CHAPTER NINE
STRAIN TRANSFORMATION

Learning Objectives

1. Learn the equations and procedures of relating strains at a point in different coordinate systems.
2. Learn the analysis in using strain gages.

To minimize the likelihood of occupants being thrown from the vehicle as a result of impact, Federal Motor Vehicle Safety Standard specifies requirements for attaching of doors. Automobile companies conduct and sponsor research to understand the stresses near bolts attaching doors to the car body. The door's own weight subjects the bolts to bending loads.

In the experiment shown in Figure 9.1, a composite plate is subjected to bending loads like those at the attachment points of the car door to a fiber glass body. Strain gages—the most popular strain-measuring devices—are used to determine the strains (and hence the stresses) in the critical region. In previous chapters, we obtained formulas for predicting strain. How do we relate the experimentally measured strains to those obtained from theory in Cartesian or polar coordinates? This chapter discusses *strain transformation*, which relates strains in different coordinate systems.

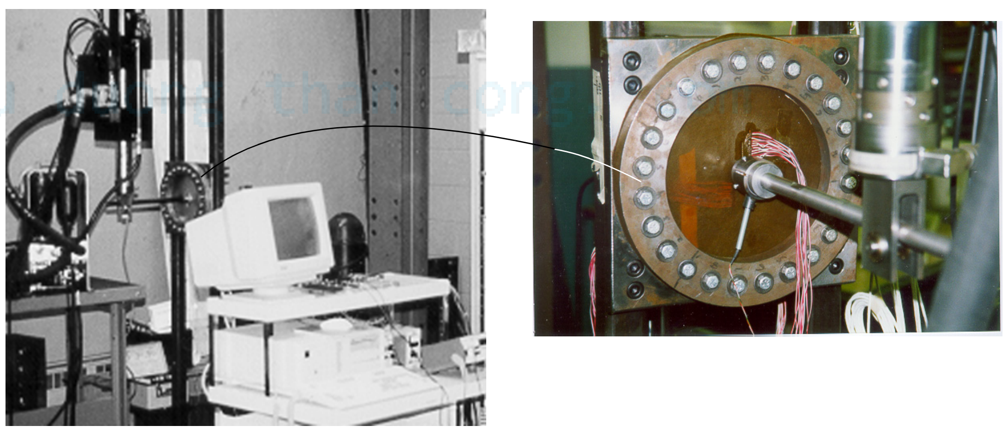


Figure 9.1 Measurement of strains using strain gages. (Courtesy Professor I. Miskioglu.)

The idea of strain transformation is very similar to stress transformation. We shall rely on this similarity to develop the key definitions and equations for strain transformation. But there are also differences, and they are critical to a successful understanding of the methods.

9.1 PRELUDE TO THEORY: THE LINE METHOD

In the wedge method of stress transformation (see Section 8.1), the key idea was to convert *stresses* into *forces*—that is, to convert a second-order tensor into a vector. We adopt a similar strategy for strain transformation. By multiplying a strain component by the length of a line, we obtain the *deformation*, which is a vector quantity. Using the small-strain approximation, we can then find the component of deformation in a given direction (see Problems 2.40-2.47). Section 9.1.1 elaborates this strategy.

We restrict ourselves to plane strain problems (see Section 2.5.1), where all strains with subscript z are zero. We further assume that the strains in the global Cartesian coordinate system (ε_{xx}, ε_{yy}, and γ_{xy}) at a point are known. We define a right-handed local coordinate system n, t, z, as shown in Figure 9.2. As in stress transformation, only those coordinate systems that can be obtained by rotation about the z axis are considered. Our objective is to find ε_{nn}, ε_{tt}, and γ_{nt}.

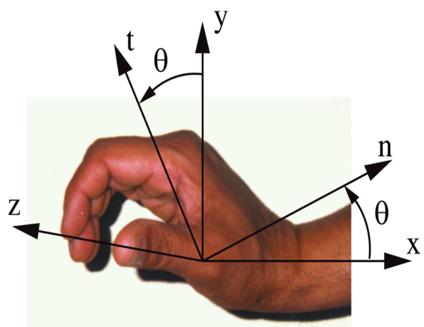


Figure 9.2 Global and local coordinate system.

Recall that normal strains are a measure of change in the *length* of a line, whereas shear strains are a measure of change in the *angle* between two lines. By finding the change in length and the rotation each axes we can find the strains in that coordinate system. This process is formally described in next section and elaborated in Example 9.1.

9.1.1 Line Method Procedure

The normal and shear strains in the local coordinate system can be found by the following steps:

Step 1: Consider each strain component one at a time and view the *n* and *t* directions as two separate lines.

Step 2: Construct a rectangle with a diagonal in the direction of the line. Relate the length of the diagonal to the lengths of the rectangle's sides.

Step 3: Draw the deformed shape assuming one side is fixed and apply the deformation on the opposite side. Find the deformation and rotation of the diagonal using small-strain approximations.

Step 4: Calculate the normal and shear strains in the *n* and *t* directions.

Step 5: Repeat steps *2* through *4* for each strain component. Superpose the results to obtain the strains in the *n* and *t* directions.

Because the line method is repetitive and tedious, we will consider problems with only one nonzero strain component. However, the same principle will be used to develop the equations of strain transformation.

EXAMPLE 9.1

At a point, the only nonzero strain component is $\varepsilon_{xx} = 200\ \mu$. Determine the strain components in the *n*, *t* coordinate system that is rotated 25° counterclockwise to the *x* axis.

PLAN

We follow the procedure described in Section 9.1.1.

SOLUTION

Step 1: View the axes of the *n*, *t* coordinate system as two lines, as shown in Figure 9.3*a*. Due to the normal strain in the *x* direction, the lines in the *n* and *t* directions deform to n_1 and t_1, as shown in Figure 9.3*a*.

Calculations in the n direction

Step 2: We can draw a rectangle with a diagonal in the *n* direction, as shown in Figure 9.3*a*. The diagonal length be Δn can be related to Δx as shown.

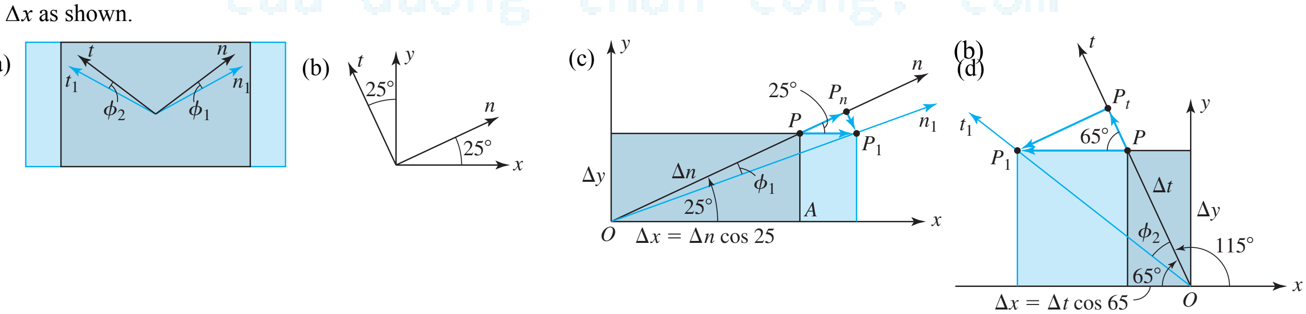


Figure 9.3 (a) (b) Movement of *n* and *t* lines. Deformation calculations (c) in the *n* direction; (d) in the *t* direction in Example 9.1.

Step 3: Let point *P* move to point P_1 due to strain ε_{xx}. The deformed shape can be drawn as shown in Figure 9.3*c*.

Step 4: A perpendicular line from P_1 to the n direction is drawn. The sides of the triangle P_nPP_1 are

$$PP_1 = \varepsilon_{xx}\Delta x = 200(10^{-6})\Delta n\cos 25° = 181.3\Delta n(10^{-6}) \quad \text{(E1)}$$

$$PP_n = PP_1\cos 25° = 164.3\Delta n(10^{-6}) \qquad P_nP_1 = PP_1\sin 25° = 76.6\Delta n(10^{-6}) \quad \text{(E2)}$$

The angle ϕ_1 can be found from triangle P_nOP_1 as

$$OP_n = OP + PP_n = \Delta n[1 + 164.3(10^{-6})] \approx \Delta n \qquad \tan\phi_1 \approx \phi_1 = \frac{P_nP_1}{OP_n} = 76.6(10^{-6})\ \text{rad} \quad \text{(E3)}$$

Calculations in the t direction

Step 2: We can draw a rectangle with a diagonal representing the t direction, as shown in Figure 9.3*d*. The diagonal length Δt can be related to Δx as shown.

Step 3: Let point P move to point P_1 due to strain ε_{xx}. The deformed shape can be drawn as shown in Figure 9.3*d*.

Step 4: A perpendicular line from P_1 to the t direction is drawn. The sides *of* triangle P_tPP_1 are

$$PP_1 = \varepsilon_{xx}\Delta x = 200(10^{-6})\Delta t\cos 65° = 84.5\Delta t(10^{-6}) \quad \text{(E4)}$$

$$P_tP_1 = PP_1\sin 65° = 76.58\Delta t(10^{-6}) \qquad PP_t = PP_1\cos 65° = 35.7\Delta t(10^{-6}) \quad \text{(E5)}$$

We can calculate the angle ϕ_2 from triangle P_tOP_1 as

$$OP_t = OP + PP_t = \Delta t[1 + 35.7(10^{-6})] \approx \Delta t \qquad \tan\phi_2 \approx \phi_2 = \frac{P_tP_1}{OP_t} = 76.6(10^{-6}) \quad \text{(E6)}$$

Step 5: The normal strain in the n and t directions are

$$\varepsilon_{nn} = \frac{PP_n}{\Delta n} = \frac{164.3\Delta n(10^{-6})}{\Delta n} = 164.3(10^{-6}) \qquad \varepsilon_{tt} = \frac{PP_t}{\Delta t} = \frac{35.7\Delta t(10^{-6})}{\Delta t} = 35.7(10^{-6}) \quad \text{(E7)}$$

$$\gamma_{nt} = -(\phi_1 + \phi_2) = -[76.6(10^{-6}) + 76.6(10^{-6})] = -153.2(10^{-6}) \quad \text{(E8)}$$

ANS. $\varepsilon_{nn} = 164.3\ \mu \qquad \varepsilon_{tt} = 35.7\ \mu \qquad \gamma_{nt} = -153.2\ \mu$

COMMENTS

1. In Figure 9.3*a* the displacement of point P to P_1 is in the positive x direction, whereas in Figure 9.3*d* the displacement is in the negative x direction. But notice that both rectangles show elongation to reflect positive ε_{xx}. Both rectangles represent the same point. This emphasizes once more the difference between displacements and deformations.
2. The negative sign in Equation (E8) reflects the increase in angle between n and t as shown in Figure 9.3.
3. We repeated the calculations for n and t directions for one strain component. For all three components we would do similar calculations six times, making the line method a repetitive, tedious process. We will develop formulas using this method to overcome the tedium.

PROBLEM SET 9.1

Line method

In Problems 9.1 through 9.4, determine the rotation of line OP and the normal strain in the direction OP due to the strain given in each problem.

Problem	Strain	Use
9.1	$\varepsilon_{xx} = 500\ \mu$	Figure P9.1
9.2	$\gamma_{xy} = 300\ \mu$	Figure P9.1
9.3	$\varepsilon_{yy} = -400\ \mu$	Figure P9.3
9.4	$\gamma_{yx} = 300\ \mu$	Figure P9.3

Figure P9.1

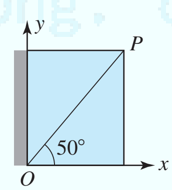
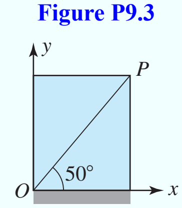
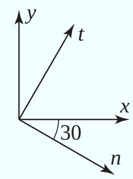
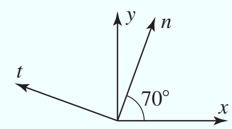
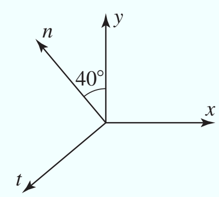


Figure P9.3



In Problems 9.5 through 9.13, at a point, the nonzero strain component are as given in each problem. Determine the strain components in the n, t coordinate system shown.

Problem	Strain	Use	
9.5	$\varepsilon_{xx} = -400\ \mu$	Figure P9.5	**Figure P9.5** 
9.6	$\varepsilon_{yy} = 600\ \mu$	Figure P9.5	
9.7	$\gamma_{xy} = -500\ \mu$	Figure P9.5	
9.8	$\varepsilon_{xx} = -600\ \mu$	Figure P9.8	**Figure P9.8** 
9.9	$\varepsilon_{yy} = -1000\ \mu$	Figure P9.8	
9.10	$\gamma_{xy} = 500\ \mu$	Figure P9.8	
9.11	$\varepsilon_{xx} = 600\ \mu$	Figure P9.11	**Figure P9.11** 
9.12	$\varepsilon_{yy} = 600\ \mu$	Figure P9.11	
9.13	$\gamma_{xy} = 600\ \mu$	Figure P9.11	

9.2 METHOD OF EQUATIONS

In this section we develop strain transformation equations using the line method.[1] We assume the point is in plane strain and the strains ε_{xx}, ε_{yy}, and γ_{xy} are known. As in stress transformation, we consider only those coordinate systems that can be obtained by rotation about the *z* axis shown in Figure 9.2. Our objective is to find the strains ε_{nn}, ε_{tt}, and γ_{nt}.

Sign Convention: The angle θ describing the orientation of the local coordinate system is positive in counterclockwise direction from the *x* axis.

We will follow the procedure outlined in Section 9.1.1 to determine the deformation and rotation of a line in the *n* direction. By substituting $90^\circ + \theta$ in place of θ in the expressions obtained in the *n* direction, we will obtain the expressions in the *t* direction.

Calculations for ε_{xx} acting alone

Step 1: View the axes of the *n*, *t* coordinate system as two lines, as shown in Figure 9.4*a*. Due to the normal strain in the *x* direction, the lines in the *n* and *t* directions deform to n_1 and t_1, as shown.

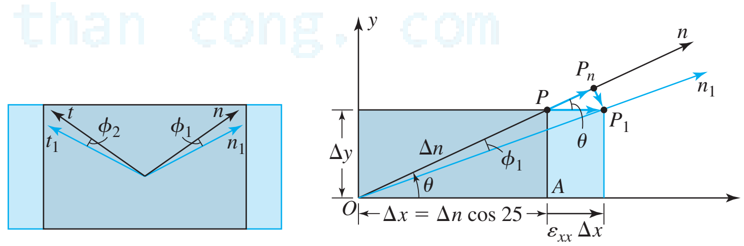


Figure 9.4 Strain transformation with ε_{xx} only. (a) (b)

Step 2: Draw a rectangle with a diagonal at an angle θ as shown in Figure 9.4*b*. The diagonal length Δ*n* can be related to Δ*x* as shown.

[1]See Problems 9.78 through 9.80 for an alternative derivation of the strain transformation equations.

Step 3: Let point P move to point P_1 due to strain ε_{xx}. Draw an exaggerated deformed shape as shown in Figure 9.4*b*. A perpendicular line from P_1 to the n direction is drawn. The sides of the triangle P_nPP_1 are

$$PP_1 = \varepsilon_{xx}\Delta x = \varepsilon_{xx}\Delta n\cos\theta \tag{9.1.a}$$

$$PP_n = PP_1\cos\theta = (\varepsilon_{xx}\cos^2\theta)(\Delta n) \tag{9.1.b}$$

$$P_nP_1 = PP_1\sin\theta = (\varepsilon_{xx}\sin\theta\ \cos\theta)\ \Delta n \tag{9.1.c}$$

Now $OP_n = OP + PP_n = OP(1 + PP_n/OP) = OP(1 + \varepsilon_{nn})$. For small strain, $\varepsilon_{nn} \ll 1$ and hence can be neglected, giving $OP_n = OP = \Delta_n$. For small strain the tangent function can be approximated by its argument. With these two approximations, we obtain the rotation ϕ_1 from triangle P_nOP_1,

$$\tan\phi_1 \approx \phi_1 = \frac{P_nP_1}{OP_n} \approx \frac{P_nP_1}{OP} = \frac{(\varepsilon_{xx}\sin\theta\ \cos\theta)\ \Delta n}{\Delta n} = \varepsilon_{xx}\sin\theta\cos\theta$$

Step 4: The normal strain in the n direction can be found as

$$\varepsilon_{nn}^{(1)} = PP_n/\Delta n = \varepsilon_{xx}\cos^2\theta \tag{9.1.d}$$

The superscript 1 differentiates the strains calculated from ε_{xx} from those calculated from ε_{yy} and γ_{xy}. We note that the t axis is a line like the n axis, but at an angle of 90º + θ instead of θ. We can obtain the normal strain in the t direction and the rotation ϕ_2 of the t axis by substituting 90º + θ in place of θ.

$$\varepsilon_{tt}^{(1)} = \varepsilon_{xx}\cos^2(90° + \theta) = \varepsilon_{xx}\sin^2\theta \tag{9.1.e}$$

$$\phi_2 = |\varepsilon_{xx}\sin(90° + \theta)\cos(90° + \theta)| = \varepsilon_{xx}\sin\theta\ \cos\theta \tag{9.1.f}$$

The angle between the n and t directions increases, as seen from the rectangle in Figure 9.4*a*. This implies that the shear strain will be negative and is given as

$$\gamma_{nt}^{(1)} = -(\phi_1 + \phi_2) = -2\varepsilon_{xx}\sin\theta\ \cos\theta \tag{9.1.g}$$

Calculations for ε_{yy} acting alone

The preceding calculations can be repeated for ε_{yy}. The calculations for Steps 1–3 are shown in Figure 9.5. Based on small strain, we once more approximate $OP_n \cong OP$ and $\tan\phi_1 \approx \phi_1$ to obtain

$$\tan\phi_1 \approx \phi_1 = \frac{P_nP_1}{OP_n} \approx \frac{P_nP_1}{OP} = \varepsilon_{yy}\sin\theta\ \cos\theta \tag{9.2.a}$$

Step 4: The normal strain in the n direction can be found as

$$\varepsilon_{nn}^{(2)} = PP_n/\Delta n = \varepsilon_{yy}\sin^2\theta \tag{9.2.b}$$

We can obtain the normal strain in the t direction and the rotation ϕ_2 of the t axis by substituting 90º + θ in place of θ.

$$\varepsilon_{tt}^{(2)} = \varepsilon_{yy}\sin^2(90° + \theta) = \varepsilon_{yy}\cos^2\theta \tag{9.2.c}$$

$$\phi_2 = |\varepsilon_{yy}\sin(90° + \theta)\cos(90° + \theta)| = \varepsilon_{yy}\sin\theta\ \cos(\tag{9.2.d}$$

The angle between the n and t directions decreases, as seen from the rectangle in Figure 9.5*a*. This implies that the shear strain will be positive:

$$\gamma_{nt}^{(2)} = \phi_1 + \phi_2 = 2\varepsilon_{yy}\sin\theta\ \cos\theta \tag{9.2.e}$$

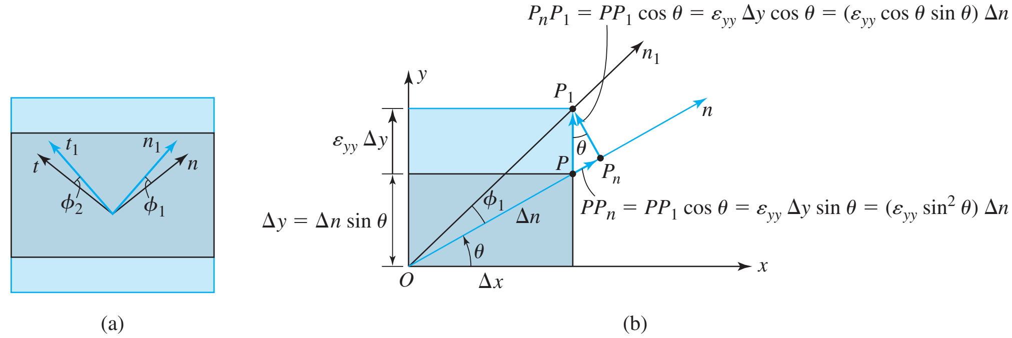


Figure 9.5 Strain transformation with ε_{yy} only.

Calculations for γ_{xy} acting alone

The preceding calculations can be repeated for γ_{xy}. The calculations for Steps 1–3 are shown in Figure 9.6. Based on small strain, we once more approximate $OP_n \cong OP$ and $\tan\phi_1 \approx \phi_1$ to obtain

$$\tan\phi_1 \approx \phi_1 = \frac{P_nP_1}{OP_n} \approx \frac{P_nP_1}{OP} = \gamma_{xy} \sin^2\theta \tag{9.3.a}$$

Step 4: The normal strain in the n direction can be found as

$$\varepsilon_{nn}^{(3)} = PP_n/\Delta n = \gamma_{xy} \sin\theta \cos\theta \tag{9.3.b}$$

We can obtain the normal strain in the t direction and the rotation ϕ_2 of the t axis by substituting $90° + \theta$ in place of θ.

$$\varepsilon_{tt}^{(3)} = \gamma_{xy} \sin(90° + \theta)\cos(90° + \theta) = -\gamma_{xy} \sin\theta \cos\theta \tag{9.3.c}$$

$$\phi_2 = \left|\gamma_{xy} \cos^2(90° + \theta)\right| = \gamma_{xy} \cos^2\theta \tag{9.3.d}$$

From Figure 9.6*a* it is seen that the movement of the line in the n direction to n_1 increases the initial angle, and the movement of the line in the t direction to t_1 decreases the initial angle. The final angle between the n_1 and t_1 directions is $\pi/2 + \phi_1 - \phi_2$. Thus from the definition of shear strain in Chapter 2, we obtain

$$\gamma_{nt}^{(3)} = \phi_1 - \phi_2 = \gamma_{xy}(\cos^2\theta - \sin^2\theta) \tag{9.3.e}$$

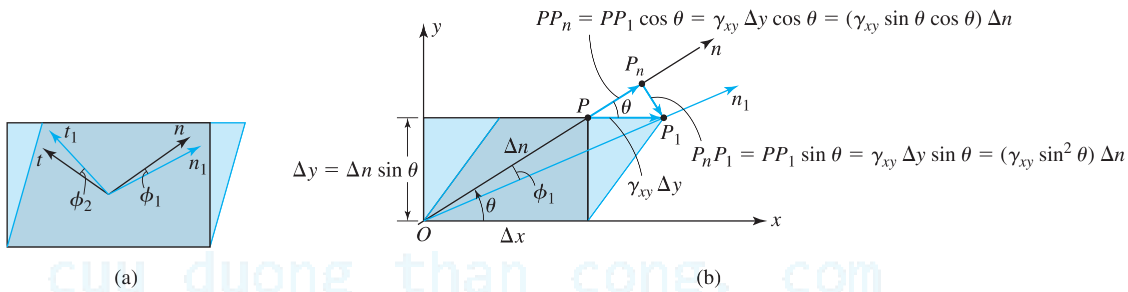


Figure 9.6 Strain transformation with γ_{xy} only.

Total strains

Step 5: As we are working with small strains, we have a linear system, and the total strain in the n and t directions is the superposition of the strains due to the individual components. That is,

$$\varepsilon_{nn} = \varepsilon_{nn}^{(1)} + \varepsilon_{nn}^{(2)} + \varepsilon_{nn}^{(3)} \qquad \varepsilon_{tt} = \varepsilon_{tt}^{(1)} + \varepsilon_{tt}^{(2)} + \varepsilon_{tt}^{(3)} \qquad \gamma_{nt} = \gamma_{nt}^{(1)} + \gamma_{nt}^{(2)} + \gamma_{nt}^{(3)}$$

We obtain the following equations:

$$\boxed{\varepsilon_{nn} = \varepsilon_{xx} \cos^2\theta + \varepsilon_{yy} \sin^2\theta + \gamma_{xy} \sin\theta \cos\theta} \tag{9.4}$$

$$\varepsilon_{tt} = \varepsilon_{xx}\sin^2\theta + \varepsilon_{yy}\cos^2\theta - \gamma_{xy}\sin\theta\cos\theta \tag{9.5}$$

$$\gamma_{nt} = -2\varepsilon_{xx}\sin\theta\,\cos\theta + 2\varepsilon_{yy}\,\sin\theta\,\cos\theta + \gamma_{xy}(\cos^2\theta - \sin^2\theta) \tag{9.6}$$

Equations (9.4), (9.5), and (9.6) are similar to the stress transformation equations, Equations (8.1), (8.2), and (8.3). However, the coefficient of the shear strain term is *half* the coefficient of the shear stress term. This is because we are using engineering strain instead of tensor strain.[2] With this difference accounted for, we can rewrite the results from stress transformation for strain transformation, as described next.

9.2.1 Principal Strains

Analogous to the case of stress transformation, we have the following definitions.

- The **principal directions** are the coordinate axes in which the shear strain is zero.
- The angles the principal axes make with the global coordinate system are called **principal angles**.
- Normal strains in the principal directions are called **principal strains**.
- The greatest principal strain is called **principal strain 1** (ε_1). By greatest principal strain we refer to the magnitude and the sign of the principal strain. Thus a strain of $-600\ \mu$ is greater than one of $-1000\ \mu$

Note that the coefficient of the shear strain in strain transformation equations is half the coefficient of shear stress in the stress transformation equations. We therefore obtain the principal angle θ_p and principal strains [see Equations (8.6) and (8.7)],

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}} \tag{9.7}$$

$$\varepsilon_{1,2} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} \pm \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \tag{9.8}$$

Here $\varepsilon_{1,2}$ represents the two strains ε_1 and ε_2. The plus sign is to be taken with ε_1 and the minus sign with ε_2. Like principal stresses, the principal strains correspond to the maximum and minimum normal strains at a point.

Adding Equations (9.4) and (9.5) and the principal strains in Equation (9.8), we obtain

$$\varepsilon_{nn} + \varepsilon_{tt} = \varepsilon_{xx} + \varepsilon_{yy} = \varepsilon_1 + \varepsilon_2 \tag{9.9}$$

Equation (9.9) shows that the sum of the normal strains at any point in an orthogonal coordinate system does not depend on the orientation of the coordinate system.

The angle of principal axis 1 from the *x* axis is reported only in describing the principal coordinate system in two-dimensional problems. Counterclockwise rotation from the *x* axis is defined as positive.

Two values of θ_p satisfy Equation (9.7), separated by 90°. The direction θ_1 corresponding to ε_1 is 90° from the direction θ_2 corresponding to ε_2. In other words, the *principal directions are orthogonal*. It is not clear whether the principal angle found from Equation (9.7) is associated with ε_1 or ε_2. We will resolve this problem as we did in stress transformation, as elaborated in Example 9.4.

In plane strain, the shear strains with subscript z are zero. Therefore the *z* direction is a principal direction and the normal strain ε_{zz} is a principal strain of zero value. In plain stress the shear strains with subscript *z* are again zero, but ε_{zz} is not zero; as shown in Figure 3.27, it is equal to $\nu(\sigma_{xx} + \sigma_{yy})$. If we add Equations (3.12a) and (3.12b) for plane stress problems, we obtain $\sigma_{xx} + \sigma_{yy} = E[(\varepsilon_{xx} + \varepsilon_{yy})/(1-\nu)]$. Thus [See Equation (3.18)],

$$\varepsilon_{zz} = -\frac{\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy})$$

[2]An alternative is to let $\gamma_{xy} = 2\varepsilon_{xy}$ and $\gamma_{nt} = 2\varepsilon_{nt}$ in Equations (9.4) through (9.6), where it is understood that ε_{xy} is the tensor shear strain and γ_{xy} is the engineering shear strain. In such a case the equations of stress and strain transformation have identical forms.

and we can write the third principal strain as

$$\varepsilon_3 = \begin{cases} 0, & \text{plane strain} \\ -\dfrac{\nu}{1-\nu}(\varepsilon_{xx}+\varepsilon_{yy}) = -\dfrac{\nu}{1-\nu}(\varepsilon_1+\varepsilon_2), & \text{plane stress} \end{cases} \tag{9.10}$$

9.2.2 Visualizing Principal Strain Directions

A circle at a given point will deform into an ellipse with the major axis in the direction of maximum normal strain (principal strain 1) and the minor axis in the direction of minimum normal strain (principal strain 2). We make use of this observation to estimate the direction of the principal strains within a 45° quadrant.

Step 1: Visualize or draw a square with a circle inside.

Step 2: Visualize or draw the deformed shape of the square due to *only* normal strains.

The deformed shape will be a rectangle. The circle within the square has now become an ellipse with the major axis either along the x direction or along the y direction, depending which normal strain is greater.

Step 3: Visualize or draw the deformed shape of the rectangle due to the shear strain.

The rectangle will deform into a rhombus, and the ellipse inside would have rotated such that the major axis is in the direction of the longer diagonal of the rhombus. The major axis can rotate at most 45° from its orientation in Step 2. The major axis represents principal direction 1 and the minor axis represents principal direction 2.

Step 4: Using the eight 45° sectors shown in Figure 9.7, report the orientation of principal direction 1. Also report principal direction 2 as two sectors counterclockwise from the sector reported for principal direction 1.

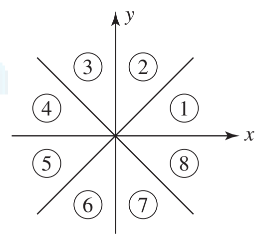


Figure 9.7 The eight sectors in which the principal axis will lie.

As in stress transformation, principal directions 1, 2, and 3 form a right-handed coordinate system. The z direction is the third principal direction. Once principal direction 1 is determined, the right-hand rule places principal direction 2 at two sectors (90°) counterclockwise from it.

EXAMPLE 9.2

At a point in plane strain, the strain components are $\varepsilon_{xx} = 200\ \mu$, $\varepsilon_{yy} = 500\ \mu$, and $\gamma_{xy} = 600\ \mu$. Estimate the orientation of the principal directions and report your results using the sectors shown in Figure 9.7.

PLAN

We will follow the steps outlined in section Section 9.2.2.

SOLUTION

Step 1: We draw a circle inside a square, as shown in Figure 9.8*a*.

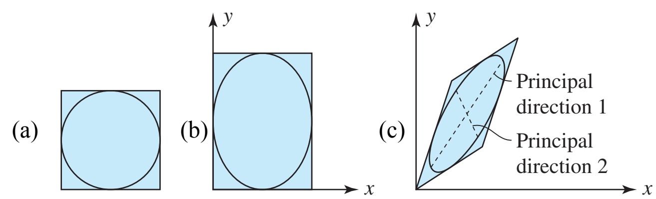


Figure 9.8 (*a*) Undeformed shape. (*b*) Deformation due to normal strains. (*c*) Additional deformation due to shear strain.

Step 2: As $\varepsilon_{yy} > \varepsilon_{xx}$, the extension in the *y* direction is greater than that in the *x* direction. The square becomes a rectangle and the circle becomes an ellipse, as shown Figure 9.8*b*.
Step 3: As γ_{xy} is positive, the angle between the *x* and *y* axes must decrease and we obtain the rhombus shown in Figure 9.8*c*.
Step 4: The two solutions follow by inspection.

ANS. Principal axis 1 is in sector 2 and principal axis 2 is in sector 4.
or
Principal axis 1 is in sector 6 and principal axis 2 is in sector 8.

COMMENTS

1. In Figure 9.8*b* the major axis is along the *y* axis. This major axis can rotate at most 45° clockwise or counterclockwise, as dictated by the shear strain. Thus principal axis 1 will be either in sector 2 or in sector 6, according to Figure 9.8*c*.
2. We will always obtain two answers for principal angle 1 as we did in stress transformation. Both answers are correct, and either can be reported.

EXAMPLE 9.3

At a point in plane strain, the strain components are $\varepsilon_{xx} = -200\ \mu$, $\varepsilon_{yy} = -400\ \mu$, and $\gamma_{xy} = -300\ \mu$. Estimate the orientation of the principal directions and report your results using the sectors shown in Figure 9.7.

PLAN

This time we will visualize but not draw any deformed shapes.

SOLUTION

Step 1: We visualize a square with a circle.
Step 2: Due to normal strains, the contraction in the *y* direction is greater than that in the *x* direction. Hence the rectangle will have a longer side in the *x* direction, that is, the major axis is along the *x* axis.
Step 3: As the shear strain is negative, the angle will increase. The major axis will rotate clockwise, and it will lie either in sector 8 or in sector 4.
Step 4: Principal axis 1 is either in sector 8 or in sector 4, giving the solution.

ANS. Principal axis 1 is in sector 8 and principal axis 2 is in sector 2.
or
Principal axis 1 is in sector 4 and principal axis 2 is in sector 6.

9.2.3 Maximum Shear Strain

As in stress transformation, we differentiate between in-plane maximum shear strain and maximum shear strain. The maximum shear strain in coordinate systems that can be obtained by rotating about the *z* axis is called **in-plane maximum shear strain. Since** the coefficient of shear strain in strain transformation equations is half the coefficient of shear stress in stress transformation equations, we obtain the in-plane maximum shear strain as

$$\left|\frac{\gamma_p}{2}\right| = \left|\frac{\varepsilon_1 - \varepsilon_2}{2}\right| \tag{9.11}$$

The **maximum shear strain** at a point is the maximum shear strain in any coordinate system given by

$$\frac{\gamma_{max}}{2} = \max\left(\left|\frac{\varepsilon_1 - \varepsilon_2}{2}\right|, \left|\frac{\varepsilon_2 - \varepsilon_3}{2}\right|, \left|\frac{\varepsilon_3 - \varepsilon_1}{2}\right|\right) \tag{9.12}$$

Equation (9.12) shows that the value of the maximum shear strain depends on the value of principal strain 3.Equation (9.10) shows that the value of principal strain 3 depends on the plane stress or plane strain problem. As in stress transformation, the maximum shear strain exists in two coordinate systems that are at 45° to the principal coordinate system.

EXAMPLE 9.4

At a point in plane strain, the strain components are $\varepsilon_{xx} = 200\ \mu$, $\varepsilon_{yy} = 1000\ \mu$, and $\gamma_{xy} = -600\ \mu$. Determine (a) the principal strains and principal angle 1; (b) the maximum shear strain; (c) the strain components in a coordinate system that is rotated 25° counterclockwise, as shown in Figure 9.9.

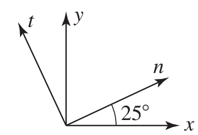


Figure 9.9

PLAN

(a) Using Equation (9.7), we can find θ_p. We can substitute θ_p into Equation (9.4) and find one of the principal strains. Using Equation (9.9), we find the other principal strain and decide which is principal strain 1. (b) We can find the maximum shear strain using Equation (9.12). (c) We can find the strains in the n and t coordinates by substituting $\theta = 25^\circ$ in Equations (9.4), (9.5), and (9.6).

SOLUTION

(a) From Equation (9.7) we obtain the principal angle,

$$\tan 2\theta_p = \frac{-600\ \mu}{200\ \mu - 1000\ \mu} = 0.75 = \tan 36.87^\circ \qquad \text{or} \qquad \theta_p = 18.43^\circ \text{ccw} \tag{E1}$$

Substituting θ_p into Equation (9.4), we obtain one of the principal strains,

$$\varepsilon_p = (200\ \mu)\cos^2 18.43^\circ + (1000\ \mu)\sin^2 18.43^\circ + (-600\ \mu)\ \sin 18.43^\circ \cos 18.43^\circ = 100\ \mu \tag{E2}$$

Now $\varepsilon_{xx} + \varepsilon_{yy} = 1200\ \mu$. From Equation (9.9) we obtain the other principal strain as $1200 - 100 = 1100\ \mu$, which is greater than the principal strain in Equation (E2). Thus $1100\ \mu$ is principal strain 1, and principal angle 1 is obtained by adding (or subtracting) 90° from Equation (E1). As the point is in plane strain, the third principal strain is zero. We report our results as

ANS. $\varepsilon_1 = 1100\ \mu \qquad \varepsilon_2 = 100\ \mu \qquad \varepsilon_3 = 0 \qquad \theta_1 = 108.4^\circ$ ccw or 71.6° cw

We can check the principal strain values using Equation (9.8),

$$\varepsilon_{1,2} = \frac{(200\ \mu) + (1000\ \mu)}{2} \pm \sqrt{\left(\frac{200\ \mu - 1000\ \mu}{2}\right)^2 + \left(\frac{-600\ \mu}{2}\right)^2} = 600\ \mu \pm 500\ \mu\text{----Checks}$$

Intuitive check orientation of principal axis 1: We visualize a circle in a square, as shown in Figure 9.10*a*. As $\varepsilon_{yy} > \varepsilon_{xx}$, the rectangle will become longer in the y direction than in the x direction and the circle will become an ellipse with major axis along the y direction, as shown in Figure 9.10*b*. As $\gamma_{xy} < 0$, the angle between the x and y directions will increase. The rectangle will become a rhombus and the major axis of the ellipse will rotate counterclockwise from the y axis. Hence we expect principal axis 1 to be in either the third sector or the seventh sector, confirming our result.

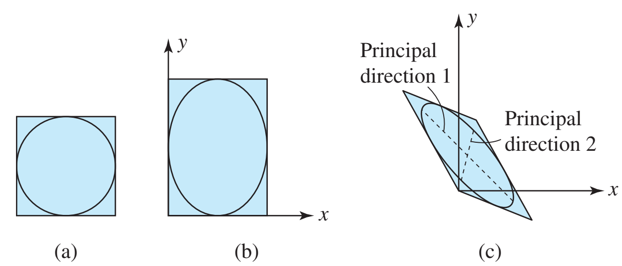


Figure 9.10 (*a*) Undeformed shape. (*b*) Deformation due to normal strains. (*c*) Additional deformation due to shear strain.

(b) We can find the maximum shear strain from Equation (9.12), that is, the maximum difference is between ε_1 and ε_3, thus the maximum shear strain is

ANS. $\gamma_{max} = 1100\ \mu$

(c) Substituting $\theta = 25^\circ$ in Equations (9.4), (9.5), and (9.6), we obtain

$$\varepsilon_{nn} = (200\ \mu)\cos^2 25^\circ + (1000\ \mu)\sin^2 25^\circ + (-600\ \mu)\sin 25^\circ \cos 25^\circ = 113.1\ \mu \tag{E3}$$

$$\varepsilon_{tt} = (200\ \mu)\sin^2 25^\circ + (1000\ \mu)\cos^2 25^\circ - (-600\ \mu)\sin 25^\circ \cos 25^\circ = 1086.9\ \mu \tag{E4}$$

$$\gamma_{nt} = -2(200\ \mu)\sin 25^\circ \cos 25^\circ + 2(1000\ \mu)\sin 25^\circ \cos 25^\circ + (-600\ \mu)(\cos^2 25^\circ - \sin^2 25^\circ) = 227.2\ \mu \tag{E5}$$

ANS. $\varepsilon_{nn} = 113.1\ \mu \qquad \varepsilon_{tt} = 1086.9\ \mu \qquad \gamma_{nt} = 227.2\ \mu$

We can use Equation (9.9) to check our results. We note that $\varepsilon_{nn} + \varepsilon_{tt} = 1200\ \mu$, which is the same value as for $\varepsilon_{xx} + \varepsilon_{yy}$, confirming the accuracy of our results.

COMMENTS

1. It can be checked that if we substitute $\theta = 25° + 180° = 205°$ or $\theta = 25° - 180° = -155°$ in Equations (9.4), (9.5), and (9.6), we will obtain the same values for ε_{nn}, ε_{tt}, and γ_{nt} as in part (c). In other words, adding or subtracting 180° from the angle θ in Equations (9.4), (9.5), and (9.6) does not affect the results. This emphasizes that the strain at a point in a given direction (coordinate system) is unique and does not depend on how we describe or measure the orientation of the line.
2. If the point were in plane stress on a material with a Poisson's ratio of $\frac{1}{3}$, then the third principle strain would be $\varepsilon_3 = -[\nu/(1-\nu)](\varepsilon_{xx} + \varepsilon_{yy}) = -600\ \mu$ and the maximum shear strain would be $\gamma_{max} = 1700\ \mu$ which is different than the value we obtained in part (b) for plane strain.

EXAMPLE 9.5

For the wooden cantilever beam shown in Figure 9.11 determine at point A (a) the principal strains and the angle of first principal direction θ_1; (b) the maximum shear strain. Use the modulus of elasticity $E = 12.6$ GPa and Poisson's ratio $\nu = 0.3$.

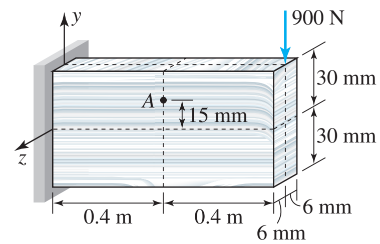


Figure 9.11 Beam and loading in Example 9.5.

PLAN

The bending stresses σ_{xx} and τ_{xy} at point A can be found using Equations (6.12) and (6.27), respectively. Using Hooke's law, the strains ε_{xx}, ε_{yy}, and γ_{xy} can be found. Using Equation (9.7), θ_p can be found and substituted into Equation (9.4) to obtain one of the principal strains. Using Equation (9.9), we find the other principal strain and decide which is principal strain 1. The maximum shear strain can be found using Equation (9.12).

SOLUTION

Bending stress calculations: Recall that A_s is the area between the free surface and the parallel line passing through point A, where shear stress is to be found. The area moment of inertia I_{zz} and the first moment Q_z of the area A_s shown in Figure 9.12*a* are

$$I_{zz} = \frac{(12\text{ mm})(60\text{ mm})^3}{12} = 0.216(10^6)\text{ mm}^4 = 0.216(10^{-6})\text{ m}^4 \tag{E1}$$

$$Q_z = (12\text{ mm})(15\text{ mm})(15\text{ mm} + 7.5\text{ mm}) = 4.050(10^3)\text{ mm}^3 = 4.050(10^{-6})\text{ m}^3 \tag{E2}$$

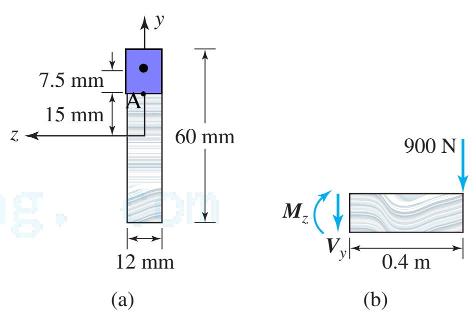


Figure 9.12 Calculation of geometric and internal quantities.

Figure 9.12*b* shows the free-body diagram of the right part of the beam after making the imaginary cut through point A in Figure 9.11. The shear force V_y and the bending moment M_z are drawn according to our sign convention. By balancing forces and moment we obtain

$$V_y = -900\text{ N} \tag{E3}$$

$$M_z = -(0.4\text{ m})(900\text{ N}) = -360\text{ N}\cdot\text{m} \tag{E4}$$

Substituting Equations (E1), (E4), and $y_A = 0.015$ m into Equation (6.12), we obtain the bending normal stress,

$$\sigma_{xx} = -\frac{M_z y}{I_{zz}} = -\frac{(-360\text{ N}\cdot\text{m})(0.015\text{ m})}{0.216(10^{-6})\text{ m}^4} = 25(10^6)\text{ N/m}^2 \tag{E5}$$

By visualizing the beam deformation, we expect σ_{xx} to be tensile consistent with the calculations above.

Substituting Equations (E1), (E2), and (E3) and $t = 0.012$ m into Equation (6.27), we obtain the magnitude of τ_{xy}. Noting that τ_{xy} must have the same sign as V_y, we obtain the sign of τ_{xy} (see Section 6.6.6) as given by

$$|\tau_{xy}| = \left|\frac{V_y Q_z}{I_{zz} t}\right| = \left|\frac{(-900\ N)[4.050(10^{-6})\ m^3]}{[0.216(10^{-6})\ m^4](0.012\ m)}\right| = 1.41(10^6)\ N/m^2 \qquad \text{or} \qquad \tau_{xy} = -1.41(10^6)\ N/m^2 \tag{E6}$$

Bending strain calculations: The shear modulus of elasticity can be found from $G = E/2(1 + \nu)$. Substituting $E = 12.6$ GPa and $\nu = 0.3$, we obtain $G = 4.85$ GPa. The only two nonzero stress components are given by Equations (E5) and (E6). Using the generalized Hooke's law [or Equation (6.29)], we obtain the bending strains,

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} = \frac{25(10^6)\ N/m^2}{12.6(10^9)\ N/m^2} = 1.984(10^{-3}) = 1984\ \mu \tag{E7}$$

$$\varepsilon_{yy} = -\frac{\nu\sigma_{xx}}{E} = -\nu\varepsilon_{xx} = -0.3(1984\ \mu) = -595.2\ \mu \tag{E8}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{-1.41(10^6)\ N/m^2}{4.85(10^9)\ N/m^2} = -0.2907(10^{-3}) = -290.7\ \mu \tag{E9}$$

(a) *Stress transformation calculations:* From Equation (9.7) we obtain the principal angle,

$$\tan 2\theta_p = \frac{-290.7\ \mu}{1984\ \mu - (-595.2\ \mu)} = -0.1124 = -\tan 6.41° \qquad \text{or} \qquad \theta_p = -3.21° \tag{E10}$$

Substituting θ_p into Equation (9.4) we obtain one of the principal strains,

$$\varepsilon_p = (1984\ \mu)\cos^2(-3.21°) + (-595.2\ \mu)\sin^2(-3.21°) + (-290.7\ \mu)\sin(-3.21°)\cos(-3.21°) = 1992\ \mu \tag{E11}$$

Now $\varepsilon_{xx} + \varepsilon_{yy} = 1389\ \mu$. From Equation (9.9) we obtain the other principal strain as $1389\ \mu - 1992\ \mu = -603\ \mu$, which is less than the principal strain in Equation (E11). Thus 1992 μ is principal strain 1, and principal angle 1 is obtained from Equation (E10). The third principal strain will be the same as the second principal strain. We report our results as

ANS. $\varepsilon_1 = 1992\ \mu \qquad \varepsilon_2 = -603\ \mu \qquad \varepsilon_3 = -603\ \mu \qquad \theta_1 = 3.21^o\text{cw}$

Check on principal strains: We can check the principal strain values using Equation (9.8),

$$\varepsilon_{1,2} = \frac{1984\ \mu + (-595.2\ \mu)}{2} \pm \sqrt{\left(\frac{1984\ \mu - (-595.2\ \mu)}{2}\right)^2 + \left(\frac{-290.7\ \mu}{2}\right)^2} = 694.4\ \mu \pm 1297.7\ \mu \quad \text{or}$$

$$\varepsilon_1 = 1992.1\ \mu \qquad \varepsilon_2 = -603.3\ \mu\text{----checks}$$

Intuitive check orientation of principal axis 1: We visualize a circle in a square. As $\varepsilon_{xx} > \varepsilon_{yy}$ the rectangle will become longer in the x direction. The circle will become an ellipse with its major axis along the x direction. As the shear strain is negative, the angle will increase. The major axis will rotate clockwise, and it will lie either in sector 8 or in sector 4, confirming our result.

(b) We can find the maximum shear strain from Equation (9.12), as the difference between ε_1 and ε_2 (or ε_3).

ANS. $\gamma_{max} = 2595\ \mu$

COMMENT

1. The example demonstrates the synthesis of the theory of symmetric bending of beams and the theory of strain transformation. A similar synthesis can be elaborated for axial and torsion members.

9.3 MOHR'S CIRCLE

As for stress transformation, Mohr's circle is graphical technique for solving problems in strain transformation. We eliminate θ from Equations (9.4) and (9.6) written in terms of double angles, to obtain

$$\left(\varepsilon_{nn} - \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{nt}}{2}\right)^2 = \left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2 \tag{9.13}$$

Comparing Equation (9.13) with the equation of a circle, $(x - a)^2 + y^2 = R^2$, we see that Equation (9.13) it represents a circle with a center that has coordinates $(a, 0)$ and radius R, where

$$a = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} \qquad R = \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \tag{9.14}$$

The circle is called Mohr's circle for strain. *Each point on Mohr's circle represents a unique direction* passing through the point at which the strains are specified. The coordinates of each point on the circle are the strains $(\varepsilon_{nn}, \gamma_{nt}/2)$. These represent the normal strain of a line in the n direction and half the shear strain, which represents the rotation of the line passing through the point at which strains ε_{xx}, ε_{yy}, and γ_{xy} are specified.

9.3.1 Construction of Mohr's Circle for Strains

The construction of Mohr's circle for strain is very similar to that for stress. However, there are two important differences. (i) In stress transformation we talked about planes, while here we talk about directions. The directions are the outward normals of the planes. (ii) The vertical axis is shear strain divided by 2. All values of shear strain that are plotted on Mohr's circle or calculated from Mohr's circle must take into account that the vertical coordinate is shear strain divided by 2.

The steps in the construction of Mohr's circle for strain are as follows.

Step 1: Draw a square with a shape deformed due to shear strain γ_{xy}. Label the intersection of the vertical plane and the x axis as V and the intersection of the horizontal plane and the y axis as H, as shown in Figure 9.13.

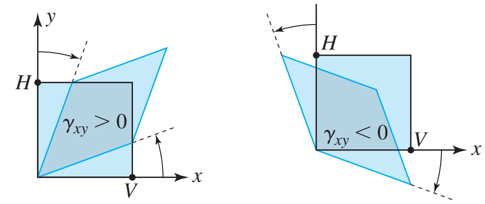


Figure 9.13 Deformed cube for construction of Mohr's circle.

Unlike in stress transformation, where V and H represented planes, here V and H refer to directions. The outward normal to the vertical plane is the x axis, and V is the label associated with it. Similarly, the outward normal to the horizontal plane is the y axis, which is represented by point H.

Step 2: Write the coordinates of points V and H,

$$V(\varepsilon_{xx}, \gamma_{xy}/2 \circlearrowleft) \qquad H(\varepsilon_{yy}, \gamma_{xy}/2 \circlearrowright), \qquad \text{for } \gamma_{xy} > 0$$

The arrow of rotation along side the shear strains corresponds to the rotation of the line on which the point lies, as shown in Figure 9.13.

Step 3: Draw the horizontal axis to represent the normal strain, with extensions (E) to the right and contractions (C) to the left, as shown in Figure 9.14*a*. Draw the vertical axis to represent half the shear strain, with clockwise rotation of a line in the upper plane and counterclockwise rotation of a line in the lower plane.

As this step emphasizes, the value of shear strain read from Mohr's circle does not tell us whether shear strain is positive or negative. Rather, it shows that the shear strain will cause a line in a given direction to rotate clockwise or counterclockwise. This point is further elaborated in Section 9.3.2.

Step 4: Locate points V and H and join the points by drawing a line. Label the point at which line VH intersects the horizontal axis as C.

Step 5: The horizontal coordinate of point C is the average normal strain. Distance CE can be found from the coordinates of points E and C and the radius R calculated using the Pythagorean theorem. With C as the center and CV or CH as the radius, draw Mohr's circle.

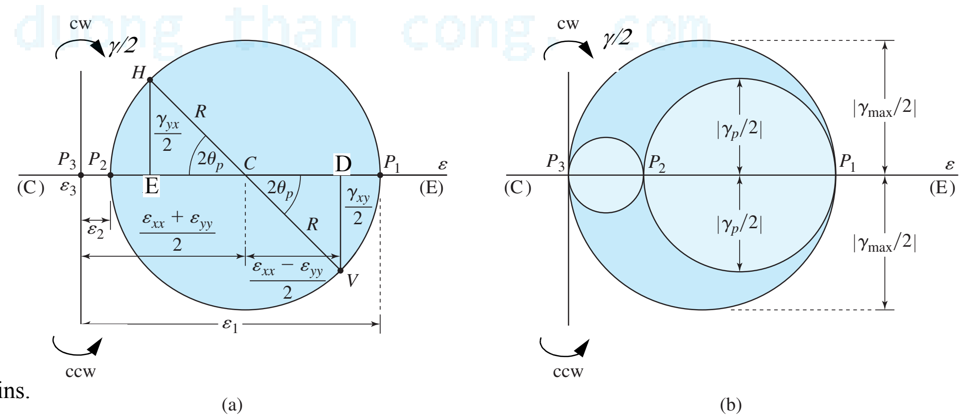


Figure 9.14 Mohr's circle for strains.

Step 6: Calculate the principal strains by finding the coordinates of points P_1 and P_2 in Figure 9.14*a*.

Step 7: Calculate principal angle θ_p from either triangle *VCD* or triangle *ECH*. Find the angle between lines *CV* and CP_1 if θ_1 is different from θ_p.

In Figure 9.14*a*, θ_p and θ_1 have the same value, but this may not always be the case, as elaborated in Example 9.6. θ_1 is the angle measured from the *x* axis, which is represented by point *V* on Mohr's circle, and principal direction 1 is represented by point P_1.

Step 8: Check your answer for θ_1 intuitively using the visualization technique of Section 9.2.2.

Step 9: The in-plane maximum shear strain $\gamma_p/2$ equals *R*, the radius of the in-plane circle shown in Figure 9.14*a*. To find the absolute maximum shear strain, locate point P_3 at the value of the third principal strain. Then draw two more circles between P_1 and P_3 and between P_2 and P_3, as shown in Figure 9.14*b*. The maximum shear strain at a point is found from the radius of the largest circle.

For plane strain P_3 is at the origin, as shown in Figure 9.14*b*. But for plane stress, the third principal strain must be found from Equation (9.10) and located before drawing the remaining two circles. Notice that the radii of the circles yield half the value of the maximum shear strain.

9.3.2 Strains in a Specified Coordinate System

The strains in a specified coordinate system are found by first locating the coordinate directions on Mohr's circle and then determining the coordinates of the point representing the directions. This is achieved as follows.

Step 10: Draw the Cartesian coordinate system and the specified coordinate system along with a square in each coordinate system, representing the undeformed state. Label points *V*, *H*, *N*, and *T* to represent the four directions, as shown in Figure 9.15*a*.

Step 11: Points *V* and *H* on Mohr's circle are known. Point *N* on Mohr's circle is located by starting from point *V* and rotating by $2\theta_V$ in the same direction, as shown in Figure 9.15*a*. Similarly, starting from point *H* on Mohr's circle, point *T* is located as shown in Figure 9.15*b*.

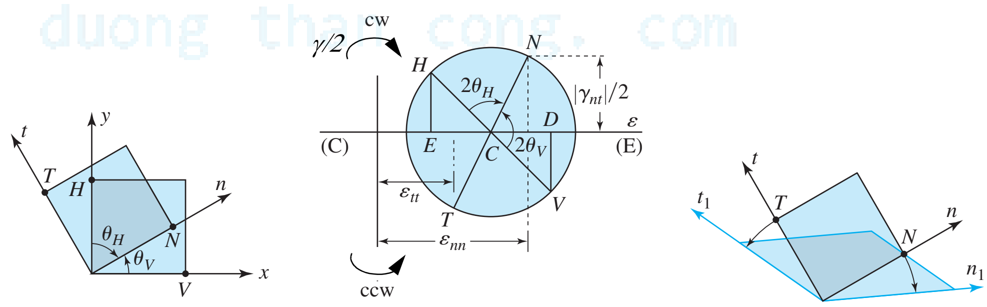


Figure 9.15 Strains in specified coordinate system.

It should be emphasized that we could start from point *H* on Mohr's circle and reach point *N* by rotating $2\theta_H$, as shown in Figure 9.15*b*. In Figure 9.15*a*, $\theta_H + \theta_V = 90^\circ$ and in Figure 9.15*b* we see that $2\theta_H + 2\theta_V$ is 180°, which once more emphasizes that each point on Mohr's circle represents a unique direction, and it is immaterial how one reaches it.

Step 12: Calculate the coordinates of points *N* and *T*.

This is the reverse of Step 2 in the construction of Mohr's circle and is a problem in geometry. As seen in Figure 9.15*b*, the coordinates of points *N* and *T* are

$$N(\varepsilon_{nn}, \gamma_{nt}/2 \curvearrowright) \qquad T(\varepsilon_{tt}, \gamma_{nt}/2 \curvearrowleft)$$

The rotation of the line at point *N* is clockwise, as it is in the upper plane, whereas the rotation of the line at point *T* is counterclockwise, as it is in the lower plane in Figure 9.15*b*.

Step 13: Determine the sign of the shear strain.

To draw the deformed shape we rotate the *n* coordinate in the direction shown for point *N* in Step 3. Similarly, we rotate the *t* coordinate in the direction shown for point *T* in Step 3, as illustrated in Figure 9.15*c*. The angle between the *n* and *t* directions increases, and hence the shear strain γ_{nt} is negative.

EXAMPLE 9.6

At a point in plane strain, the strain components are $\varepsilon_{xx} = 200\ \mu$, $\varepsilon_{yy} = 1000\ \mu$, and $\gamma_{xy} = -600\ \mu$. Using Mohr's circle, determine (a) the principal strains and principal angle 1; (b) the maximum shear strain; (c) the strain components in a coordinate system that is rotated 25° counterclockwise, as shown in Figure 9.16.

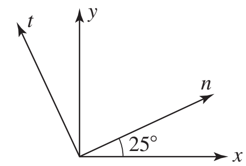


Figure 9.16

PLAN

We can follow the steps outlined for the construction of Mohr's circle and for the calculation of the various quantities as outlined in this section.

SOLUTION

Step 1: The shear strain is negative, and hence the angle between the x and y axes should increase. We draw the deformed shape of a square due to shear strain γ_{xy}. We label the intersection of the vertical plane and the x axis as V and the intersection of the horizontal plane and the y axis as H, as shown in Figure 9.17.

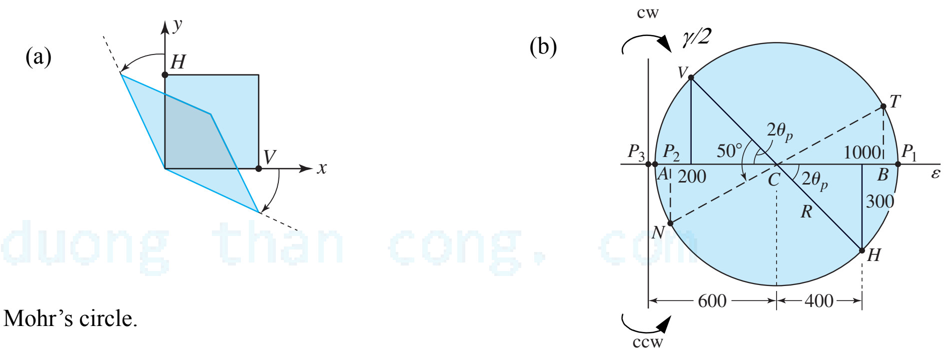


Figure 9.17 (a) Deformed cube. (b) Mohr's circle.

Step 2: Using Figure 9.17*a*, we can write the coordinates of points V and H,

$$V(200, 300 \circlearrowright) \qquad H(1000, 300 \circlearrowleft) \tag{E1}$$

Step 3: We draw the axes for Mohr's circle as shown in Figure 9.17*b*.
Step 4: Locate points V and H and join the points by drawing a line.
Step 5: Point C, the center of Mohr's circle, is midway between points A and B—that is, at 600 μ. The distance BC can thus be found as 400 μ, as shown in Figure 9.17*b*. From the Pythagorean theorem we can find the radius R,

$$R = \sqrt{CB^2 + BH^2} = \sqrt{400^2 + 300^2} = 500 \tag{E2}$$

Step 6: The principal strains are the coordinates of points P_1 and P_2 in Figure 9.17*b*. By adding the radius CP_1 to the coordinate of point C, we can obtain the principal strains, $\varepsilon_1 = 600 + 500 = 1100$ and $\varepsilon_2 = 600 - 500 = 100$. Note that for plane strain the third principal strain is zero.

ANS. $\varepsilon_1 = 1100\ \mu \qquad \varepsilon_2 = 100\ \mu \qquad \varepsilon_3 = 0$

Step 7: Using triangle BCH we can find the principal angle θ_p,

$$\tan 2\theta_p = \frac{BH}{BC} = \frac{300}{400} \qquad \text{or} \qquad 2\theta_p = 36.87° \tag{E3}$$

Principal angle 1 can be found from θ_p as shown in Figure 9.18.

ANS. $\theta_1 = 71.6°$ cw or $\theta_1 = 108.4°$ ccw

Step 8: Intuitive check: We visualize a circle in a square, as shown in Figure 9.18*b*. As $\varepsilon_{yy} > \varepsilon_{xx}$ the rectangle will become longer in the y direction than in the x direction, and the circle will become an ellipse with the major axis along the y direction, as shown in Figure 9.18*c*. As $\gamma_{xy} < 0$, the angle between the x and y directions will increase. The rectangle will become a rhombus, and the major axis of the ellipse will rotate counterclockwise from the y axis, as shown in Figure 9.18*d*. Hence we expect principal axis 1 to be either in the third sector or in the seventh sector, confirming the result.

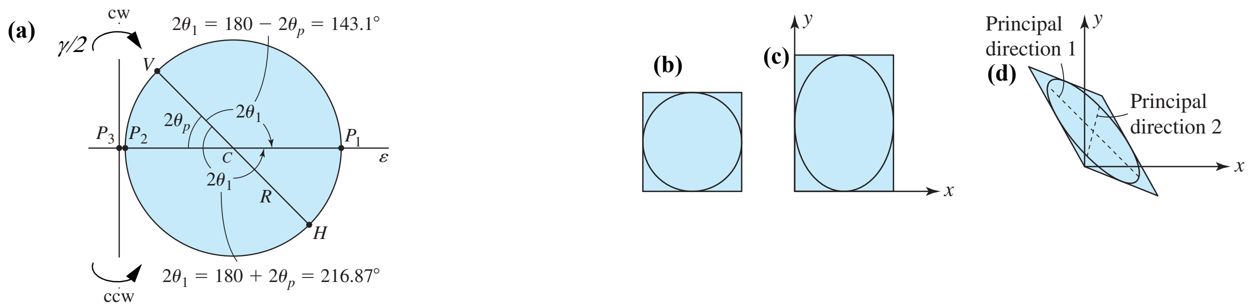


Figure 9.18 (a) Two values of principal angle 1. (b) Un-deformed shape. (c) Deformation due to normal strains. (d) Additional deformation due to shear strain.

Step 9: The circles between P_1 and P_2 and between P_2 and P_3 will be inscribed in the circle between P_1 and P_3. Thus the maximum shear strain at the point can be determined from the circle between P_1 and P_3,

$$\frac{\gamma_{max}}{2} = \frac{\varepsilon_1 - \varepsilon_3}{2} = \frac{1100}{2} \quad \text{or} \quad \textbf{ANS.} \quad \gamma_{max} = 1100\ \mu \tag{E4}$$

Step 10: We can draw the Cartesian coordinate system and the specified coordinate system with a square representing the undeformed state. Label points *V*, *H*, *N*, and *T* to represent the four directions, as shown in Figure 9.19.

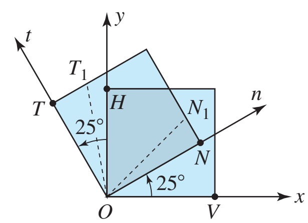


Figure 9.19 *n*, *t* coordinate system in Example 9.6

Step 11: Starting from point *V* on Mohr's circle, we rotate by 50° counterclockwise and obtain point *N* on Mohr's circle in Figure 9.17. Similarly, by starting from point *H* and rotating by 50° counterclockwise, we obtain point *T* on Mohr's circle in Figure 9.17.

Step 12: Angle *ACN* and angle *BCT* can be found as $50 - 2\theta_p = 13.13°$. From triangle *ACN* in Figure 9.21, the coordinates of point *N* are

$$\varepsilon_{nn} = 600 - 500\cos 13.13° = 113.1 \qquad \gamma_{nt}/2 = 500\sin 13.13° = 113.58 \tag{E5}$$

From triangle *BCT*, the coordinates of point *T* are

$$\varepsilon_{tt} = 600 + 500\cos 13.13° = 1086.9 \qquad \gamma_{nt}/2 = 500\sin 13.13° = 113.58 \tag{E6}$$

Step 13: In Figure 9.19 line *ON* rotates in the counterclockwise direction to ON_1, as seen in Equation (E5), and line *OT* rotates in the clockwise direction to OT_1, as seen in Equation (E6). Angle N_1OT_1 is less than angle *NOT*, and hence the shear strain in the *n*, *t* coordinate system is positive.

$$\textbf{ANS.} \quad \varepsilon_{nn} = 113.1\ \mu \qquad \varepsilon_{tt} = 1086.9\ \mu \qquad \gamma_{nt} = 227.2\ \mu$$

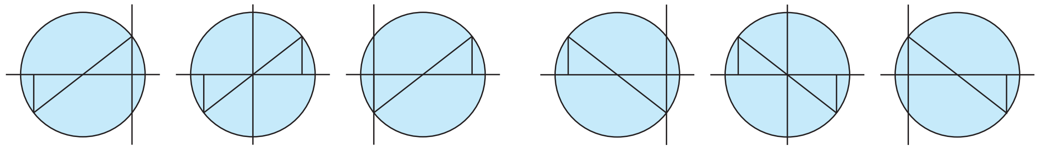
COMMENT

1. Example 9.4 and this example solve the same problem. But unlike with the method of equations used in Example 9.4, this example shows that we do not need an equation to solve the problem by Mohr's circle. Once Mohr's circle is constructed, the problem of strain transformation becomes a problem of geometry.

QUICK TEST 9.1

Time: 15 minutes/Total: 20 points

Grade yourself with the answers given in Appendix E. Each question is worth two points.
In Questions 1 through 3, associate the strain states with the appropriate Mohr's circle given.



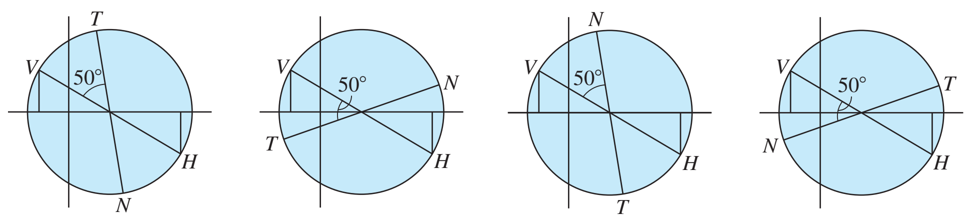
Circle *A* Circle *B* Circle *C* Circle *D* Circle *E* Circle *F*

1. $\varepsilon_{xx} = -600\ \mu$, $\varepsilon_{yy} = 0$, and $\gamma_{xy} = -600\ \mu$.

2. $\varepsilon_{xx} = 0$, $\varepsilon_{yy} = 600\ \mu$, and $\gamma_{xy} = 600\ \mu$.

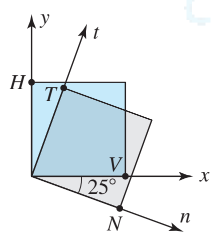
3. $\varepsilon_{xx} = 300\ \mu$, $\varepsilon_{yy} = -300\ \mu$, and $\gamma_{xy} = -600\ \mu$.

In Questions 4 and 5, the Mohr's circles corresponding to the states of strain $\varepsilon_{xx} = -500\ \mu$, $\varepsilon_{yy} = 1100\ \mu$, and $\gamma_{xy} = -1200\ \mu$ are shown. Identify the circle you would use to find the strains in the *n*, *t* coordinate system in each question.

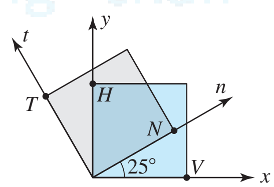


Circle *A* Circle *B* Circle *C* Circle *D*

4.

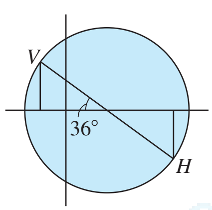


5.

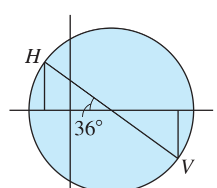


In Questions 6 and 7, the Mohr's circles for a state of strain are given. Determine the two possible values of principal angle 1 (θ_1) in each question.

6.

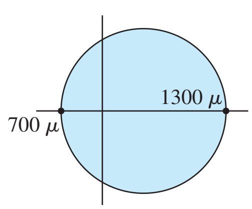


7.

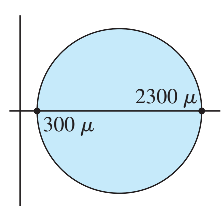


In Questions 8 through 10, the Mohr's circles for points in plane strain are given. Report principal strain 1 and maximum shear strain in each question.

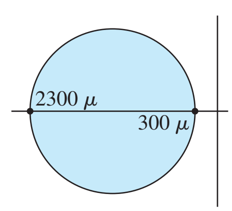
8.



9.



10.



9.4 GENERALIZED HOOKE'S LAW IN PRINCIPAL COORDINATES

In Section 3.5 it was observed that the generalized Hooke's law is valid for any orthogonal coordinate system. We have seen that the principal coordinates for stresses and strains are orthogonal.

It has been shown mathematically and confirmed experimentally that for isotropic materials the principal directions for strains are the same as the principal directions for stresses. In Example 9.9, we will see that the principal directions for stresses and strains are different when the material is orthotropic. For isotropic materials, we can write the generalized Hooke's law relating principal stresses to principal strains as follows:

$$\varepsilon_1 = \frac{\sigma_1 - \nu(\sigma_2 + \sigma_3)}{E} \tag{9.14.a}$$

$$\varepsilon_2 = \frac{\sigma_2 - \nu(\sigma_3 + \sigma_1)}{E} \tag{9.14.b}$$

$$\varepsilon_3 = \frac{\sigma_3 - \nu(\sigma_1 + \sigma_2)}{E} \tag{9.14.c}$$

Note that there are no equations for shear stresses and shear strains, as both these quantities are zero in the principal coordinate system. Now that we know that, at a point, principal axis 1 for stresses and strains is the same for isotropic material, we can extend our intuitive check to stress transformation. This can be done by viewing σ_{xx}, σ_{yy}, and τ_{xy} as analogous to ε_{xx}, ε_{yy}, and γ_{xy} in the visualization procedure outlined in Section 9.2.2.

EXAMPLE 9.7

The stresses $\sigma_{xx} = 4$ ksi (T), $\sigma_{yy} = 10$ ksi (C), and $\tau_{xy} = 4$ ksi were calculated at a point on a free surface of an isotropic material. Determine (a) the orientation of principal axis 1 for stresses, using Mohr's circle for stress; (b) the orientation of principal axis 1 for strains, using Mohr's circle for strain. Use the following material constants: $E = 7500$ ksi, $G = 3000$ ksi, and $\nu = 0.25$.

PLAN

By substituting the stresses and material constants into the generalized Hooke's law in Cartesian coordinates, we can find the strains ε_{xx}, ε_{yy}, and γ_{xy}. We can draw Mohr's circle for stress to find principal direction 1 for stress, and we can draw Mohr's circle for strain to find principal direction 1 for strain.

SOLUTION

As the point is on a free surface, the state of stress is plane stress; hence $\sigma_{zz} = 0$. Substituting the stresses and the material constants into Equations (3.12a), (3.12b), and (3.12d), we obtain

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu}{E}\sigma_{yy} = \frac{4\text{ ksi}}{7500\text{ ksi}} - \frac{0.25}{7500\text{ ksi}}(-10\text{ ksi}) = 0.867(10^{-3}) = 867\ \mu \tag{E1}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu}{E}\sigma_{xx} = \frac{-10\text{ ksi}}{7500\text{ ksi}} - \frac{0.25}{7500\text{ ksi}}(4\text{ ksi}) = -1.467(10^{-3}) = -1467\ \mu \tag{E2}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{4\text{ ksi}}{3000\text{ ksi}} = 1.333(10^{-3}) = 1333\ \mu \tag{E3}$$

(a) We draw the stress cube and record the coordinates of planes *V* and *H*,

$$V(4, 4\circlearrowleft) \qquad H(-10, 4\circlearrowright) \tag{E4}$$

We then draw Mohr's circle for stress, as shown in Figure 9.20*a*. The angle θ_p can be found from triangle *BCH* (or *ACV*) and is given by

$$\tan 2\theta_p = \frac{4}{7} \qquad \text{or} \qquad \theta_p = 14.87° \tag{E5}$$

For this example $\theta_1 = \theta_p$ and we obtain the result for the orientation of principal axis 1.

ANS. $\theta_1 = 14.87°$ ccw

(b) Since γ_{xy} is positive, the angle between the *x* and *y* coordinates decreases, as shown by the deformed shape in Figure 9.20*b*. Noting that the vertical coordinate is $\gamma/2$, we record the coordinates of points *V* and *H*,

$$V(867, 666.7\circlearrowleft) \qquad H(-1467, 666.7\circlearrowright) \tag{E6}$$

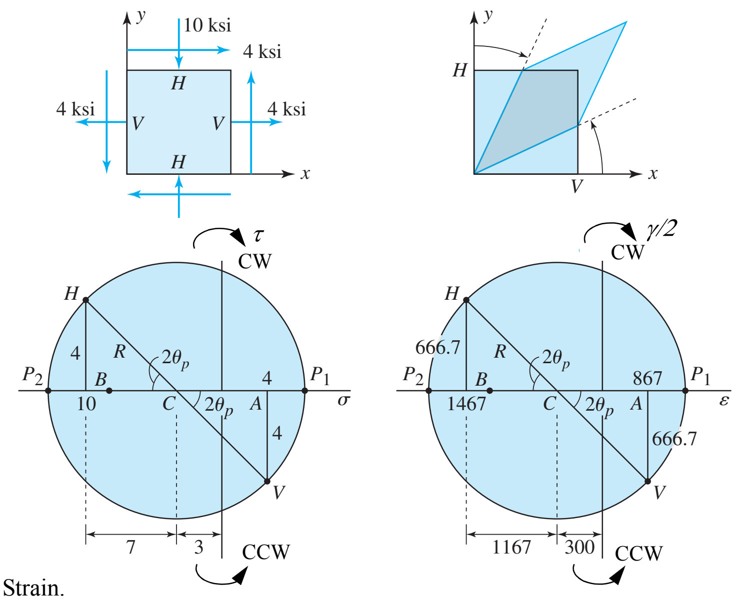


Figure 9.20 Mohr's circles in Example 9.7. (*a*) Stress. (*b*) Strain.

(a) (b)

We then draw Mohr's circle for strain, as shown in Figure 9.20*b*. The angle θ_p can be found from triangle *BCH* (or *ACV*) and is given by

$$\tan 2\theta_p = \frac{666.7}{1167} \quad \text{or} \quad \theta_p = 14.87° \tag{E7}$$

For this example $\theta_1 = \theta_p$ and we obtain the result for the orientation of principal axis 1.

ANS. $\theta_1 = 14.87°$ ccw

COMMENTS

1. The example highlights that for isotropic materials the principal axes for stresses and strains are the same.
2. The principal stresses can be found from Mohr's circle for stress as

$$\sigma_1 = -3 \text{ ksi} + 8.06 \text{ ksi} = 5.06 \text{ ksi} \qquad \sigma_2 = -3 \text{ ksi} - 8.06 \text{ ksi} = -11.06 \text{ ksi}$$

Noting that $\sigma_3 = 0$ because of the plane stress state, we obtain the principal strains from Equations (9.21a) and (9.21b),

$$\varepsilon_1 = \frac{5.06 \text{ ksi} - 0.25(-11.06 \text{ ksi})}{7500 \text{ ksi}} = 1044\ \mu \qquad \varepsilon_2 = \frac{(-11.06 \text{ ksi}) - 0.25(5.06 \text{ ksi})}{7500 \text{ ksi}} = -1644\ \mu \tag{E8}$$

3. From Mohr's circle for strain we obtain the same values,

$$\varepsilon_1 = -300\ \mu + 1344\ \mu = 1044\ \mu \qquad \varepsilon_2 = -300\ \mu - 1344\ \mu = -1644\ \mu \tag{E9}$$

The preceding highlights that the sequence of using the generalized Hooke's law and Mohr's circle does not affect the calculation of the principal strains.

4. We can conduct an intuitive check on the orientation of principal axis 1 for strain. We visualize a circle in a square, as shown in this example (Figure 9.21). Since $\varepsilon_{xx} > \varepsilon_{yy}$, the rectangle will become longer in the *x* direction than in the *y* direction, and the circle will become an ellipse with its major axis along the *x* direction. Since $\gamma_{xy} > 0$, the angle between the *x* and *y* directions will decrease. The rectangle will become a rhombus, and the major axis of the ellipse will rotate counterclockwise from the *x* axis. Hence we expect principal axis 1 to be either in the first sector or in the fifth sector of Figure 9.6. The result given in Equation (E7) puts principal axis 1 in sector 1, which is one of our intuitive answers.

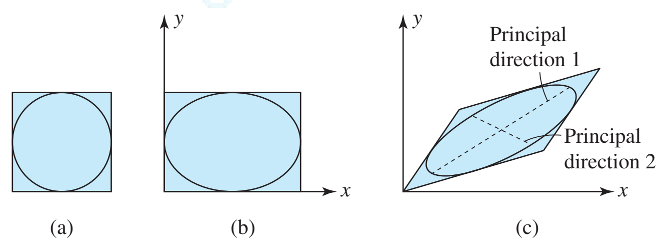


Figure 9.21 Estimating principal directions in Example 9.7 (*a*) Undeformed shape. (*b*) Deformation due to normal strains. (*c*) Additional deformation due to shear strain.

EXAMPLE 9.8

For an isotropic materials show that $G = E/2(1 + \nu)$.

PLAN

We can start with a state of pure shear and find principal stresses in terms of shear stress τ and principal strains in terms of shear strain γ. Using Equation (9.14.a) we can relate principal strain 1 to the principal stresses and then obtain a relationship between shear stress τ and shear strain γ. This relationship will have only E and ν in it. Comparing this to the relationship $\tau = G\gamma$, we can obtain the relationship between E, ν, and G.

SOLUTION

We start by assuming that all stress components except $\tau_{xy} = \tau$ are zero in the Cartesian coordinate system. We draw the stress cube and Mohr's circle in Figure 9.22*a* and find the principal stresses in terms of τ,

$$\sigma_1 = +\tau \qquad \sigma_2 = -\tau \tag{E1}$$

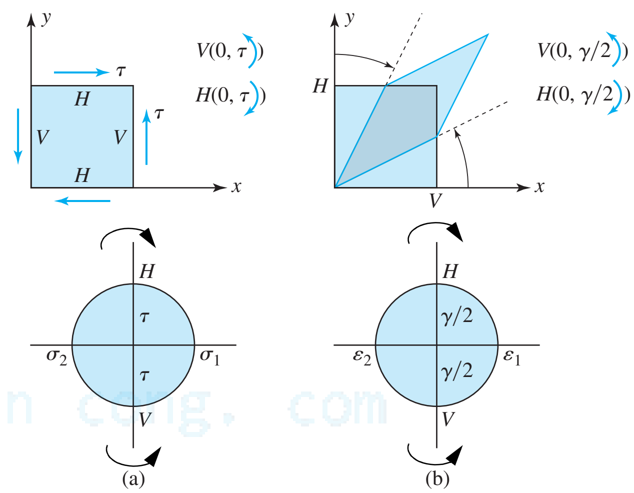


Figure 9.22 Mohr's circles for pure shear in Example 9.8. (*a*) Stress. (*b*) Strain.

We then start with all strains except $\gamma_{xy} = \gamma$ as zero. Using Mohr's circle in Figure 9.22*b*, we find the principal strains,

$$\varepsilon_1 = +\gamma/2 \qquad \varepsilon_2 = -\gamma/2 \tag{E2}$$

Noting that $\sigma_3 = 0$, we substitute Equations (E1), (E2), and (E3) into Equation (9.14.a) to obtain

$$\frac{\gamma}{2} = \frac{\tau - \nu(-\tau + 0)}{E} = \frac{1+\nu}{E}\tau \qquad \text{or} \qquad \tau = \frac{E}{2(1+\nu)}\gamma \tag{E3}$$

Comparing Equation (E5) to $\tau = G\gamma$, we obtain $G = E/2(1 + \nu)$.

COMMENTS

1. Principal axes 1 in Mohr's circles for stress and for strain are seen to be at 90° counterclockwise from plane V. This implies that for isotropic materials the principal direction for stresses is the same as the principal direction for strains.
2. The state of pure shear can be produced by applying tensile stress in one direction (σ_1) and a compressive stress of equal magnitude in a perpendicular direction (σ_2). Then on a 45° plane a state of pure shear will be seen.

EXAMPLE 9.9

The stresses $\sigma_{xx} = 4$ ksi (T), $\sigma_{yy} = 10$ ksi (C), and $\tau_{xy} = 4$ ksi were calculated at a point on a free surface of an orthotropic composite material. An orthotropic material has the following stress–strain relationship at a point in plane stress:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E_x} - \frac{\nu_{yx}}{E_y}\sigma_{yy} \qquad \varepsilon_{yy} = \frac{\sigma_{yy}}{E_y} - \frac{\nu_{xy}}{E_x}\sigma_{xx} \qquad \gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} \qquad \frac{\nu_{yx}}{E_y} = \frac{\nu_{xy}}{E_x} \tag{9.15}$$

Determine (a) the orientation of principal axis 1 for stresses using Mohr's circle for stress; (b) the orientation of principal axis 1 for strains using Mohr's circle for strain. Use the following values for the material constants: $E_x = 7500$ ksi, $E_y = 2500$ ksi, $G_{xy} = 1250$ ksi, and $\nu_{xy} = 0.3$.

PLAN

By substituting the stresses and material constants into Equation (9.15), we can find the strains ε_{xx}, ε_{yy}, and γ_{xy}. We can draw Mohr's circle for stress to find principal direction 1 for stress, and we can draw Mohr's circle for strain to find principal direction 1 for strain.

SOLUTION

From $\nu_{yx}/E_y = \nu_{xy}/E_x$, we obtain

$$\nu_{yx} = \frac{E_y \nu_{xy}}{E_x} = \frac{(2500 \text{ ksi})(0.3)}{(7500 \text{ ksi})} = 0.1 \tag{E1}$$

Substituting the stresses and the material constants into Equation (9.15), we obtain

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E_x} - \frac{\nu_{yx}}{E_y}\sigma_{yy} = \frac{4 \text{ ksi}}{7500 \text{ ksi}} - \frac{0.1}{2500 \text{ ksi}}(-10 \text{ ksi}) = 0.933(10^{-3}) = 933\ \mu \tag{E2}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E_y} - \frac{\nu_{xy}}{E_x}\sigma_{xx} = \frac{(-10 \text{ ksi})}{2500 \text{ ksi}} - \frac{0.3}{(7500 \text{ ksi})}(4 \text{ ksi}) = -4.160(10^{-3}) = -4160\ \mu \tag{E3}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} = \frac{4 \text{ ksi}}{1250 \text{ ksi}} = 3.200(10^{-3}) = 3200\ \mu \tag{E4}$$

(a) We draw the stress cube and record the coordinates of points V and H. We then draw Mohr's circle for stress, as shown in Figure 9.23*a*, and obtain

$$\tan 2\theta_p = \frac{4}{7} \quad \text{or} \quad \theta_p = 14.87° \text{ ccw} \tag{E5}$$

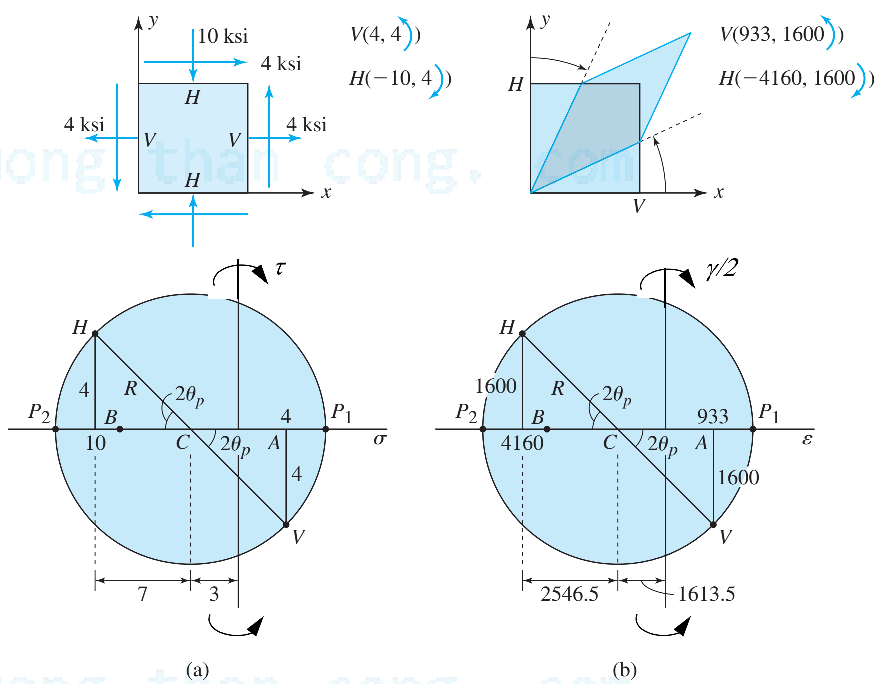


Figure 9.23 Mohr's circles in Example 9.9. (*a*) Stress. (*b*) Strain.

For this example $\theta_1 = \theta_p$, and we obtain the result for the orientation of principal axis 1.

ANS. $\theta_1 = 14.87°$ ccw

(b) Since γ_{xy} is positive, the angle between the x and y coordinates decreases, as shown by the deformed shape in Figure 9.23*b*. Noting that the vertical coordinate is $\gamma/2$, we record the coordinates of points V and H. We then draw Mohr's circle for strain, as shown in Figure 9.23*b*. The angle θ_p can be found from triangle BCH (or ACV):

$$\tan 2\theta_p = \frac{1600}{2546.5} \quad or \quad \theta_p = 16.1° \text{ ccw} \tag{E6}$$

For this example $\theta_1 = \theta_p$, and we obtain the result for the orientation of principal axis 1.

ANS. $\theta_1 = 16.1°$ ccw

COMMENTS

1. The stress state in this example is the same as in Example 9.7. In Example 9.7 we concluded that for isotropic materials the principal directions for stresses and strains are the same. Equations (E5) and (E6) show that for orthotropic materials the principal directions for stresses and strains are different.
2. In Example 9.7, if we change the material constants for the isotropic material, then the stress values will be different, but the result for the principal angle for stress will not change. If we change the material constants for orthotropic materials, then we not only change the stress values but we may also change the principal angle for stress. This is because we may change the degree of orthotropicness—that is, the degree of difference in the material constants in the x and y directions.
3. The preceding two comments highlight some of the reasons why intuition based on isotropic materials can be misleading when working with composite materials. In such cases mathematical rigor can provide answers that once confirmed by experiment, can form a new knowledge base for the development of intuitive understanding.

PROBLEM SET 9.2

Visualization of principal axis

In Problems 9.14 through 9.18, the state of strain at a point in plane strain is as given in each problem. Estimate the orientation of the principal directions and report your results using the sectors shown in Figure 9.7.

	Strains		
Problem	ε_{xx}	ε_{yy}	γ_{xy}
9.14	–400 μ	600 μ	–500 μ
9.15	–600 μ	–800 μ	500 μ.
9.16	800 μ	600 μ	–1000 μ
9.17	0	600 μ,	–500 μ
9.18	–1000 μ	–500 μ	700 μ

Method of Equations and Mohr's circle

9.19 Starting from Equation (9.4), show that maximum or minimum normal strain will exist in the direction of θ_p, as given by Equation (9.7). (*Hint:* See the similar derivation in stress transformation.)

9.20 Show that the values of the maximum and minimum normal strains are given by Equation (9.8). (*Hint:* See the similar derivation in stress transformation.)

9.21 Show that angle θ_p as given by Equation (9.7) is the principal angle, that is, shear strain is zero in a coordinate system that is at an angle θ_p to the Cartesian coordinate system. (*Hint:* See the similar derivation in stress transformation.)

9.22 Show that the coordinate system of maximum in-plane shear strain is 45° to the principal coordinate system. (*Hint:* See the similar derivation in stress transformation.)

9.23 Show that the maximum in-plane shear strain is given by Equation (9.11). (*Hint:* See the similar derivation in stress transformation.)

9.24 Starting from Equations (9.4) and (9.6), obtain the expression of Mohr's circle given by Equation (9.13). (*Hint:* See the similar derivation in stress transformation.)

9.25 Solve Problem 9.5 by the method of equations.

9.26 Solve Problem 9.5 by Mohr's circle.

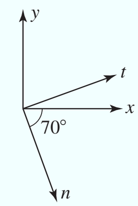
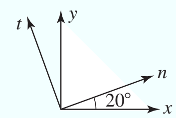
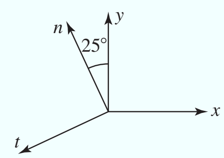
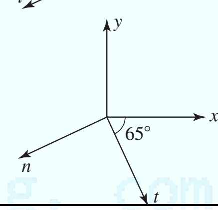
9.27 Solve Problem 9.6 by the method of equations.

9.28 Solve Problem 9.6 by Mohr's circle.

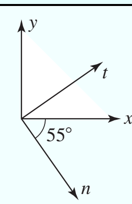
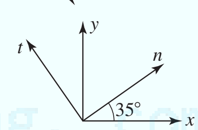
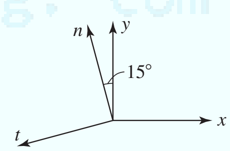
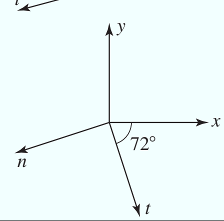
9.29 Solve Problem 9.7 by the method of equations.

9.30 Solve Problem 9.7 by Mohr's circle.

In Problems 9.31 through 9.34, at a point in plane strain, the strain components in the x, y coordinate system are as given. Using the associated figure, determine (a) the principal strains and principal angle 1; (b) the maximum shear strain; (c) the strain components in the n, t coordinate system.

Problem	Strains ε_{xx}	ε_{yy}	γ_{xy}	
9.31	–400 μ	600 μ	–500 μ	**Figure P9.31** 
9.32	–600 μ	–800 μ,	500 μ	**Figure P9.32** 
9.33	250 μ	850 μ,	1600 μ	**Figure P9.33** 
9.34	–1800μ	–3600 μ	–1500 μ	**Figure P9.34** 

In Problems 9.35 through 9.38, at a point in plane strain, the strain components in the n, t coordinate system are as given. Using the associated figure, determine (a) the principal strains; (b) the maximum shear strain; (c) the strain components in the x, y coordinate system.

Problem	Strains ε_{nn}	ε_{tt}	γ_{nt}	
9.35	2000 μ	–800 μ	750 μ	**Figure P9.35** 
9.36	–2000 μ	–800 μ	–600 μ	**Figure P9.36** 
9.37	350 μ	700 μ	1400 μ	**Figure P9.37** 
9.38	–3600 μ	2500 μ	–1000 μ	**Figure P9.38** 

In Problems 9.39 through 9.42, the principal strains ε_1 and ε_2 and the direction of principal direction 1 θ_1 from the x axis are given. Determine strains ε_{xx}, ε_{yy}, and γ_{xy} at the point.

	Principal Strains		*Principal Angle 1*
Problem	ε_1	ε_2	θ_1
9.39	1200 μ	300 μ	27.5°
9.40	900 μ	–600 μ	–20°
9.41	–200 μ	–2000 μ	105°
9.42	1400 μ	–600 μ	–75°

Generalized Hooke's law in principal coordinates

In Problems 9.43 through 9.45, the stresses in a thin body (plane stress) in the xy plane are as shown on each stress element. The modulus of elasticity E and Poisson's ratio ν are given in each problem. Using the associated figure, determine (a) the principal strains and principal angle 1 at the point; (b) the maximum shear strain at the point.

Problem	***E***	ν
9.43	70 GPa	ν = 0.25
9.44	70 GPa	ν = 0.25
9.45	30,000 ksi	0.28

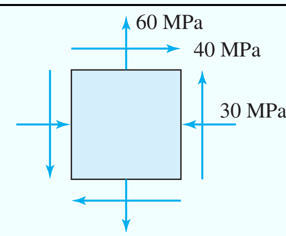


Figure P9.43

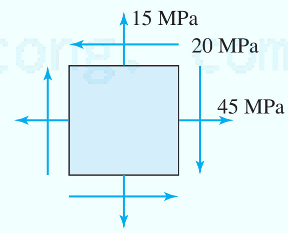


Figure P9.44

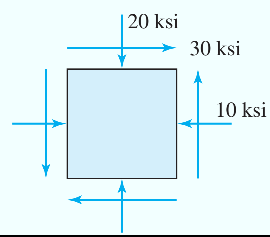


Figure P9.45

In Problems 9.46 through 9.48, the stresses in a thick body (plane strain) in the xy plane are as shown on each stress element. The modulus of elasticity E and Poisson's ratio ν are given in each problem. Using the associated figure, determine (a) the principal strains and principal angle 1 at the point; (b) The maximum shear strain at the point.

Problem	***E***	ν
9.46	105 GPa	ν = 0.35

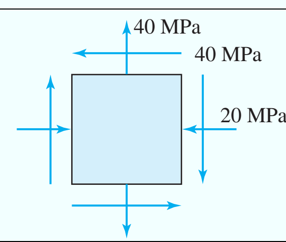


Figure P9.46

Problem	E	ν
9.47	70 GPa	$\nu = 0.25$
9.48	30,000 ksi	0.28

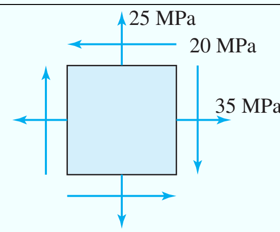


Figure P9.47

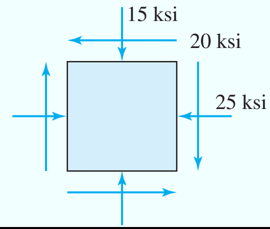


Figure P9.48

Orthotropic materials

In Problems 9.49 through 9.52, the properties of an orthotropic material and the stresses or strain are given at a point on a free surface. Using Equation (9.15), determine the principal direction for stresses and strains.

Problem	E_x	E_y	G_{xy}	ν_{xy}	***Stresses / Strains***
9.49	7500 ksi	2500 ksi	1250 ksi	0.25	$\varepsilon_{xx} = -400\ \mu$, $\varepsilon_{yy} = 600\ \mu$, and $\gamma_{xy} = -500\ \mu$
9.50	7500 ksi	2500 ksi	1250 ksi	0.25	$\sigma_{xx} = 10$ ksi (T), $\sigma_{yy} = 7$ ksi (C), and $\tau_{xy} = 5$ ksi.
9.51	50 GPa	18 GPa	9 GPa	0.25	$\varepsilon_{xx} = 800\ \mu$, $\varepsilon_{yy} = 200\ \mu$, and $\gamma_{xy} = 300\ \mu$.
9.52	50 GPa	18 GPa	9 GPa	0.25	$\sigma_{xx} = 70$ MPa (C), $\sigma_{yy} = 49$ MPa (C), and $\tau_{xy} = -30$ MPa

9.5 STRAIN GAGES

Strain gages are strain-measuring devices based on the changes in resistance in a wire with changes in its length. Since strain causes a length change, the change in resistance can be correlated to the strain in the wire by conducting an experiment. By bonding a wire to a stressed part, we can *assume* that the deformation of the wire is the same as that of the material. Hence, by measuring changes in the resistance of a wire, we can get the strains in the material. Strain gages are a sophisticated application of this technique.

Strain gages are usually manufactured by etching a thin foil of material, as shown in Figure 9.24. The back-and-forth pattern increases the sensitivity of the gage by providing a long length of wire in a very small area. Strain gages can be as small in length as $\frac{8}{1000}$ in., which for many engineering calculations is equivalent to measuring strain at a point.

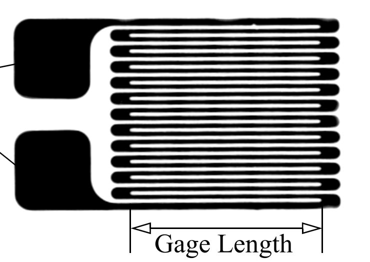


Figure 9.24 Typical strain gage.

Since we are measuring changes in the length of a wire, a strain gage measures only normal strains directly and not shear strains. In this section it will be shown how shear strains are calculated from the measured normal strains. Because of the finite sizes of strain gages, strain gages give an average value of strain at a point. To protect the strain gage from damage, no force is applied on its top. Hence strain gages are bonded to a free surface; that is, measurements take place in plane stress. We record the following observations:

1. Strain gages measure only normal strains directly.
2. Strain gages are bonded to a free surface. That is, the strains are in a state of plane stress and not plane strain.
3. Strain gages measure average strain at a point

In plane stress there are three *independent* strain components ε_{xx}, ε_{yy}, and γ_{xy}. To determine these, we need three observations at a point. In other words, we need to find normal strains in three directions. Figure 9.25 shows an assembly of three strain gages called a *strain rosette*. The strain gage readings ε_a, ε_b, and ε_c can be related to ε_{xx}, ε_{yy}, and γ_{xy} by Equation (9.4) as

$$\varepsilon_a = \varepsilon_{xx}\cos^2\theta_a + \varepsilon_{yy}\sin^2\theta_a + \gamma_{xy}\sin\theta_a\cos\theta_a \quad (9.16.a)$$

$$\varepsilon_b = \varepsilon_{xx}\cos^2\theta_b + \varepsilon_{yy}\sin^2\theta_b + \gamma_{xy}\sin\theta_b\cos\theta_b \quad (9.16.b)$$

$$\varepsilon_c = \varepsilon_{xx}\cos^2\theta_c + \varepsilon_{yy}\sin^2\theta_c + \gamma_{xy}\sin\theta_c\cos\theta_c \quad (9.16.c)$$

The three equations can be solved for the three unknowns ε_{xx}, ε_{yy}, and γ_{xy} since θ_a, θ_b, and θ_c are known.

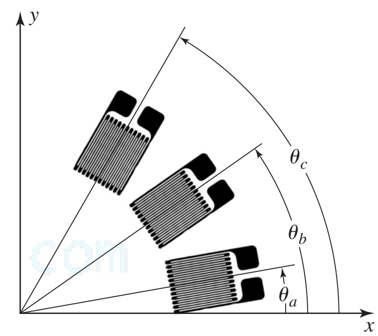


Figure 9.25 Strain rosette.

The angles at which strain gages are attached are chosen to reduce the algebra in the calculation of ε_{xx}, ε_{yy}, and γ_{xy}. Figure 9.26 shows two popular choices of angles in a strain rosette. Notice in Figure 9.26*b* that angle θ_c can be 120° or −60° (or 300° or −240°). This emphasizes that Equation (9.4) does not change if 180° is added to or subtracted from angle θ. (See Problem 9.53.) An alternative explanation is that normal strain is a measure of the deformation of a line and deformation is the relative movement of two points on a line. Hence the value does not depend on whether the two points on the line have positive or negative coordinates. We can summarize our observation simply:

- A change in strain gage orientation by ±180° makes no difference in the strain values.

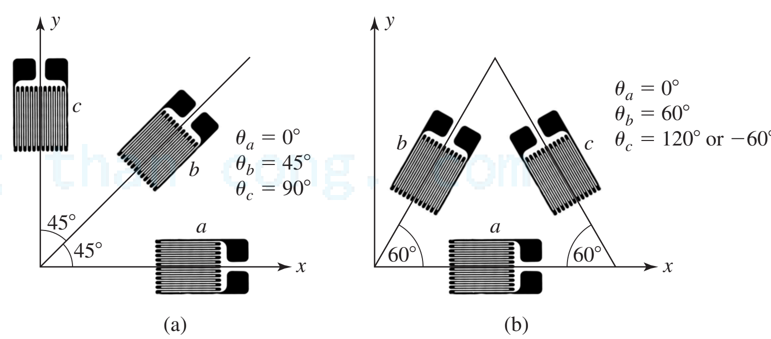


Figure 9.26 Strain rosettes. (*a*) 45°. (*b*) 60°.

Once strains ε_{xx}, ε_{yy}, and γ_{xy} are found, then the principal strains can be found. The principal stresses can be found next, if needed, from the generalized Hooke's law in principal coordinates. Alternatively, the stresses σ_{xx}, σ_{yy}, and τ_{xy} may be found first from the generalized Hooke's law, and then the principal stresses can be found. But it is important to remember that the point where strains are being measured is in plane stress, and hence $\sigma_{zz} = 0$. The strain in the z direction is the third principal strain and can be found from Equation (9.10).

EXAMPLE 9.10

Strains $\varepsilon_a = 900\ \mu$in./in., $\varepsilon_b = 200\ \mu$in./in., and $\varepsilon_c = 700\ \mu$in./in. were recorded by the three strain gages shown in Figure 9.27 at a point on the free surface of a material that has a modulus of elasticity $E = 30{,}000$ ksi and a Poisson ratio $\nu = 0.3$. Determine the principal stresses, principal angle 1, and the maximum shear stress at the point.

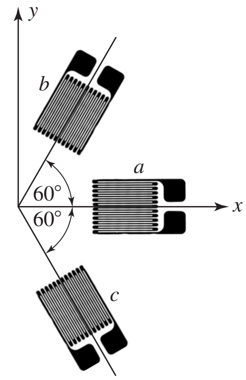


Figure 9.27 Strain rosette in Example 9.10.

PLAN: METHOD 1

We note that $\varepsilon_a = \varepsilon_{xx}$. We can find strains ε_{yy} and γ_{xy} from the two equations obtained by substituting $\theta_b = +60°$ and $\theta_c = -60°$ into Equation (9.4). We can then find principal strains 1 and 2 and principal angle 1 by using either Mohr's circle or the method of equations. Principal strain 3 can be found from Equation (9.10), and the maximum shear strain from the radius of the biggest circle. Using the generalized Hooke's law in principal coordinates we can find the principal stresses.

SOLUTION

Strain gages: The strain in the x direction is given by the strain gage a reading. Thus

$$\varepsilon_{xx} = 900\ \mu \tag{E1}$$

Substituting $\theta_b = +60°$ and $\theta_c = -60°$ into Equation (9.4), we obtain

$$\varepsilon_b = (900)\cos^2 60 + \varepsilon_{yy}\sin^2 60 + \gamma_{xy}\sin 60\cos 60 = 200 \qquad \text{or} \qquad 0.75\varepsilon_{yy} + 0.433\gamma_{xy} = -25 \tag{E2}$$

$$\varepsilon_c = (900)\cos^2(-60) + \varepsilon_{yy}\sin^2(-60) + \gamma_{xy}\sin(-60)\cos(-60) = 700 \qquad \text{or} \qquad 0.75\varepsilon_{yy} - 0.433\gamma_{xy} = 475 \tag{E3}$$

Solving Equations (E2) and (E3), we obtain

$$\varepsilon_{yy} = 300\ \mu \qquad \gamma_{xy} = -577.4\ \mu \tag{E4}$$

Mohr's circle for strain: We draw the deformed shape as shown in Figure 9.28*a* and write the coordinates of points V and H as

$$V(900, 288.7\ \circlearrowright) \qquad H(300, 288.7\ \circlearrowleft) \tag{E5}$$

We then draw Mohr's circle for strain shown in Figure 9.28b and calculate the principal strains. From the Pythagorean theorem we can find the radius R,

$$R = \sqrt{CB^2 + BV^2} = \sqrt{300^2 + 288.7^2} = 416.4 \tag{E6}$$

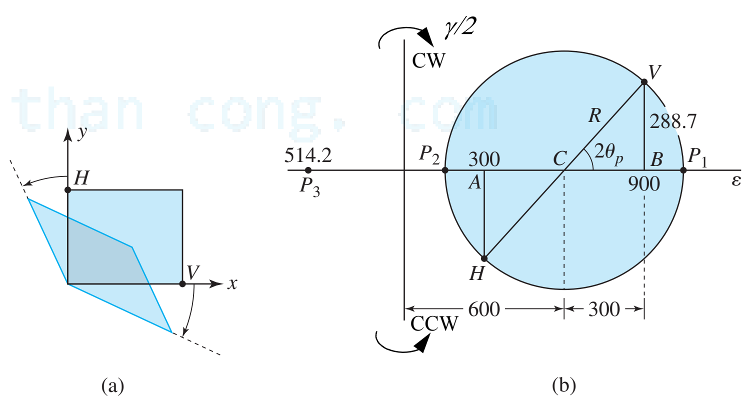


Figure 9.28 Mohr's circle in Example 9.4.

The principal strains are the coordinates of points P_1 and P_2 in Figure 9.28*b*,

$$\varepsilon_1 = 600 + 416.4 = 1016.4 \qquad \varepsilon_2 = 600 - 416.4 = 183.6 \tag{E7}$$

As the point is on a free surface, the state is in plane stress. Hence the third principal strain from Equation (9.10) is

$$\varepsilon_3 = \varepsilon_{zz} = -\frac{0.3}{1-0.3}(900+300) = -514.2\ \mu \qquad \text{(E8)}$$

Using triangle *BCV* in Figure 9.28*b*, we can find the principal angle θ_p,

$$\cos 2\theta_p = \frac{CB}{CV} = \frac{300}{416.4} \qquad \text{(E9)}$$

From Figure 9.28*b* we see that $\theta_1 = \theta_p$, and the direction is clockwise:

$$2\theta_p = 43.9° \qquad \theta_1 = \theta_p = 21.9°\ \text{cw} \qquad \text{(E10)}$$

Intuitive check: We visualize a circle in a square, as in Figure 9.29*a*. Since $\varepsilon_{xx} > \varepsilon_{yy}$, the rectangle will become longer in the *x* direction than in the *y* direction, and the circle will become an ellipse with its major axis along the *x* direction, as shown in Figure 9.29*b*. Since $\gamma_{xy} < 0$, the angle between the *x* and *y* directions will increase. The rectangle will become a rhombus, and the major axis of the ellipse will rotate clockwise from the *x* axis, as shown in Figure 9.29*c*. Hence we expect principal axis 1 to be either in the eighth sector or in the fourth sector, confirming the result given in Equation (E10).

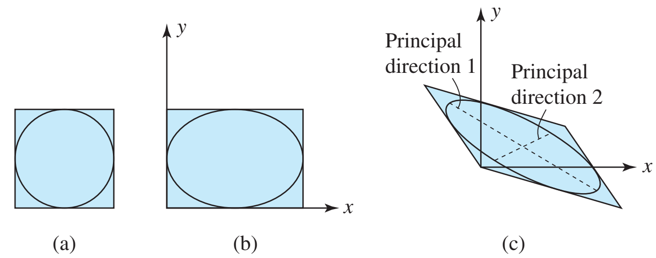


Figure 9.29 Estimating principal directions in Example 9.10. (*a*) Un-deformed shape. (*b*) Deformation due to normal strains. (*c*) Additional deformation due to shear strain.

Locating point P_3, which corresponds to the third principal strain in Figure 9.28*b*, we note that the circle between P_1 and P_3 will be a bigger circle than between P_2 and P_3, or between P_1 and P_2. Thus the maximum shear strain at the point can be determined from the circle between P_1 and P_3,

$$\frac{\gamma_{max}}{2} = \frac{\varepsilon_1 - \varepsilon_3}{2} = 765.3 \qquad \gamma_{max} = 1531\ \mu \qquad \text{(E11)}$$

Hooke's law: For plane stress $\sigma_3 = 0$. From Equations (9.14.a) and (9.14.b) we obtain

$$\varepsilon_1 = \frac{\sigma_1 - \nu\sigma_2}{30{,}000\ \text{ksi}} = 1016(10^{-6}) \qquad \text{or} \qquad \sigma_1 - 0.3\sigma_2 = 30.48\ \text{ksi} \qquad \text{(E12)}$$

$$\varepsilon_2 = \frac{\sigma_2 - \nu\sigma_1}{30{,}000\ \text{ksi}} = 184(10^{-6}) \qquad \text{or} \qquad \sigma_2 - 0.3\sigma_1 = 5.52\ \text{ksi} \qquad \text{(E13)}$$

Solving Equations (E12) and (E13), we obtain $\sigma_1 = 35.31$ ksi and $\sigma_2 = 16.11$ ksi. For isotropic materials the principal direction for stresses and strains is the same.

ANS. $\sigma_1 = 35.3\ \text{ksi(T)} \qquad \sigma_2 = 16.1\ \text{ksi(T)} \qquad \sigma_3 = 0 \qquad \theta_1 = 21.9^o\ \text{CW}$

The shear modulus of elasticity is

$$G = \frac{E}{2(1+\nu)} = 11{,}538\ \text{ksi} \qquad \text{(E14)}$$

The maximum shear stress can be found from Hooke's law as

$$\tau_{max} = G\gamma_{max} = (11,538)(1531)(10^{-6}) = 17.65\ \text{ksi} \qquad \text{(E15)}$$

Check: We can also find the maximum shear stress as half the maximum difference between principal stresses. That is, from Equation (8.13), $\tau_{max} = (35.3 - 0)/2 = 17.65\ \text{ksi}$, confirming Equation (E15).

ANS. $\tau_{max} = 17.65\ \text{ksi}$

COMMENT

This example combines three concepts: the use of strain gages to find strain components in Cartesian coordinates, the use of Mohr's circle for finding principal strains, and the use of Hooke's law in principal coordinates for finding principal stresses.

PLAN: METHOD 2

We can find ε_{xx}, ε_{yy}, and γ_{xy} from the values of the strains recorded by the strain gages, as we did in Method 1. We can use Hooke's law in Cartesian coordinates to find σ_{xx}, σ_{yy}, and τ_{xy}. Using Mohr's circle for stress (or the method of equations), we can then find the principal stresses, principal angle 1, and the maximum shear stress.

SOLUTION

Strain gages: From Equations (E1) and (E4),

$$\varepsilon_{xx} = 900\ \mu \qquad \varepsilon_{yy} = 300\ \mu \qquad \gamma_{xy} = -577.4\mu \tag{E16}$$

Hooke's law: We note that for plane stress $\sigma_{zz} = 0$. Using Equations (3.12a) and (3.12b), we can write

$$\varepsilon_{xx} = \frac{\sigma_{xx} - \nu\sigma_{yy}}{30{,}000\text{ ksi}} = 900(10^{-6}) \quad \text{or} \quad \sigma_{xx} - 0.3\sigma_{yy} = 27\text{ ksi} \tag{E17}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy} - \nu\sigma_{xx}}{30{,}000\text{ ksi}} = 300(10^{-6}) \quad or \quad \sigma_{yy} - 0.3\sigma_{xx} = 9\text{ ksi} \tag{E18}$$

Solving Equations (E17) and (E18), we obtain $\sigma_{xx} = 32.63$ ksi and $\sigma_{yy} = 18.79$ ksi From Equations (3.12d) and (E14) we obtain the shear stress as

$$\tau_{xy} = G\gamma_{xy} = (11,538)(-577.4)(10^{-6}) = -6.66\text{ ksi} \tag{E19}$$

Mohr's circle for stress

We draw the stress cube as shown in Figure 9.30*a* and record the coordinates of points *V* and *H* as

$$V(32.63, 6.66\circlearrowright) \qquad H(18.79, 6.66\circlearrowleft) \tag{E20}$$

We then draw Mohr's circle for stress as shown in Figure 9.30*b* and calculate the principal stresses. From the Pythagorean theorem we can find the radius *R*,

$$R = \sqrt{CB^2 + BV^2} = \sqrt{6.92^2 + 6.66^2} = 9.60 \tag{E21}$$

The principal stresses are the coordinates of points P_1 and P_2 in Figure 9.30*b*. As the point is on free surface, the state is in plane stress. Hence the third principal stress is zero,

$$\sigma_1 = 25.71 + 9.60 = 35.31\text{ ksi} \qquad \sigma_2 = 25.71 - 9.60 = 16.11\text{ ksi} \qquad \sigma_3 = 0 \tag{E22}$$

Using triangle *BCV* in Figure 9.30*b* we can find the principal angle θ_p, From Figure 9.30*b* we see that $\theta_1 = \theta_p$ and the direction is clockwise:

$$\cos 2\theta_p = \frac{CB}{CV} = \frac{6.92}{9.6} \quad \text{or} \quad 2\theta_p = 43.9° \quad \theta_1 = \theta_p = 21.9°\text{ cw} \tag{E23}$$

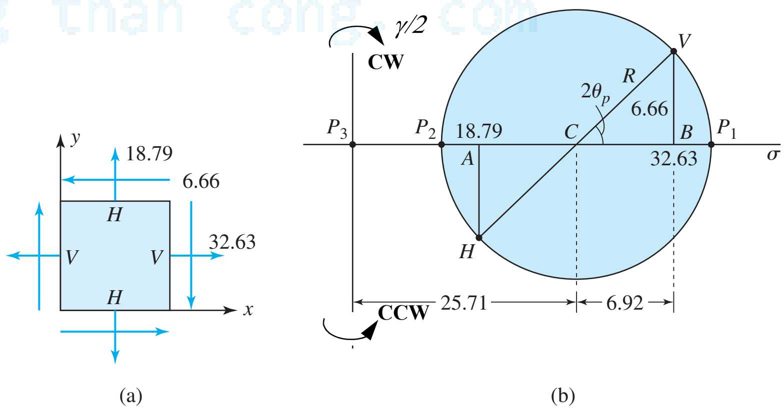


Figure 9.30 Mohr's circle in Example 9.10.

ANS. $\sigma_1 = 35.3\text{ ksi(T)} \qquad \sigma_2 = 16.1\text{ ksi(T)} \qquad \sigma_3 = 0 \qquad \theta_1 = 21.9^o\text{ cw}$

The biggest circle will be between P_1 and P_3. The maximum shear stress is the radius of this circle and can be calculated as $\tau_{max} = (35.3\text{ ksi} - 0)/2 = 17.65\text{ ksi}$.

ANS. $\tau_{max} = 17.65\text{ ksi}$

COMMENT

1. As in Method 1, three concepts are combined, but the sequence in which the problem is solved is different. In Method 1 we used Mohr's circle (for strain) first and Hooke's law (in principal coordinates) second. In Method 2 we used Hooke's law (Cartesian coordinates) first and Mohr's circle (for stress) second. The number of calculations differs only with respect to ε_3, which is not calculated in Method 2.

EXAMPLE 9.11

The strain gage at point A recorded a value of $\varepsilon_A = -200\ \mu$. Determine the load P that caused the strain for the three cases shown in Figure 9.31. In each case the strain gage is 30° clockwise to the longitudinal axis (x axis). Use $E = 10{,}000$ ksi, $G = 4000$ ksi, and $\nu = 0.25$.

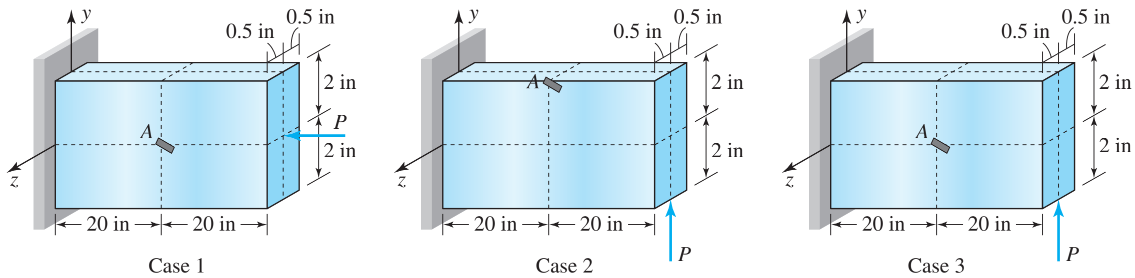


Figure 9.31 Three beams in Example 9.11.

PLAN

The axial stress σ_{xx} in case 1, the bending normal stress σ_{xx} in case 2, and the bending shear stress τ_{xy} in case 3 can be found in terms of P using Equations (4.8), (6.12), and (6.27), respectively. All other stress components are zero. Strains ε_{xx}, ε_{yy}, and γ_{xy} can be found in terms of P for each case, using the generalized Hooke's law. Substituting the strains and $\theta_A = -30°$ into Equation (9.4), the strain in the gage can be found in terms of P and equated to the given value of $-200\ \mu$ to obtain the value of P.

SOLUTION

Stress calculations: Recollect that A_s is the area between the free surface and point A, where shear stress is to be found. The cross-sectional area A, the area moment of inertia I_{zz}, and the first moment Q_z of the area A_s shown in Figure 9.32*a* can be calculated as

$$A = (1)(4) = 4\ \text{in}^2 \qquad I_{zz} = \frac{(1\ \text{in.})(4\ \text{in.})^3}{12} = 5.33\ \text{in.}^4 \qquad Q_z = (1\ \text{in.})(2\ \text{in.})(1\ \text{in.}) = 2\ \text{in.}^3 \tag{E1}$$

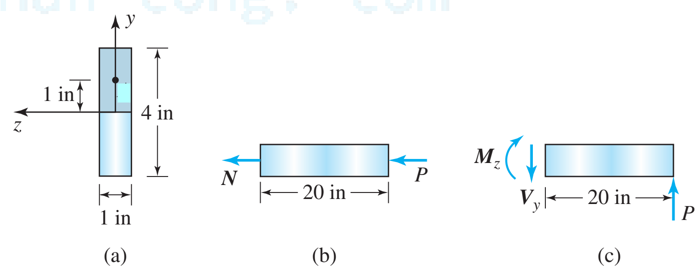


Figure 9.32 Calculation of geometric and internal quantities in Example 9.11

Figure 9.32*b* and *c* shows the free-body diagrams of the axial member and the beam after making the imaginary cut through point A. Using force and moment equilibrium equations, we find the internal forces and moment,

$$N = -P\ \text{kips} \qquad V_y = P\ \text{kips} \qquad M_z = 20P\ \text{in.}\cdot\text{kips} \tag{E2}$$

Substituting Equations (E1) and (E2) into Equation (4.8), we find the axial stress in case 1,

$$\sigma_{xx} = \frac{N}{A} = \frac{-P\ \text{kips}}{4\ \text{in.}^2} = -0.25P\ \text{ksi} \tag{E3}$$

Substituting Equations (E1), (E2), and $y = 2$ in into Equation (6.12), we find the bending normal stress in case 2,

$$\sigma_{xx} = -\frac{M_z y}{I_{zz}} = -\frac{(20P\ \text{in.}\cdot\text{kips})(2\ \text{in.})}{5.33\ \text{in.}^4} = -7.5P\ \text{ksi} \tag{E4}$$

Substituting Equations (E1), (E2), and $t = 1$ into Equation (6.27), we find the magnitude of τ_{xy} in case 3,

$$|\tau_{xy}| = \left|\frac{V_y Q_z}{I_{zz} t}\right| = \left|\frac{(P\ \text{kips})(2\ \text{in.}^3)}{(5.33\ \text{in.}^4)(1\ \text{in.})}\right| = 0.375P\ \text{ksi} \tag{E5}$$

Noting that τ_{xy} must have the same sign as V_y, we obtain the sign of τ_{xy} (see Section 6.6.6),

$$\tau_{xy} = 0.375P\ \text{ksi} \tag{E6}$$

Strain calculations: The only two nonzero stress components are given by Equations (E3), (E4), and (E6) for each case. Using the generalized Hooke's law [or Equations (4.13) and (6.29)], we obtain the strains for each case. Substituting the strains and $\theta_A = -30°$ into Equation (9.4) and equating the result to $-200\ \mu$ give the value of load P for each case:

- *Case 1:*

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} = \frac{-0.25P\text{ ksi}}{10000\text{ ksi}} = -25P\,\mu \qquad \varepsilon_{yy} = -\frac{\nu\sigma_{xx}}{E} = -\nu\varepsilon_{xx} = 6.25P\,\mu \qquad \gamma_{xy} = 0 \tag{E7}$$

$$\varepsilon_A = (-25P\,\mu)\cos^2(-30°) + (6.25P\,\mu)\sin^2(-30°) = -17.19P\,\mu = -200\,\mu \qquad \text{or} \qquad P = 11.6\text{ kips} \tag{E8}$$

ANS. $P = 11.6\text{ kips}$

- *Case 2:*

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} = \frac{-7.5P\text{ ksi}}{10000\text{ ksi}} = -750P\,\mu \qquad \varepsilon_{yy} = -\frac{\nu\sigma_{xx}}{E} = -\nu\varepsilon_{xx} = 187.5P\,\mu \qquad \gamma_{xy} = 0 \tag{E9}$$

$$\varepsilon_A = (-750P\,\mu)\cos^2(-30°) + (187.5P\,\mu)\sin^2(-30°) = -515.63P\,\mu = -200\,\mu \qquad \text{or} \qquad P = 0.39\text{ kips} \tag{E10}$$

ANS. $P = 0.39\text{ kips}$

- *Case 3:*

$$\varepsilon_{xx} = 0 \qquad \varepsilon_{yy} = 0 \qquad \gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{0.375P\text{ ksi}}{4000\text{ ksi}} = 93.75P\,\mu \tag{E11}$$

$$\varepsilon_A = (93.75P\,\mu)\sin(-30°)\cos(-30°) = -40.59P\,\mu = -200\,\mu \qquad \text{or} \qquad P = 4.93\text{ kips} \tag{E12}$$

ANS. $P = 4.93\text{ kips}$

COMMENTS

1. This example demonstrates one of the basic principles used in the design of *load transducers*, also called *load cells*. Load transducers are used for measuring, applying, and controlling forces and moments on a structure. This example showed how one may measure a force by using strain gage readings and mechanics of materials formulas. The electrical signal from the strain gage can be processed and correlated with the intensity of the force and moment. It can be used to apply and control these quantities.
2. In this example the strain in the gage was caused by a single force. When there are multiple forces or moments acting on a structure, then to correlate strain gage readings to the applied forces and moments we need to supplement the formulas of mechanics and materials with the formulas for the Wheatstone bridge. See Section 9.6 for additional details on the Wheatstone bridge.
3. In Examples 9.5 and 9.10 and in this example we saw the use of the generalized Hooke's law. An alternative is to use formulas that are derived from the generalized Hooke's law. This is one important reason for memorizing the generalized Hooke's law.

PROBLEM SET 9.3

Strain gages

9.53 Show that upon substituting $\theta \pm 180°$ in place of θ, the strain transformation equation, Equation (9.4), is unchanged.

9.54 At a point on a free surface the strain components in the x, y coordinates are calculated as $\varepsilon_{xx} = 400\ \mu$in./in., $\varepsilon_{yy} = -200\ \mu$in./in., and $\gamma_{xy} = 500\ \mu$rad. Predict the strains that the strain gages shown in Figure P9.54 would record.

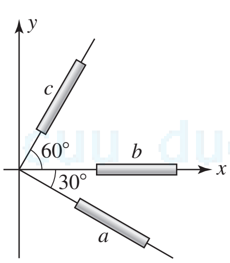


Figure P9.54

9.55 At a point on a free surface the strains recorded by the three strain gages shown in Figure P9.54 are $\varepsilon_a = 200\ \mu$in./in., $\varepsilon_b = 100\ \mu$in./in., and $\varepsilon_c = -400\ \mu$in./in. Determine strains ε_{xx}, ε_{yy}, and γ_{xy}.

9.56 At a point on a free surface of an aluminum machine component (E = 10,000 ksi and G = 4000 ksi) the stress components in the x, y coordinates were calculated by the finite-element method as σ_{xx} = 22 ksi (T), σ_{yy} = 15 ksi (C), and $\tau_{xy} = -10$ ksi. Predict the strains that the strain gages shown in Figure P9.56 would show.

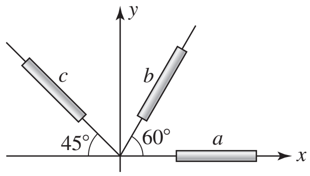


Figure P9.56

9.57 At a point on a free surface of aluminum (E = 10,000 ksi and G = 4000 ksi) the strains recorded by the three strain gages shown in Figure P9.56 are ε_a = –600 μin./in., ε_b = 500 μin./in., and ε_c = 400 μin./in. Determine stresses σ_{xx}, σ_{yy}, and τ_{xy}.

9.58 At a point on a free surface of a machine component (E = 80 GPa and G = 32 GPa) the stress components in the x, y coordinates were calculated by the finite-element method as σ_{xx} = 50 MPa (T), σ_{yy} = 20 MPa (C), and τ_{xy} = 96 MPa. Predict the strains that the strain gages shown in Figure P9.58 would show.

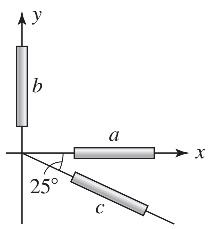


Figure P9.58

9.59 At a point on a free surface of a machine component (E = 80 GPa and G = 32 GPa) the strains recorded by the three strain gages shown in Figure P9.58 are ε_a = 1000 μm/m, ε_b = 1500 μm/m, and ε_c = –450 μm/m. Determine stresses σ_{xx}, σ_{yy}, and τ_{xy}.

9.60 On a free surface of steel (E = 210 GPa and ν = 0.28) the strains recorded by the three strain gages shown in Figure P9.60 are ε_a = –800 μm/m, ε_b = –300 μm/m, and ε_c = –700 μm/m. Determine the principal *strains,* principal angle 1, and the maximum shear *strain*

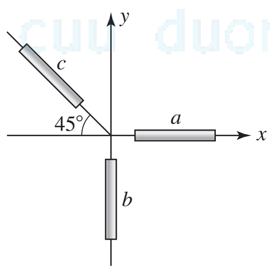


Figure P9.60

9.61 On a free surface of steel (E = 210 GPa and ν = 0.28) the strains recorded by the three strain gages shown in Figure P9.60 are ε_a = 200 μm/m, ε_b = 100 μm/m, and ε_c = 0. Determine the principal *stresses*, principal angle 1, and the maximum shear *stress*.

9.62 On a free surface of an aluminum machine component (E = 10,000 ksi and ν = 0.25) the strains recorded by the three strain gages shown in Figure P9.62 are ε_a = –100 μin./in., ε_b = 200 μin./in., and ε_c = 300 μin./in. Determine the principal *strains,* principal angle 1, and the maximum shear *strain*.

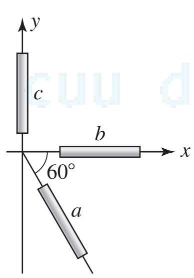


Figure P9.62

9.63 On a free surface of an aluminum machine component (E = 10,000 ksi and ν = 0.25) the strains recorded by the three strain gages shown in Figure P9.62 are ε_a = 500 μin./in., ε_b = 500 μin./in., and ε_c = 500 μin./in. Determine the principal *stresses*, principal angle 1, and the maximum shear *stress*.

Strain gages on structural elements

9.64 An aluminum (E = 70 GPa, G = 28 GPa) 50-mm × 50-mm square bar is axially loaded with a force F = 100 kN as shown in shown in Figure P9.64. Determine the strain that will be recorded by the strain gage.

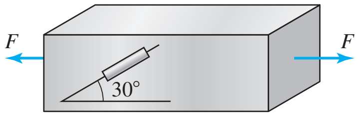


Figure P9.64

9.65 An aluminum (E = 70 GPa, G = 28 GPa) 50-mm × 50-mm square bar is axially loaded as shown in shown in Figure P9.64. Determine applied force F when the gage shows a reading of 200 μ.

9.66 A circular steel (E = 30,000 ksi, ν = 0.3) bar has a diameter of 2 in. and is axially loaded as shown in Figure P9.66. If the applied axial force F = 100 kips, determine the strain the gage will show.

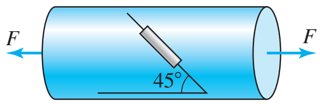


Figure P9.66

9.67 A circular steel (E = 30,000 ksi, ν = 0.3) bar has a diameter of 2 in. and is axially loaded as shown in Figure P9.66. Determine the applied axial force F when the strain gage shows a reading of 1000 μin./in.

9.68 A circular shaft of 2-in diameter has a torque applied to it as shown in Figure P9.68.The shaft material has a modulus of elasticity of 30,000 ksi and a Poisson's ratio of 0.3. Determine the strain that will be recorded by a strain gage.

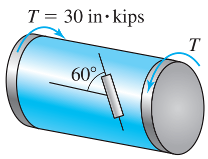


Figure P9.68

9.69 A circular shaft of 50-mm diameter has a torque applied to it as shown in Figure P9.69. The shaft material has a modulus of elasticity E = 70 GPa and a shear modulus G = 28 GPa. If the applied torque T = 500 N-m, determine the strain that the gage will show.

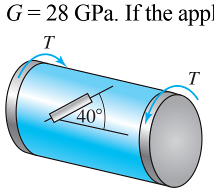


Figure P9.69

9.70 A circular shaft of 50-mm diameter has a torque applied to it as shown in Figure P9.69. The shaft material has a modulus of elasticity E = 70 GPa and a shear modulus G = 28 GPa. If the strain gage shows a reading of –600 μ, determine the applied torque T.

9.71 The steel cylindrical pressure vessel (E = 210 GPa and ν = 0.28) shown in Figure P9.71 has a mean diameter of 1000 mm. The wall of the cylinder is 10 mm thick and the gas pressure is 200 kPa. Determine the strain recorded by the two strain gages attached on the surface of the cylinder.

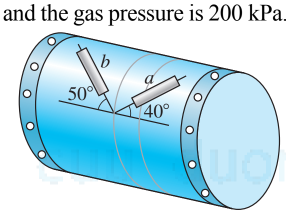


Figure P9.71

9.72 An aluminum beam (E = 70 GPa and ν = 0.25) is loaded by a force P = 10 kN and moment M = 5 kN·m at the free end, as shown in Figure P9.72. If the two strain gages shown are at an angle of 25° to the longitudinal axis, determine the strains in the gages.

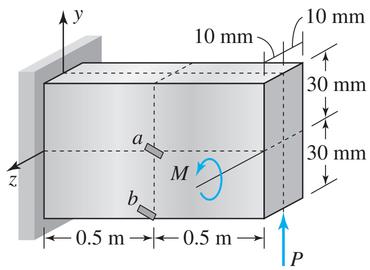


Figure P9.72

9.73 An aluminum beam (E = 70 GPa and ν = 0.25) is loaded by a force P and a moment M at the free end, as shown in Figure P9.72. Two strain gages at 30° to the longitudinal axis recorded the following strains: $\varepsilon_a = -386\ \mu$m/m and $\varepsilon_b = 4092\ \mu$m/m. Determine the applied force P and applied moment M.

9.74 A steel rod (E = 210 GPa and ν = 0.28) of 50-mm diameter is loaded by axial forces P = 100 kN, as shown in Figure P9.74. Determine the strain that will be recorded by the strain gage.

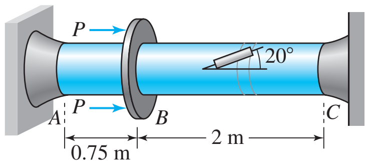


Figure P9.74

9.75 The strain gage mounted on the surface of the solid axial steel rod (E = 210 GPa and ν = 0.28) illustrated in Figure P9.74 showed a strain of $-214\ \mu$m/m. If the diameter of the shaft is 50 mm, determine the applied axial force P.

9.76 A steel shaft (E = 210 GPa and ν = 0.28) of 50-mm diameter is loaded by a torque T = 10 kN·m, as shown in Figure P9.76. Determine the strain that will be recorded by the strain gage.

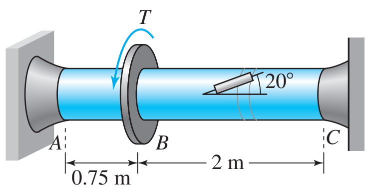


Figure P9.76

9.77 The strain gage mounted on the surface of the solid steel shaft (E = 210 GPa and ν = 0.28) shown in Figure P9.76 recorded a strain of 1088 μm/m. If the diameter of the shaft is 75 mm, determine the applied torque T.

Stretch yourself

In Problems 9.78 through 9.80, Equations (9.17.a) and (9.17.b) are transformation equations relating the x, y coordinates to the n, t coordinates of a point (Figure P9.77). Equations (9.17.c) and (9.17.d) are transformation equations relating displacements u and v in the x and y directions to the displacements u_n and u_t in the n and t directions, respectively. Solve each problem using Equations (9.24a) through (9.24d).

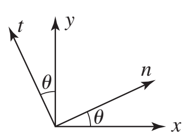


Figure P9.77

$$n = x\cos\theta + y\sin\theta \qquad (9.17.a)$$

$$t = -x\sin\theta + y\cos\theta \qquad (9.17.b)$$

$$u_n = u\cos\theta + \mathrm{v}\sin\theta \qquad (9.17.c)$$

$$u_t = -u\sin\theta + \mathrm{v}\cos\theta \qquad (9.17.d)$$

9.78 Starting with $\varepsilon_{nn} = \partial u_n/\partial n$ and using Equations (9.24*a*) through (9.24*d*) and the chain rule for differentiation, derive Equation (9.4).

9.79 Starting with $\varepsilon_{tt} = \partial u_t/\partial t$ and using Equations (9.24*a*) through (9.24*d*) and the chain rule for differentiation, derive Equation (9.5).

9.80 Starting with $\gamma_{nt} = \partial u_t/\partial n + \partial u_n/\partial t$ and using Equations (9.24*a*) through (9.24*d*) and the chain rule for differentiation, derive Equation (9.6).

9.81 Starting from Equation (9.15), show that for isotropic materials $E_x = E_y$ and $G_{xy} = E_x/2(1 + \nu)$.

Computer problems

9.82 The displacements u and v in the x and y directions are given by the equations

$$u = [0.5(x^2 - y^2) + 0.5xy + 0.25x]10^{-3}\ \text{mm} \qquad \mathrm{v} = [0.25(x^2 - y^2) - xy]10^{-3}\ \text{mm}$$

Assuming plane strain, determine the principal strains, principal angle 1, and the maximum shear strain every 30° on a circle of radius 1 around the origin. Use a spreadsheet or write a computer program for the calculation.

9.83 On an aluminum beam (E = 70 GPa and ν = 0.25) two strain gages were attached to monitor loads P and w, which vary slowly over time (Figure P9.83). The strain gage readings are given in Table 9.83. Determine the values of P and w at the times the strains were measured.

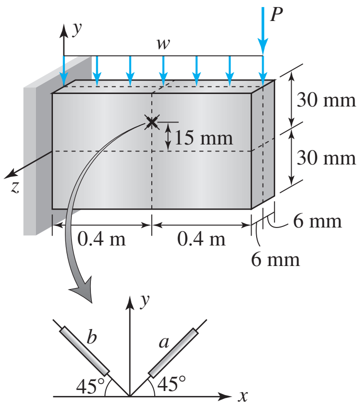


Figure P9.83

TABLE P9.83 Strain values

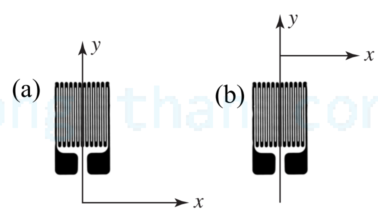
	ε_a (μ)	ε_b (μ)
1	1501	2368
2	1433	2276
3	1385	2193
4	1483	2336
5	1470	2331
6	1380	2191
7	1448	2282
8	1496	2366
9	1398	2223
10	1411	2228

QUICK TEST 9.2

Time: 15 minutes/Total: 20 points

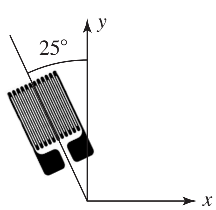
Grade yourself with the answers given in Appendix E. Each question is worth two points.

1. The strain gage recorded a strain of 800 μ. What is ε_{yy} for the two cases shown?

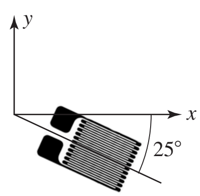


In Questions 2 through 4, report the smallest positive and the smallest negative angle θ that can be substituted in the strain transformation equation relating the strain gage reading to strains in Cartesian coordinates.

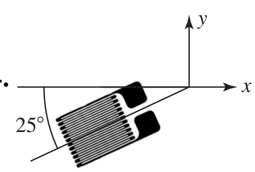
4.



3.

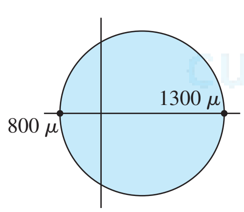


4.

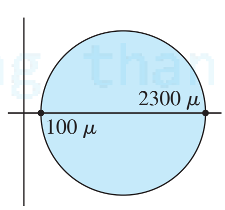


In Questions 5 through 7, Mohr's circles for strains for points in plane stress are as shown. The modulus of elasticity of the material is E = 10,000 ksi and Poisson's ratio is 0.25. What is the maximum shear strain in each question?

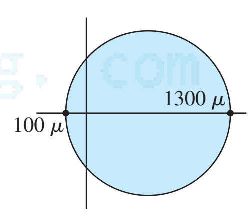
4.



6.



7.



In Questions 8 through 10, answer true or false. If false, then give the correct explanation.

8. In plane strain there are two principal strains, but in plane stress there are three principal strains.

9. Since strain values change with the coordinate system, the principal strains at a point depend on the coordinate system used in finding the strains.

10. The principal coordinate axis for stresses and strains is always the same, irrespective of the stress–strain relationship.

MoM in Action: Load Cells

Load cells are everywhere in our lives, even if we do not call them by their name. A *load cell* is a device that measures, controls, or applies a force or a moment. Bathroom scales, tire pressure gauges, hydraulic presses, and pressure pads are all load cells, based on variety of mechanical principles.

Scales for weighing were the first kind of load cell. They have been in existence at least since Archimedes (Syracuse, Greece, 287– 212 BCE) stated the lever principle. These mechanical load cells can measure weights with high precision (Figure 9.33*a*) over a large range, provided the fulcrum point is a fine knife edge. A pointer, attached at the fulcrum point, allows the readings to be calibrated and amplified (Figure 9.33*a*). In this way measurements can be made from a few milligrams in chemistry laboratories to thousands of kilograms on truck scales. In spring scales, it is the extension of a spring that is calibrated. However, the familiar pointers from bathroom scales are now being replaced by digital readings.

Today most load cells are constructed using strain gages. Trucks that had to come to a stop for weighing by mechanical scales now simply drive over scales that have been strain gaged. The popularity of strain gages comes from two facts, one in the mechanics of materials and the other electrical: the formulas relating the force or moment on structural members to the strains (see Example 9.11) are very reliable; and the signal from strain gages can be processed for reading, storage, or control. A vast variety of load cells are manufactured ready for use; others are custom build for specific applications. Figure 9.33*b* shows a load cells built around axial-member.

(a) **(b)**

Figure 9.33 Load cells: (a) weighing scale (b) tension/compression (Courtesy Celsum Technologies Ltd.).

Load cells are used to maintain proper tension in manufacturing rolls of paper or metal sheets. They are also used for monitoring tension in the cables and compression in the towers of a suspension bridge. Load cells embedded in masonry can detect cracks in structures during construction and operation. Accurate drug dosages can be delivered by calibrating the weight of fluid to load cell readings. The field of robotics and assembly-line automation also uses a vast variety of load cells, from earthbound applications to the *Rovers* on the Moon and Mars.

For all their complexity and variety, from mundane applications to the cutting edge, the heart of a load cell is the predictable deformation of a structural member, according to the simple formulas we have studied in this book. Such is the breath and importance of mechanics of materials.

*9.6 CONCEPT CONNECTOR

The history of strain gages is interesting in its own right. As we see in Section 9.6.1, however, it also heralds the pitfalls for modern universities in maintaining the delicate balance between pure research for knowledge and its potential commercial benefits. Section 9.6.2 then looks ahead at how strain gage resistance is measured using a Wheatstone bridge.

9.6.1 History: Strain Gages

Two Americans invented the strain gage nearly simultaneously. In 1938 Arthur C. Ruge of the Massachusetts Institute of Technology (MIT) wanted to measure low-level strains in an elevated thin-walled water tank during an earthquake. He solved this problem by inventing the strain gage. When Ruge sought to register his invention with the MIT patent committee in 1939, the committee felt that the invention was unlikely to have significant commercial use and released the invention to him. Around the same time, Edward E. Simmons, then a graduate student at the California Institute of Technology, was studying the stress–strain characteristics of metals during impact. He invented the strain gage independently, as part of a dynamometer for measuring the power of impact. Caltech and Simmons waged a legal battle for the rights to the patent, but Simmons won because, as a student, he was not a salaried employee. Ruge and Simmons subsequently resolved their patent claims to each one's satisfaction.

Today strain gages are the most popular strain-measuring devices. Strain gages are also used in applications involving measurements or control of forces and moments. Pressure transducers, force transducers, torque transducers, load cells, and dynamometers are all examples of industrial applications of strain gages, whereas a bathroom scale is an example of a household product using strain gages. The popularity of strain gages comes from their cost-effectiveness in measuring strains as small as 1 μmm/mm to strains as large as 50,000 μmm/mm over a large range of temperatures.

The sensitivity of a strain gage is called the *gage factor,* which is the ratio of percentage change in resistance to percentage change in length (strain). Metal foil gages have gage factors of between 2 and 4. Ideally we would like a linear relationship between changes in resistance to strain—in other words, a constant gage factor over the range of measurements. To keep the value as close as possible to a constant, strain gages are constructed with different materials for different applications. The most common material is constantan or Advance, an alloy of copper (55%) and nickel (45%). The thermal conductivity of the two metals is such that the gage does not undergo significant thermal expansion over a large range of temperatures (–75°C to 175°C); the gage is thus said to be *self-temperature-compensated*. Annealed constantan is useful in large strain measurements (as high as 20%). For high-temperature applications, an alloy of iron (70%), chromium (20%), and aluminum (10%), called Armour D, is used. Strain gages using semiconductors (doped silicon wafers) have gage factors of between 50 to 200 and are used for small-strain measurements, but they require extreme care during installation because of the brittle nature of the silicon wafers.

9.6.2 Wheatstone Bridge Application to Strain Gages

Early strain gages were built by taking a very thin wire and going back and forth a number of times over a small area. This construction technique is based on the observation that the resistance R of a wire is related to its length L, its cross-sectional area A, and the material-specific resistance ρ by the expression $R = \rho L/A$. For a given value of strain, a longer wire results in a larger change in L, and hence a larger change in the resistance, which can be measured more easily. At the same time, the small cross-sectional area reduces the transverse effect of Poisson's ratio. Winding the long wire in a small region therefore leads to a better average strain value. Though the idea of using a long thin conductor in a small region still dictates the design of modern strain gages, the manufacturing process has changed. *Photoetching*, in which material is removed chemically to produce a desired pattern, has replaced winding a wire.

By measuring the change in resistance and knowing the gage factor, one can find the strain from a strain gage. The most common means of measuring changes in resistance is the Wheatstone bridge circuit, shown in Figure 9.34. The bridge was invented by Samuel Hunter Christie in 1833 and made popular by Charles Wheatstone in 1843.

The voltage V_0 in Figure 9.34 can be related to V as follows:

$$V_0 = V\frac{R_1R_3 - R_2R_4}{(R_1 + R_2)(R_3 + R_4)}$$

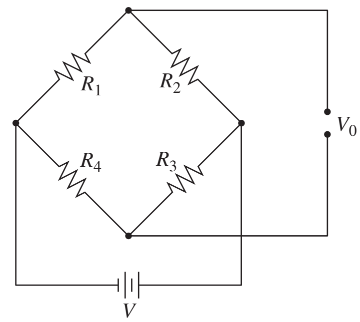


Figure 9.34 Wheatstone bridge circuit.

Clearly, if $R_1R_3 = R_2R_4$ then the voltage V_0 is be zero, and the bridge is said to be *balanced*. Suppose that one of the resistor is a strain gage—say, R_1. Before the material is loaded (and strained) the bridge is balanced. When the load is applied, the resistance R_1 changes. By adjusting the values of the other resistances by a known amount, we can again balance the bridge, and from $R_1R_3 = R_2R_4$ the resistance of R_1 can again be found. The strain can then be calculated from the change in R_1. A Wheatstone bridge is so important in strain measurements because it is sensitive to very small changes in resistance. And since we need to use only one of the resistances to balance the bridge, strains due to different causes can be separated by creative combinations of two or more gages.

9.7 CHAPTER CONNECTOR

In this chapter we studied the relationship of strains in different coordinate systems, and we found methods to determine the maximum normal strains and maximum shear strains. We noted that the principal axes form an orthogonal coordinate system. Hence we can determine the principal stresses from the principal strains by using the generalized Hooke's law. These principal stresses will be used in Chapter 10 to determine whether a material would fail.

We also learned about strain gages as a means of measuring strains at a point on a material. In Chapters 4 through 7 we studied one-dimensional structural elements and developed theories that let us compute the strains in an x, y, z coordinate system that an applied load produces. From these predicted strains, we are able to determine what a strain gage will record at any orientation. This same idea, of relating external loads to the reading of a strain gage, can be used in monitoring and controlling the applied forces and moments on a structure.

POINTS AND FORMULAS TO REMEMBER

- Strain transformation equations relate strains *at a point* in different coordinate systems:

$$\varepsilon_{nn} = \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta \quad (9.4)$$

$$\gamma_{nt} = -2\varepsilon_{xx}\sin\theta\cos\theta + 2\varepsilon_{yy}\sin\theta\cos\theta + \gamma_{xy}(\cos^2\theta - \sin^2\theta) \quad (9.6)$$

- Directions of the principal coordinates are the axes in which the shear strain is zero.
- Normal strains in principal directions are called *principal strains.*
- The greatest principal strain is called *principal strain 1.*
- The angles the principal axis makes with the global coordinate system are called *principal angles.*
- The angle of principal axis 1 from the *x* axis is only reported in describing the principal coordinate system in two-dimensional problems. Counterclockwise rotation from the *x* axis is defined as positive.
- Principal directions are orthogonal.
- Maximum and minimum normal strains at a point are the principal strains.
- The maximum shear strain in coordinate systems that can be obtained by rotating about one of the three axes (usually the *z* axis) is called *in-plane maximum shear strain*.
- The maximum shear strain at a point is the absolute maximum shear strain that can be obtained in a coordinate system by considering rotation about all three axes.
- Maximum shear strain exists in two coordinate systems that are 45° to the principal coordinate system.

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}} \quad (9.7) \qquad \varepsilon_{1,2} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} \pm \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \quad (9.8) \qquad \left|\frac{\gamma_p}{2}\right| = \left|\frac{\varepsilon_1 - \varepsilon_2}{2}\right| \quad (9.11)$$

- where θ_p is the angle to either principal plane 1 or 2, ε_1 and ε_2 are the principal stresses, γ_p is the in-plane maximum shear stress.
- $$\varepsilon_{nn} + \varepsilon_{tt} = \varepsilon_{xx} + \varepsilon_{yy} = \varepsilon_1 + \varepsilon_2 \quad (9.9)$$
- $$\varepsilon_3 = \begin{cases} 0, & \text{plane strain} \\ -\dfrac{\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy}) & \text{plane stress} \end{cases} \quad (9.10) \qquad \frac{\gamma_{max}}{2} = \left|\max\left(\frac{\varepsilon_1 - \varepsilon_2}{2}, \frac{\varepsilon_2 - \varepsilon_3}{2}, \frac{\varepsilon_3 - \varepsilon_1}{2}\right)\right| \quad (9.12)$$
- Each point on Mohr's circle represents a unique direction passing through the point at which the strains are specified. The coordinates of each point on the circle are the strains (ε_{nn}, $\gamma_{nt}/2$).
- The maximum shear strain at a point is the radius of the biggest of the three circles that can be drawn between the three principal strains.
- The principal directions for stresses and strains are the same for isotropic materials.
- Generalized Hooke's law in principal coordinates:
- $$\varepsilon_1 = \frac{\sigma_1 - \nu(\sigma_2 + \sigma_3)}{E} \quad (9.14.a) \qquad \varepsilon_2 = \frac{\sigma_2 - \nu(\sigma_3 + \sigma_1)}{E} \quad (9.14.b) \qquad \varepsilon_3 = \frac{\sigma_3 - \nu(\sigma_1 + \sigma_2)}{E} \quad (9.14.c)$$
- Strain gages measure only normal strains directly.
- Strain gages are bonded to a free surface, i.e., the strains are in a state of plane stress and not plane strain.
- Strain gages measure average strain at a point.
- The change in strain gage orientation by ±180° makes no difference to the strain values.