\ \text{ksi} \tag{E7}$$

- On cross section *D*, at point *F*, the bending normal stress is

$$\sigma_{DF} = -\frac{(136\ \text{in.}\cdot\text{kips})(-4.4\ \text{in.})}{47.7\ \text{in.}^4} = +12.6\ \text{ksi} \tag{E8}$$

From the results in Equations (E5), (E6), (E7), and (E8), it is clear that the maximum tensile bending normal stress occurs at point *F* on the cross section at *D*. The maximum compressive bending normal stress occurs instead at point *F* on the cross section at *A*.

ANS. $\sigma_{DF} = 12.6\ \text{ksi (T)}$ $\sigma_{AF} = 19.9\ \text{ksi (C)}$

COMMENTS

1. In practice we need not write each step. Equations (E3) through (E4) suffice for drawing the shear and moment diagrams.
2. In Figure 6.45 we see that at point *D*, where shear force is zero, there is a local maximum in the bending moment. But the maximum moment in the beam is at point *A*, where a point moment is applied.
3. If we were determining the *magnitude* of the maximum bending normal stress, then we need to evaluate stress one point only—where the moment is a maximum (cross section A) and where *y* is also a maximum (point *F*).
4. We could use the template shown in Figure 6.46 to determine the direction of the jump in the moment. It can be verified that the moment jumps at *A* and *C* will be as before. This shows that the direction of M_{ext} on the template is immaterial. Thus there is no need to memorize the template, which can be drawn before starting on the shear and moment diagrams.

Template Equation: $\boldsymbol{M}_2 = \boldsymbol{M}_1 - M_{ext}$

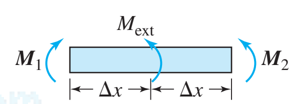


Figure 6.46 Alternative template and equation.

EXAMPLE 6.12

Consider the beam shown in Figure 6.38. Select the lightest W- or S-shaped beams from these given in Appendix E if the allowable bending normal stress is 53 MPa in tension or compression.

PLAN

We can draw the shear and moment diagrams using the procedure described in Section 6.4.3. From the moment diagram we can find the maximum moment. Using the allowable bending normal stress of 53 MPa and Equation (6.22), we can find the minimum sectional modulus. Using Section C.6, we can make a list of the beams for which the sectional modulus is just above the one we determined and choose the lightest beam we can use.

SOLUTION

Step 1: The reaction forces at points *A* and *D* were determined in Example 6.10.

Step 2: The beam with all forces and moments acting on it is shown in Figure 6.47. Vertical lines at points *A*, *B*, *C*, and *D* are shown as drawn.

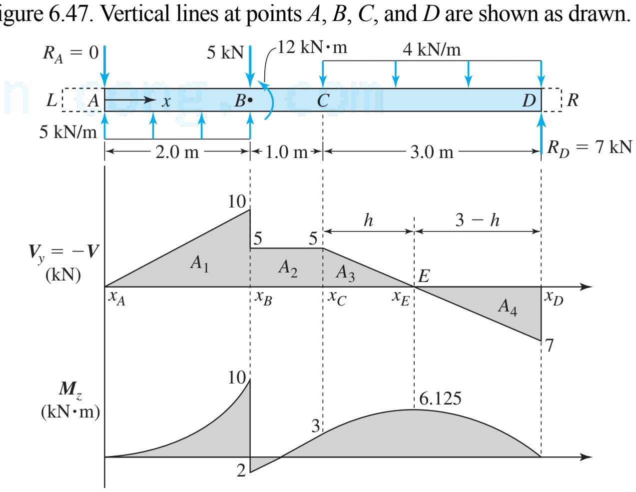


Figure 6.47 Shear and moment diagrams in Example 6.12.

Step 3: We draw imaginary extensions *LA* and *DR* to the beam.

Step 4: We label the axes for V and M_z and record the units.

Shear force diagram in Figure 6.47

Steps 5, 6, 7: In segment LA, $V_1 = 0$. Since R_A is zero, there is no jump at A and we start our diagram at zero.

In segment AB, $p_y = +5$ kN/m. Hence we add the area of 5 × 2 = 10 kN to obtain $V_2 = 10$ kN just before point B and draw a straight line between the values of V at A and B, as shown in Figure 6.47. At B the point force of 5 kN is downward. Thus we jump downward by 5 kN to obtain $V_2 = 5$ kN just after point B.

In segment BC, $p_y = 0$; hence the value of V does not change until point C.

In segment CD, $p_y = -4$ kN/m. Hence we subtract the area of 4 × 3 = 12 kN to obtain $V_2 = -7$ kN just before point D and draw a straight line between the values of V at C and D, as shown in Figure 6.47. The reaction force at D is upward, so we jump upward by 7 kN to obtain $V_2 = 0$ kN just after point D. That is, in the imaginary segment DR the shear force is zero as expected, confirming the correctness of our construction.

Step 8: At point E, where $V_y = 0$, we draw another vertical line. To find the location of point E, we use the two similar triangles on either side of point E to obtain h:

$$\frac{5\text{ m}}{h} = \frac{7\text{ m}}{3\text{ m} - h} \qquad \text{or} \qquad h = 1.25\text{ m} \tag{E1}$$

Step 9: We calculate the areas A_1 through A_4

$$A_1 = \frac{1}{2}10 \times 2 = 10 \qquad A_2 = 5 \times 1 = 5 \qquad A_3 = \frac{1}{2}5h = 3.125 \qquad A_4 = \frac{1}{2}7(3-h) = 6.125 \tag{E2}$$

Bending moment diagram in Figure 6.47

Steps 10, 11, 12: In segment LA the bending moment is zero, and hence $M_1 = 0$. As there is no point moment at A, we start our moment diagram at zero.

As V is positive in segment AB, we add the area A_1 to obtain $M_2 = +10$ kN·m just before B. As V is linear in AB, the integral will result in a quadratic function between A and B. As the magnitude of the shear force is increasing, the incline of the tangent to the moment curve must increase as we move from point A toward point B, resulting in the concave curve shown between A and B. Alternatively, since p_y is positive in AB, the moment curve is concave.

Comparing the moment 12 kN·m at B with M_{ext} in the template in Figure 6.42, we obtain $M_{ext} = -12$ kN·m. Hence from the template equation $M_2 = 10 - 12 = -2$ kN·m just to the right of point B.

As V is positive in segment BC, we add the area A_2 to obtain the value of $M_2 = +3$ kN·m just before C. As V is constant between B and C, the integral will result in a linear function, so we draw a straight line between B and C.

As V is positive in segment CE, we add the area A_3 to obtain the value of $M_2 = +6.125$ kN·m just before E. As V is linear between C and E, the integral will result in a quadratic function. As the magnitude of the shear force is decreasing, the incline of the tangent to the moment curve must also decrease as we move from point C toward E, resulting in the convex curve between C and E, as shown. Alternatively, since p_y is negative in CE, the moment curve is convex.

As V is positive in ED, we add the area A_4 to obtain the value of $M_2 = 0$ just before D. As V is linear between E and D, the integral will result in a quadratic function. As the magnitude of the shear force is increasing, the incline of the tangent to the moment curve must also increase as we move from point E toward D, resulting in the convex curve between E and D, as shown. Alternatively, since p_y is negative in ED, the moment curve is convex.

As there is no point moment at D, there will be no jump in the moment at D. Hence we obtain a zero value for the moment in the imaginary segment DR as expected, confirming the correctness of our construction.

From the moment diagram in Figure 6.47 the maximum moment is $M_{max} = 10$ kN·m. Noting that the allowable bending normal stress is 53 MPa, Equation (6.22) yields

$$\sigma_{max} = \frac{10(10^3)\text{ N}}{S} \le 53(10^6)\text{ N/m}^2 \qquad \text{or} \qquad S \ge 188.7(10^3)\text{ mm}^3 \tag{E3}$$

From Section C.6 we obtain the following list of W- and S-shaped beams that have a section modulus close to that given in Equation (E3):

W150 × 29.8	$S = 219 \times 10^3\text{ mm}^3$	and	S200 × 27.4	$S = 236 \times 10^3\text{ mm}^3$
W200 × 22.5	$S = 194.2 \times 10^3\text{ mm}^3$		S180 × 30	$S = 198.3 \times 10^3\text{ mm}^3$

The lightest beam is W200 × 22.5 as it has a mass of only 22.5 kg/m.

ANS. W200 × 22.5

COMMENTS

1. This example demonstrate the use of the section modulus in selecting the beam cross section from a set of standard shapes. But the section modulus can also be used with nonstandard shapes.
2. The alternative template shown in Figure 6.46 could have been used in this example. It would result in the same jumps as shown in Figure 6.47.
3. Again, only Equations (E1) through (E2) are needed to obtain the shear and moment diagrams. From here on, the shear and moment diagrams will be drawn without additional explanations.

QUICK TEST 6.1

Time: 20 minutes/Total: 20 points

Answer true or false and justify each answer in one sentence. Grade yourself with the answers given in Appendix E. Assume linear elastic, homogeneous material unless stated otherwise.

1. If you know the geometry of the cross section and the bending normal strain at one point on a cross section, then the bending normal strain can be found at any point on the cross section.

2. If you know the geometry of the cross section and the maximum bending normal stress on a cross section, then the bending normal stress at any point on the cross section can be found.

3. A rectangular beam with a 2-in. × 4-in. cross section should be used with the 2-in. side parallel to the bending (transverse) forces.

4. The best place to drill a hole in a beam is through the centroid.

5. In the formula $\sigma_{xx} = -M_z y / I_{zz}$, y is measured from the bottom of the beam.

6. The formula $\sigma_{xx} = -M_z y / I_{zz}$ can be used to find the normal stress on a cross section of a tapered beam.

7. The equations $\int_A \sigma_{xx}\, dA = 0$ and $M_z = -\int_A y\sigma_{xx}\, dA$ cannot be used for nonlinear materials.

8. The equation $M_z = -\int_A y\sigma_{xx}\, dA$ can be used for nonhomogeneous cross sections.

9. The internal shear force jumps by the value of the applied transverse force as one crosses it from left to right.

10. The internal bending moment jumps by the value of the applied concentrated moment as one crosses it from left to right.

PROBLEM SET 6.3

Equilibrium of shear force and bending moment

6.59 (a) Write the equations for shear force and bending moments as a function of x for the entire beam shown in Figure P6.59; (b) show that your results satisfy Equations (6.17) and (6.18).

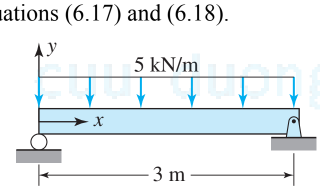


Figure P6.59

6.60 (a) Write the equations for shear force and bending moments as a function of x for the entire beam shown in Figure P6.60; (b) show that your results satisfy Equations (6.17) and (6.18).

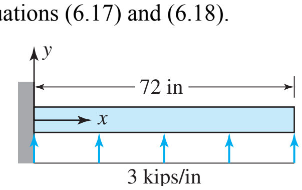


Figure P6.60

6.61 (a) Write the equations for shear force and bending moments as a function of x for the entire beam shown in Figure P6.61; (b) show that your results satisfy Equations (6.17) and (6.18).

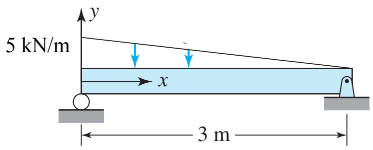


Figure P6.61

6.62 (a) Write the equations for shear force and bending moments as a function of x for the entire beam shown in Figure P6.62; (b) show that your results satisfy Equations (6.17) and (6.18).

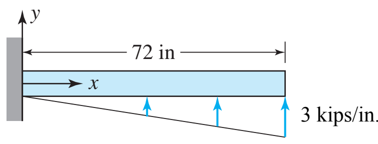


Figure P6.62

6.63 (a) Write the equations for shear force and bending moments as a function of x for the entire beam shown in Figure P6.63; (b) show that your results satisfy Equations (6.17) and (6.18).

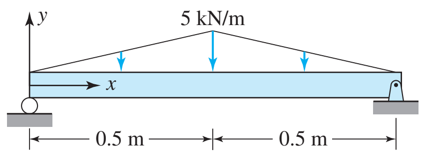


Figure P6.63

6.64 (a) Write the equations for shear force and bending moments as a function of x for the entire beam shown in Figure P6.64; (b) show that your results satisfy Equations (6.17) and (6.18).

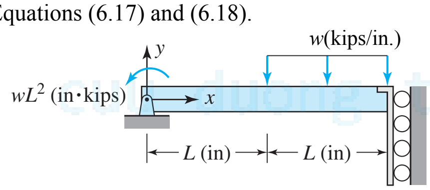


Figure P6.64

6.65 (a) Write the equations for shear force and bending moments as a function of x for the entire beam shown in Figure P6.65; (b) show that your results satisfy Equations (6.17) and (6.18).

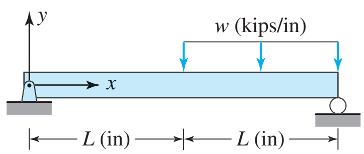


Figure P6.65

6.66 (a) Write the equations for shear force and bending moments as a function of x for the entire beam shown in Figure P6.66; (b) show that your results satisfy Equations (6.17) and (6.18).

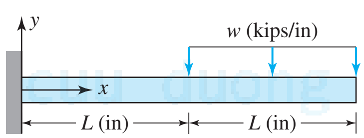


Figure P6.66

6.67 Consider the beam shown in Figure P6.67. (a) Write the shear force and moment equations as a function of x in segments AB and BC. (b) Show that your results satisfy Equations (6.17) and (6.18). (c) What are the shear force and bending moment values just before and just after point B?

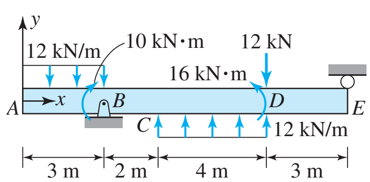


Figure P6.67

6.68 Consider the beam shown in Figure P6.67. (a) Write the shear force and moment equations as a function of x in segments CD and DE. (b) Show that your results satisfy Equations (6.17) and (6.18). (c) What are the shear force and bending moment values just before and just after point D?

6.69 Consider the beam shown in Figure P6.69. (a) Write the shear force and moment equations as a function of x in segments AB and BC. (b) Show that your results satisfy Equations (6.17) and (6.18). (c) What are the shear force and bending moment values just before and just after point B?

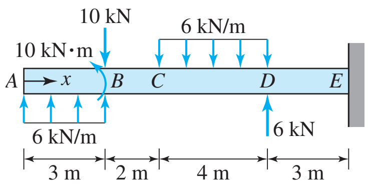


Figure P6.69

6.70 Consider the beam shown in Figure P6.69. (a) Write the shear force and moment equations as a function of x in segments CD and DE. (b) Show that your results satisfy Equations (6.17) and (6.18). (c) What are the shear force and bending moment values just before and just after point D?

6.71 During skiing, the weight of a person is often all on one ski. The ground reaction is modeled as a distributed force $p(x)$ and a concentrated fore P is modeled as shown in Figure P6.71. (a) Find shear force and bending moment as a function of x across the ski. (b) The ski is 50 mm wide and the thickness of the ski varies as shown. Determine the maximum bending normal stress. Use of spread sheet recommended.

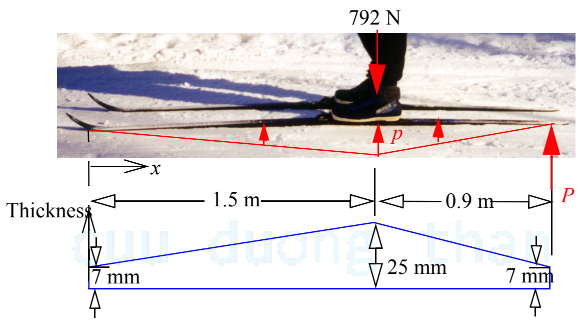


Figure P6.71

Shear and moment diagrams

6.72 Draw the shear and moment diagrams for the beam and loading shown in Figure P6.72.

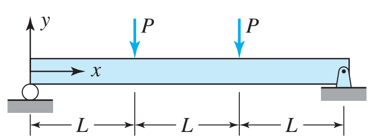


Figure P6.72

6.73 Draw the shear and moment diagrams for the beam and loading shown in Figure P6.73.

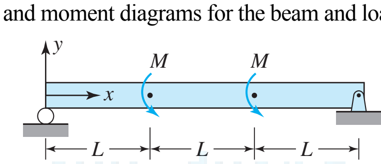


Figure P6.73

6.74 For the beam shown in Figure P6.59, draw the shear force bending moment diagram. Determine the maximum values of shear force and bending moment.

6.75 For the beam shown in Figure P6.60, draw the shear force bending moment diagram. Determine the maximum values of shear force and bending moment.

6.76 A man whose mass is 80 kg is sitting in the middle of a flat bottom boat, as shown in Figure 6.76. The weight of the boat per unit length between A and B is 130 N/m. To a first approximation assume the resisting water pressure acts between A and B and is uniform. Draw the shear force and bending moment diagram between A and B.

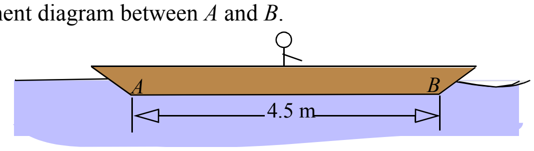


Figure P6.76

6.77 Draw the shear and moment diagrams for the beam and loading shown in Figure P6.77. Determine the maximum values of shear force and bending moment.

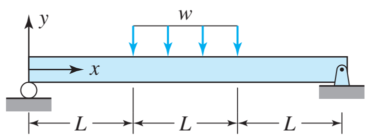


Figure P6.77

6.78 Draw the shear and moment diagrams for the beam and loading shown in Figure P6.78. Determine the maximum values of shear force and bending moment.

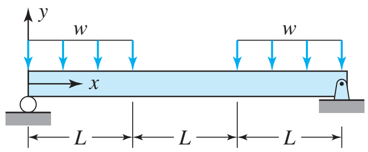


Figure P6.78

6.79 Determine the maximum values of shear force and bending moment for the beam shown in Figure P6.67.

6.80 Determine the maximum values of shear force and bending moment for the beam shown in Figure P6.69.

6.81 Determine the maximum values of shear force and bending moment for the beam shown in Figure P6.81.

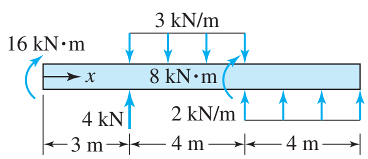


Figure P6.81

6.82 Determine the maximum values of shear force and bending moment for the beam shown in Figure P6.82.

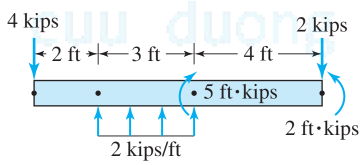


Figure P6.82

6.83 Determine the maximum values of shear force and bending moment. for the beam shown in Figure P6.83.

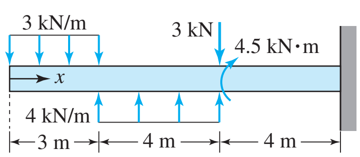


Figure P6.83

6.84 Determine the maximum values of shear force and bending moment for the beam shown in Figure P6.84.

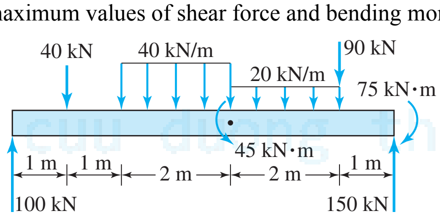


Figure P6.84

6.85 Determine the maximum value of the shear force and bending moment for the beam shown in Figure 6.85.

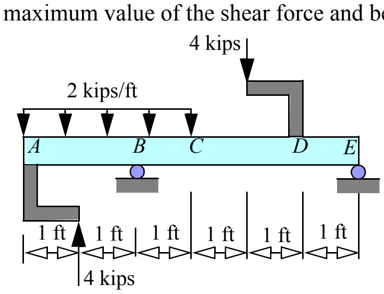


Figure P6.85

6.86 Determine the maximum values of shear force and bending moment for the beam shown in Figure P6.86.

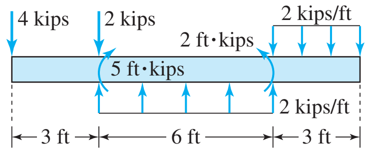


Figure P6.86

Maximum bending normal stress

6.87 A diver weighing 200 lb stands at the edge of the diving board, as shown in Figure 6.87. The diving board cross section is 16 in. x 1 in. Determine the maximum bending normal stress in the diving board.

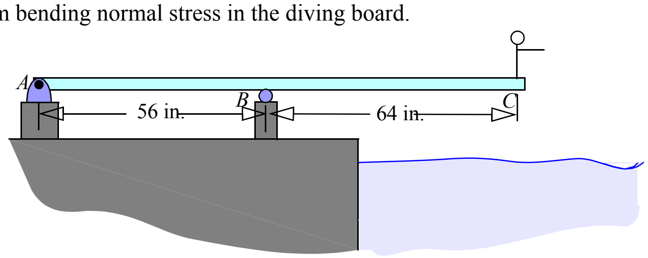


Figure P6.87

6.88 A diver weighing 200 lb stands at the edge of the diving board, as shown in Figure 6.87. The diving board cross section is 16 in. x 1 in. and has a weight of 60 lb. Model the weight of the diving board as a uniform distributed load of 0.5 lb/in. along the length. Determine the maximum bending normal stress in the diving board.

6.89 The diving board shown in Figure 6.87 has a cross section of 18 in. x 1 in. The allowable bending normal stress 10 ksi. What is the maximum force to the nearest pound that the board can sustain when the diver jumps on it before a dive. Neglect the weight of the diving board.

6.90 A father and his son are playing on a seesaw, as shown in Figure 6.90. The wooden plank of the see saw is 12 ft x 10 in. x 1.5 in. and is hinged in the middle. The weights of the father and son are 225 lb and 80 lb, respectively. The mass of the father m_F and mass of the son m_s times the acceleration a are the inertial forces acting on them at the time the plank is horizontal. Neglecting the weight of the plank, determine the maximum bending normal stress.

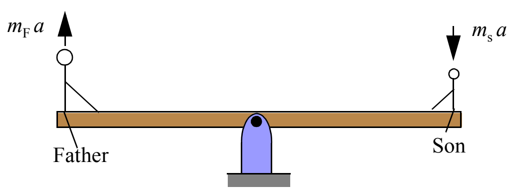


Figure P6.90

6.91 A mother and her daughter are on either side of the seesaw with the teenager son standing in the middle as shown in Figure 6.91. The wooden plank of the seesaw is 3.5 m x 250 mm x 40 mm and is hinged in the middle. The mass of the mother, son, and daughter are m_m= 70 kg, m_s= 80 kg, and m_d= 40 kg, respectively. At the time the plank is horizontal, inertial forces of mass times the acceleration a acts on the mother and daughter. Neglecting the weight of the plank, determine the maximum bending normal stress.

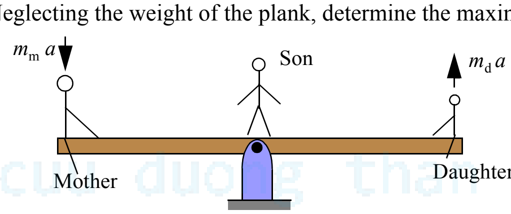
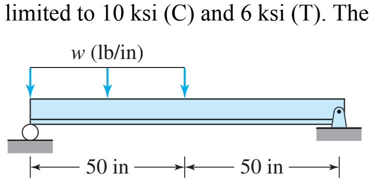


Figure P6.91

Design problems

6.92 A beam, its loading, and its cross section are as shown Figure P6.92. Determine the intensity w of the distributed load if the maximum bending normal stress is limited to 10 ksi (C) and 6 ksi (T). The second area moment of inertia is I_{zz} = 47.73 in.^{4}.



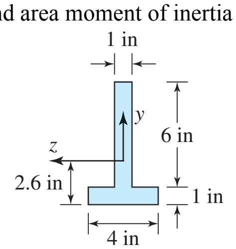


Figure P6.92

6.93 Two pieces of lumber are glued together to form the beam shown in Figure P6.93. Determine the intensity w of the distributed load if the maximum tensile bending normal stress in the glue is limited to 800 psi (T) and the maximum bending normal stress in wood is limited to 1200 psi.

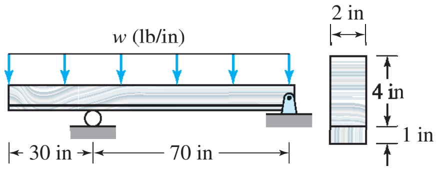


Figure P6.93

6.94 The beam shown in Figure P6.64 has a load w = 25 lb/in. and L = 72 in. Select the lightest W- or S-shaped beam from Section C.6 if the allowable bending normal stress is 21 ksi in tension and compression.

6.95 The beam shown in Figure P6.65 has a load w = 0.4 kips/in. and L = 48 in. Select the lightest W- or S-shaped beam from Section C.6 if the allowable bending normal stress is 16 ksi in tension and compression.

6.96 The beam shown in Figure P6.66 has a load w = 0.15 kips/in. and L = 48 in. Select the lightest W- or S-shaped beams from Section C.6 if the allowable bending normal stress is 21 ksi in tension and compression.

6.97 Consider the beam shown in Figure P6.67. Select the lightest W- or S-shaped beam from Section C.6 if the allowable bending normal stress is 180 MPa in tension and compression.

6.98 Consider the beam shown in Figure P6.69. Select the lightest W- or S-shaped beam from Section C.6 if the allowable bending normal stress is 225 MPa in tension and compression.

6.99 The wind pressure on a signpost is approximated as a uniform pressure, as shown Figure P6.99. A similar signpost is to be designed using a hollow square steel beam for the post. The outer dimension of the square is to be 12 in. If the allowable bending normal stress is 24 ksi and the pressure p = 33 lb/ft^2, determine the inner dimension of the lightest hollow beam to the nearest $\frac{1}{8}$ in.

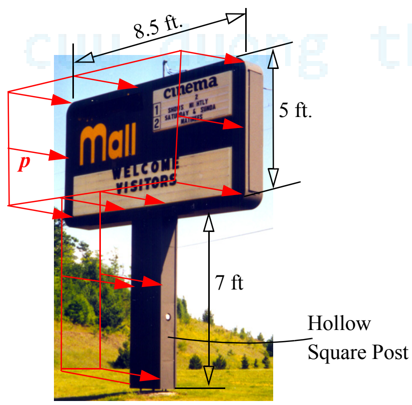


Figure P6.99

Stress concentration

6.100 The allowable bending normal stress in the stepped circular beam shown in Figure P6.100 is 200 MPa and P = 200 N. Determine the smallest fillet radius that can be used at section B. Use stress concentration graphs given in Section C.4.

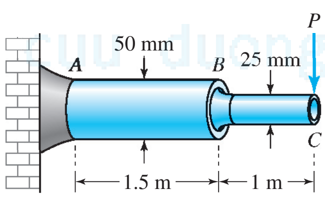


Figure P6.100

6.101 The allowable bending normal stress in the stepped circular beam shown in Figure P6.101 is 48 ksi. Determine the maximum intensity of the distributed load w assuming the fillet radius is: (a) 0.3 in.; (b) 0.5 in. Use stress concentration graphs given in Section C.4.

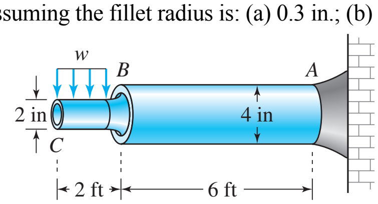


Figure P6.101

Fatigue

6.102 The fillet radius is 5 mm in the stepped aluminum circular beam shown in Figure P6.100. What should be the peak value of the cyclic load P to ensure a service life of one-half million cycles? Use the S–N curve shown in Figure 3.36.

6.103 The beam in Figure P6.101 is made from a steel alloy that has the S–N curve shown in Figure 3.36. The peak intensity of the cyclic distributed load is w = 80 lbs/in. and the fillet radius is 0.36 in. What is the predicted service life of the beam?

Stretch Yourself

6.104 A simply supported 3-m-long beam has a uniformly distributed load of 10 kN/m over the entire length of the beam. If the beam has the composite cross section shown in Figure P6.104, determine the maximum bending normal stress in each of the three materials. Use E_{al} = 70 GPa, E_w = 10 GPa, and E_s = 200 GPa. [Hint: Use Equations (6.14) and (6.16)].

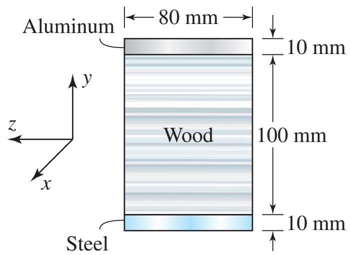


Figure P6.104

6.105 A steel (E_{steel} = 200 GPa) tube of outside diameter of 240 mm is attached to a brass (E_{brass} = 100 GPa) tube to form the cross section shown in Figure P6.105. Determine the maximum bending normal stress in steel and brass. [Hint: Use Equation (6.16)]

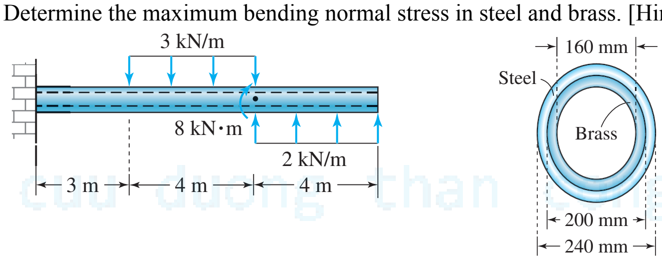


Figure P6.105

6.6 SHEAR STRESS IN THIN SYMMETRIC BEAMS

In Section 6.2.6 we observed that the maximum bending shear stress has to be nearly an order of magnitude less than the maximum bending normal stress for our theory to be valid. But shear stress plays an important role in bending, particularly when beams are constructed by joining a number of beams together to increase stiffness. In this section we develop a theory that can be used for calculating the bending shear stress.

Figure 6.48*a* and *b* shows the bending of four wooden strips that are *separate* and *glued together*, respectively. In Figure 6.48*a* each wooden strip slides relative to the other in the longitudinal direction. But in Figure 6.48*b* the relative sliding is prevented by the shear resistance of the glue—that is, the shear stress in the glue. One may thus hypothesize that in any beam there will be shear stresses on imaginary surfaces parallel to the axis of the beam.

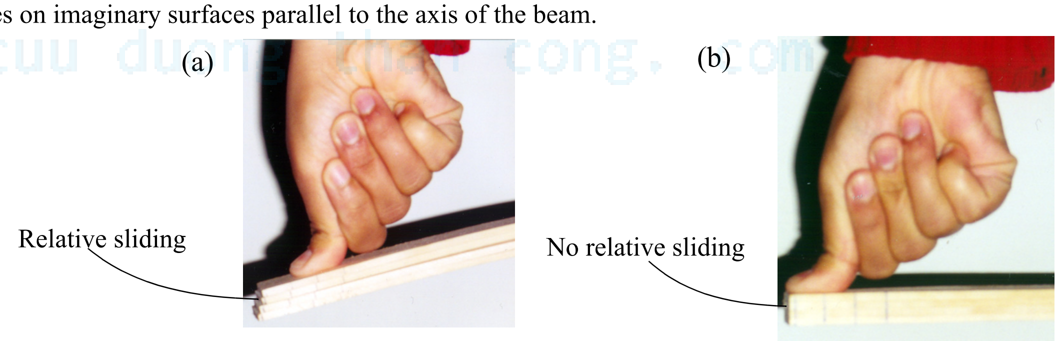


Figure 6.48 Effect of shear stress in bending. (a) Separate beams. (b) Glued beams.

Notice that the beam in Figure 6.48*a* has significantly more curvature (it bends more) than that in Figure 6.48*b*, even though the forces exerted in both cases are approximately the same. This phenomenon of increasing stiffness (see Problem

6.20) at the expense of introducing shear stress is exploited in the design of lightweight structures. In metal beams, the flanges are designed for carrying most of the normal stress in bending, and the webs are designed for carrying most of the shear stress (see Figure 6.28). In sandwich beams two stiff panels are separated by a soft core material. The stiff panels are designed to carry the normal stress and the soft core is designed to carry the shear stress.

6.6.1 Shear Stress Direction

Before developing formulas, it is worthwhile to understand the character of the shear stresses in bending and to determine their direction by inspection.

Consider the beam in Figure 6.49*a*. The beam is constructed by gluing five pieces of wood together. From the evidence of the photographs in Figure 6.48, we know that shear stress will exist at each glued surface to resist the relative sliding of the wood strips. If we take a small element Δx of strips 3 and 5, we obtain Figure 6.49*c* and *d*. On the glued surface between wooden strips 2 and 3 there will be a shear stress τ_{zx} as the outward normal of the surface is in the z direction and the internal shear force is in the x direction. On the glued surface between wooden strips 4 and 5 there will be a shear stress τ_{yx}, as the outward normal of the surface is in the y direction and the internal shear force is in the x direction.

Because of the bending load P, a normal stress distribution across the cross section will develop as shown in Figure 6.49*b*. From Equation (6.12), we know the bending normal stress σ_{xx} will vary along the length of the beam as the moment $\boldsymbol{M_z}$ varies. The bending normal stress distribution is such that there is no resultant axial force over the *entire* cross section. But if we only take a *part* of the cross section, as in Figure 6.49*b* and *c,* then there will be an axial force generated that varies along the length as shown in Figure 6.49*c* and *d.*

On the small element Δx, the equivalent shear force from the bending shear stresses must balance the change in the equivalent normal axial force, as shown Figure 6.54*e* and *f*. Thus in bending, the shear stress must balance the variations in the normal stress σ_{xx} along the length of the beam.[9]

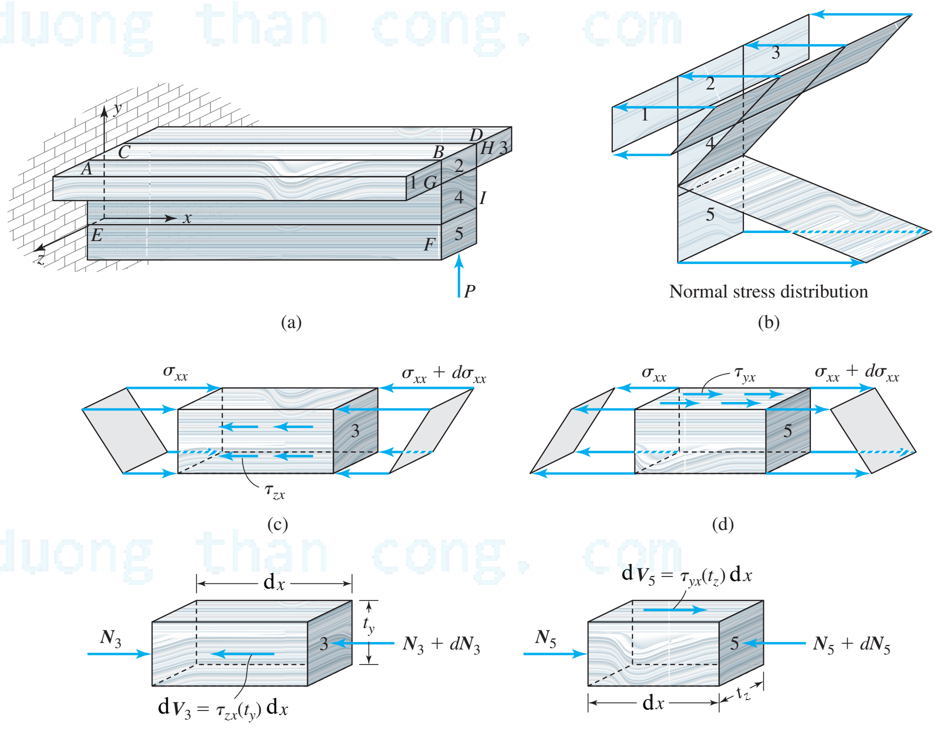


Figure 6.49 Shear stress on different surfaces in bending.

The preceding shows that shear stress develops on surfaces cut parallel to the axis of the beam. But from the symmetry of shear stresses $\tau_{xy} = \tau_{yx}$ and $\tau_{xz} = \tau_{zx}$. These stresses, τ_{xy} and τ_{xz}, are on the cross sections perpendicular to the axis of the beam.

[9]From the field of elasticity it is known that in the absence of body forces, the equilibrium at a point requires $\partial\sigma_{xx}/\partial x + \partial\tau_{yx}/\partial y + \partial\tau_{zx}/\partial z = 0$ (see Problem 1.105). Thus if σ_{xx} varies with x, then τ_{yx} (or τ_{xy}) must vary with y, and τ_{zx} (or τ_{xz}) must vary with z. See Problem 6.136 for additional details.

We know from Equation (6.10) that on the cross section of the beam the resultant of the shear stress τ_{xy} distribution is the shear force V_y. Thus the direction (sign) of τ_{xy} should be the same as that of V_y. But the shear force V_z that would be statically equivalent to τ_{xz} must be zero, as there is no external force in the *z* direction. This means that τ_{xz} must reverse sign (and direction) on the cross section if the net force from it is zero. We also know that the *y* axis is the axis of symmetry, and the loading is in the plane of symmetry. Therefore all stresses including τ_{xz} must be symmetric about the *y* axis. In other words, the shear stress τ_{xz} will reverse its direction as one crosses the *y* axis on the cross section. This sometimes implies that the shear stress τ_{xz} will be zero at the *y* axis.

Consider now a circular cross section that is glued together from nine wooden strips, as shown in Figure 6.50*a*. Once more shear stresses will develop along each glued surface, to resist relative sliding between two adjoining wooden strips, and the shear stress value must balance the change in axial force due to the variation in σ_{xx}. The outward normal of the surface will be in a different direction for each glued surface on which we consider the shear stress. If we define a tangential coordinate *s* that is in the direction of the tangent to the center line of the cross section, then the outward normal to the glued surface will be in the *s* direction and the shear stress will be τ_{sx}. Once more by the symmetry of shear stresses, $\tau_{xs} = \tau_{sx}$. At a point if the *s* direction and the *y* direction are the same, then τ_{xs} will equal $\pm\tau_{xy}$. If the *s* direction and the *z* direction are the same at a point, then τ_{xs} will equal $\pm\tau_{xz}$.

It should be noted that in Figure 6.49*e* and *f* and in Figure 6.50*b* the shear force that balances the change in the axial force ***N*** is shown on only one surface. The surface on the other end of the free-body diagram is always assumed to be a free surface. That is, the shear stress is zero on these other surfaces. The origin of the *s* coordinate is chosen to be one of the free surfaces and will be used in the next section in developing shear stress formulas. In a beam cross section the top and bottom and the side surfaces are always assumed to be surfaces on which shear stress is zero.

In Figure 6.49*e* and *f* and in Figure 6.50*b* we notice that the shear force expression contains the product of the shear stress and the thickness *t* of the cross section at that point. This product is the **shear flow** *q*.

$$q = \tau_{xs} t \tag{6.23}$$

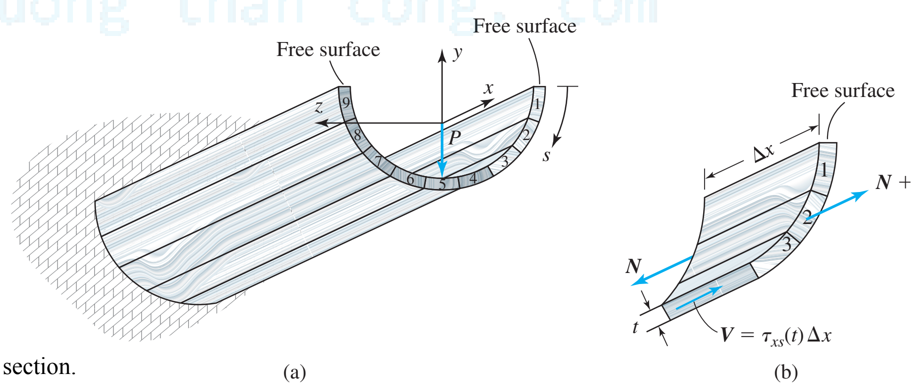


Figure 6.50 Shear stress in circular cross section.

The units of the shear flow are force per unit length. The terminology is from fluid flow in channels, but it is used extensively to discuss shear stresses in thin cross sections, probably because of the image of an actual flow helps in discussing shear stress directions, as elaborated further in the next section and Example 6.13.

6.6.2 Shear Flow Direction by Inspection

The shear flow and shear stress along the center line of the cross section are drawn in a direction that satisfies the following rules:

1. The resultant force in the *y* direction is in the same direction as V_y.
2. The resultant force in the *z* direction is zero.
3. It is symmetric about the *y* axis. This requires that shear flow change direction as one crosses the *y* axis on the center line. Sometimes this will imply that shear stress is zero at the point(s) where the center line intersects the *y* axis.

EXAMPLE 6.13

Assuming a positive shear force V_y, sketch the direction of the shear flow along the center line on the thin cross sections shown in Figure 6.51.

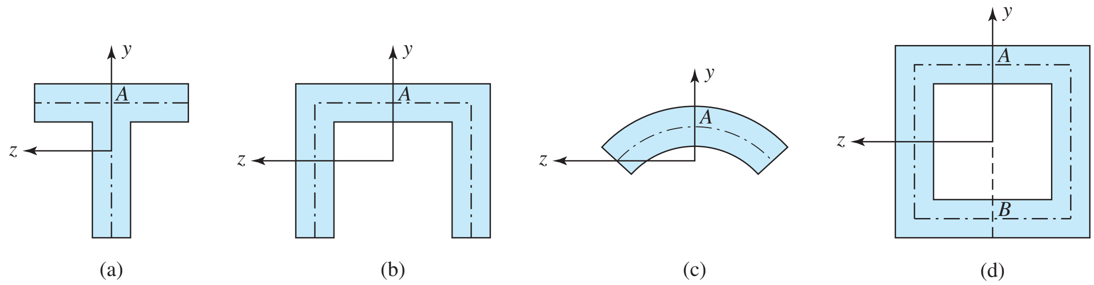


Figure 6.51 Cross sections in Example 6.13.

PLAN

With the outward normal of the cross section in the positive *x* direction, the positive shear force V_y will be in the positive *y* direction according to the sign convention in Section 6.2.6. We can determine the direction of the flow in each cross section to satisfy the rules described at the end of Section 6.6.2.

SOLUTION

(a) On the cross section shown in Figure 6.52*a* the shear flow (shear stress) from *C* to *A* will be in the positive *y* direction, since V_y on the cross section is in the positive *y* direction. At point *A* in the flange the flow will break in two and go in opposite directions, as shown in Figure 6.52*a*. The resultant force due to shear flow from *A* to *D* will cancel the force due to shear flow from *A* to *E*, satisfying the condition of zero resultant force in the *z* direction and the condition of symmetric flow about the *y* axis.

(b) On the cross section shown in Figure 6.52*b*, the shear flow from *C* to *E* and from *D* to *F* will be in the positive *y* direction. This satisfies the condition of symmetry about the *y* axis and is consistent with direction of V_y. In the flange the two flows will approach point *A* from opposite directions. The resultant force due to shear flow from *A* to *E* will cancel the force due to shear flow from *A* to *F*, satisfying the condition of zero resultant force in the *z* direction and the condition of symmetric flow about the *y* axis.

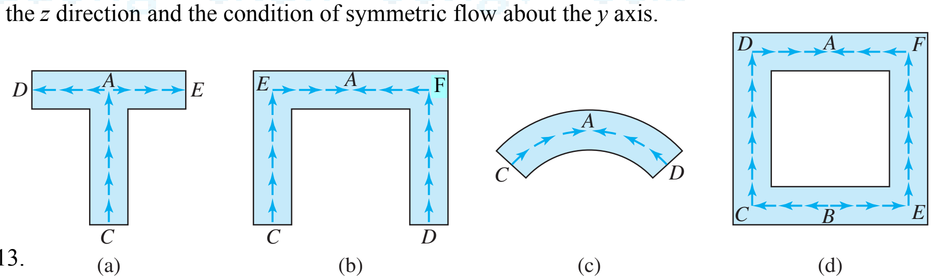


Figure 6.52 Shear flow in Example 6.13.

(c) On the cross section shown in Figure 6.52*c* the shear flows from points *C* and *D* will approach point *A* in opposite directions. This ensures the condition of symmetry, and the condition of zero force in the *z* direction is met.

(d) The shear flow from *C* to *D* and the shear flow from *E* to *F* have to be in the positive *y* direction to satisfy the condition of symmetry about the *y* axis and to have the same direction as V_y. At points *A* and *B* the shear flows must change direction to ensure symmetric shear flows about the *y* axis. The force from the shear flows in *BC* and *DA* will cancel the force from the shear flows in *BE* and *FA*, ensuring the condition of a zero force in the *z* direction.

COMMENTS

1. The shear flow (shear stress) is zero at the following points because these points are on the free surface: points *C*, *D*, and *E* in Figure 6.52*a*; points *C* and *D* in Figure 6.52*b* and *c*.
2. At point *A* in Figure 6.52*b*, *c*, and *d* the shear flow will be zero, but it will not be zero in Figure 6.52*a* as we can appreciate by analogy to fluid flow. In Figure 6.52*a* the shear flows at point *A* in branches *AD* and *AE* add up to the value of shear flow at point *A* in branch *CA*. With no other branch at point *A* in Figure 6.52*b*, *c*, and *d* the values of the shear flow are equal and opposite, which is possible only if the value of shear flow is zero.
3. The term *flow* invokes an image that helps in visualizing the direction of shear stress.
4. By examining the direction of the stress components in the Cartesian system, we can determine whether a stress component is positive or negative τ_{xy} or τ_{xz}, as shown in Figure 6.53. Note that τ_{xy} is positive in all cases, a consequence of positive shear force V_y. But τ_{xz} can be positive or negative, depending on the location of the point.

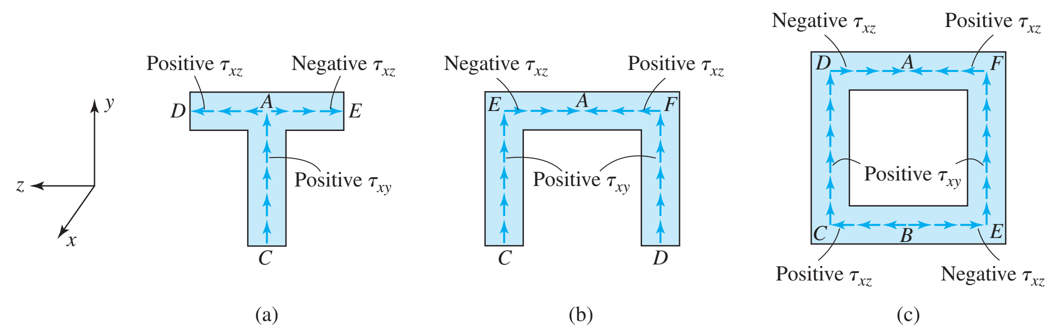


Figure 6.53 Directions and signs of stress components in Example 6.13.

6.6.3 Bending Shear Stress Formula

The previous section and Example 6.13 highlight that the bending shear stress is τ_{xy} in the web and τ_{xz} in the flange, whereas for symmetric curvilinear cross sections it depends on the location of the point. To develop a single formula applicable to all situations, we define a tangential coordinate s in the direction of the tangent to the center line of the cross section, starting from a free surface. In this section we derive the formula for bending shear stress τ_{xs}.

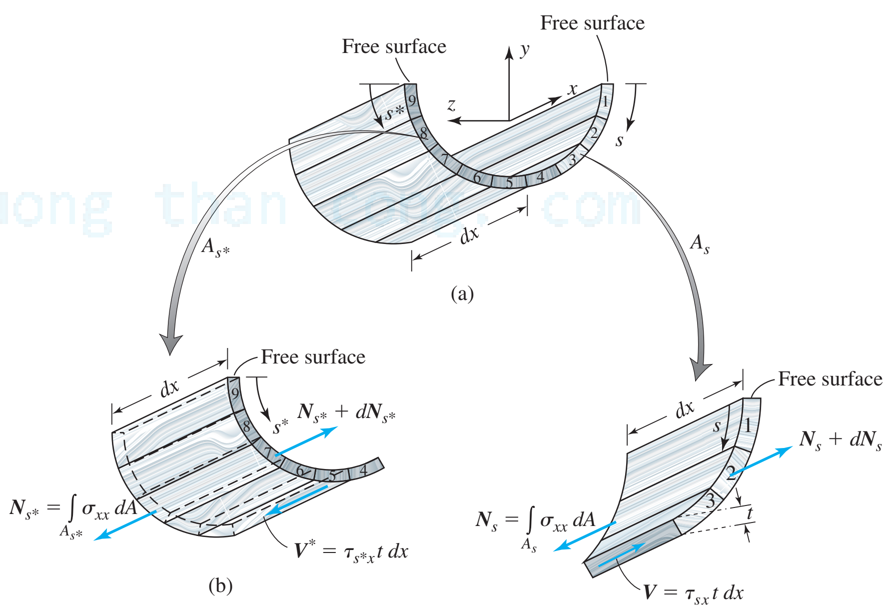


Figure 6.54 Differential element of beam for shear stress calculations.

Consider a differential element of a wooden beam with circular cross section, as shown in Figure 6.54*a*. Consider the shear stress acting on the surface between wooden pieces 3 and 4. We can consider two possible free-body diagrams, shown in Figure 6.54*b* and *c*. The axial force N_s (or N_{s^*}) acting on the part of cross section A_s (or A_{s^*}) varies because of the variation of the bending stress σ_{xx} along the length of the beam. If the shear stress does not change across the thickness, the shear force V (or V^*) is equal to the product of the shear stress multiplied by the area $t\,dx$, as shown. The assumption of constant shear stress in the thickness direction is a good approximation if the thickness is small.

Assumption 9: The beam is thin perpendicular to the center line of the cross section.

By equilibrium of forces in Figure 6.54*c*, we obtain $N_s + dN_s - N_s + \tau_{sx} t\ dx\ =\ 0$ or

$$\tau_{sx} t\ =\ -\frac{dN_s}{dx}\ =\ -\frac{d}{dx}\int_{A_s}\sigma_{xx}\ dA \tag{6.24}$$

Substituting Equation (6.12) into Equation (6.24) and noting that the moment M_z and the area moment of inertia I_{zz} do not vary over the cross section, we obtain

$$\tau_{sx}t = \frac{d}{dx}\left(\frac{M_z}{I_{zz}}\int_{A_s} y\ dA\right) = \frac{d}{dx}\left(\frac{M_z Q_z}{I_{zz}}\right) \tag{6.25}$$

where Q_z is referred to as the first moment of the area A_s and is defined as

$$Q_z = \int_{A_s} y\ dA \tag{6.26}$$

Assumption 10: The beam is not tapered.

Assumption 10 implies that I_{zz} and Q_z are not a function of x, and these quantities can be taken outside the derivative sign. We obtain

$$\tau_{sx}t = (Q_z/I_{zz})dM_z/dx$$

Substituting Equation (6.18), we obtain the formula for bending shear stress:

$$\tau_{sx} = \tau_{xs} = -\left(\frac{V_y Q_z}{I_{zz}t}\right) \tag{6.27}$$

In Equation (6.27) the shear force V_y can be found either by equilibrium or by drawing the shear force diagram. Also, t is the thickness at the point where the shear stress is being found, and I_{zz} is known from the geometry of the cross section. The direction of s, identification of the area A_s, and the calculation of Q_z are the critical new elements. We record the following observations before discussing in detail the calculation of Q_z.

- Area A_s is the area between the free surface and the point where the shear stress is being evaluated.
- s is the direction from the free surface in the area A_s used in the calculation of Q_z.

6.6.4 Calculating Q_z

Figure 6.55 shows the area A_s between the top free surface and the point at which the shear stress is being found (line s–s). From Equation (6.26) we note that Q_z is the first moment of the area A_s about the z axis. The integral in Equation (6.26) is the numerator in the definition of the centroid of the area A_s. Analogous to the moment due to a force, the first moment of an area can be found by placing the area A_s at its centroid and finding the moment about the neutral axis. That is, Q_z is the product of area A_s and the distance of the centroid of the area A_s from the neutral axis, as shown in Figure 6.55. Alternatively, Q_z can be found by using the bottom surface as the free surface, shown as Q_{z*} in Figure 6.55.

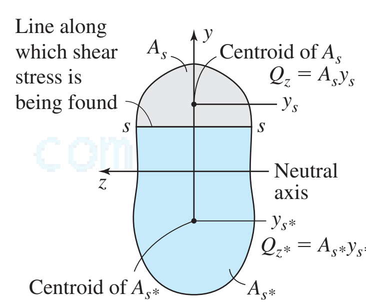


Figure 6.55 Calculation of Q_z.

At the top surface, which is a free surface, the value of Q_z is zero, as the area A_s is zero. When we reach the bottom surface after starting from the top, the value of Q_z is once more zero because $A_s = A$, and from Equation (6.9), $\int_A y\ dA = 0$. If Q_z starts with a zero value at the top and ends with a zero value at the bottom, then it must reach a maximum value somewhere on the cross section.

To see where Q_z reaches a maximum value, consider the change in Q_z as the line s–s moves downward in Figure 6.55. Toward the neutral axis, the moment of the area Q_z increases as we add the moments from the additional areas. When the line s–s crosses the neutral axis, then the new additional area below the axis produces a negative moment because the centroid of this area is in the negative y direction. In other words, Q_z increases up to the neutral axis and then starts decreasing. Thus Q_z is maximum at the neutral axis. From Equation (6.27), it follows that bending shear stress is maximum at the neutral axis of a cross section. In summary

- Q_z is zero at the top and bottom surface.
- Q_z is maximum at the neutral axis.
- The maximum bending shear stress on a cross section is at the neutral axis.
- The maximum bending shear stress in the beam will be at the neutral axis on a cross section where V_y is maximum.

We can write $A = A_s + A_{s*}$ in Equation (6.9) and write the integral as $\int_{A_s} y\,dA + \int_{A_{s*}} y\,dA = 0$ to obtain $Q_z + Q_{z*} = 0$. This implies that Q_z and Q_{z*} will have the same magnitude but opposite signs. Thus, if we used Q_z or Q_{z*} in Equation (6.27), we would get the same magnitude of the shear stress, but which would give the correct sign (or direction)?[10] The answer is that both will give the correct sign, provided the s direction in Equation (6.27) is measured from the free surface used in the calculation of Q_z.

We can find the magnitude and the direction of the bending shear stress in two ways:

1. Use Equation (6.27) to find the magnitude of the shear stress. Use the rules described in Section 6.6.2 to determine the direction of the shear stress.
2. Alternatively, follow the sign convention described in Section 6.2.6 to determine the shear force V_y. The shear stress is found from Equation (6.27), and the direction of the shear stress is determined using the subscripts, as elaborated in Section 1.3.

6.6.5 Shear Flow Formula

The formula for shear flow can be obtained by substituting Equation (6.27) into Equation (6.23) to get

$$q = -\frac{V_y Q_z}{I_{zz}} \tag{6.28}$$

Equation (6.28) can be used in two ways. It can be used for finding the magnitude of the shear flow at a point, and the direction of shear flow can then be found by inspection following the rules described in Section 6.6.2. Alternatively, the sign convention for the shear force V_y is followed and the shear flow is determined from Equation (6.28). A positive value of shear flow implies that the flow is in the positive s direction, where s is measured from the free surface used in the calculation of Q_z.

One application of Equation (6.28) is the determination of the spacing between mechanical fasteners holding strips of beams together. Nails or screws are examples of mechanical fasteners used in wooden beams. Bolts or rivets are examples of mechanical fasteners used in metal beams. Figure 6.56 shows two strips of beams held together by a row of mechanical fasteners. Suppose the fasteners are spaced at intervals Δs, and each fastener can support a shear force V_F. Then the row of fasteners can support an average shear force per unit length of $V_F/\Delta s$, which can be approximated as the shear flow in the beam, or, $q \approx V_F/\Delta s$.

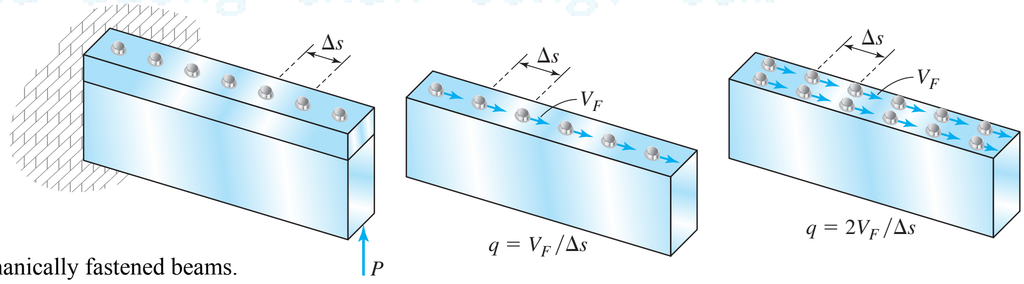


Figure 6.56 Spacing in mechanically fastened beams.

[10]In many textbooks the bending shear stress formula gives only the correct magnitude. The correct sign (or direction) has to be found by inspection. In this book inspection as well as subscripts in the formulas will be used in determining the direction of shear stress.

Thus once we know the shear flow from Equation (6.28), we can find the spacing in a row of fasteners as $\Delta s \approx V_F / q$. If there is more than one row of fasteners holding two pieces of wood together, then each row of fasteners can carry an average shear flow of $V_F/\Delta s$. Thus the total shear flow carried by two rows is $2V_F/\Delta s$, which is then approximated by the shear flow in the beam, $q = 2V_F/\Delta s$. Thus once we know the shear flow from Equation (6.28), we can use it to determine the spacing between the fasteners, once the shear force that the fasteners can support is known. Alternatively, if the spacing is known, then we can find the shear force carried by each fastener. Example 6.18 further elaborates on this discussion.

6.6.6 Bending Stresses and Strains

In symmetric bending about the z axis, the significant stress components in Cartesian coordinates are σ_{xx} and τ_{xy} in the web and σ_{xx} and τ_{xz} in the flange. We can find σ_{xx} from Equation (6.12), but from Equation (6.27) we get τ_{xs}. How do we get τ_{xy} or τ_{xz} from τ_{xs}? There are two alternatives.

1. Follow the sign convention for the shear force to determine V_y. Using Equation (6.27), get $\tau_{xs.}$ Note that the positive s direction is from the free surface to the point where the shear stress is found. Draw the stress cube using the argument of subscripts as described in Section 1.3. Now look at the shear stress in the Cartesian coordinates and determine the direction and sign of the stress component (τ_{xy} or τ_{xz}).
2. Alternatively, use Equation (6.27) to find the magnitude of τ_{xs}, and determine the direction of the shear stress by inspection, as described in Section 6.6.1. Draw the stress cube. Now look at the shear stress in the Cartesian coordinates and determine the direction and sign of the stress component (τ_{xy} or τ_{xz}).

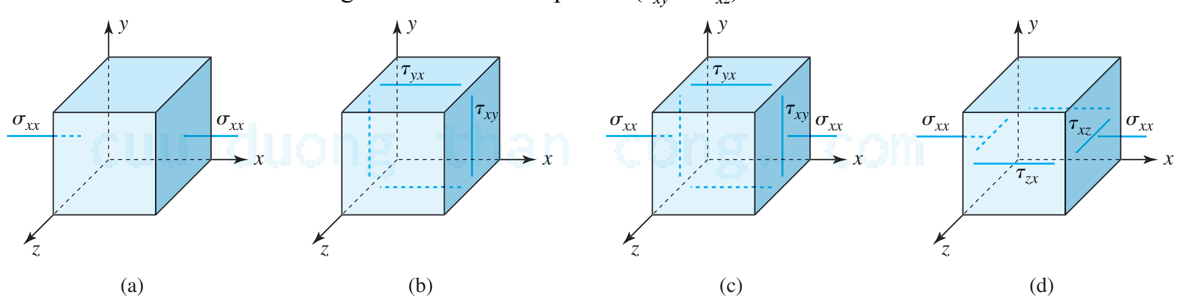


Figure 6.57 Stress elements in symmetric bending of beams: (*a*) top or bottom; (*b*) neutral axis; (*c*) any point on web. (*d*) any point on flange.

In beam bending problems there are four possible stress elements, as shown in Figure 6.57. At the top and bottom surfaces of the beam the bending shear stress τ_{xy} is zero, and the bending normal stress σ_{xx} is maximum at a cross section. The state of stress at the top and bottom is shown on in Figure 6.57*a*. No arrows are shown in the figures, as the normal stress could be tensile or compressive. At the neutral axis σ_{xx} is zero and τ_{xy} is maximum in a cross section, as shown by the stress element in Figure 6.57*b*. At any point on the web σ_{xx} and τ_{xy} are nonzero, whereas at any point in the flange σ_{xx} and τ_{xz} are nonzero, as shown in Figure 6.57*c* and *d*.

From the generalized Hooke's law given by Equations (3.12*a*) through (3.12*f*), we obtain the strains

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} \qquad \varepsilon_{yy} = -\frac{\nu\sigma_{xx}}{E} = -\nu\varepsilon_{xx} \qquad \varepsilon_{zz} = -\frac{\nu\sigma_{xx}}{E} = -\nu\varepsilon_{xx} \qquad \gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{xz} = \frac{\tau_{xz}}{G} \tag{6.29}$$

The normal strains in the y and z directions are due to the Poisson effect.

Consolidate your knowledge

1. With the book closed, derive Equation (6.27).

EXAMPLE 6.14

A positive shear force V acts on the thin rectangular cross section shown in Figure 6.58. Determine the shear stress τ_{xs} due to bending about the z axis as a function of s and sketch it.

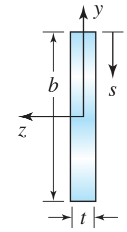


Figure 6.58 Cross section in Example 6.14.

PLAN

We can find Q_z by taking the first moment of the area between the top surface and the surface located at an arbitrary point s. By substituting Q_z as a function of s in Equation (6.27), we can obtain τ_{xs} as a function of s.

SOLUTION

We can draw the area A_s between the top surface and some arbitrary location s in Figure 6.59a and determine the first moment about the z axis to find Q_z.

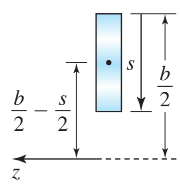
$$Q_z = st\left(\frac{b}{2} - \frac{s}{2}\right) = \frac{st(b-s)}{2} \quad \text{(E1)}$$

Substituting Equation (E1) and the area moment of inertia $I_{zz} = tb^3/12$ into Equation (6.27), we obtain

$$\tau_{xs} = -\frac{Vst(b-s)/2}{(tb^3/12)t} \quad \text{(E2)}$$

ANS. $\tau_{xs} = \dfrac{-6Vs(b-s)}{b^3 t}$

Suppose we take the positive x direction normal to this page. Since we obtain a negative sign for the shear stress, the direction of the shear stress has to be in the negative s direction, as shown in Figure 6.59*b*.



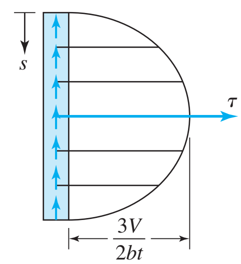


Figure 6.59 (a) Calculation of Q_z in Example 6.14. (b) Shear stress distribution in Example 6.14

COMMENTS

1. Figure 6.59*b* shows that the shear stress is zero at the top ($s = 0$) and the bottom ($s = b$) and is maximum at the neutral axis, as expected. The maximum bending shear stress at a cross section can be written $\tau_{max} = 1.5V/A$, where A is the cross-sectional area.
2. The shear force is in the positive y direction. Hence the shear stress on the cross section should be in the positive direction, as shown in Figure 6.59*b*.
3. Note that the s direction is in the negative y direction. Hence $\tau_{xy} = -\tau_{xs}$, which is confirmed by the direction of shear stress in Figure 6.59*b*.
4. Substituting $\tau_{xy} = -\tau_{xs} = 6Vs(b-s)/b^3 t$ into Equation (6.13) and noting that $dA = t\,ds$, we obtain by integration

$$V_y = \int_0^b \frac{6Vs(b-s)}{b^3 t} t\,ds = \frac{6V}{b^3}\left(\frac{bs^2}{2} - \frac{s^3}{3}\right)\Big|_0^b = V$$

which once more confirms our results.

EXAMPLE 6.15

A positive shear force $V_y = 30$ kN acts on the thin cross sections shown in Figure 6.60 (not drawn to scale). Determine the shear stress at points B, C, D, and E. Report the answers as τ_{xy} or τ_{xz}.

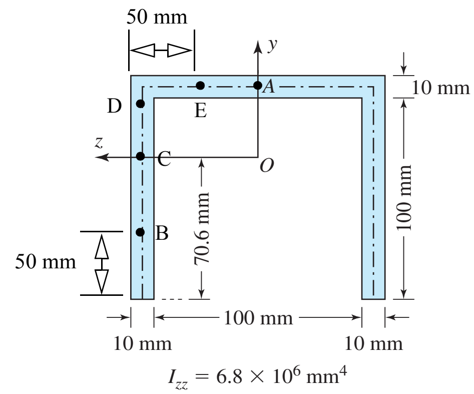


Figure 6.60 Cross sections in Example 6.15.

PLAN

V_y, I_{zz} and t in Equation (6.27) are known. Hence the shear stress will be determined if Q_z is determined at the given points. We make an imaginary cut perpendicular to the center line, and we draw the area A_s between the bottom and the imaginary cut and calculate Q_z.

SOLUTION

Figure 6.61 shows the areas A_s that can be used for finding Q_z at points B, C, D, and E. The distance from the centroid of the areas A_s to the z axis can be found and multiplied by the area A_s to obtain Q_z at each point:

$$(Q_z)_B = (0.05\text{ m})(0.01\text{ m})[-(0.0706\text{ m} - 0.025\text{ m})] = -22.8(10^{-6})\text{ m}^3 \quad \text{(E1)}$$

$$(Q_z)_C = (0.0706\text{ m})(0.01\text{ m})[-0.0706/2\text{ m}] = -24.92(10^{-6})\text{ m}^3 \quad \text{(E2)}$$

$$(Q_z)_D = (0.10\text{ m})(0.01\text{ m})[-(0.0706\text{ m} - 0.05\text{ m})] = -20.6(10^{-6})\text{ m}^3 \quad \text{(E3)}$$

$$(Q_z)_E = (0.10\text{ m})(0.01\text{ m})[-(0.0706\text{ m} - 0.05\text{ m})] + (0.05\text{ m})(0.01\text{ m})(0.105\text{ m} - 0.0706\text{ m}) \quad \text{or} \quad (Q_z)_E = -3.4(10^{-6})\text{ m}^3 \quad \text{(E4)}$$

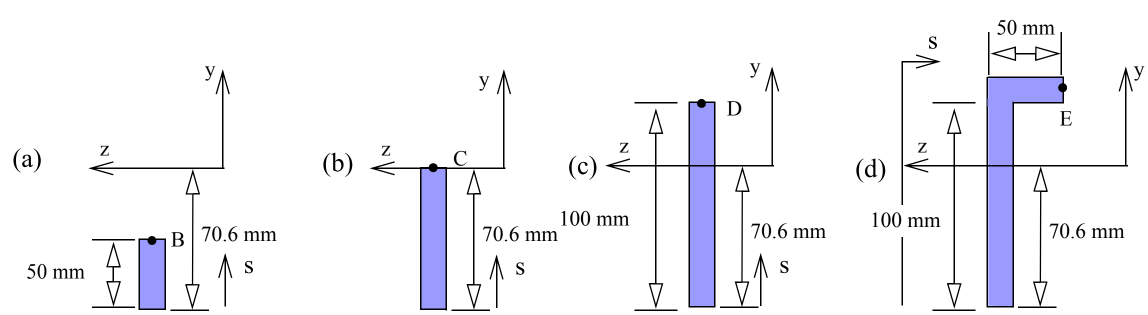


Figure 6.61 Area As for calculations of Q_z in Example 6.15.

The shear stress at the points can be found from Equation (6.27):

$$(\tau_{xs})_B = -\left[\frac{V_y(Q_z)_B}{I_{zz}t_B}\right] = -\frac{[30(10^3)\text{ N}][-22.8(10^{-6})\text{ m}^3]}{[6.8(10^{-6})\text{ m}^4](0.01\text{ m})} = 10.5(10^6)\text{ N/m}^2 \quad \text{(E5)}$$

$$(\tau_{xs})_C = -\left[\frac{V_y(Q_z)_C}{I_{zz}t_C}\right] = -\frac{[30(10^3)\text{ N}][-24.92(10^{-6})\text{ m}^3]}{[6.8(10^{-6})\text{ m}^4](0.01\text{ m})} = 10.99(10^6)\text{ N/m}^2 \quad \text{(E6)}$$

$$(\tau_{xs})_D = -\left[\frac{V_y(Q_z)_D}{I_{zz}t_D}\right] = -\frac{[30(10^3)\text{ N}][-20.6(10^{-6})\text{ m}^3]}{[6.8(10^{-6})\text{ m}^4](0.01\text{ m})} = 9.09(10^6)\text{ N/m}^2 \quad \text{(E7)}$$

$$(\tau_{xs})_E = -\left[\frac{V_y(Q_z)_E}{I_{zz}t_E}\right] = -\frac{[30(10^3)\text{ N}][-3.4(10^{-6})\text{ m}^3]}{[6.8(10^{-6})\text{ m}^4](0.01\text{ m})} = 1.5(10^6)\text{ N/m}^2 \quad \text{(E8)}$$

In Figure 6.61, the s direction is in the positive y direction at points B, C, and D and negative z direction at point E. Thus, the stress results are

ANS. $(\tau_{xy})_B = 10.5\text{ MPa}$ $\quad (\tau_{xy})_C = 11.0\text{ MPa}$ $\quad (\tau_{xy})_D = 9.1\text{ MPa}$ $\quad (\tau_{xy})_B = -1.5\text{ MPa}$

COMMENTS

1. The signs of the bending shear stress components in this example are consistent with those in Figure 6.53b where they were determined by inspection.
2. In Equation (E4) we added the first moment of the area of the horizontal piece in Figure 6.61d to the $(Q_z)_D$ calculated in Equation (E3). We could also have written the integral over the entire area A_s as a sum of integrals over its parts.
3. The maximum bending shear stress will occur at point *C*—that is, at the neutral axis.
4. The bending shear stress at *A* will be zero, because the shear flow will go in opposite direction at the axis of symmetry.

EXAMPLE 6.16

A positive shear force $V_y = 30$ N acts on the thin cross sections shown in Figure 6.62 (not drawn to scale). Determine the shear flow along the center lines and sketch it.

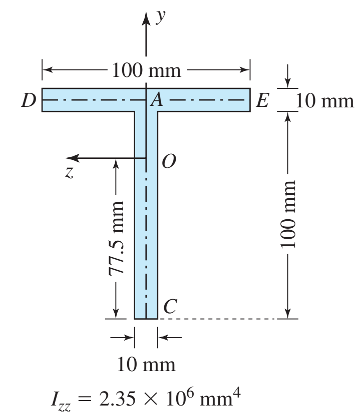


Figure 6.62 Cross sections in Example 6.16.

PLAN

V_y and I_{zz} are known in Equation (6.28). Hence the shear flow along the center line will be determined if Q_z is determined along the centerline. Noting that the cross section is symmetric about the *y* axis, the shear flow needs to be found only on one side of the *y* axis.

SOLUTION

(a) Figure 6.63 shows the areas A_s that can be used for finding the shear flows in *DA* and *CA* of the cross section in Figure 6.62*a*. The parameters s_1 and s_2 are defined from the free surface to the point where the shear flow is to be found. The distance from the centroid of the areas A_s to the *z* axis can be found and Q_z calculated as

$$Q_1 = s_1(0.01\text{ m})(0.105\text{ m} - 0.0775\text{ m}) = 0.275 s_1(10^{-3})\text{ m}^3 \quad \text{(E1)}$$

$$Q_2 = (s_2(0.01\text{ m}))[-(0.0775\text{ m} - s_2/2)] = -(0.775 s_2 - 5s_2^2)(10^{-3})\text{ m}^3 \quad \text{(E2)}$$

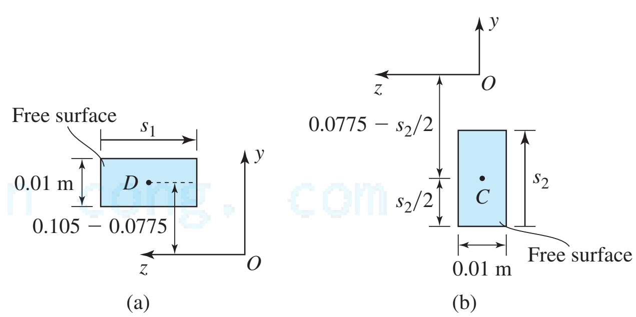


Figure 6.63 Calculation of Q_z in part (a) of Example 6.16.

Substituting V_y, I_{zz}, and Equations (E1) and into Equation (6.28), we find the shear flow in *DA* and *CA* of the cross section in Figure 6.62*a*:

$$q_1 = -\frac{(30\text{ N})[0.275 s_1(10^{-3})\ m^3]}{2.35(10^{-6})\text{ m}^4} = -3.51 s_1\text{ kN/m} \quad \text{(E3)}$$

$$q_2 = -\frac{(30\text{ N})[-(0.775 s_2 - 5s_2^2)(10^{-3})\text{ m}^3]}{2.35(10^{-6})\text{ m}^4} = (9.89 s_2 - 63.83 s_2^2)\text{ kN/m} \quad \text{(E4)}$$

ANS. $q_1 = -3.51 s_1$ kN/m $\quad q_2 = (9.89 s_2 - 63.83 s_2^2)$ kN/m

The shear flow q_1 is negative, implying that the direction of the flow is opposite to the direction of s_1. The values of q_1 can be calculated and plotted as shown in Figure 6.64*a*. By symmetry the flow in *AE* can also be plotted. The values of q_2 are positive between *C* and *A*, implying the flow is in the direction of s_2. The values of q_2 can be calculated from and plotted as shown in Figure 6.64*a*.

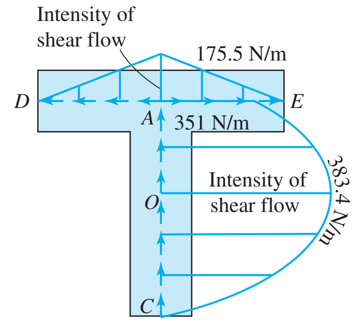


Figure 6.64 Shear flows on cross sections in Example 6.16.

COMMENTS

1. In Example 6.13 the direction of flow was determined by inspection, whereas in this example it was determined using formulas. A comparison of Figures 6.64 and 6.52*a* shows the same results. Thus, inspection can be used to check our results. Alternatively, we could calculate the magnitude of the shear flow (or stress) from formulas and then determine the direction of the shear flow by inspection.
2. In Figure 6.64 the flow value at point *A* in *CA* is 351 N/m, which is the sum of the flows in *AD* and *AE*. Thus the behavior of shear flow is similar to that of fluid flow in a channel.
3. Figure 6.64 shows that the shear flow in the flanges varies linearly. The shear flow in the web varies quadratically, and its maximum value is at the neutral axis.

EXAMPLE 6.17

A beam is loaded as shown in Figure 6.65. The cross section of the beam is shown on the right and has an area moment of inertia $I_{zz} = 40.83$ in.^{4} (a) Determine the maximum bending normal and shear stresses. (b) Determine the bending normal and shear stresses at point *D* on a section just to the right of support *A*. Point *D* is just below the flange. (c) Show the results of parts (a) and (b) on stress cubes.

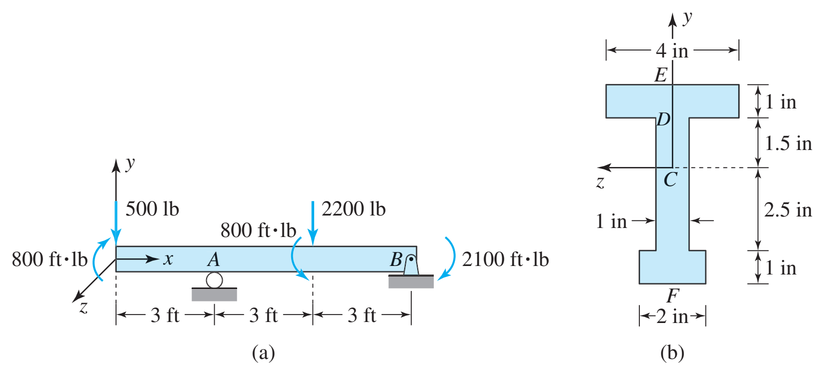


Figure 6.65 Beam and loading in Example 6.17.

PLAN

We can draw the shear force and bending moment diagrams and determine the maximum bending moment $\boldsymbol{M}_{max}$, the maximum shear force $(V_y)_{max}$, and the value of the bending moment $\boldsymbol{M}_A$ and the shear force $(V_y)_A$ just to the right of support *A*. Using Equations (6.12) and (6.27), we can determine the required stresses and show the results on a stress cube.

SOLUTION

By considering the free-body diagram of the entire beam, we can find the reaction forces at *A* and *B* and draw the shear force and bending moment diagrams in Figure 6.66. The areas under the shear force curve are

$$A_1 = 500 \times 3 = 1500 \qquad A_2 = 1000 \times 3 = 3000 \qquad A_3 = 1200 \times 3 = 3600 \tag{E1}$$

From Figure 6.66 we can find the maximum shear force and moment, as well as the values of shear force and moment just to the right of support *A*:

$$(V_y)_{\max} = 1200 \text{ lbs} \qquad \boldsymbol{M}_{max} = 2300 \text{ ft}\cdot\text{lbs} \qquad (\boldsymbol{V}_y)_A = -1000 \text{ lbs} \qquad \boldsymbol{M}_A = -700 \text{ ft}\cdot\text{lbs} \tag{E2}$$

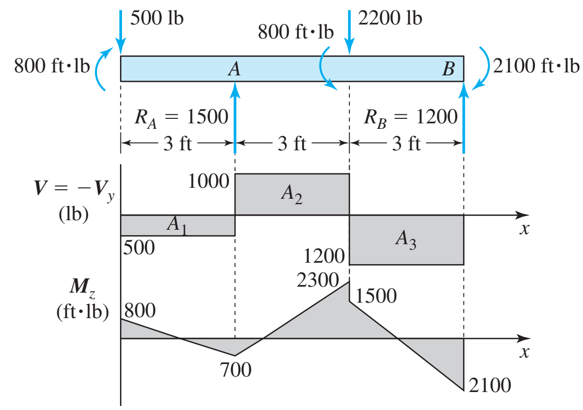


Figure 6.66 Shear force and bending moment diagrams in Example 6.17.

(a) Point F is the point farthest away from the neutral axis. Hence the maximum bending normal stress will occur at point F. Substituting $y_F = -3.5$ and M_{max} into Equation (6.12), we obtain

$$\sigma_{max} = -\frac{[(2300)(12)\ \text{in.}\cdot\text{lbs}](-3.5\ \text{in.})}{40.83\ \text{in.}^4} = 2365.7\ \text{lbs/in.}^2 \qquad \text{(E3)}$$

ANS. $\sigma_{max} = 2366\ \text{psi (T)}$

The maximum bending shear stress will occur at the neutral axis in the section where V_y is maximum. We can draw the area A_s between the top surface and the neutral axis (NA) as shown in Figure 6.67 and determine the first moment about the z axis to find Q_z, in Equation (E4)

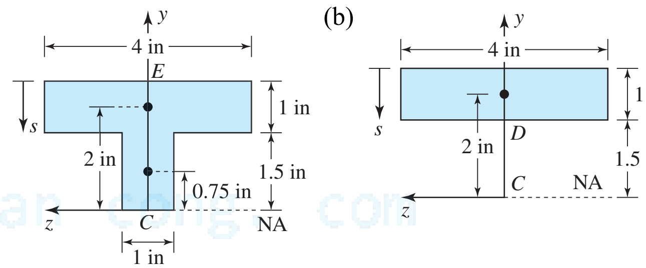


Figure 6.67 Calculation of Q_z in Example 6.17 (a) at neutral axis (b) at D.

$$Q_{NA} = (4\ \text{in.})(1\ \text{in.})(2\ \text{in.}) + (1.5\ \text{in.})(1\ \text{in.})(0.75\ \text{in.}) = 9.125\ \text{in.}^3 \qquad \text{(E4)}$$

Substituting V_{max} and Equation (E4) into Equation (6.27), we obtain

$$(\tau_{xs})_{max} = -\frac{(1200\ \text{lbs})(9.125\ \text{in.}^3)}{(40.83\ \text{in.}^4)(1\ \text{in.})} = -268.2\ \text{lbs/in.}^2 \qquad \text{(E5)}$$

ANS. $(\tau_{xs})_{max} = -268\ \text{psi}$

(b) Substituting $y_D = 1.5$ in.and M_A into Equation (6.12), we obtain the value of the normal stress at point D on a section just right of A as

$$(\sigma_{xx})_D = -\frac{[(-700)(12)\ \text{in.}\cdot\text{lbs}](1.5\ \text{in.})}{(40.83\ \text{in.}^4)} = 308.6\ \text{lbs/in.}^2 \qquad \text{(E6)}$$

ANS. $(\sigma_{xx})_D = 309\ \text{psi (T)}$

We can draw the area A_s between the free surface at the top and point D as shown in Figure 6.67b and find Q_z at D.

$$Q_D = (4\ \text{in.})(1\ \text{in.})(2\ \text{in.}) = 8\ \text{in.}^3 \qquad \text{(E7)}$$

Substituting $(V_y)_A$ and Equation (E7) into Equation (6.27), we obtain the value of the shear stress at point D on a section just right of A as

$$(\tau_{xs})_D = -\frac{(-1000\ \text{lbs})(8\ \text{in.}^3)}{(40.83\ \text{in.}^4)(1\ \text{in.})} = 196\ \text{lbs/in.}^2 \qquad \text{(E8)}$$

ANS. $(\tau_{xs})_D = 196\ \text{psi}$

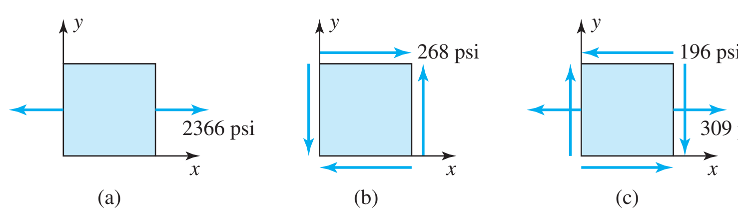


Figure 6.68 Stress elements in Example 6.17. (*a*) Maximum bending normal stress. (*b*) Maximum bending shear stress. (*c*) Bending and normal shear stresses at point D just to the right of A.

(c) In Figures 6.67*a* and *b* the coordinate *s* is in the opposite direction to *y* at points *D* and the neutral axis. Hence at both these points $\tau_{xy} = -\tau_{xs}$, and from Equations (E5) and (E8) we obtain

ANS. $(\tau_{xy})_{max} = 268$ psi $\quad (\tau_{xy})_D = -196$ psi

We can show these results along with the normal stress values on the stress elements in Figure 6.68.

COMMENTS

1. The maximum value of V is –1200 lbs but $V = -V_y$. Hence the maximum value of V_y is a positive value, as given in Equation (E2).
2. V_y is positive in Equation (E2), thus we expect $(\tau_{xy})_{max}$ to be positive. Just after support *A* the shear force V_y is negative, thus we expect that $(\tau_{xy})_D$ will be negative, as shown in Figure 6.68.
3. Note that the maximum bending shear stress in the beam given by Equation (E5) is nearly an order of magnitude smaller than the maximum bending normal stress given by Equation (E3). This is consistent with the requirement for validity of our beam theory, as was remarked in Section 6.2.6. If in some problem the maximum bending shear stress were nearly the same as the maximum bending normal stress, then that would indicate that the assumptions of beam theory are not valid and the theory needs to be modified to account for shear stress.

EXAMPLE 6.18

A wooden cantilever box beam is to be constructed by nailing four pieces of lumber in one of the two ways shown in Figure 6.69. The allowable bending normal and shear stresses in the wood are 750 psi and 150 psi, respectively. The maximum force that the nails can support is 100 lb. Determine the maximum value of load *P* to the nearest pound, the spacing of the nails to the nearest half inch, and the preferred nailing method.

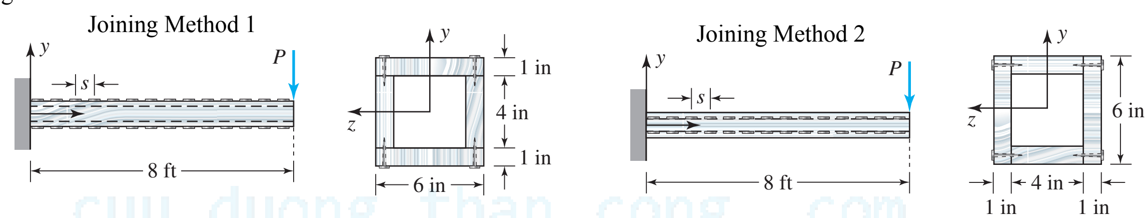


Figure 6.69 Wooden beams in Example 6.18.

PLAN

The maximum bending normal and shear stresses for both beams can be found in terms of *P*. These maximum values can be compared to the allowable stress values, and the limiting value on force *P* can be found. The shear flow at the junction of the wood pieces can be found using Equation (6.28). The spacing of the nails for each joining method can be found by dividing the allowable force in the nail by the shear flow. The method that gives the greater spacing between the nails is better as fewer nails will be needed.

SOLUTION

We can draw the shear force and bending moment diagrams for the beams as shown in Figure 6.70a and calculate the maximum shear force and moment

$$(V_y)_{max} = -P \text{ lb} \qquad M_{max} = -8P \text{ ft} \cdot \text{lb} = -96P \text{ in} \cdot \text{lb} \tag{E1}$$

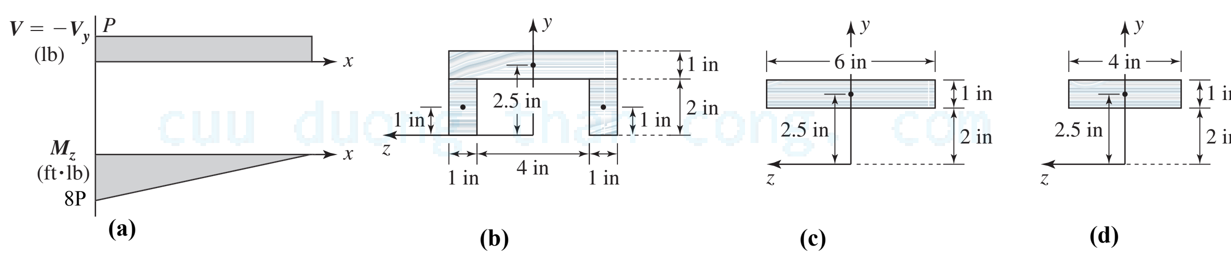


Figure 6.70 (a) Shear and moment diagrams. (b) Q_z at neutral axis. (c) Q_z at nails in joining method 1. (d) Q_z at nails in joining method 2

The area moment of inertia about the z axis can be found as

$$I_{zz} = \frac{1}{12}(6 \text{ in.})(6 \text{ in.})^3 - \frac{1}{12}(4 \text{ in.})(4 \text{ in.})^3 = 86.67 \text{ in.}^4 \tag{E2}$$

Substituting Equations (E1) and (E2) and $y_{max} = \pm 3$ in, we can find the magnitude of the maximum bending normal stress from Equation (6.12) in terms of *P*. Using the allowable normal stress as 750 psi, we can obtain one limiting value on *P*,

$$\sigma_{max} = \left|\frac{M_{max} y_{max}}{I_{zz}}\right| = \frac{(96P)(3 \text{ in.})}{86.67 \text{ in.}^4} \le 750 \text{ lbs/in.}^2 \qquad \text{or} \qquad P \le 225.7 \text{ lbs} \tag{E3}$$

Figure 6.70*b* shows the area A_s for the calculation of Q_z at the neutral axis:

$$Q_{NA} = (6 \text{ in.})(1 \text{ in.})(2.5 \text{ in.}) + 2(2 \text{ in.})(1 \text{ in.})(1 \text{ in.}) = 19 \text{ in.}^3 \quad \text{(E4)}$$

We also note that at the neutral axis, the thickness perpendicular to the center line is $t = 1$ in. + 1 in. = 2 in. Substituting Equations (E1), (E2), and (E4) and $t = 2$ in. into Equation (6.27), we can obtain the magnitude of the bending shear stress in terms of *P*. Using the allowable shear stress as 150 psi, we can obtain another limiting value on *P*.

$$\tau_{max} = \left|\frac{P(19 \text{ in.}^3)}{(86.67 \text{ in.}^4)(2 \text{ in.})}\right| \le 150 \text{ lbs/in.}^2 \qquad \text{or} \qquad P \le 1368 \text{ lbs} \quad \text{(E5)}$$

If the maximum value of P is determined from Equation (E3), then Equation (E5) will be satisfied. Rounding downward we determine the maximum value of force P to the nearest pound.

ANS. $P_{max} = 225$ lbs

To find the shear flow on the surface joined by the nails, we make imaginary cuts through the nails and draw the area A_s, as shown in Figure 6.70*c* and *d*. We can then find Q_z for each joining method:

$$Q_1 = (6 \text{ in.})(1 \text{ in.})(2.5 \text{ in.}) = 15 \text{ in.}^3 \qquad Q_2 = (4 \text{ in.})(1 \text{ in.})(2.5 \text{ in.}) = 10 \text{ in.}^3 \quad \text{(E6)}$$

From Equations (E1) and (E3) we obtain the shear force as $V_y = -225$ lb. Substituting this value along with Equations (E2) and (E6) into Equation (6.28), we obtain the magnitude of the shear flow for each joining method,

$$q_1 = \left|\frac{V_y Q_1}{I_{zz}}\right| = \left|\frac{(225 \text{ lbs})(15 \text{ in.}^3)}{86.67 \text{ in.}^4}\right| = 39.94 \text{ lbs/in.} \quad \text{(E7)}$$

$$q_2 = \left|\frac{V_y Q_2}{I_{zz}}\right| = \left|\frac{(225 \text{ lbs})(10 \text{ in.}^3)}{86.67 \text{ in.}^4}\right| = 26.96 \text{ lbs/in.} \quad \text{(E8)}$$

This shear flow is to be carried by two rows of nails for each of the joining methods. Thus each row resists half of the flow. Using this fact, we can find the spacing between the nails,

$$\frac{100 \text{ lbs}}{\Delta s_1} = \frac{q_1}{2} \qquad \text{or} \qquad \Delta s_1 = \frac{2(100 \text{ lbs})}{39.94 \text{ lbs/in.}} = 5.1 \text{ in.} \quad \text{(E9)}$$

$$\frac{100}{\Delta s_2} = \frac{q_2}{2} \qquad \text{or} \qquad \Delta s_2 = \frac{2(100 \text{ lbs})}{26.96 \text{ lbs/in.}} = 7.7 \text{ in.} \quad \text{(E10)}$$

As $\Delta s_2 > \Delta s_1$, fewer nails will be used in joining method 2. Rounding downward to the nearest half inch, we obtain the nail spacing.

ANS. Use joining method 2 with a nail spacing of 7.5 in.

COMMENTS

1. In this particular example only, the magnitudes of the stresses were important; the sign did not play any role. This will not always be the case, particularly in later chapters when we consider combined loading and stresses on different planes.
2. From visualizing the imaginary cut surface of the nails, we observe that the shear stress component in the nails is τ_{yx} in joining method 1 and τ_{zx} in joining method 2.
3. In Section 6.6.1 we observed that the shear stresses in bending balance the changes in axial force due to σ_{xx}. The shear stresses in the nails balance σ_{xx}, which acts on a greater area in joining method 1 (6 in. wide) than in joining method 2 (4 in. wide). This is reflected in the higher value of Q_z, which led to a higher value of shear flow for joining method 1 than for joining method 2, as shown by Equations (E7) and (E8).
4. The observations in comment 3 are valid as long as σ_{xx} is the same for both joining methods at any location. If I_{zz} and y_{max} were different, then it is possible to arrive at a different answer. See Problem 6.126

PROBLEM SET 6.4

Bending normal and shear stresses

6.106 For a positive shear force V_y, (a) sketch the direction of the shear flow along the center line on the thin cross sections shown in Figure P6.106. (b) At points *A*, *B*, *C*, and *D*, determine whether the stress component is τ_{xy} or τ_{xz} and whether it is positive or negative.

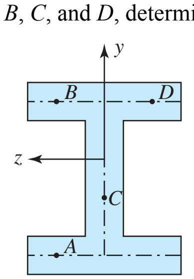


Figure P6.106

6.107 For a positive shear force V_y, (a) sketch the direction of the shear flow along the center line on the thin cross sections shown in Figure P6.107. (b) At points *A*, *B*, *C*, and *D*, determine whether the stress component is τ_{xy} or τ_{xz} and whether it is positive or negative

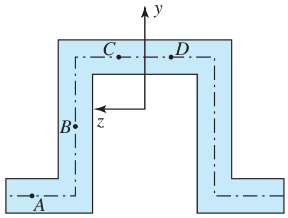


Figure P6.107

6.108 For a positive shear force V_y, (a) sketch the direction of the shear flow along the center line on the thin cross sections shown in Figure P6.108. (b) At points *A*, *B*, *C*, and *D*, determine whether the stress component is τ_{xy} or τ_{xz} and whether it is positive or negative.

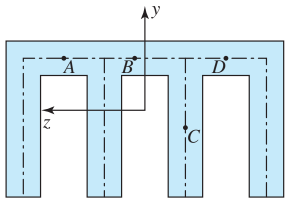


Figure P6.108

6.109 For a positive shear force V_y, (a) sketch the direction of the shear flow along the center line on the thin cross sections shown in Figure P6.109. (b) At points *A*, *B*, *C*, and *D*, determine whether the stress component is τ_{xy} or τ_{xz} and whether it is positive or negative.

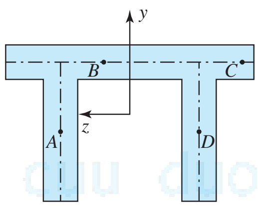


Figure P6.109

6.110 For a positive shear force V_y, (a) sketch the direction of the shear flow along the center line on the thin cross sections shown in Figure P6.109. (b) At points *A*, *B*, *C*, and *D*, determine whether the stress component is τ_{xy} or τ_{xz} and whether it is positive or negative.

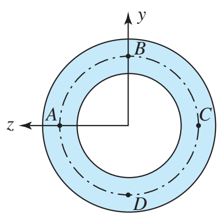


Figure P6.110

6.111 For a positive shear force V_y, (a) sketch the direction of the shear flow along the center line on the thin cross sections shown in Figure P6.111. (b) At points *A*, *B*, *C*, and *D*, determine whether the stress component is τ_{xy} or τ_{xz} and whether it is positive or negative.

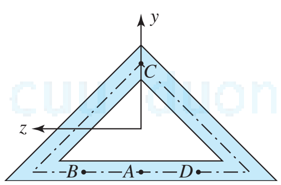


Figure P6.111

6.112 For a positive shear force V_y, (a) sketch the direction of the shear flow along the center line on the thin cross sections shown in Figure P6.112. (b) At points *A*, *B*, *C*, and *D*, determine whether the stress component is τ_{xy} or τ_{xz} and whether it is positive or negative.

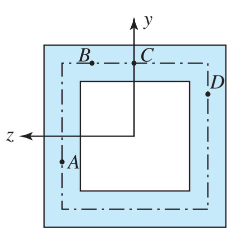


Figure P6.112

6.113 A cross section (not drawn to scale) of a beam that bends about the z axis is shown in Figure 6.113. The shear force acting at the cross section is 5 kips. Determine the bending shear stress at points *A*, *B*, *C*, and *D*. Report your answers as positive or negative τ_{xy} or τ_{xz}. Point *B* is just below the flange.

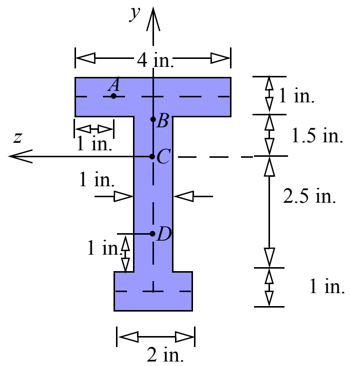


Figure P6.113

6.114 A cross section (not drawn to scale) of a beam that bends about the z axis is shown in Figure 6.71. The shear force acting at the cross section is -10 kN. Determine the bending shear stress at points *A*, *B*, *C*, and *D* Report your answers as positive or negative τ_{xy} or τ_{xz}. Point *B* is just below the flange.

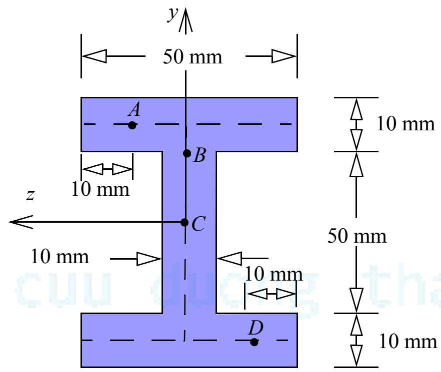


Figure 6.71

6.115 A cross section of a beam that bends about the z axis is shown in Figure 6.115. The internal bending moment and shear force acting at the cross section are M_z = 50 in.-kips and V_y = 10 kips, respectively. Determine the bending normal and shear stress at points *A*, *B*, and *C* and show it on stress cubes.Point *B* is just below the flange.

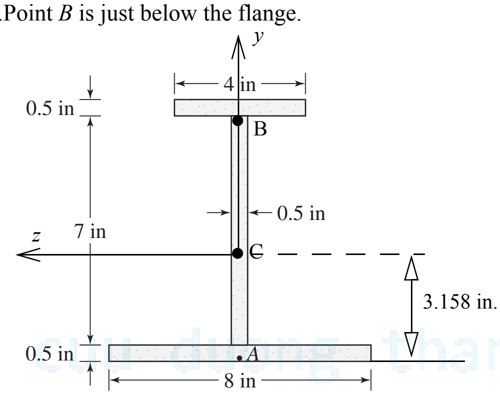


Figure P6.115

6.116 Determine the magnitude of the maximum bending normal stress and bending shear stress in the beam shown in Figure P6.116.

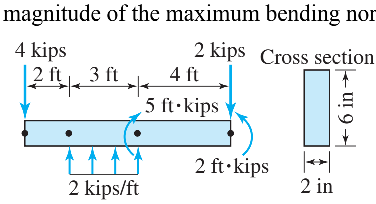


Figure P6.116

6.117 For the beam, loading, and cross section shown in Figure P6.117, determine (a) the magnitude of the maximum bending normal stress, and shear stress; (b) the bending normal stress and the bending shear stress at point *A*. Point *A* is just below the flange on the cross section just right of the 4 kN force. Show your result on a stress cube. The area moment of inertia for the beam was calculated to be $I_{zz} = 3.6 \times 10^6$ mm^4.

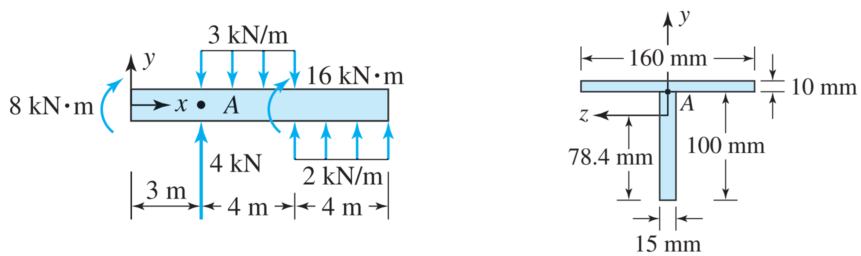


Figure P6.117

6.118 For the beam, loading, and cross section shown in Figure P6.118, determine (a) the magnitude of the maximum bending normal stress and shear stress; (b) the bending normal stress and the bending shear stress at point *A*. Point *A* is on the cross section 2 m from the right end. Show your result on a stress cube. The area moment of inertia for the beam was calculated to be $I_{zz} = 453\ (10^6)\ \text{mm}^4$.

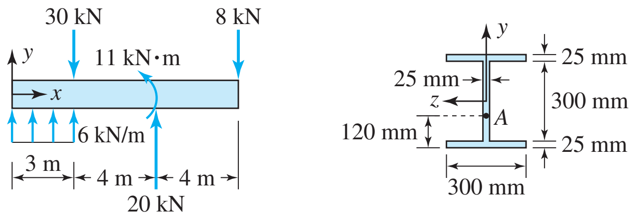


Figure P6.118

6.119 Determine the maximum bending normal and shear stress in the beam shown in Figure 6.119*a*. The beam cross section is shown in Figure 6.119*b*.

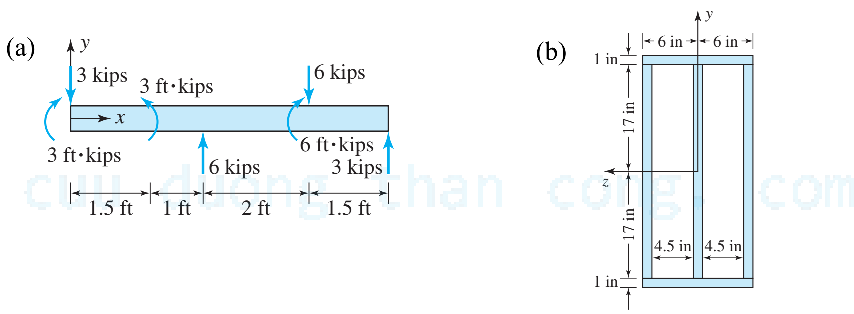


Figure P6.119

6.120 Two pieces of lumber are nailed together as shown in Figure P6.120. The nails are uniformly spaced 10 in apart along the length. Determine the average shear force in each nail in segments *AB* and *BC*.

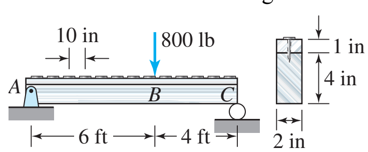


Figure P6.120

6.121 A cantilever beam is constructed by nailing three pieces of lumber, as shown in Figure P6.121. The nails are uniformly spaced at intervals of 75 mm. Determine the average shear force in each nail.

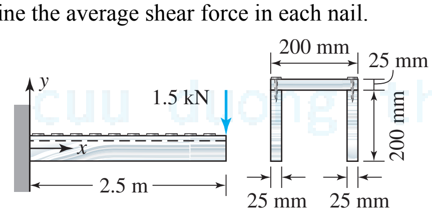


Figure P6.121

6.122 A cantilever beam is constructed by nailing three pieces of lumber, as shown in Figure P6.122. The nails are uniformly spaced at intervals of 75 mm. (a) Determine the shear force in each nail. (b) Which is the better nailing method, the one shown in Problem 6.99 or the one in this problem?

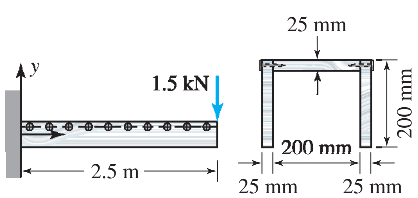


Figure P6.122

Design problems

6.123 The planks in a park bench are made from recycled plastic and are bolted to concrete supports, as shown in Figure P6.123. For the purpose of design the front plank is modeled as a simply supported beam that carries all the weight of two individuals. Assume that each person has a mass 100 kg and the weight acts at one-third the length of the plank, as shown. The allowable bending normal stress for the recycled plastic is 10 MPa and allowable bending shear stress is 2 MPa. The width d of the planks that can be manufactured is in increments of 2 cm, from 12 to 20 cm. To design the lightest bench, determine the corresponding values of the thickness t to the closest centimeter for the various values of d.

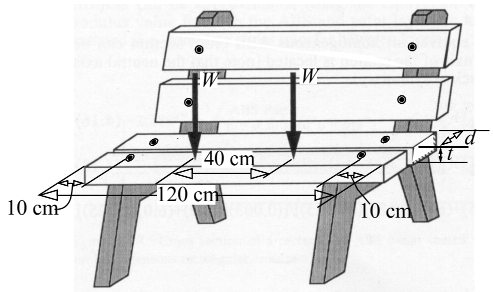


Figure P6.123

6.124 Two pieces of wood are glued together to form a beam, as shown in Figure P6.124. The allowable bending normal and shear stresses in wood are 3 ksi and 1 ksi, respectively. The allowable bending normal and shear stresses in the glue are 600 psi (T) and 250 psi, respectively. Determine the maximum moment M_{ext} that can be applied to the beam.

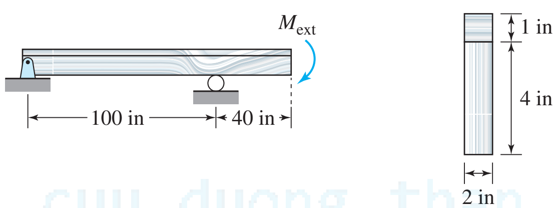


Figure P6.124

6.125 A wooden cantilever beam is to be constructed by nailing two pieces of lumber together, as shown in Figure P6.125. The allowable bending normal and shear stresses in the wood are 7 MPa and 1.5 MPa, respectively. The maximum force that the nail can support is 300 N. Determine the maximum value of load P to the nearest Newton and the spacing of the nails to the nearest centimeter.

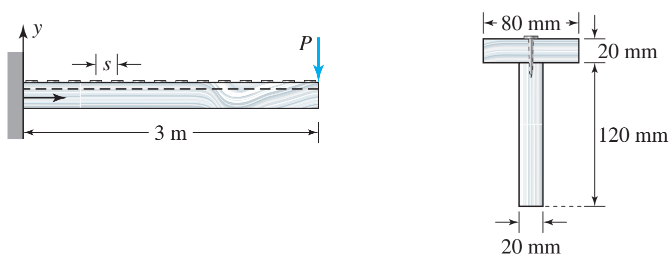


Figure P6.125

6.126 A wooden cantilever box beam is to be constructed by nailing four 1-in.× 6-in. pieces of lumber in one of the two ways shown in Figure P6.126. The allowable bending normal and shear stresses in the wood are 750 psi and 150 psi, respectively. The maximum force that a nail can support is 100 lb. Determine the maximum value of load P to the nearest pound, the spacing of the nails to the nearest half inch, and the preferred nailing method

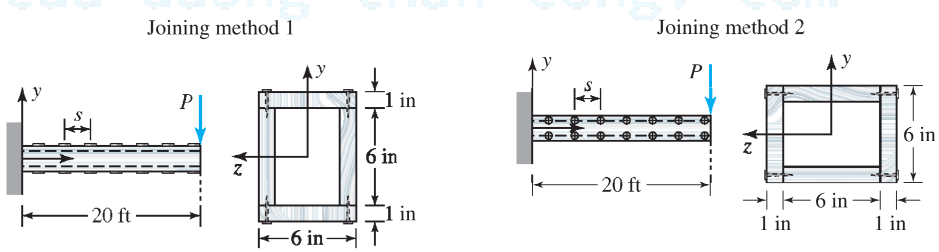


Figure P6.126

Historical problems

6.127 Leonardo da Vinci conducted experiments on simply supported beams and drew the following conclusion: "If a beam 2 braccia long (L) supports 100 libbre (W), a beam 1 braccia long ($L/2$) will support 200 libbre ($2W$). As many times as the shorter length is contained in the longer (L/α), so many times more weight (αW) will it support than the longer one." Prove this statement to be true by considering the two simply supported beams in Figure P6.127 and showing that $W_2 = \alpha W$ for the same allowable bending normal stress.

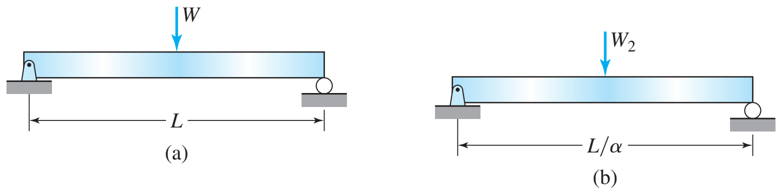


Figure P6.127

6.128 Galileo believed that the cantilever beam shown in Figure P6.128*a* would break at point *B*, which he considered to be a fulcrum point of a lever, with *AB* and *BC* as the two arms. He believed that the material resistance (stress) was uniform across the cross section. Show that the stress value σ that Galileo obtained from Figure P6.128*b* is three times smaller than the bending normal stress predicted by Equation (6.12).

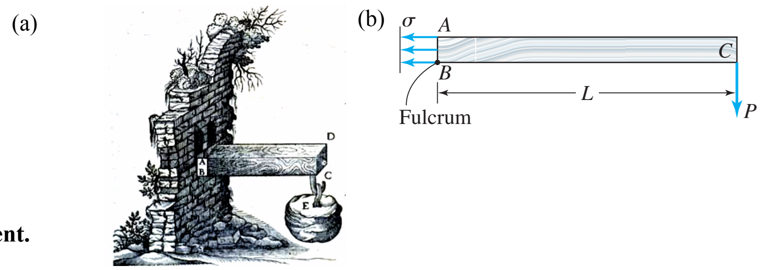


Figure P6.128 Galileo's beam experiment.

6.129 Galileo concluded that the bending moment due to the beam's weight increases as the square of the length at the built-in end of a cantilever beam. Show that Galileo's statement is correct by deriving the bending moment at the built-in end in the cantilever beam in terms of specific weight γ, cross-sectional area A, and beam length L.

6.130 In the simply supported beam shown in Figure P6.130. Galileo determined that the bending moment is maximum at the applied load and its value is proportional to the product ab. He then concluded that to break the beam with the smallest load P, the load should be placed in the middle. Prove Galileo's conclusions by drawing the shear force and bending moment diagrams and finding the value of the maximum bending moment in terms of P, a, and b. Then show that this value is largest when $a = b$.

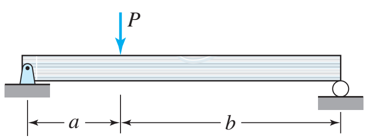


Figure P6.130

6.131 Mariotte, in an attempt to correct Galileo's strength prediction, hypothesized that the stress varied in proportion to the distance from the fulcrum, point *B* in Figure P6.128. That is, it varied linearly from point *B*. Show that the maximum bending stress value obtained by Mariotte is twice that predicted by Equation (6.12).

Stretch Yourself

6.132 A beam is acted upon by a distributed load $p(x)$. Let $\mathbf{M}_A$ and $\mathbf{V}_A$ represent the internal bending moment and the shear force at A. Show that the internal moment at B is given by

$$\mathbf{M}_B = \mathbf{M}_A - \mathbf{V}_A(x_B - x_A) + \int_{x_A}^{x_B}(x_B - x)p(x)\,dx \qquad (6.30)$$

6.133 The displacement in the x direction in a beam cross section is given by $u = u_0(x) - y(dv/dx)(x)$. Assuming small strains and linear, elastic, isotropic, homogeneous material with no inelastic strains, show that

$$\mathbf{N} = EA\frac{du_0}{dx} - EAy_c\frac{d^2v}{dx^2} \qquad \mathbf{M}_z = -EAy_c\frac{du_0}{dx} + EI_{zz}\frac{d^2v}{dx^2}$$

where y_c is the y coordinate of the centroid of the cross section measured from some arbitrary origin, A is the cross-sectional area, I_{zz} is the area moment of inertia about the z axis, and $\mathbf{N}$ and $\mathbf{M}_z$ are the internal axial force and the internal bending moment. Note that if y is measured from the centroid of the cross section, that is, if $y_c = 0$, then the axial and bending problems decouple. In such a case show that $\sigma_{xx} = \mathbf{N}/A - \mathbf{M}_z y/I_{zz}$.

6.134 Show that the bending normal stresses in a homogeneous, linearly elastic, isotropic symmetric beam subject to a temperature change $\Delta T(x, y)$ is given by

$$\sigma_{xx} = -\frac{M_z y}{I_{zz}} + \frac{M_T y}{I_{zz}} - E\alpha\Delta T(x, y) \tag{6.31}$$

where $M_T = E\alpha\int_A y\Delta T(x, y)\ dA$, α is the coefficient of thermal expansion, and E is the modulus of elasticity.

6.135 In unsymmetrical bending of beams, under the assumption of plane sections remaining plane and perpendicular to the beam axis, the displacement u in the x direction can be shown to be $u = -y\, dv/dx - z\, dw/dx$, where y and z are measured from the centroid of the cross section, and v and w are the deflections of the beam in the y and z directions, respectively. Assume small strain, a linear, elastic, isotropic, homogeneous material, and no inelastic strain. Using Equations (1.8b) and (1.8c), show that

$$M_y = EI_{yz}\frac{d^2 v}{dx^2} + EI_{yy}\frac{d^2 w}{dx^2} \qquad M_z = EI_{zz}\frac{d^2 v}{dx^2} + EI_{yz}\frac{d^2 w}{dx^2} \qquad \sigma_{xx} = -\left(\frac{I_{yy}M_z - I_{yz}M_y}{I_{yy}I_{zz} - I_{yz}^2}\right)y - \left(\frac{I_{zz}M_y - I_{yz}M_z}{I_{yy}I_{zz} - I_{yz}^2}\right)z \tag{6.32}$$

Note that if either y or z is a plane of symmetry, then $I_{yz} = 0$. From Equation (6.32), this implies that the moment about the z axis causes deformation in the y direction only and the moment about the y axis causes deformation in the z direction only. In other words, the bending problems about the y and z axes are decoupled.

6.136 The equation $\partial\sigma_{xx}/\partial x + \partial\tau_{yx}/\partial y = 0$ was derived in Problem 1.105. Into this equation, substitute Equations (6.12) and (6.18) and integrate with y for beam cross section in Figure P6.136 and obtain the equation below.

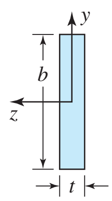


Figure P6.136

$$\tau_{yx} = \frac{6V_y(b^2/4 - y^2)}{b^3 t}$$

Computer problems

6.137 A cantilever, hollow circular aluminum beam of 5-ft length is to support a load of 1200 lb. The inner radius of the beam is 1 in. If the maximum bending normal stress is to be limited to 10 ksi, determine the minimum outer radius of the beam to nearest $\frac{1}{16}$ in.

6.138 Table P6.138 shows the values of the distributed loads at several points along the axis of the rectangular beam shown in Figure P6.138. Determine the maximum bending normal and shear stresses in the beam.

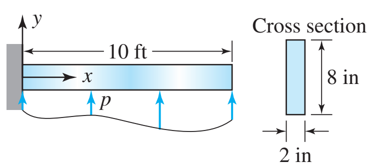


Figure P6.138

Table P6.138 Data for Problem 6.138

x (ft)	p(x) (lb/ft)
0	275
1	348
2	398
3	426
4	432
5	416
6	377
7	316
8	233
9	128
10	0

6.139 Let the distributed load $p(x)$ in Problem 6.138 be represented by $p(x) = a + bx + cx^2$. Using the data in Table P6.138, determine the constants a, b, and c by the least-squares method. Then find the maximum bending moment and the maximum shear force by analytical integration and determine the maximum bending normal and shear stresses.

6.7* CONCEPT CONNECTOR

Historically, an understanding of the strength of materials began with the study of beams. It did not, however, follow a simple course. Instead, much early work addressed mistakes, regarding the location of the neutral axis and the stress distribution across the cross section. The predicted values for the fracture loads on a beam did not correlate well with experiment. To make the pioneers' struggle in the dark more difficult, near fracture the stress–strain relationship is nonlinear, which alters the stress distribution and the location of the neutral axis.

6.7.1 History: Stresses in Beam Bending

The earliest known work on beams was by Leonardo da Vinci (1452–1519). In addition to his statements on simply supported beams, which are described in Problem 6.127, he correctly concluded that in a cantilevered, untapered beam the cross section farthest from the built-in end deflects the most. But it was Galileo's work that had the greatest early influence.

Galileo Galilei (1564–1642) (Figure 6.72) was born in Pisa. In 1581 he enrolled at the University of Pisa to study medicine, but the work of Euclid, Archimedes, and Leonardo attracted him to mathematics and mechanics. In 1589 he became professor of mathematics at the university, where he conducted his famous experiments on falling bodies, and the field of dynamics was born. He concluded that a heavier object takes the same time as a lighter object to fall through the same height, in complete disagreement with the popular Aristotelian mechanics. He paid the price for his views, for the proponents of Aristotelian mechanics made his stay at the university untenable, and he left in early 1592.

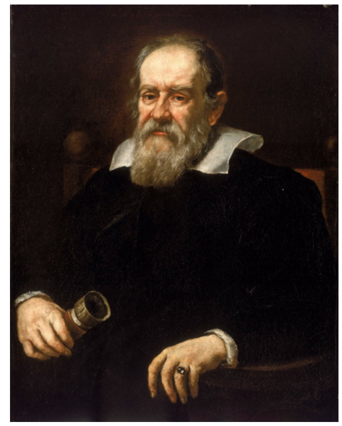


Figure 6.72 Galileo Galilei.

Fortunately, by the end of 1592 he was appointed professor of mathematics at the University of Padua. During this period he discovered his interest in astronomy. Based on sketchy reports, he built himself a telescope. On seeing moons in orbit around Jupiter in 1610, he found evidence for the Copernican theory, which held that the Earth is not the center of the universe. In 1616 Copernicus was condemned by the Church, and the Inquisition warned Galileo to leave theology to the Church. In 1632, however, he published his views, under the mistaken belief that the new pope, Maffeo Barberini, who was Galileo's admirer, would be more tolerant. Galileo was now condemned by the Inquisition and put under house arrest for the last eight years of his life. During this period he wrote *Two New Sciences,* in which he describes his work in mechanics, including the mechanics of materials. We have seen his contribution toward a concept of stress in Section 1.6. Here we discuss briefly his contributions on the bending of beams.

Figure P6.128 shows Galileo's illustration of the bending test. We described two of his insights in Problems 6.129 and 6.130. Three other conclusions of his, too, have influenced the design of beams ever since. First, a beam whose width is greater than its thickness offers greater resistance standing on its edge than lying flat, because the area moment of inertia is then greater. Second, the resisting moment—and thus the strength of the beam—increases as the cube of the radius for circular beams. Thus, the section modulus increases as the cube of the radius. Finally, the cross-sectional dimensions must increase at a greater rate than the length for constant strength cantilever beam bending due to its own weight. However, as we saw in Problem 6.128, Galileo incorrectly predicted the load-carrying capacity of beams, because he misjudged the stress distribution and the location of the neutral axis.

Credit for an important correction to the stress distribution goes to Edme Mariotte (1620–1684), who also discovered the eye's blind spot. Mariotte became interested in the strength of beams while trying to design pipes for supplying water to the palace of Versailles. His experiments with wooden and glass beams convinced him that Galileo's load predictions were greatly exaggerated. His own theory incorporated linear elasticity, and he concluded that the stress distribution is linear, with a zero stress value at the bottom of the beam. Mariotte's predicted values did not correlate well with experiment either, however. To explain why, he argued that beams loaded over a long time would have failure loads closer to his predicted values. While true, this is not the correct explanation for the discrepancy.

As we saw in Problem 6.131, the cause lay instead in an incorrect assumption about the location of the neutral axis. This incorrect location hindered many pioneers, including Claude-Louis Navier (1785-1836), who also helped develop the formulas for fluid flow, and the mathematician Jacob Bernoulli (1654-1705). (We saw some of Navier's contribution in Section 1.6 and will discuss Bernoulli's in Chapter 7 on beam deflection.) As a result, engineers used Galileo's theory in designing beams of brittle material such as stone, but Mariotte's theory for wooden beams.

Antoine Parent (1666–1716) was the first to show that Mariotte's stress formula does not apply to beams with circular cross section. Born in Paris, Parent studied law on the insistence of his parents, but he never practiced it, because he wanted to do mathematics. He also proved that, for a linear stress distribution across a rectangular cross section, the zero stress point is at the center, provided the material behavior is elastic. Unfortunately Parent published his work in a journal that he himself edited and published, not in the journal by the French Academy, and it was not widely read. More than half a century later, Charles Augustin Coulomb, whose contributions we saw in Section 5.5, independently deduced the correct location of the neutral axis. Coulomb showed that the stress distribution is such that the net axial force is zero (Equation (6.2)), independent of the material.

Jean Claude Saint-Venant (see Chapter 5) rigorously examined kinematic Assumptions 1 through 3. He demonstrated that these are met exactly only for zero shear force: the beam must be subject to couples only, with no transverse force. However, the shear stresses in beams had still not received much attention.

As mentioned in Section 1.6, the concept of shear stress was developed in 1781, by Coulomb, who believed that shear was only important in short beams. Louis Vicat's experiment in 1833 with short beams gave ample evidence of the importance of shear. Vicat (1786-1861), a French engineer, had earlier invented artificial cement. D. J. Jourawaski (1821-1891), a Russian railroad engineer, was working in 1844 on building a railroad between St. Petersburg and Moscow. A 180-ft-long bridge had to be built over the river Werebia, and Jourawaski had to use thick wooden beams. These thick beams were failing along the length of the fibers, which were in the longitudinal direction. Jourawaski realized the importance of shear in long beams and developed the theory that we studied in Section 6.6.

In sum, starting with Galileo, it took nearly 250 years to understand the nature of stresses in beam bending. Other historical developments related to beam theory will appear in Section 7.6.

6.8 CHAPTER CONNECTOR

In this chapter we established formulas for calculating normal and shear stresses in beams under symmetric bending. We saw that the calculation of bending stresses requires the internal bending moment and the shear force at a section. We considered only statically determinate beams. For these, the internal shear force and bending moment diagrams can be found by making an imaginary cut and drawing an appropriate free-body diagram. Alternatively, we can draw a shear force–bending moment diagram. The free-body diagram is preferred if stresses are to be found at a specified cross section. However, shear force–moment diagrams are the better choice if maximum bending normal or shear stress is to be found in the beam.

The shear force–bending moment diagrams can be drawn by using the graphical interpretation of the integral, as the area under a curve. Alternatively, internal shear force and bending moments can be found as a function of the *x* coordinate along the beam and plotted. Finding the bending moment as a function of *x* is important in the next chapter, where we integrate the moment–curvature relationship. Once we know how to find the deflection in a beam, we can solve problems of statically indeterminate beams.

We also saw that, to understand the character of bending stresses, we can draw the bending normal and shear stresses on a stress element. In many cases, the correct direction of the stresses can be obtained by inspection. Alternatively, we can follow the sign convention for drawing the internal shear force and bending moment on free-body diagrams, determine the direction using the subscripts in the formula. It is important to understand *both* methods for determining the direction of stresses. Shear–moment diagrams yield the shear force and the bending moment, following our sign convention. Drawing the bending stresses on a stress element is also important in stress or strain transformation, as described later.

In Chapter 8, on stress transformation, we will consider problems in which we first find bending stresses, using the stress formulas in this chapter. We then find stresses on inclined planes, including planes with maximum normal and shear stress. In Chapter 9, on strain transformation, we will find the bending strains and then consider strains in different coordinate systems, including coordinate systems in which the normal and shear strains are maximum. In Section 10.1 we will consider the combined loading problems of axial, torsion, and bending and the design of simple structures that may be determinate or indeterminate.

POINTS AND FORMULAS TO REMEMBER

- Our Theory is limited to (1) slender beams; (2) regions away from the neighborhood of stress concentration; (3) gradual variation in cross section and external loads; (4) loads acting in the plane of symmetry in the cross section; and (5) no change in direction of loading during bending.

- $$M_z = -\int_A y\sigma_{xx}\, dA \quad (6.1) \qquad u = -y\frac{dv}{dx}, \quad v = v(x) \quad (6.5)$$

- $$\text{small strain, } \varepsilon_{xx} = -\frac{y}{R} = -y\frac{d^2v}{dx^2} \quad (6.6a, b)$$

- where M_z is the internal bending moment that is drawn on the free-body diagram to put a point with positive y coordinate in compression; u and v are the displacements in the x and y directions, respectively; σ_{xx} and ε_{xx} are the bending (flexure) normal stress and strain; y is the coordinate measured from the neutral axis to the point where normal stress and normal strain are defined, and d^2v/dx^2 is the curvature of the beam.
- The normal bending strain ε_{xx} is a linear function of y.
- The normal bending strain ε_{xx} will be maximum at either the top or the bottom of the beam.
- Equations (6.1), (6.6a), and (6.6b) are independent of the material model.
- The following formulas are valid for material that is linear, elastic, and isotropic, with no inelastic strains.
- For homogeneous cross section:

- $$M_z = EI_{zz}\frac{d^2v}{dx^2} \quad (6.11) \qquad \sigma_{xx} = -\frac{M_z y}{I_{zz}} \quad (6.12)$$

- where y is measured from the centroid of the cross section, and I_{zz} is the second area moment about the z axis passing through the centroid.
- EI_{zz} is the bending rigidity of a beam cross section.
- Normal stress σ_{xx} in bending varies linearly with y on a homogeneous cross section.
- Normal stress σ_{xx} is zero at the centroid ($y = 0$) and maximum at the point farthest from the centroid for a homogeneous cross section.
- The shear force V_y will jump by the value of the applied external force as one crosses it from left to right.
- M_z will jump by the value of the applied external moment as one crosses it from left to right.

- $$V_y = \int_A \tau_{xy}\, dA \quad (6.13) \qquad \tau_{xs} = -\frac{V_y Q_z}{I_{zz} t} \quad (6.27)$$

- where Q_z is the first moment of the area A_s about the z axis passing through the centroid, t is the thickness perpendicular to the centerline, A_s is the area between the free surface and the line at which the shear stress is being found, and the coordinate s is measured from the free surface used in computing Q_z.
- The direction of shear flow on a cross section must be such that (1) the resultant force in the y direction is in the same direction as V_y; (2) the resultant force in the z direction is zero; and (3) it is symmetric about the y axis.
- Q_z is zero at the top and bottom surfaces and is maximum at the neutral axis.
- Shear stress is maximum at the neutral axis of a cross section in symmetric bending of beams.
- The bending strains are
- $$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} \qquad \varepsilon_{yy} = -\frac{\nu\sigma_{xx}}{E} = -\nu\varepsilon_{xx} \qquad \varepsilon_{zz} = -\frac{\nu\sigma_{xx}}{E} = -\nu\varepsilon_{xx} \qquad \gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{xz} = \frac{\tau_{xz}}{G} \quad (6.29)$$