CHAPTER TEN
DESIGN AND FAILURE

Learning objectives

1. Learn the computation of stresses and strains on a structural member under combined axial, torsion, and bending loads.
2. Develop the design and analysis skills for structures constructed from one-dimensional members.

In countless engineering applications, the structural members are subjected a combination of loads. The propeller on a boat (Figure 10.1*a*) subjects the shaft to an *axial force* as it pushes the water backward, but also a *torsional load* as it turns through the water. Gravity subjects the Washington Monument (Figure 10.1*b*) to a *distributed axial load*, while the wind pressure of a storm subjects the monument to *bending loads*. In still other cases, we have to take into account that a structure is composed of more than one member. For example, wind pressure on a highway sign (Figure 10.1*c*) subjects the base of the sign to both *bending* and *torsional loads*. This chapter synthesizes and applies the concepts developed in the previous nine chapters to the design of structures subjected to combined loading.

(a) (b) (c)

Figure 10.1 Examples of combined loadings.

10.1 COMBINED LOADING

We have developed separately the theories for axial members (Section 4.2), for the torsion of circular shafts (Section 5.2), and for symmetric bending about the *z* axis (Section 6.2). All these are linear theories, which means that the superposition principle applies. In many problems a structural member is subject simultaneously to axial, torsional, and bending loads. The solution to the combined loading problems thus involves a superposition of stresses and strains at a point.

Equations (10.1), (10.2), (10.3a), and (10.3b), listed here for convenience as Table 10.1 summarizes the stress formulas derived in earlier chapters. Equations (10.4a) and (10.4b) extend of the formulas for symmetric bending about the *z* axis [Equations (10.3a), and (10.3b)] to symmetric bending about the *y* axis as we shall see in Section 10.1.3.

TABLE 10.1 **Stresses and strains in one-dimensional structural members**

	Stresses		Strains
Axial	$\sigma_{xx} = \dfrac{N}{A}$	(10.1)	$\varepsilon_{xx} = \dfrac{\sigma_{xx}}{E}$ $\varepsilon_{yy} = -\dfrac{\nu\sigma_{xx}}{E}$ $\varepsilon_{zz} = -\dfrac{\nu\sigma_{xx}}{E}$
	$\sigma_{yy} = 0$ $\sigma_{zz} = 0$		$\gamma_{xy} = 0$ $\gamma_{yz} = 0$ $\gamma_{xz} = 0$
	$\tau_{xy} = 0$ $\tau_{yz} = 0$ $\tau_{xz} = 0$		
Torsion	$\tau_{x\theta} = \dfrac{T\rho}{J}$	(10.2)	$\gamma_{x\theta} = \dfrac{\tau_{x\theta}}{G}$
	$\sigma_{xx} = 0$ $\sigma_{yy} = 0$ $\sigma_{zz} = 0$		$\varepsilon_{xx} = 0$ $\varepsilon_{yy} = 0$ $\varepsilon_{zz} = 0$
	$\tau_{yz} = 0$		$\gamma_{yz} = 0$
Symmetric bending about *z* axis	$\sigma_{xx} = -\dfrac{M_z y}{I_{zz}}$	(10.3a)	$\varepsilon_{xx} = \dfrac{\sigma_{xx}}{E}$ $\varepsilon_{yy} = -\dfrac{\nu\sigma_{xx}}{E}$ $\varepsilon_{zz} = -\dfrac{\nu\sigma_{xx}}{E}$
	$\tau_{xs} = -\dfrac{V_y Q_z}{I_{zz} t}$	(10.3b)	$\gamma_{xs} = \dfrac{\tau_{xs}}{G}$
	$\sigma_{yy} = 0$ $\sigma_{zz} = 0$ $\tau_{yz} = 0$		$\gamma_{yz} = 0$
Symmetric bending about *y* axis	$\sigma_{xx} = -\dfrac{M_y z}{I_{yy}}$	(10.4a)	$\varepsilon_{xx} = \dfrac{\sigma_{xx}}{E}$ $\varepsilon_{yy} = -\dfrac{\nu\sigma_{xx}}{E}$ $\varepsilon_{zz} = -\dfrac{\nu\sigma_{xx}}{E}$
	$\tau_{xs} = -\dfrac{V_z Q_y}{I_{yy} t}$	(10.4b)	$\gamma_{xs} = \dfrac{\tau_{xs}}{G}$
	$\sigma_{yy} = 0$ $\sigma_{zz} = 0$ $\tau_{yz} = 0$		$\gamma_{yz} = 0$

To understand the principal of superposition for stresses, consider a thin hollow cylinder (Figure 10.2) subjected to combined axial, torsional, and bending loads. We first draw the stress cubes at four points *A*, *B*, *C*, and *D*. The stress direction on the stress cube can then be determined by inspection or using subscripts (as in Sections 5.2.5, 6.2.5, 6.6.1, and 6.6.3). The magnitude of the stress components follows from the formulas in Table 10.1.

We will use the following notation for the magnitude of the stress components:

- σ_{axial}—axial normal stress.
- $\sigma_{bend\text{-}y}$—normal stress due to bending about y axis.
- $\sigma_{bend\text{-}z}$—normal stress due to bending about z axis.
- τ_{tor}—torsional shear stress. (10.5)
- $\tau_{bend\text{-}y}$—shear stress due to bending about y axis.
- $\tau_{bend\text{-}z}$—shear stress due to bending about z axis.

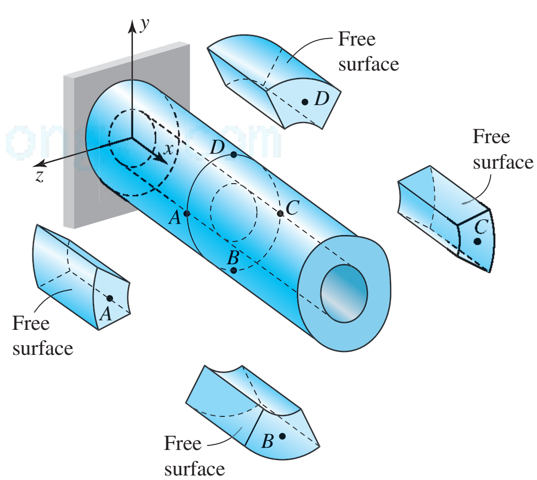


Figure 10.2 Thin hollow cylinder.

Because the surface of the shaft is a free surface, it is stress free. Hence, irrespective of the loading, no stresses act on this surface at the four points *A*, *B*, *C*, and *D* in Figure 10.2. The free surfaces at points *B* and *D* have outward normals in the *y* direction. Recall that the first subscript in each stress component is the direction of the outward normal to the surface on which the stress component acts. Thus $\tau_{yx,}$ which acts on this surface, has to be zero. Since $\tau_{xy} = \tau_{yx}$, it follows that τ_{xy} at points B and D will be zero *irrespective* of the loading. Similarly, the free surfaces at points A and C have outward normals in the z direction, and hence $\tau_{zx} = 0$. Thus, τ_{xz} is also zero at these points, irrespective of the loading.

10.1.1 Combined Axial and Torsional Loading

Figure 10.3 show the axial and torsional stresses on stress cubes at points A, B, C, and D due to individual loads. When both axial and torsional loads are present together, we do not simply add the two stress components. Rather we superpose or add the two *stress states.*

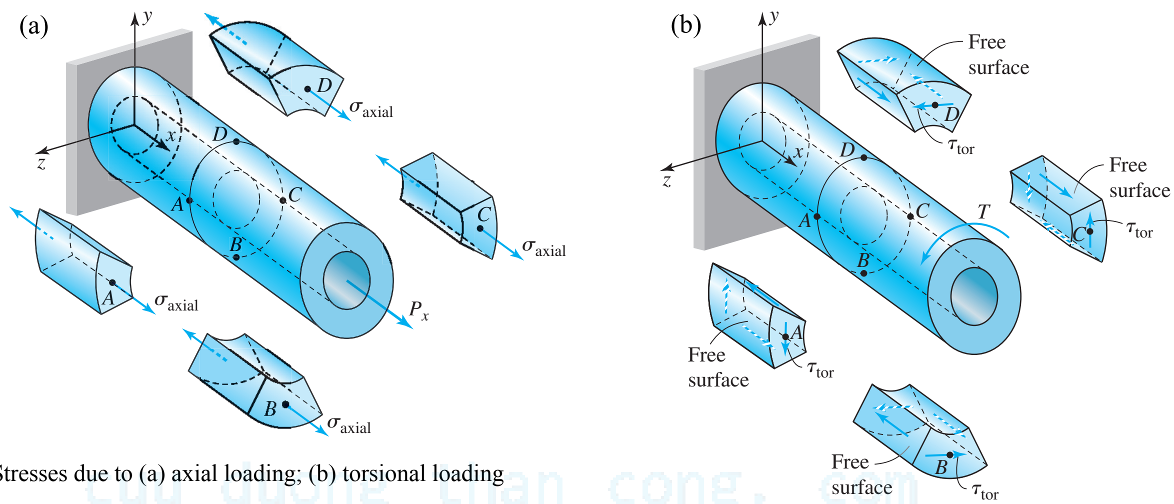


Figure 10.3 Stresses due to (a) axial loading; (b) torsional loading

What do we mean by superposing the stress states? To answer the question, consider two stress components σ_{xx} and τ_{xy} at point C. In axial loading, $\sigma_{xx} = \sigma_{axial}$ and $\tau_{xy} = 0$; in torsional loading $\sigma_{xx} = 0$ and $\tau_{xy} = \tau_{tor}$. When we add (or subtract), we add (or subtract) the same component in each loading. Hence, the total state of stress at point C is $\sigma_{xx} = \sigma_{axial} + 0 = \sigma_{axial}$ and $\tau_{xy} = 0 + \tau_{tor} = \tau_{tor}$. The state of stress at point C in combined loading (Figure 10.4) is thus very different from the states of stress in individual loadings (Figures 10.3*a* and *b*). Think how different is the Mohr's circle associated with the state of stress at point C in Figure 10.4 with those associated in Figures 10.3*a* and *b*. Example 10.1 further elaborates the differences in stress states and associated Mohr circle.

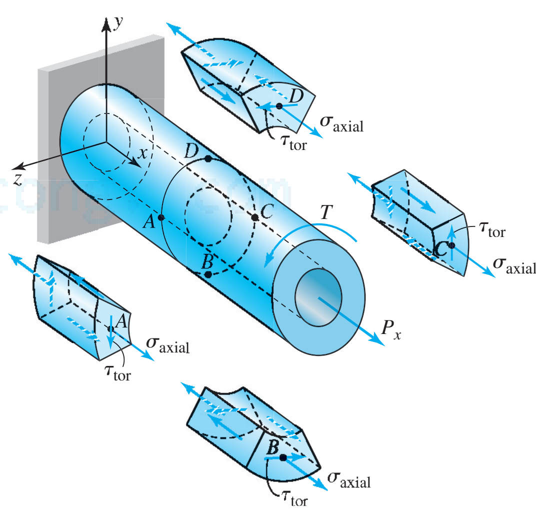


Figure 10.4 Stresses in combined axial and torsional loading.

10.1.2 Combined Axial, Torsional, and Bending Loads about *z* Axis

Figure 10.5*a* shows the thin hollow cylinder subjected to a load that bends the cylinder about the *z* axis. Points *B* and *D* are on the free surface. Hence the bending shear stress is zero at these points. Points *A* and *C* are on the neutral axis, and hence the bending normal stress is zero at these points. The nonzero stress components can be found from the formulas in Table 10.1, as shown on the stress cubes in Figure 10.5*a*. If we superpose the stress states for bending at the four points shown in Figure 10.5*a* and the stress states for the combined axial and torsional loads at the same points shown in Figure 10.4, we obtain the stress states shown in Figure 10.5*b*.

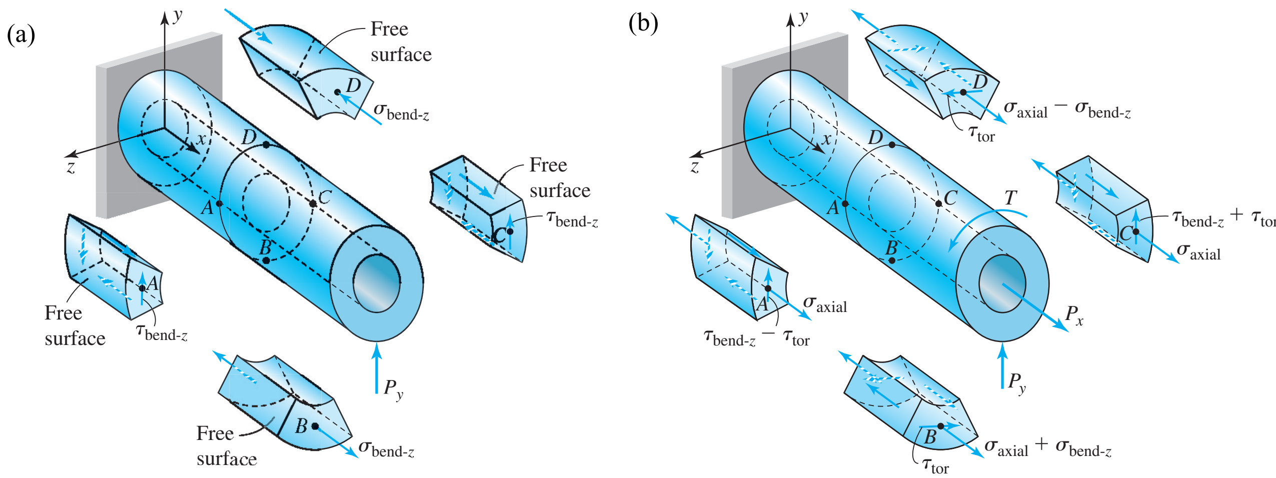


Figure 10.5 Stresses due to (a) bending about *z* axis; (b) Combined axial, torsional, and bending about *z* axis

In Figure 10.5*a*, the bending normal stress at point *D* is compressive, whereas the axial stress in Figure 10.4 is tensile. Thus, the resultant normal stress σ_{xx} is the difference between the two stress values, as shown in Figure 10.5*b*. At point *B* both the bending normal stress and the axial stress are tensile, and thus the resultant normal stress σ_{xx} is the sum of the two stress values. If the axial normal stress at point *D* is greater than the bending normal stress, then the total normal stress at point *D* will be in the direction as shown in Figure 10.5*b*. If the bending normal stress is greater than the axial stress, then the total normal stress will be compressive and would be shown in the opposite direction in Figure 10.5*b*.

At point *A* the torsional shear stress in Figure 10.4 is downward, whereas the bending shear stress in Figure 10.5*a* is upward. Thus, the resultant shear stress τ_{xy} is the difference between the two stress values, as shown in Figure 10.5*b*. At point *C* both the torsional shear stress and the bending shear stress are upward, and thus the resultant shear stress τ_{xy} is the sum of the two stress values. If the bending shear stress at point *A* is greater than the torsional shear stress, then the total shear stress at point *A* will be in the direction of positive τ_{xy}, as shown in Figure 10.5*b*. If the torsional shear stress is greater than the bending shear stress, then the total shear stress will be negative τ_{xy} and will be in the opposite direction in Figure 10.5*b*.

10.1.3 Extension to Symmetric Bending about *y* Axis

Before we combine the stresses due to bending about the *y* axis, consider the extension of the formulas derived for symmetric bending about the *z* axis. Assume that the *xz* plane is *also* a plane of symmetry, so that the loads lie in the plane of symmetry. Equations (10.4*a*) and (10.4*b*) for bending about the *y* axis can be obtained by interchanging the subscripts *y* and *z* in Equations (10.3*a*) and (10.3*b*). The sign conventions for the internal moment ***My*** and the shear force V_z in Equations (10.4*a*) and (10.4*b*) are then simple extensions of M_z and V_y, as shown in Figure 10.6.

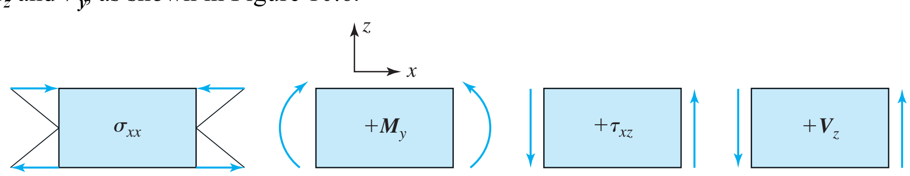


Figure 10.6 Sign convention for internal bending moments and shear force in bending about *y* axis.

Sign Convention: The positive internal moment M_y on a free-body diagram must be such that it puts a point in the positive *z* direction into compression.

Sign Convention: The positive internal shear force V_z on a free-body diagram is in the direction of positive shear stress τ_{xz} on the surface.

The direction of shear stress in Equation (10.4*b*) can be determined either by using the subscripts or by inspection, as we did for symmetric bending about the *z* axis. To use the subscripts, recall that the *s* coordinate is defined from the free surface (see Section 6.6.1) used in the calculation of Q_y. The shear flow (or shear stress) due to bending about the *y* axis only is drawn along the centerline of the cross section. Its direction must satisfy the following rules:

1. The resultant force in the *z* direction is in the same direction as V_z.
2. The resultant force in the *y* direction is zero.
3. It is symmetric about the *z* axis. This requires that shear flow change direction as one crosses the *y* axis on the centerline. Sometimes this will imply that shear stress is zero at points where the centerline intersects the *z* axis.

10.1.4 Combined Axial, Torsional, and Bending Loads about *y* and *z* Axes

Figure 10.7*a* shows the thin hollow cylinder subjected to a load that bends the cylinder about the *y* axis. Points *A* and *C* are on the free surface, and hence bending shear stress is zero at these points. Points *B* and *D* are on the neutral axis, and hence the bending normal stress is zero at these points. The nonzero stress components can be found from the formulas in Table 10.1, as shown on the stress cubes in Figure 10.7*a*. If we superpose the stress states for bending at the four points shown in Figure 10.5*a* add the stress states for the combined axial and torsional loads at the same points shown in Figure 10.5*b,* we obtain the stress states shown in Figure 10.7*b*.

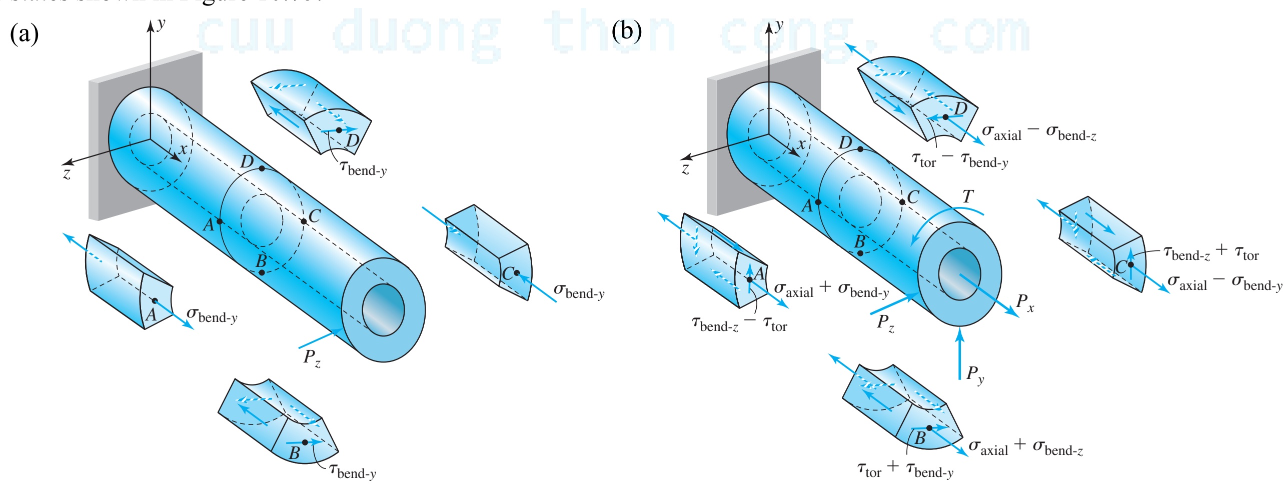


Figure 10.7 Stresses due to (a) bending about *y* axis; (b) combined axial, torsional, and bending about *y* and *z* axis.

Thus the complex stress states shown in Figure 10.7*b* can be obtained by first calculating the stresses due to individual loadings. We then simply superpose the stress states at each point.

10.1.5 Stress and Strain Transformation

To obtain strains in combined loading, we can superpose the strains given in Table 10.1. Alternatively, we can superpose the stresses, as discussed in the preceding sections and then use the generalized Hooke's law to convert these stresses to strains. The second approach is often preferable, because we may need to transform torsional shear stress $\tau_{x\theta}$ (see Section 5.2.5) and bending shear stress τ_{xs} (see Section 6.6.6) into the *x*, *y*, *z* coordinate system. (Remember that our stress and strain transformation equations were developed in the cartesian coordinates.) In Figure 10.7*b*, at points *A* and *C* the shear stress shown is positive τ_{xy}, at point *B* the shear stress shown is negative τ_{xz}, and at point *D* the shear stress shown is positive τ_{xz}. In general, it is important to show the stresses on a stress element before proceeding to stress or strain transformation.

In studying individual loading, we often had prefixes to stresses such as maximum axial normal stress, maximum torsional shear stress, maximum bending normal stress, maximum bending shear stress, or maximum in-plane shear stress. In this chapter, however, we are considering combined loading. Hence the *maximum normal stress* at a point will refer to the *principal stress* at the point, and the *maximum shear stress* will refer to the *absolute maximum shear stress*. This implies that *allowable normal stress* refers to the *principal stresses* and *allowable shear stress* refers to the *absolute maximum shear stress*. The allowable tensile normal stress refers to principal stress 1, assuming it is tensile. The allowable compressive normal stress refers to principal stress 2, assuming it is compressive.

10.1.6 Summary of Important Points in Combined Loading

We can now summarize the points to keep in mind when solving problems involving combined loading.

1. The problem of stress under combined loading can be simplified by first determining the states of stress due to individual loadings.
2. The superposition principle applies to stresses at a given point. That is, a stress component resulting from one loading can be added to or subtracted from a similar stress component from another loading. Stress components at *different* points cannot be added or subtracted. Neither can stress components that act on different planes or in different directions.
3. The stress formulas in Table 10.1 give the magnitude and the direction for each stress component, but only if the internal forces and moments are drawn on the free-body diagrams according to the prescribed sign conventions. If the directions of internal forces and moments are instead drawn so as to equilibrate external forces and moments, then the directions of the stress components must be determined by inspection.
4. In a given structure, the structural members may have different orientations. In using subscripts to determine the direction and signs of stress components, we therefore establish a local x, y, z coordinate system for each structural member such that the x direction is normal to the cross section. That is, the x direction is along the axis of the structural member.
5. Table 10.1 shows that stresses σ_{yy} and σ_{zz} are zero for the four cases listed, emphasizing that the theories are for one-dimensional structural members. Additional stress components are zero at free surfaces.
6. The state of stress in combined loading should be shown on a stress cube before applying stress or strain transformation.
7. The strains at a point can be obtained from the superposed stress values using the generalized Hooke's law. Since the normal stresses σ_{yy} and σ_{zz} are always zero in our structural members, the nonzero strains ε_{yy} and ε_{zz} are due to the Poisson effect; that is, $\varepsilon_{yy} = \varepsilon_{zz} = -\nu\varepsilon_{xx}$.

10.1.7 General Procedure for Combined Loading

A general procedure for calculating stresses in combined loading is as follows:

Step 1: Identify the equations in Table 10.1 relevant for the problem, and use the equations as a checklist for the quantities that must be calculated.

Step 2: Calculate the relevant geometric properties (A, I_{yy}, I_{zz}, J) of the cross section containing the points where stresses have to be found.

Step 3: At points where shear stress due to bending is to be found, draw a line perpendicular to the centerline through the point and calculate the first moments of the area (Q_y, Q_z) between the free surface and the drawn line. Record the s direction from the free surface toward the point where the stress is being calculated.

Step 4: Make an imaginary cut through the cross section and draw the free-body diagram. If subscripts are to be used in determining the directions of the stress components, draw the internal forces and moments according to our sign conventions. Use equilibrium equations to calculate the internal forces and moments.

Step 5: Using the equations identified in Step 1, calculate the individual stress components due to each loading. Draw the torsional shear stress $\tau_{x\theta}$ and the bending shear stress τ_{xs} on a stress cube using subscripts or by inspection. By examining the shear stresses in the x, y, z coordinate system, obtain τ_{xy} and τ_{xz} with proper signs.

Step 6: Superpose the stress components to obtain the total stress components at a point.

Step 7: Show the calculated stresses on a stress cube.

Step 8: Interpret the stresses shown on the stress cube in the x, y, z coordinate system before processing these stresses for the purpose of stress or strain transformation.

EXAMPLE 10.1

A hollow shaft that has an outside diameter of 100 mm, and an inside diameter of 50 mm is loaded as shown in Figure 10.8. For the three cases shown, determine the principal stresses and the maximum shear stress at point A. Point A is on the surface of the shaft.

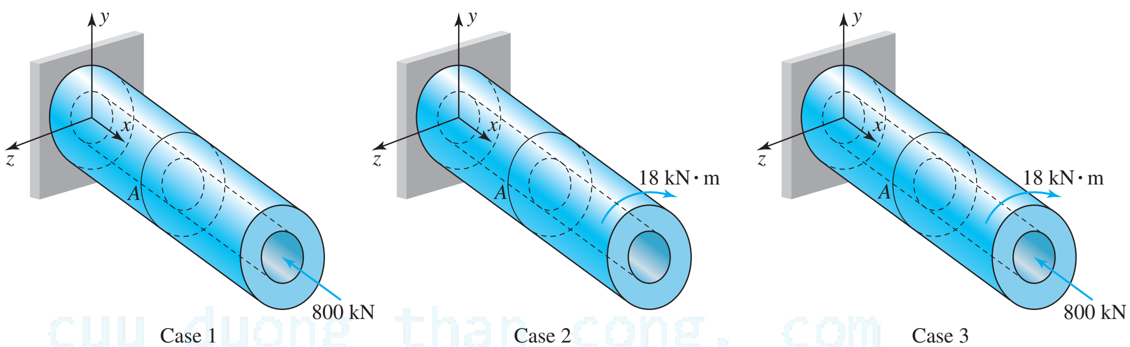


Figure 10.8 Hollow cylinder in Example 10.1.

PLAN

The axial normal stress in case 1 can be found from Equation 10.1. The torsional shear stress in case 2 can be found from Equation 10.2. The state of stress in case 3 is the superposition of the stress states in cases 1 and 2. The calculated stresses at point A can be drawn on a stress cube. Using Mohr's circle or the method of equations, we can find the principal stresses and the maximum shear stress in each case.

SOLUTION

Step 1: Equations (10.1) and (10.2) are used for calculating the axial stress and the torsional shear stress.

Step 2: The cross-sectional area A and the polar area moment J of a cross section can be found as

$$A = \frac{\pi}{4}[(100\text{ mm})^2 - (50\text{ mm})^2] = 5.89(10^3)\text{ mm}^2 \qquad J = \frac{\pi}{32}[(100\text{ mm})^4 - (50\text{ mm})^4] = 9.20(10^6)\text{ mm}^4 \tag{E1}$$

Step 3: This step is not needed as there is no bending.

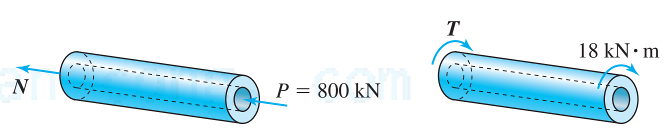


Figure 10.9 Free-body diagrams in Example 10.1.

Step 4: We draw the free-body diagrams in Figure 10.9 after making imaginary cuts. The internal axial force and the internal torque are drawn according to our sign convention. By equilibrium we obtain

$$\boldsymbol{N} = -800\text{ kN} \qquad \boldsymbol{T} = -18\text{ kN}\cdot\text{m} \tag{E2}$$

Step 5:

Case 1: The axial stress is uniform across the cross section and can be found from Equation 10.1,

$$\sigma_{xx} = \frac{N}{A} = \frac{-800(10^3)\text{ N}}{5.89(10^{-3})\text{ m}^2} = -135.8(10^6)\text{ N/m}^2 = -135.8\text{ MPa} \tag{E3}$$

Case 2: The torsional shear stress varies linearly and is maximum on the surface ($\rho = 0.05$ m) of the shaft. It can be found from Equation 10.2,

$$\tau_{x\theta} = \frac{T\rho}{J} = \frac{[-18(10^3)\ \text{N}\cdot\text{m}](0.05\ \text{m})}{9.20(10^{-6})\ \text{m}^4} = -97.83(10^6)\ \text{N/m}^2 = -97.83\ \text{MPa} \tag{E4}$$

Steps 6, 7: We draw the stress cube and show the stresses calculated in Equations (E3) and (E4).

Case 1: The axial stress is compressive, as shown Figure 10.10*a*.

Case 2: From Equation (E4) we note that $\tau_{x\theta}$ is negative. The θ direction in positive counterclockwise with respect to the x axis, as shown in Figure 10.10*b*. At point *A* the outward normal to the surface is in the positive x direction and the positive θ direction at *A* is downward. Hence a negative $\tau_{x\theta}$ will be upward at point *A*, as shown in Figure 10.10*b*.

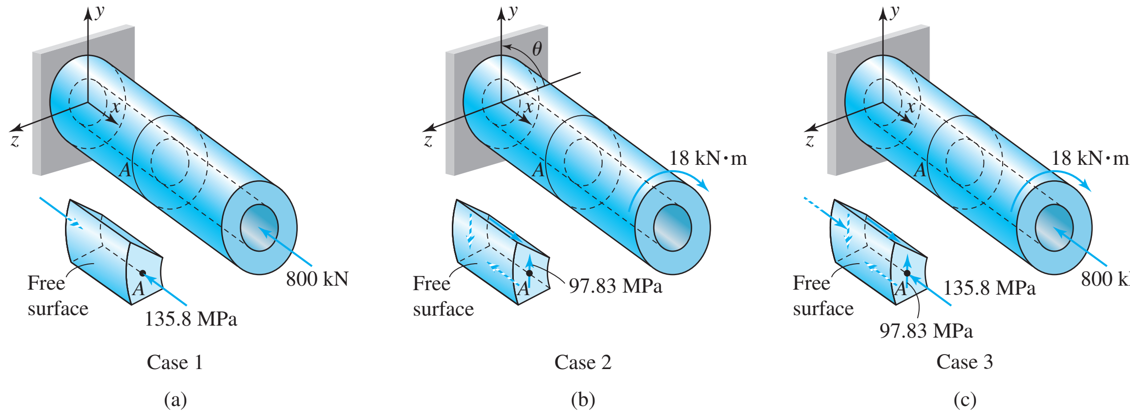


Figure 10.10 Stresses on stress cubes in Example 10.1.

Intuitive check: Figure 10.11 shows the hollow shaft with the applied torque on the right end and the reaction torque at the wall on the left end. The left part of the shaft would rotate counterclockwise with respect to the right part. Thus the surface of the cube at point *A* would be moving downward. The shear stress would oppose this impending motion by acting upward at point *A*, as shown in Figure 10.11, confirming the direction shown in Figure 10.10*b*.

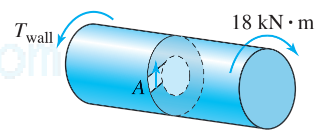


Figure 10.11 Direction of shear stress by inspection.

Case 3: The state of stress is a superposition of the states of stress shown on the stress cubes for cases 1 and 2 and is illustrated in Figure 10.10*c*.

Step 8: We can redraw the stress cubes in two dimensions and follow the procedure for constructing Mohr's circle for each case, as shown in Figure 10.12. The radius of the Mohr's circle can be found and the principal stresses and maximum shear stress calculated.

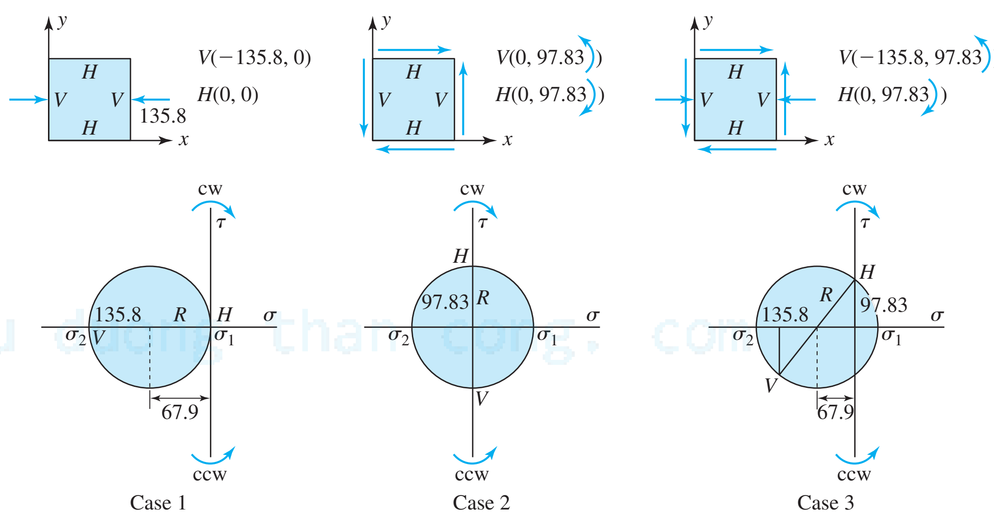


Figure 10.12 Mohr's circles in Example 10.1.

- *Case 1:* $R = 67.9\ \text{MPa}$.

ANS. $\sigma_1 = 0$ $\sigma_2 = 135.8\ \text{MPa (C)}$ $\sigma_3 = 0$ $\tau_{max} = 67.9\ \text{MPa}$

- *Case 2:* $R = 97.83\ \text{MPa}$.

ANS. $\sigma_1 = 97.8\ \text{MPa (T)}$ $\sigma_2 = 97.8\ \text{MPa (C)}$ $\sigma_3 = 0$ $\tau_{max} = 97.8\ \text{MPa}$

• *Case 3:* $R = \sqrt{(67.9\ \text{MPa})^2 + (97.83\ \text{MPa})^2} = 119.1$, thus $\sigma_{1,2} = -67.9\ \text{MPa} \pm 119.1\ \text{MPa}$

ANS. $\sigma_1 = 51.2\ \text{MPa (T)}$ $\sigma_2 = 187\ \text{MPa (C)}$ $\sigma_3 = 0$ $\tau_{max} = 119.1\ \text{MPa}$

COMMENTS

1. The results for the three cases show that the principal stresses and the maximum shear stress for case 3 cannot be obtained by superposition of the principal stresses and the maximum shear stress calculated for cases 1 and 2. Figure 10.12 emphasizes this graphically. Mohr's circle of case 3 cannot be obtained by superposing Mohr's circle for cases 1 and 2. The superposition principle is not applicable to principal stresses because the *principal planes* for the three cases are different. We cannot add (or subtract) stresses on different planes. If we had calculated the stresses for the three cases on the same plane, then we could apply the superposition principle.
2. Substituting $\sigma_{xx} = -135.8$ MPa, $\tau_{xy} = +97.8$ MPa, and $\sigma_{yy} = 0$ into Equation (8.7), we can find σ_1 and σ_2 for case 3

$$\sigma_{1,2} = \frac{(-135.8\ \text{MPa}) + 0}{2} \pm \sqrt{\left(\frac{-135.8\ \text{MPa}}{2}\right)^2 + (97.8\ \text{MPa})^2} = -67.9\ \text{MPa} \pm 119.1\ \text{MPa} \quad \text{(E5)}$$

Noting that $\sigma_3 = 0$, we can find τ_{max} from Equation (8.13),

$$\tau_{max} = \left|\max\left(\frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_2 - \sigma_3}{2}, \frac{\sigma_3 - \sigma_1}{2}\right)\right| \quad \text{(E6)}$$

The results of Equations (E5) and (E6) are same as those obtained from the Mohr's circle.

EXAMPLE 10.2

A hollow shaft has an outside diameter of 100 mm and an inside diameter of 50 mm, is shown in Figure 10.13. Strain gages are mounted on the surface of the shaft at 30° to the axis. For each case determine the applied axial load P and the applied torque T_{ext} if the strain gage readings are $\varepsilon_a = -500\ \mu$ and $\varepsilon_b = 400\ \mu$. Use $E = 200$ GPa, $G = 80$ GPa, and $\nu = 0.25$.

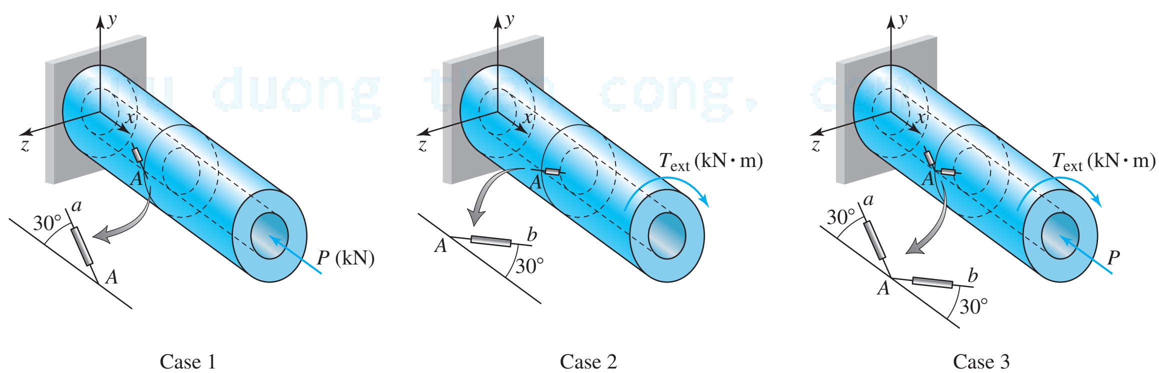


Figure 10.13 Hollow cylinder in Example 10.2.

PLAN

The stresses at point A in terms of P and T_{ext} can be found as in Example 10.1. Using the generalized Hooke's law, we can find the strains in terms of P and T_{ext}. From the strain transformation equation, Equation (9.4), the normal strain in direction of the strain gage can be found in terms of P and T_{ext}. The values of P and T_{ext} can be determined from the given strain gage readings.

SOLUTION

Step 1: Equations 10.1 and 10.2 will be used for calculating the axial stress and the torsional shear stress.
Step 2: From Example 10.1, the cross-sectional area A and the polar area moment J of a cross section are

$$A = 5.89(10^3)\ \text{mm}^2 \qquad J = 9.20(10^6)\ \text{mm}^4 \quad \text{(E1)}$$

Step 3: This step is not needed as there is no bending.

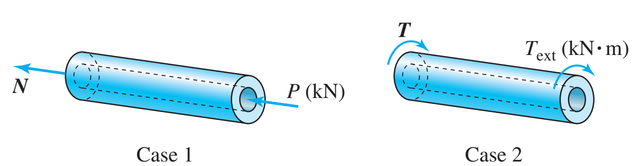


Figure 10.14 Free-body diagrams in Example 10.2.

Step 4: We make an imaginary cut and draw the free-body diagrams in Figure 10.14. By equilibrium we obtain

$$\boldsymbol{N} = -P\ \text{kN} \qquad \boldsymbol{T} = -T_{ext}\ \text{kN·m} \quad \text{(E2)}$$

Step 5:

Case 1: The axial stress is uniform across the cross section and can be found from Equation (10.1),

$$\sigma_{xx} = \frac{N}{A} = \frac{-P(10^3)\text{ N}}{5.89(10^{-3})\text{ m}^2} = -0.17P(10^6)\text{ N/m}^2 \quad \text{(E3)}$$

Case 2: The torsional shear stress on the surface ($\rho = 0.05$ m) of the shaft can be found from Equation (10.2),

$$\tau_{x\theta} = \frac{T\rho}{J} = \frac{[-T_{ext}(10^3)\text{ N}\cdot\text{m}](0.05\text{ m})}{9.20(10^{-6})\text{ m}^4} = -5.435T_{ext}(10^6)\text{ N/m}^2 \quad \text{(E4)}$$

Steps 6, 7: Figure 10.15 shows the stresses on the stress elements calculated using Equations (E4) and (E5), as in Example 10.1.

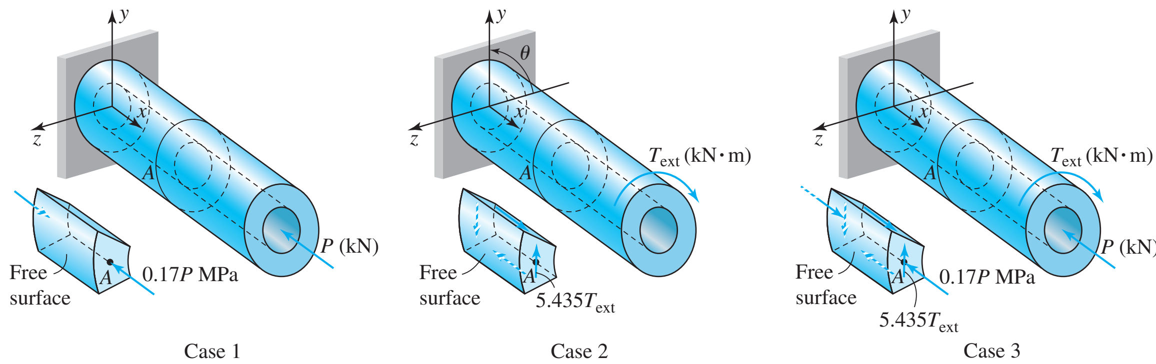


Figure 10.15 Stresses on stress cubes in Example 10.2.

Step 8:

Case 1: We note that the only nonzero stress is the axial stress given in Equation (E4). From the generalized Hooke's law we obtain the strains,

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} = \frac{-0.170P(10^6)\text{ N/m}^2}{200(10^9)\text{ N/m}^2} = -0.85P(10^{-6}) = -0.85P\ \mu \qquad \varepsilon_{yy} = -\nu\varepsilon_{xx} = -0.25(-0.85P\ \mu) = 0.213P\ \mu \quad \text{(E5)}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = 0 \quad \text{(E6)}$$

Case 2: From Figure 10.15 we note that the shear stress $\tau_{xy} = +5.435T_{ext}$. The normal stresses are all zero. From the generalized Hooke's law we obtain the strains,

$$\varepsilon_{xx} = 0 \qquad \varepsilon_{yy} = 0 \qquad \gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{5.435T_{ext}(10^6)\text{ N/m}^2}{80(10^9)\text{ N/m}^2} = 67.94T_{ext}\ \mu \quad (10.6)$$

Case 3: The state of strain is the superposition of the state of strain for cases 1 and 2,

$$\varepsilon_{xx} = -0.85P\ \mu \qquad \varepsilon_{yy} = 0.213P\ \mu \qquad \gamma_{xy} = 67.94T_{ext}\ \mu \quad \text{(E7)}$$

Load calculations

Case 1: Substituting $\theta_a = 150°$ or $-30°$ and ε_{xx}, ε_{yy}, and γ_{xy}, into the strain transformation equation, Equation (9.4), we can find the normal strain in terms of P and equated it to the given value of $\varepsilon_a = -500\ \mu$. The value of P can be found as

$$\varepsilon_a = (-0.85P\ \mu)\cos^2(-30^\circ) + (0.213P\ \mu)\sin^2(-30^\circ) = -500\ \mu \qquad \text{or} \qquad (-0.638\ \mu + 0.053\ \mu)P = -500\ \mu \quad \text{(E8)}$$

ANS. $P = 855$ kN

Case 2: Substituting $\theta_b = 30°$ and ε_{xx}, ε_{yy}, and γ_{xy}, into the strain transformation equation, Equation (9.4), we can find the normal strain in terms of T_{ext} and equate it to the given value of $\varepsilon_b = 400\ \mu$. The value of T_{ext} can be found as

$$\varepsilon_b = (67.94T_{ext}\ \mu)\sin(30^\circ)\cos(30^\circ) = 400\ \mu \qquad \text{or} \qquad 29.42\ \mu T_{ext} = 400\ \mu \quad \text{(E9)}$$

ANS. $T_{ext} = 13.6$ kN·m

Case 3: Substituting $\theta_a = -30°$, $\theta_b = 30$, and Equations (E12), (E13), and (E14) into the strain transformation equation, Equation (9.4), and using the given strain values, we obtain

$$\varepsilon_a = (-0.85P\ \mu)\cos^2(-30°) + (0.213P\ \mu)\sin^2(-30°) + (67.94T_{ext}\ \mu)\sin(-30°)\cos(-30°) = -500\mu \text{ or}$$

$$-0.585P - 29.42T_{ext} = -500 \quad \text{(E10)}$$

$$\varepsilon_b = (-0.85P\ \mu)\cos^2(30°) + (0.213P\ \mu)\sin^2(30°) + (67.94T_{ext}\ \mu)\sin(30°)\cos(30°) = 400\ \mu$$

$$-0.585P + 29.42T_{ext} = 400 \quad \text{(E11)}$$

Equations (E10) and (E11) can be solved simultaneously to obtain the result.

ANS. $P = 85.4\ \text{kN}$ $T_{ext} = 15.3\ \text{kN}\cdot\text{m}$

COMMENTS

1. The values of P and T_{ext} for combined loading are different than the values obtained for individual loadings. The next comment explains why.
2. If we had been given P and T_{ext} and were required to predict the strains in the gages, we could have calculated strains along the strain gage direction for individual loads and superposed to get the total strain in the gages for combined loading. But as the results in this example demonstrate, the strains in the gages (or the total strain) for combined loading cannot be separated into strain due to axial load and strain due to torsion. Loads P and T_{ext} affect both strain gages simultaneously, and these effects cannot be decoupled into effects of individual loadings.
3. In this example and the previous one we solved the problem by separating axial and torsion problems and calculated internal axial force and internal torque using separate free-body diagrams. We could have used a single free-body diagram, as shown in Figure 10.16, to calculate the internal quantities. In subsequent examples we shall construct a single free-body diagram for the calculation of the internal quantities,

$$N = -P\ \text{kN} \qquad T = -T_{ext}\ \text{kN}\cdot\text{m}$$

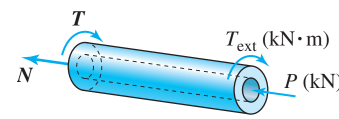


Figure 10.16 Single free-body diagram for combined loading.

This choice is not only less tedious but may be necessary. A single force may produce axial, torsion, and bending, which cannot be separated on a free-body diagram.

EXAMPLE 10.3

A box column is constructed from $\frac{1}{4}$-in.-thick sheet metal and subjected to the loads shown in Figure 10.17. (a) Determine the normal and shear stresses in the x, y, z coordinate system at points A and B and show the results on stress cubes. (b) A surface crack at point B is oriented as shown. Determine the normal and shear stresses on the plane containing the crack.

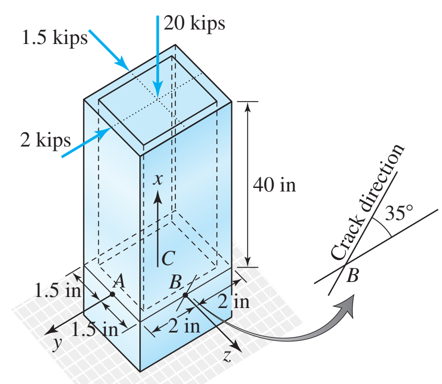


Figure 10.17 Beam and loading in Example 10.3.

PLAN

(a) We can follow the procedure in Section 10.1.7. The 20-kips force is an axial force, whereas the 2-kips and 1.5-kips forces produce bending about the z and y axes, respectively. Thus Equations 10.1, (10.3*a*), (10.3*b*), (10.4*a*), and (10.4*b*) will be used for calculating stresses. These formulas can be used as a checklist of the quantities that must be calculated in finding the individual stress components. By superposition the total stress at points A and B can be obtained. (b) Using the method of equations or Mohr's circle, the normal and shear stresses on the plane containing the crack can be found from the stresses determined at point B.

SOLUTION

Step 1: Equations 10.1, (10.3*a*), (10.3*b*), (10.4*a*), and (10.4*b*) will be used for calculating the stress components.
Step 2: The geometric properties of the cross section can be found as

$$A = (4\ \text{in.})(3\ \text{in.}) - (3.5\ \text{in.})(2.5\ \text{in.}) = 3.25\ \text{in.}^2 \qquad \text{(E1)}$$

$$I_{yy} = \frac{1}{12}(4\ \text{in.})(3\ \text{in.})^3 - \frac{1}{12}(3.5\ \text{in.})(2.5\ \text{in.})^3 = 4.443\ \text{in.}^4 \qquad I_{zz} = \frac{1}{12}(3\ \text{in.})(4\ \text{in.})^3 - \frac{1}{12}(2.5\ \text{in.})(3.5\ \text{in.})^3 = 7.068\ \text{in.}^4 \qquad \text{(E2)}$$

Step 3: At points A and B we draw a line perpendicular to the centerline of the cross section. We may then obtain the area A_s needed for the calculations of Q_y and Q_z at points A and B as shown in Figure 10.18:

$$t_A = t_B = 0.25\text{ in.} + 0.25\text{ in.} = 0.5\text{ in.} \quad \text{(E3)}$$

$$(Q_y)_A = 2(1.5\text{ in.})(0.25\text{ in.})(0.75\text{ in.}) + (3.5\text{ in.})(0.25\text{ in.})(1.5\text{ in.} - 0.125\text{ in.}) = 1.766\text{ in.}^3 \qquad (Q_z)_A = 0 \quad \text{(E4)}$$

$$(Q_y)_B = 0 \qquad (Q_z)_B = (2\text{ in.})(2\text{ in.})(0.25\text{ in.})(1\text{ in.}) + (2.5\text{ in.})(0.25\text{ in.})(2\text{ in.} - 0.125\text{ in.}) = 2.172\text{ in.}^3 \quad \text{(E5)}$$

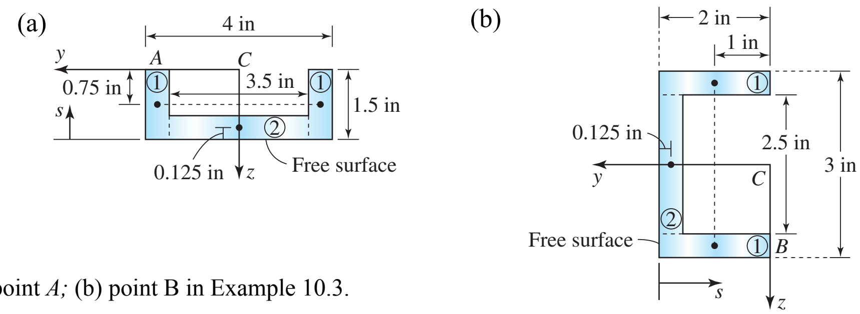


Figure 10.18 Calculation of Q_y and Q_z at (a) point A; (b) point B in Example 10.3.

Step 4: We can make an imaginary cut through the cross section containing points A and B and draw the free-body diagram shown in Figure 10.19. Internal forces and moments are drawn according to our sign convention. From the equilibrium equations, the internal forces and moments can be found,

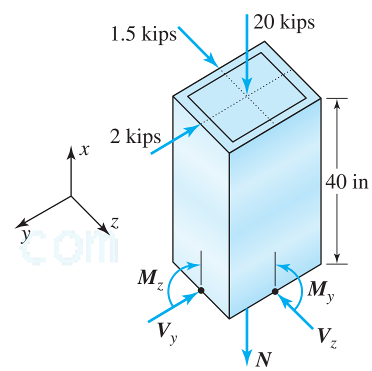


Figure 10.19 Free-body diagram in Example 10.3.

$$\boldsymbol{N} = -20\text{ kips} \qquad \boldsymbol{V_y} = -2.0\text{ kips} \qquad \boldsymbol{V_z} = 1.5\text{ kips} \qquad \boldsymbol{M_y} = 60\text{ in.}\cdot\text{kips} \qquad \boldsymbol{M_z} = -80\text{ in.}\cdot\text{kips} \quad \text{(E6)}$$

Step 5: The stress components due to each loading are calculated next.

Axial stress calculations: The axial stresses at points A and B can be found from Equation (10.1) as

$$(\sigma_{xx})_{A,B} = \frac{\boldsymbol{N}}{A} = \frac{(-20\text{ kips})}{3.25\text{ in.}^2} = -6.154\text{ ksi} \quad \text{(E7)}$$

Stresses due to bending about the y axis: We note that $z_A = 0$ and $z_B = 1.5$. From Equation (10.4a), we obtain

$$(\sigma_{xx})_A = 0 \qquad (\sigma_{xx})_B = -\frac{\boldsymbol{M_y} z_B}{I_{yy}} = -\frac{(60\text{ in.}\cdot\text{kips})(1.5\text{ in.})}{4.443\text{ in.}^4} = -20.258\text{ ksi} \quad \text{(E8)}$$

From Equation (10.4b) we obtain the shear stress at A and B,

$$(\tau_{xs})_A = -\frac{\boldsymbol{V_z} Q_y}{I_{yy} t} = -\frac{(1.5\text{ kips})(1.766\text{ in.}^3)}{(4.443\text{ in.}^4)(0.5\text{ in.})} = -1.192\text{ ksi} \qquad (\tau_{xs})_B = 0 \qquad (\tau_{xy})_B = 0 \quad \text{(E9)}$$

From Figure 10.18*a* we note that the s direction is in the negative z direction at point A. Thus

$$(\tau_{xz})_A = -(\tau_{xs})_A = 1.19\text{ ksi} \quad \text{(E10)}$$

Stresses due to bending about the z axis: We note that $y_A = 2$ and $y_B = 0$. From Equation (10.3a), we obtain

$$(\sigma_{xx})_A = -\frac{M_z y_A}{I_{zz}} = -\frac{(-80\text{ in.}\cdot\text{kips})(2\text{ in.})}{7.068\text{ in.}^4} = 22.638\text{ ksi} \qquad (\sigma_{xx})_B = 0 \quad \text{(E11)}$$

From Equation (10.3*b*), we obtain the shear stresses at points A and B,

$$(\tau_{xs})_A = 0 \qquad (\tau_{xz})_A = 0 \qquad (\tau_{xs})_B = -\frac{\boldsymbol{V_y} Q_z}{I_{zz} t} = -\frac{(-2\text{ kips})(2.172\text{ in.}^3)}{(7.068\text{ in.}^4)(0.5\text{ in.})} = 1.229\text{ ksi} \quad \text{(E12)}$$

From Figure 10.18*b* we note that the s direction is in the negative y direction at point B. Thus

$$(\tau_{xy})_B = -(\tau_{xs})_B = -1.23\text{ ksi} \quad \text{(E13)}$$

Step 6: Superposition

Normal stress calculations: The normal stress at point A can be obtained by superposing the values in Equations (E7), (E8), and (E11),

$$(\sigma_{xx})_A = -6.154 \text{ ksi} + 0 + 22.638 \text{ ksi} = 16.484 \text{ ksi} \quad \text{(E14)}$$

ANS. $(\sigma_{xx})_A = 16.5 \text{ ksi (T)}$

Similarly, the normal stress at point *B* can be obtained by superposition of Equations (E7), (E8), and (E11),

$$(\sigma_{xx})_B = -6.154 \text{ ksi} - 20.258 \text{ ksi} + 0 = -26.412 \text{ ksi} \quad \text{(E15)}$$

ANS. $(\sigma_{xx})_B = 26.4 \text{ ksi (C)}$

Intuitive check on normal stress calculations: The axial stress σ_{axial} due to a 20-kips force will be compressive. Figure 10.20 shows the exaggerated deformed shapes due to bending about the *y* and *z* axes. (These deformed shapes can actually be visualized without drawing the figures.) From Figure 10.20*a* it can be seen that the line passing through *A* will be in tension. That is, the normal stress due to bending about the *z* axis $\sigma_{\text{bend-}z}$ will be tensile. From 10.24*b* it can be seen that point *A* is on the neutral (bending) axis. Hence the normal stress due to bending about the *y* axis $\sigma_{\text{bend-}y} = 0$. Thus the total normal stress at point *A* is $(\sigma_{xx})_A = \sigma_{\text{bend-}z} - \sigma_{\text{axial}}$. Substituting the magnitude of $\sigma_{\text{bend-}z} = 22.638$ ksi and $\sigma_{\text{axial}} = 6.154$ ksi, we obtain the result in Equation (E14).

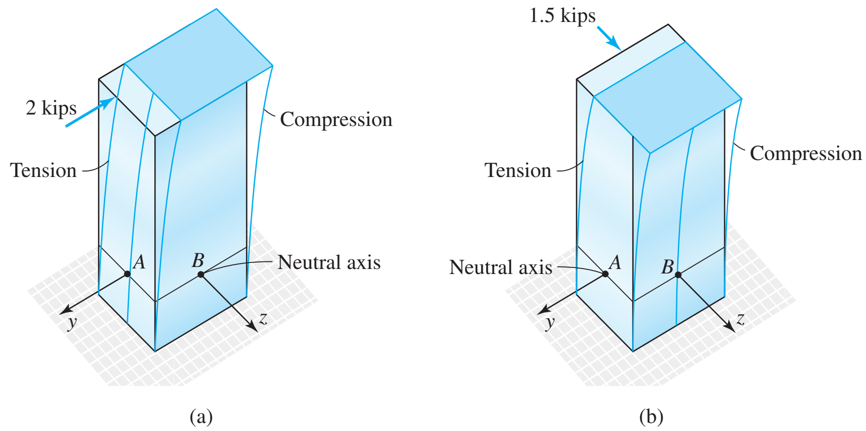


Figure 10.20 Determination of normal stress components by inspection for bending about (*a*) *z* axis; (*b*) *y* axis.

From Figure 10.20*b* it can be seen that the line passing though *B* will be in compression. That is, the normal stress due to bending about the *y* axis $\sigma_{\text{bend-}y}$ will be compressive. From Figure 10.20*a* it can be seen that point *B* is on the neutral (bending) axis; hence $\sigma_{\text{bens-}z} = 0$. Thus the total normal stress at point *B* can be written as $(\sigma_{xx})_B = -\sigma_{\text{axial}} - \sigma_{\text{bend-}y}$. Substituting the magnitude of $\sigma_{\text{bend-}y} = 20.258$ ksi and $\sigma_{\text{axial}} = 6.154$ ksi, we obtain the result in Equation (E15).

Shear stress calculations: The shear stresses at point *A* can be obtained by superposing the values in Equations (E10) and (E12). The shear stress at point *B* can be obtained by superposing the values in Equations (E9) and (E13).

ANS. $(\tau_{xz})_A = 1.2 \text{ ksi} \qquad (\tau_{xy})_B = -1.2 \text{ ksi}$

Intuitive check on shear stress calculations: By inspection we deduce that the shear force on the bottom segment containing points *A* and *B* is in the negative *y* and positive z direction, as shown in Figure 10.21. We obtain the shear stress distribution (see Section 6.6.1) as shown. The direction of shear stress at point *A* and *B* are consistent with our results. Points *A* and *B* are on free surfaces with outward normals in y and z, respectively. Hence, $(\tau_{xy})_A = 0$ and $(\tau_{xz})_B = 0$. Thus the total shear stresses at *A* and *B* are $(\tau_{xz})_A = \tau_{\text{bend-}y}$ and $(\tau_{xy})_B = -\tau_{\text{bend-}z}$, consistent with our answers.

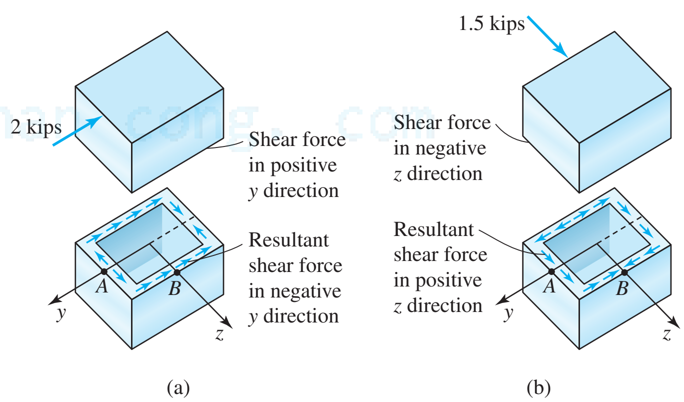


Figure 10.21 Direction of shear stress components by inspection for bending about (*a*) *z* axis; (*b*) *y* axis.

Step 7: The stresses at points *A* and *B* can now be drawn on a stress cube, as shown in Figure 10.22.

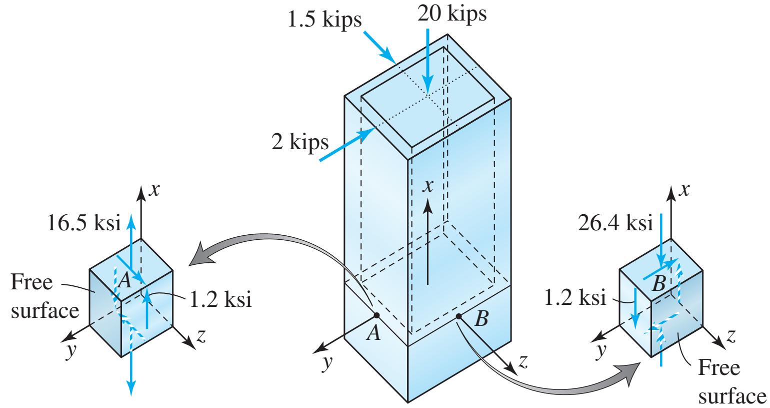


Figure 10.22 Stress cubes in Example 10.3.

Step 8: Figure 10.23 shows the plane containing the crack. From geometry we conclude that the angle that the outward normal makes with the x axis is 35°. Substituting $\theta = 35°$, $(\sigma_{xx})_B = -26.4$ ksi, $(\tau_{xy})_B = -1.2$ ksi, and $(\sigma_{yy})_B = 0$ into Equations (8.1) and (8.2), we obtain the normal and shear stresses on the plane containing the crack,

$$\sigma_{nn} = (-26.4 \text{ ksi})\cos^2 35° + 2(-1.2 \text{ ksi})\sin 35° \cos 35° = -18.84 \text{ ksi} \quad \text{(E16)}$$

$$\tau_{nt} = -(-26.4 \text{ ksi})\cos 35° \sin 35° + (-1.2 \text{ ksi})(\cos^2 35° - \sin^2 35°) = 11.99 \text{ ksi} \quad \text{(E17)}$$

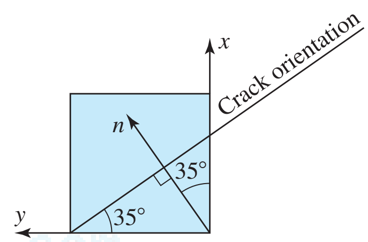


Figure 10.23 Angle of normal to plane containing crack.

ANS. $\sigma_{nn} = 18.84 \text{ ksi (C)}$ $\tau_{nt} = 11.99 \text{ ksi}$

COMMENTS

1. It may seem that the intuitive checks take as much effort as the calculation of the stresses by the procedural approach. But much of the description and diagrams here are for purpose of explanation only. Most of the intuitive check is by inspection. In the process you will develop an intuitive sense of the stresses under combined loading.
2. In place of three-dimensional free-body diagram of Figure 10.19, you may prefer drawing two perspectives of the free-body diagram shown in Figure 10.24. Figure 10.24*a* is constructed by looking down the y axis, whereas Figure 10.24*b* is the perspective looking down the z axis. Equation (E6) can be obtained from equilibrium.

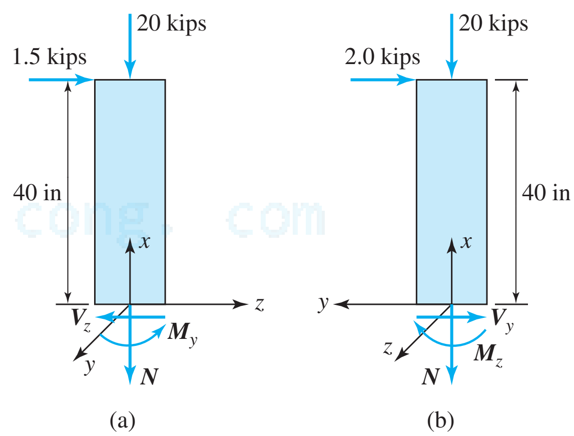


Figure 10.24 Two-dimensional free-body diagrams in Example 10.3.

3. In calculating bending stresses by inspection, be sure to use the correct area moment of inertia in the formula for rectangular cross sections: I_{yy} is not the same as I_{zz}. The subscripts emphasize that the moment of inertia to be used is the value about the bending axis.
4. The stresses on the plane containing the crack are used to assess whether a crack will grow and break the body.

EXAMPLE 10.4

A thin cylinder with an outer diameter of 100 mm and a thickness of 10 mm is loaded as shown in Figure 10.25. At point A, which is on the surface of the cylinder, determine the normal and shear stresses in the x, y, z coordinate system. Show your results on a stress cube.

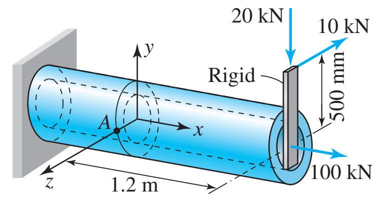


Figure 10.25 Geometry and loading in Example 10.4.

PLAN

We can follow the procedure outlined in Section 10.1.7. The 100 kN is an axial force. The 20-kN force will produce bending about the z axis. The 10-kN force will produce bending about the y axis and will also produce torque. Thus we need all the stress equations listed in Table 10.1. We can use these equations as a checklist of the quantities to calculate. and determine the stress at point A by superposition.

SOLUTION

Step 1: All the stress equations in Table 10.1 will be used.
Step 2: The geometric properties of the cross section can be found as

$$A = \pi[(50\ \text{mm})^2 - (40\ \text{mm})^2] = 2.827(10^3)\ \text{mm}^2 \qquad J = \frac{\pi}{2}[(50\ \text{mm})^4 - (40\ \text{mm})^4] = 5.796(10^6)\ \text{mm}^4 \tag{E1}$$

$$I_{yy} = I_{zz} = \frac{J}{2} = 2.898(10^6)\ \text{mm}^4 \tag{E2}$$

Step 3: At points A we draw a line perpendicular to the centerline of the cross section to obtain the area A_s needed for the calculations of Q_y and Q_z at point A, as shown in Figure 10.26*a*.

$$t_A = 20\ \text{mm} \qquad (Q_y)_A = 0 \tag{E3}$$

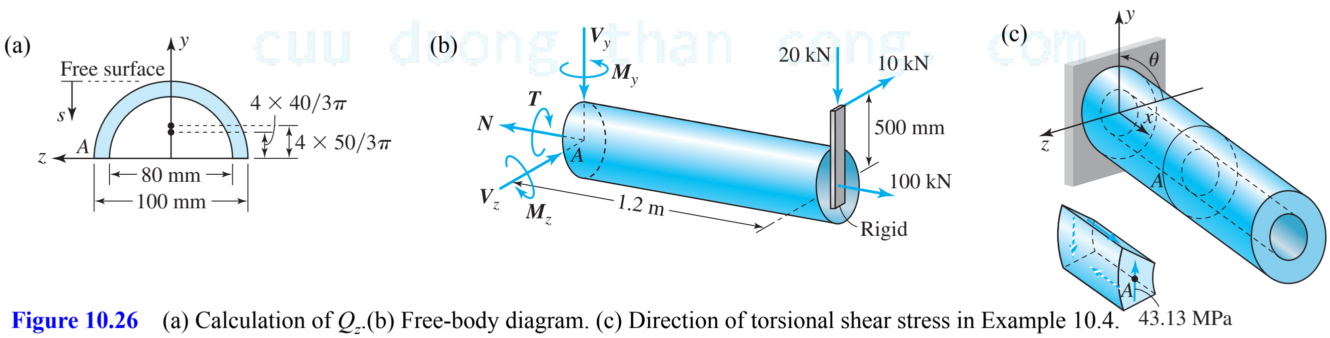


Figure 10.26 (a) Calculation of Q_z.(b) Free-body diagram. (c) Direction of torsional shear stress in Example 10.4.

To find $(Q_z)_A$, we use the formula $4r/3\pi$, given in Table C.2, for the location of the centroid for a half-disc of radius r. From Figure 10.26*a,* subtracting the first moment of the area of the inner disc of radius 40 mm from the first moment of the outer disc of radius 50 mm, we obtain

$$(Q_z)_A = \left[\frac{\pi(50\ \text{mm})^2}{2}\right]\left[\frac{(4\ \text{mm})(50\ \text{mm})}{3\pi}\right] - \left[\frac{\pi(40\ \text{mm})^2}{2}\right]\left[\frac{(4\ \text{mm})(40\ \text{mm})}{3\pi}\right] = 40.667(10^3)\ \text{mm}^3 \tag{E4}$$

Step 4: We draw the free-body diagram shown in Figure 10.26*b* by making an imaginary cut at $x = 0$. The internal forces and moments are drawn according to our sign convention and can be obtained by equilibrium,

$$\boldsymbol{N} = 100\ \text{kN} \quad \boldsymbol{V}_y = -20\ \text{kN} \quad \boldsymbol{V}_z = -10\ \text{kN} \quad \boldsymbol{T} = -5\ \text{kN·m} \quad \boldsymbol{M}_y = -12\ \text{kN·m} \quad \boldsymbol{M}_z = -24\ \text{kN·m} \tag{E5}$$

Step 5: The stress components due to each loading are calculated next.
Axial stress calculations: From Equation (10.1) we obtain

$$(\sigma_{xx})_A = \frac{N}{A} = \frac{100(10^3)\ \text{N}}{2.827(10^{-3})\ \text{m}^2} = 35.373(10^6)\ \text{N/m}^2 = 35.373\ \text{MPa} \tag{E6}$$

Torsional shear stress calculations: Noting $\rho_A = 50\ (10^{-3})$ m we obtain from Equation (10.2),

$$(\tau_{x\theta})_A = \frac{T\rho_A}{J} = \frac{[-5(10^3)\ \text{N}\cdot\text{m}][50(10^{-3})\ \text{m}]}{5.796(10^{-6})\ \text{m}^4} = -43.133(10^6)\ \text{N/m}^2 \tag{E7}$$

The shear stress can be drawn on a stress cube using the subscripts, as shown in Figure 10.26*c*. The direction of shear stress in the x, y coordinate system is

$$(\tau_{xy})_A = 43.133\ \text{MPa} \tag{E8}$$

Stresses due to bending about the y axis: Noting that $z_A = 50 \times 10^{-3}$ m, we obtain from Equation (10.4a),

$$(\sigma_{xx})_A = -\frac{M_y z_A}{I_{yy}} = -\frac{[-12(10^3)\ \text{N}\cdot\text{m}][50(10^{-3})\ \text{m}]}{2.898(10^{-6})\ \text{m}^4} = 207.04(10^6)\ \text{N/m}^2 = 207.04\ \text{MPa} \quad \text{(E9)}$$

From Equation (10.4*b*) we obtain

$$(\tau_{xs})_A = 0 \qquad \text{or} \qquad (\tau_{xy})_A = 0 \quad \text{(E10)}$$

Stresses due to bending about the z axis: Noting that $y_A = 0$, we obtain from Equation (10.3a)

$$(\sigma_{xx})_A = 0 \quad \text{(E11)}$$

From Equation (10.3b) we obtain

$$(\tau_{xs})_A = -\frac{V_y(Q_z)_A}{I_{zz}t_A} = -\frac{[-20(10^3)\ \text{N}][40.667(10^{-6})\ \text{m}^3]}{[2.898(10^{-6})\ \text{m}^4][20(10^{-3})\ \text{m}]} = 14.033(10^6)\ \text{N/m}^2 \quad \text{(E12)}$$

From Figure 10.26*a* we note that the *s* direction is in the negative *y* direction at point *A*. Thus,

$$(\tau_{xy})_A = -(\tau_{xs})_A = -14.033\ \text{MPa} \quad \text{(E13)}$$

Step 6: Superposition

Normal stress calculations: The normal stress at point A can be obtained by superposing the values in Equations (E6), (E9), and (E12),

$$(\sigma_{xx})_A = 35.373\ \text{MPa} + 207.04\ \text{MPa} + 0 = 242.412\ \text{MPa} \quad \text{(E14)}$$

ANS. $(\sigma_{xx})_A = 242.4\ \text{MPa (T)}$

Intuitive check on normal stress calculations: The axial stress σ_{axial} due to a 100-kN force will be tensile. Figure 10.27 shows the exaggerated deformed shapes due to bending about the *y* and *z* axes. From Figure 10.27*a*, it can be seen that line *AB* and hence the normal stress due to bending about the *y* axis $\sigma_{\text{bend-}y}$ will be tensile at point *A*. Hence the normal stress due to bending about the *z* axis $\sigma_{\text{bend-}z} = 0$ at point *A* is on the neutral (bending) axis in Figure 10.27*b*. Thus the total normal stress at point *A* can be written as $(\sigma_{xx})_A = \sigma_{\text{axial}} + \sigma_{\text{bend-}y}$, confirming our results.

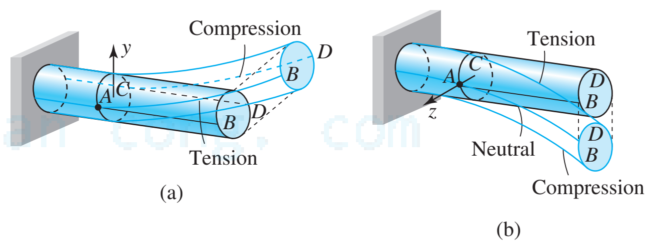


Figure 10.27 Determination of normal stress components by inspection from bending about (*a*) y axis; (*b*) z axis.

Shear stress calculations: The shear stress at point *A* can be obtained by superposing the values in Equations (E8), (E10), and (E13),

$$(\tau_{xy})_A = 43.133\ \text{MPa} + 0 - 14.033\ \text{MPa} = 29.10\ \text{MPa} \quad \text{(E15)}$$

ANS. $(\tau_{xy})_A = 29.10\ \text{MPa}$

Intuitive check on shear stress calculations: Figure 10.28 shows the direction of shear stress due to torsion (see Section 5.2.5) and bending about *y* and *z* axis (see Section 6.6.1). We see that at point *A* the torsional shear stress τ_{tor} is upward, the shear stress due to bending about the *z* axis $\tau_{\text{bend-}z}$ is downward, and the shear stress due to bending about *y* axis $\tau_{\text{bend-}y}$ is zero. Thus the total shear stress at *A* can be written as $(\tau_{xy})_A = \tau_{\text{tor}} - \tau_{\text{bend-}z}$, confirming our results

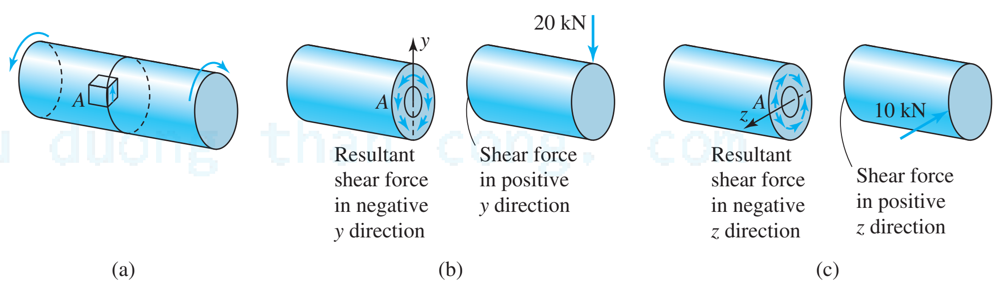


Figure 10.28 Direction of shear stress components by inspection from (*a*) torsion; (*b*) bending about *z* axis; (*c*) bending about *y* axis.

Step 7: Figure 10.29 shows the result of stresses on a stress cube.

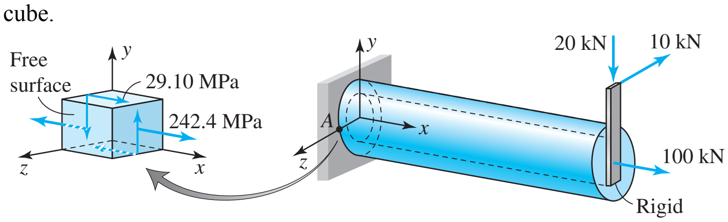


Figure 10.29 Results on stress cube in Example 10.4

COMMENTS

1. The stresses shown on the stress cube in Figure 10.29 can be processed further if necessary. We could find principal stresses as in Example 10.1, stresses on a plane as in Example 10.3, or strains along the direction of a gage as in Example 10.2.
2. The advantage of the procedure outlined in Section 10.1.7 is that it is methodical. It breaks a complex problem into a sequence of simple steps, as shown in this and previous examples. The shortcoming of this procedural approach is that it does not exploit any simplification that may be intrinsic to the problem.
3. Solving a problem by inspection has two distinct advantages: it helps build an intuitive understanding of stress behavior, and it can reduce the computational effort significantly.
4. The possibility of error is higher when solving the problem primarily by inspection because less is worked out on paper. For example, internal forces and moments are equal and opposite on the two surfaces created by an imaginary cut. It is easy to confuse one surface with another if we try to visualize all in our head, particularly in calculating of shear stress. Rough sketches can help.
5. You may find it more effective to solve part of the problem by a procedural approach and part by inspection. For example, you could solve for normal stresses but not shear stresses by inspection.

Consolidate your knowledge

1. Write a procedure you would use for solving combined loading problems.

EXAMPLE 10.5

The cylinder of 800-mm outer diameter shown in Figure 10.30 has a wall thickness of 15 mm. In addition to the axial and torsional loads, the cylinder is pressurized to 150 kPa. Determine the normal and shear stresses at point *A* on the center line of the cross section, and show them on a stress element in a cylindrical coordinate system.

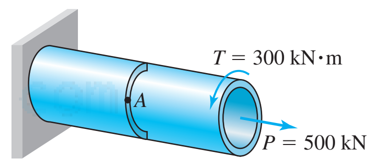


Figure 10.30 Cylinder and loading in Example 10.5*.

PLAN

The stress state at any point is from three sources: axial stress [Equation (10.1)]; torsional shear stress [Equation (10.2)]; and axial and hoop stresses due to pressure on the thin cylinder [Equations (4.28) and (4.29)].

SOLUTION

Step 1: Equations (10.1), (10.2), (4.28) and (4.29) will be used to find the stress components.
Step 2: The outer radius (R_o), the inner radius (R_i), the mean radius (R_m), the cross sectional area (A), and polar moment of inertia (J) can be found as

$$R_o = 0.4 \text{ m} \qquad R_i = 0.385 \text{ m} \qquad R_m = 0.3925 \text{ m} \tag{E1}$$

$$A = \pi(R_o^2 - R_i^2) = \pi[(0.4 \text{ m})^2 - (0.385 \text{ m})^2] = 36.99(10^{-3}) \text{ m}^2 \tag{E2}$$

$$J = \frac{\pi}{2}(R_o^4 - R_i^4) = \frac{\pi}{2}[(0.4 \text{ m})^4 - (0.385 \text{ m})^4] = 5.701(10^{-3}) \text{ m}^2 \tag{E3}$$

Step 3: This step is not needed as there is no bending.

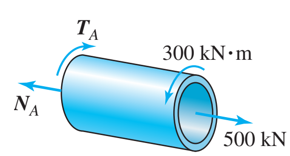


Figure 10.31 Free-body diagram in Example 10.5*.

Step 4: By equilibrium of free-body diagram in Figure 10.31 we obtain

$$N_A = 500 \text{ kN} \qquad \boldsymbol{T_A} = 300 \text{ kN·m} \tag{E4}$$

Step 5: The stress components due to each loading are calculated next.
Axial stress calculation: From Equation (10.1) we obtain

$$\sigma_{xx} = \frac{N_A}{A} = \frac{500(10^3) \text{ N}}{36.99(10^{-3}) \text{ m}^2} = 13.52(10^6) \text{ N/m}^2 = 13.52 \text{ MPa} \tag{E5}$$

Torsional shear stress: Noting $\rho_A = 0.3925$ m we obtain from Equation (10.2)

$$\tau_{x\theta} = \frac{T_A \rho_A}{J} = \frac{[300(10^3)\ \text{N}\cdot\text{m}](0.3925\ \text{m})}{5.701(10^{-3})\ \text{m}^4} = 20.65(10^6)\ \text{N/m}^2 = 20.65\ \text{MPa} \quad \text{(E6)}$$

Stresses due to pressure of thin-walled cylinders: Noting that the pressure is $p = 150$ kPa and t = 0.015 m we obtain from Equations (4.28) and (4.29)

$$\sigma_{xx} = \frac{pR_m}{2t} = \frac{[150(10^3)\ \text{N/m}^2](0.3925\ \text{m})}{2(0.015\ \text{m})} = 1.96\times 10^6\ \text{N/m}^2 = 1.96\ \text{MPa} \quad \text{(E7)}$$

$$\sigma_{\theta\theta} = \frac{pR_m}{t} = 3.92\ \text{MPa} \quad \text{(E8)}$$

Step 6: Superposition
The normal stress at point *A* can be obtained by superposing the values in Equations (E5) and (E7),

$$\sigma_{xx} = 13.52\ \text{MPa} + 1.96\ \text{MPa} = 15.48\ \text{MPa} \quad \text{(E9)}$$

Figure 10.32 shows the stresses in Equations (E6), (E8), and (E9) on a stress element.

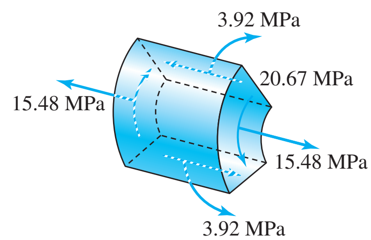


Figure 10.32 Stress element in Example 10.5*.

COMMENTS

1. From the stress state in Figure 10.32, we could find principal stresses, or strains in any direction.
2. We could have used Equation (5.26) in place of Equation (10.2), as the shaft is thin-walled, to obtain the same results

$$\tau = \frac{\boldsymbol{T}_A}{2tA_E} = \frac{[300(10^3)\ \text{N}\cdot\text{m}]}{2(0.015\ \text{m})[\pi(0.3925\ \text{m})^2]} = 20.67\ \text{MPa}$$

PROBLEM SET 10.1

Combined axial and torsion forces

10.1 Determine the normal and shear stresses in the seam of the shaft passing through point *A*, as shown in Figure P10.1 The seam is at an angle of 60º to the axis of a solid shaft of 2-in. diameter.

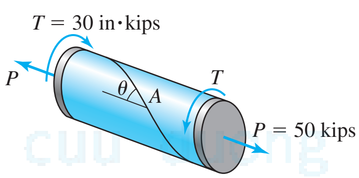


Figure P10.1

10.2 A 4-in.-diameter solid circular steel shaft is loaded as shown in Figure P10.2. Determine the shear stress and the normal stress on a plane passing through point *E*. Point *E* is on the surface of the shaft.

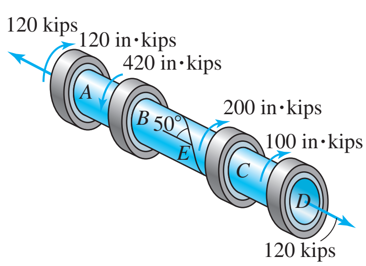


Figure P10.2

10.3 A solid shaft of 75-mm diameter is loaded as shown in Figure P10.3. The strain gage is 20º to the axis of the shaft and the shaft material has a modulus of elasticity E = 250 GPa and a Poisson ratio $\nu = 0.3$. If $T = 20$ kN·m and $P = 50$ kN, what strain will the strain gage show?

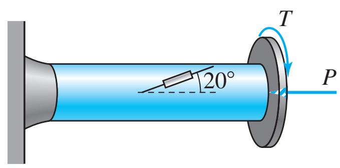


Figure P10.3

10.4 A solid shaft of 75-mm diameter is loaded as shown in Figure P10.3. The strain gage is 20º to the axis of the shaft and the shaft material has a modulus of elasticity E = 250 GPa and a Poisson ratio $\nu = 0.3$. If the strain gage shows a reading of −450 μm/m and $T = 10$ kN, determine the axial load P

10.5 A solid shaft of 75-mm diameter is loaded as shown in Figure P10.3. The strain gage is 20º to the axis of the shaft and the shaft material has a modulus of elasticity E = 250 GPa and a Poisson ratio $\nu = 0.3$. If the strain gage shows a reading of –300 μm/m and P = 55 kN, determine the applied torque T.

10.6 A solid shaft of 2-in. diameter is loaded as shown in Figure P10.6 The shaft material has a modulus of elasticity E = 30,000 ksi and a Poisson ratio $\nu = 0.3$. Determine the strains the gages would show if $P = 70$ kips and $T = 50$ in.·kips.

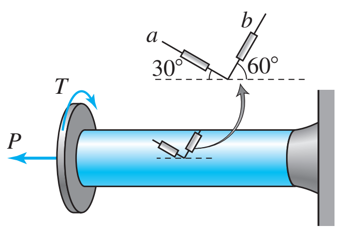


Figure P10.6

10.7 A solid shaft of 2-in. diameter is loaded as shown in Figure P10.6 The shaft material has a modulus of elasticity E = 30,000 ksi and a Poisson ratio $\nu = 0.3$. The strain gages mounted on the surface of the shaft recorded the strain values $\varepsilon_a = 2078\ \mu$ and $\varepsilon_b = -1410\ \mu$. Determine the axial force P and the torque T.

10.8 Two solid circular steel ($E_s = 200$ GPa, $G_s = 80$ GPa) shafts and a solid circular bronze shaft ($E_{br} = 100$ GPa, $G_{br} = 40$ GPa) are securely connected and loaded as shown Figure P10.8 Determine the maximum normal and shear stresses in the shaft.

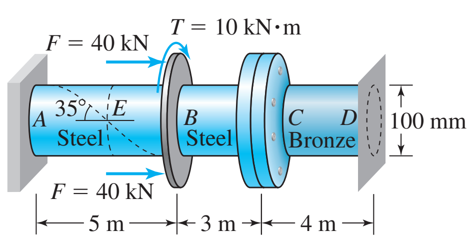


Figure P10.8

10.9 Determine the normal and shear stresses on a plane 35º to the axis of the shaft at point E in Figure P10.8. Point E is on the surface of the shaft.

Combined axial and bending forces

10.10 A 6-in. × 4-in. rectangular hollow member is constructed from a $\frac{1}{2}$-in.-thick sheet metal and loaded as shown in Figure P10.10. Determine the normal and shear stresses at points A and B and show them on the stress cubes for $P_1 = 72$ kips, $P_2 = 0$, and $P_3 = 6$ kips.

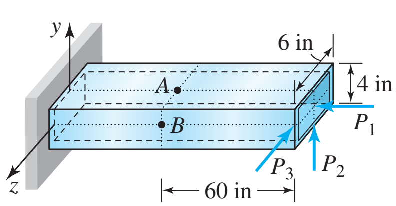


Figure P10.10

10.11 Determine the principal stresses and the maximum shear stress at points A and B in Figure P10.10 for $P_1 = 72$ kips, $P_2 = 3$ kips, and $P_3 = 0$.

10.12 Determine the strain shown by the strain gages in Figure P10.12 if $P_1 = 3$ kN, $P_2 = 40$ kN, the modulus of elasticity is 200 GPa, and Poisson's ratio is 0.3. The strain gages are parallel to the axis of the beam.

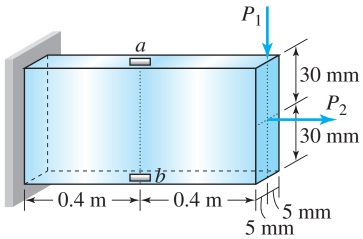


Figure P10.12

10.13 The strain gages shown in Figure P10.12 recorded the strain values $\varepsilon_a = 1000\ \mu$ and $\varepsilon_b = -750\ \mu$. Determine loads P_1 and P_2. The modulus of elasticity is 200 GPa and Poisson's ratio is 0.3.

10.14 Determine the strain shown by the strain gages in Figure P10.14 if $P_1 = 3$ kN, $P_2 = 40$ kN, the modulus of elasticity is 200 GPa, and Poisson's ratio is 0.3.

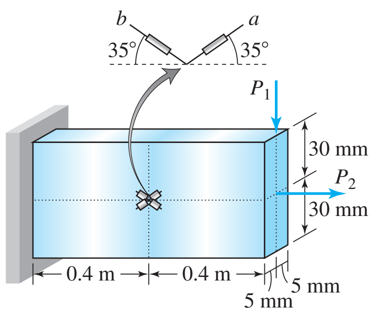


Figure P10.14

10.15 The strain gages shown in Figure P10.14 recorded the strain values $\varepsilon_a = 133\ \mu$ and $\varepsilon_b = 159\ \mu$. Determine loads P_1 and P_2. The modulus of elasticity is 200 GPa and Poisson's ratio is 0.3.

10.16 Determine the strain recorded by the gages at points A and B in Figure P10.16. Both gages are at 30º to the axis of the beam. The modulus of elasticity $E = 30{,}000$ ksi and $\nu = 0.3$.

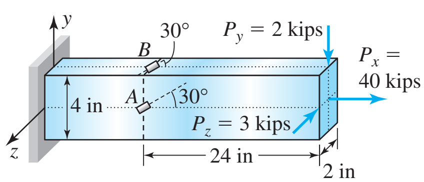


Figure P10.16

Combined axial, torsion, and bending forces

10.17 A thin cylinder with an outer diameter of 100 mm and a thickness of 10 mm is loaded as shown in Figure P10.17. Points A and B are on the surface of the shaft. Determine the normal and shear stresses at points A and B in the x, y, z coordinate system and show your results on stress cubes.

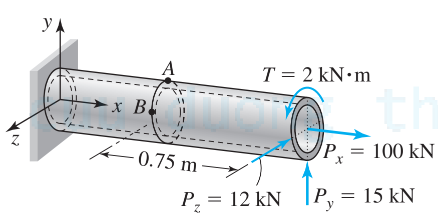


Figure P10.17

10.18 Determine the principal stresses and the maximum shear stress at point B on the shaft shown in Figure P10.17.

In problems 10.19 through 10.27, a pipe has outer diameter 120 mm and a thickness of 10 mm. All forces except the one given are zero. (a) Using the notation in Equation 10.5, determine by inspection the total normal and shear stresses at points A and B which are on the surface of the pipe. (b) Calculate the stresses and show on the stress cube.

Problem	Loads	Use
10.19	$P_x = 9$ kN	Figure P10.19, a =1.2 m, b =1.5m
10.20	$P_y = 12$ kN	Figure P10.19, a =1.2 m, b =1.5m
10.21	$P_z = 15$ kN	Figure P10.19, a =1.2 m, b =1.5m
10.22	$P_x = 9$ kN	Figure P10.22, a = 0.64 m, b = 0.5 m, c =0.3 m
10.23	$P_y = 12$ kN	Figure P10.22, a = 0.64 m, b = 0.5 m, c =0.3 m
10.24	$P_z = 15$ kN	Figure P10.22, a = 0.64 m, b = 0.5 m, c =0.3 m
10.25	$P_x = 9$ kN	Figure P10.25, a = 0.5 m, b = 0.8 m, c =0.7 m
10.26	$P_y = 12$ kN	Figure P10.25, a = 0.5 m, b = 0.8 m, c =0.7 m
10.27	$P_z = 15$ kN	Figure P10.25, a = 0.5 m, b = 0.8 m, c =0.7 m

Figure P10.19

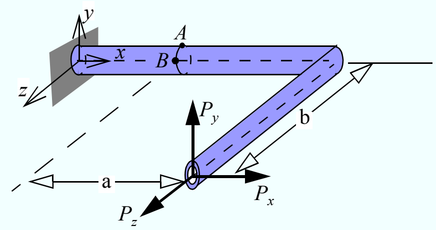


Figure P10.22

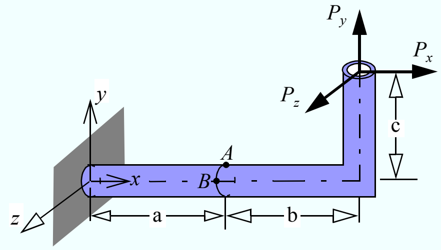
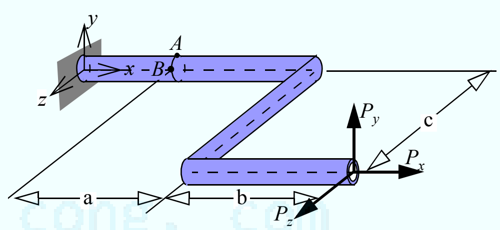


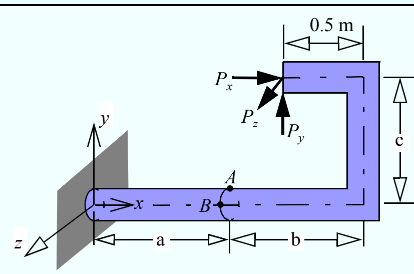
Figure P10.25



In problems 10.28 through 10.30, a pipe has outer diameter 120 mm and a thickness of 10 mm. All forces except the one given are zero. Determine the maximum normal and shear stress at points A and B.

Problem	Loads	Use
10.28	$P_x = 10$ kN	Figure P10.28, a = 0.7 m, b = 0.9 m, c =0.7 m
10.29	$P_y = 15$ kN	Figure P10.28, a = 0.7 m, b = 0.9 m, c =0.7 m
10.30	$P_z = 20$ kN	Figure P10.28, a = 0.7 m, b = 0.9 m, c =0.7 m

Figure P10.28



10.31 A pipe with an outside diameter of 2.0 in. and a wall thickness of $\frac{1}{4}$ in. is loaded as shown in Figure P10.31. Determine the normal and shear stresses at points *A* and *B* in the *x*, *y*, *z* coordinate system and show them on a stress cube. Points *A* and *B* are on the surface of the pipe. Use a =48 in. and b = 60 in.

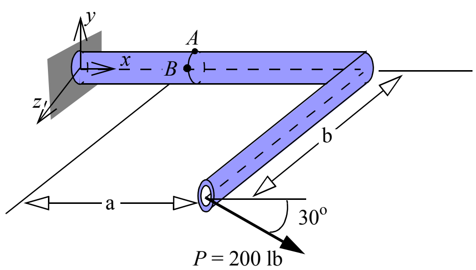


Figure P10.31

10.32 Determine the maximum normal stress and the maximum shear stress at point *B* on the pipe shown in Figure P10.31.

10.33 A pipe with an outside diameter of 40 mm and a wall thickness of 10 mm is loaded as shown in Figure P10.33. Determine the normal and shear stresses at points *A* and *B* in the *x*, *y*, *z* coordinate system and show them on a stress cube. Points *A* and *B* are on the surface of the pipe. Use a = 0.25 m, b = 0.4 m, and c = 0.1 m.

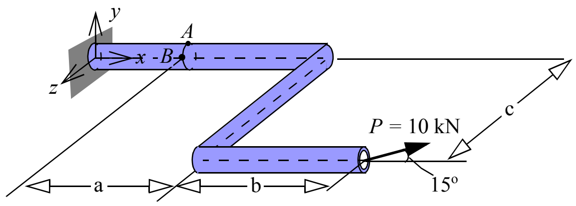


Figure P10.33

10.34 Determine the maximum normal stress and the maximum shear stress at point *B* on the pipe shown in Figure P10.33.

10.35 A bent pipe of 2-in. outside diameter and a wall thickness of $\frac{1}{4}$ in. is loaded as shown in Figure P10.35. If a = 16 in., b = 16 in.,and c = 10 in., determine the stress components at point *A*, which is on the surface of the shaft. Show your answer on a stress cube.

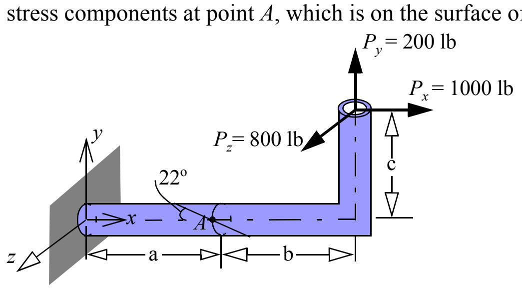


Figure P10.35

10.36 Determine the normal and shear stresses on a seam through point *A* that is 22° to the axis of the pipe shown in Figure P10.35.

10.37 The hollow steel shaft shown in Figure P10.37 has an outside diameter of 4 in. and an inside diameter of 3 in. Two pulleys of 24-in. diameter carry belts that have the given tensions. The shaft is supported at the walls using flexible bearings, permitting rotation in all directions. Determine the maximum normal and shear stresses in the shaft.

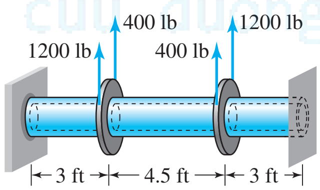


Figure P10.37

10.38 A thin hollow cylinder with an outer diameter of 120 mm and a wall thickness of 15 mm is loaded as shown in Figure P10.38. In *x*, *y*, *z* coordinate system, determine the normal and shear stresses at points *A* and *B* on the surface of the cylinder and show your results on stress cubes.

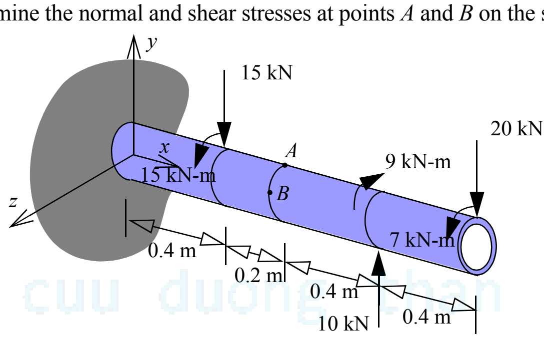


Figure P10.38

10.39 A 6 in. x 1 in. rectangular structural member is loaded as shown in Figure P10.39. Determine the maximum normal and shear stress in the member.

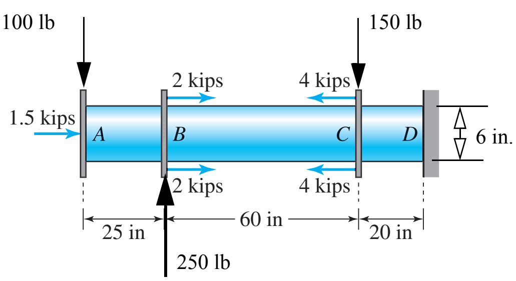


Figure P10.39

10.40 A thin cylinder is subjected to a uniform pressure of 300 psi and torques as shown in Figure P10.40. The cylinder has a outer radius of 10 in. and a wall thickness of 0.25 in. Determine the normal and shear stresses at point A and show them on a stress element in cartesian coordinates.

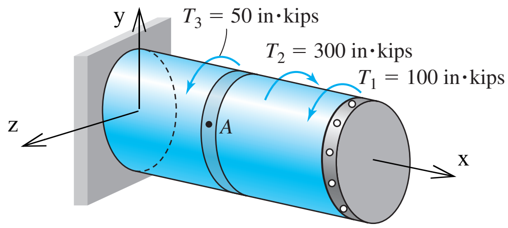


Figure P10.40

10.2 ANALYSIS AND DESIGN OF STRUCTURES

Most real structures are composed of many members joined together. Analyzing or designing these structures requires that we create a mathematical model to approximate the actual structure. Many decisions go into the creation of a model, including the proper modeling of joints and supports. Approximating joints by pins simplifies the analysis significantly, because pin joints do not transmit moments. In other words, we are neglecting the joints' intrinsic moment-carrying ability. Thus the model will predict higher internal forces, moments, and hence stresses than actually exist. This makes the pin joint approximation a conservative assumption.

Analyzing complex mathematical models requires numerical solutions. Here we shall consider simple structures made up of only a few members. Some members of the structure may be subjected to one type of combined loading, whereas other members may be subjected to another combination of loading.

There are two major steps in the solution of problems related to the analysis and design of structures:

1. Analysis of internal forces and moments that act on individual members.
2. Computation of stresses on members under combined loading.

For statically determinate structures, the internal forces and moments can be found using the principles of statics, as shown in Example 10.6. For statically indeterminate structures, we will also need the deformation equations developed earlier to complement the analysis skills learned in statics as we will see in Example 10.7.

10.2.1 Failure Envelope

In design, unlike analysis, the variable can assume a multitude of values. Each set of values corresponds to a different design. Which set of values gives the *best* design? The word *best*, of course, implies an objective. In engineering the objective is generally to minimize weight or cost. Algorithms that minimize an *objective function* (such as weight) subject to *constraints* (such as limits on stresses and displacements) are called *optimization methods*. Although optimization methods are beyond the scope of this book, we can develop an appreciation of the methods using the concept of failure envelope. A **failure envelope** separates the acceptable *design space* from the unacceptable values of the variables affecting design. More rigorous optimization algorithms would search the design space systematically, to minimize (or maximize) the objective function.

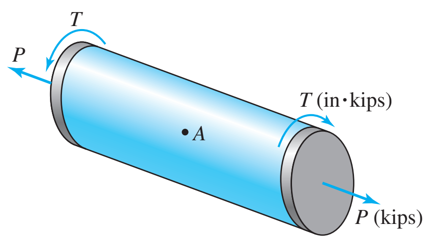
Consider a circular shaft that is subjected to axial loads and torsion, as shown in Figure 10.33*a*. Suppose the design limitation is that the maximum shear stress should not exceed 15 ksi. Further suppose that calculations show that the maximum shear stress at point A on the surface of the shaft is given by $\tau_{max} = 0.3183\sqrt{P^2 + 4T^2}$ ksi. Now τ_{max} should be less than or equal to 15 ksi, which gives us the following result: $P^2 + 4T^2 \le 2220$. We can now make a plot of T versus P, as shown in Figure 10.33*b*.

The shaded area consists of all possible values of T and P for which the maximum shear stress will be less than 15 ksi and hence represents our acceptable design space. The region beyond the shaded area represents values of T and P for which the shear stress is greater than 15 ksi and hence represents the failure space. On the curve $P^2 + 4T^2 = 2220$ all values of P and T

would result in a maximum shear stress of 15 ksi, and we are at incipient failure. This curve represents the failure envelope, which separates the design space from the failure space.

We can generalize this approach to a design problem containing *n* variables, which could be geometric variables, material constants, or loads, as in Figure 10.33. We may need to find the values of the variables to meet several design constraints. If one took each design variable and plotted it on an axis, then one would obtain a *n*-dimensional space containing all possible combinations of the *n* variables. Some of these combinations of the variables would result in failure. A failure envelope separates the space of acceptable values of these variables from the unacceptable values. On the failure envelope the values of the design variables correspond to impending failure. The sum total of all the design constraints defines the failure envelope. We shall also use the concept of failure envelope in Section 10.3 to describe failure theories in which the variables are principal stresses that are plotted on an axis.

Within the failure envelope we can compare different designs with respect to other criteria, such as cost, weight, and aesthetics. Example 10.8 elaborates the construction and use of the failure envelope.



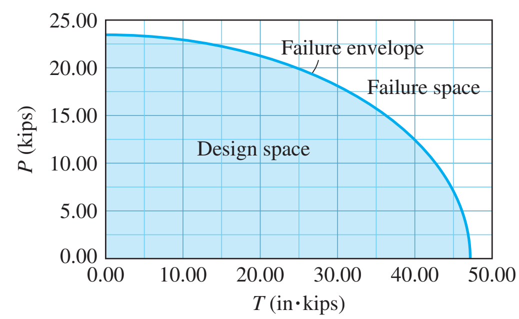


Figure 10.33 Failure envelope.

EXAMPLE 10.6

A hoist is to be designed for lifting a maximum weight $W = 300$ lb. Space considerations have established the length dimensions shown in Figure 10.34. The hoist will be constructed using lumber and assembled using steel bolts. The dimensions of the lumber cross sections are listed in Table 10.2. The bolted joints will be modeled as pins in single shear. Same-size bolts will be used in all joints. The allowable normal stress in the wood is 1.2 ksi and the allowable shear stress in the bolts is 6 ksi. Design the lightest hoist by choosing the lumber from Table 10.2 and the bolt size to the nearest $\frac{1}{8}$-in. diameter.

TABLE 10.2 **Dimensions of lumber**[a]

Cross-Section Dimensions	Cross-Section Dimensions
2 in. × 4 in.	4 in. × 8 in.
2 in. × 6 in.	6 in. × 6 in.
2 in. × 8 in.	6 in. × 8 in.
4 in. × 4 in.	8 in. × 8 in.
4 in. × 6 in.	

[a] The dimensions for finished lumber are slightly smaller and must be properly accounted in actual design.

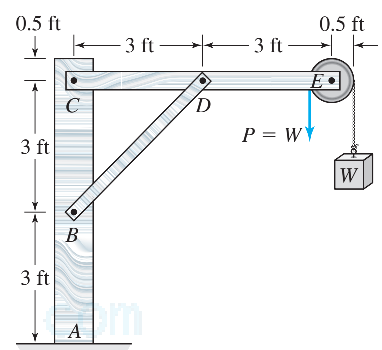


Figure 10.34 Hoist in Example 10.6.

PLAN

We analyze the problem in two steps. First we find the forces and moments on individual members, and then we find the stresses.

1. *BD* is a two-force axial member that will be in compression. Members *ABC* and *CDE* are multiforce members subjected to axial and bending loads. Free-body diagrams of members *ABC* and *CDE* will permit calculation of the forces at pin *C*, the axial force in *BD*—forces on pins *B* and *D* are thus known, as well as the reaction forces and the reaction moment at *A*.
2. We compute the maximum stresses from the forces calculated in step 1 and, using the limiting values on the maximum stresses, compute the dimensions of the pin and the wooden members. From the possible set of dimensions that satisfy the limiting criteria we choose those that will result in the lightest structure.

SOLUTION

Calculation of forces and moments on structural members: Figure 10.35 shows the free-body diagrams of members *CDE* and *ABC*. Using the moment equilibrium at point *C* in Figure 10.35*a*, we obtain

$$(N_{BD}\sin 45°)(3\text{ ft}) - (300\text{ lbs})(5.5\text{ ft}) - (300\text{ lbs})(6.5\text{ ft}) = 0 \qquad \text{or} \qquad N_{BD} = 1697\text{ lbs} \tag{E1}$$

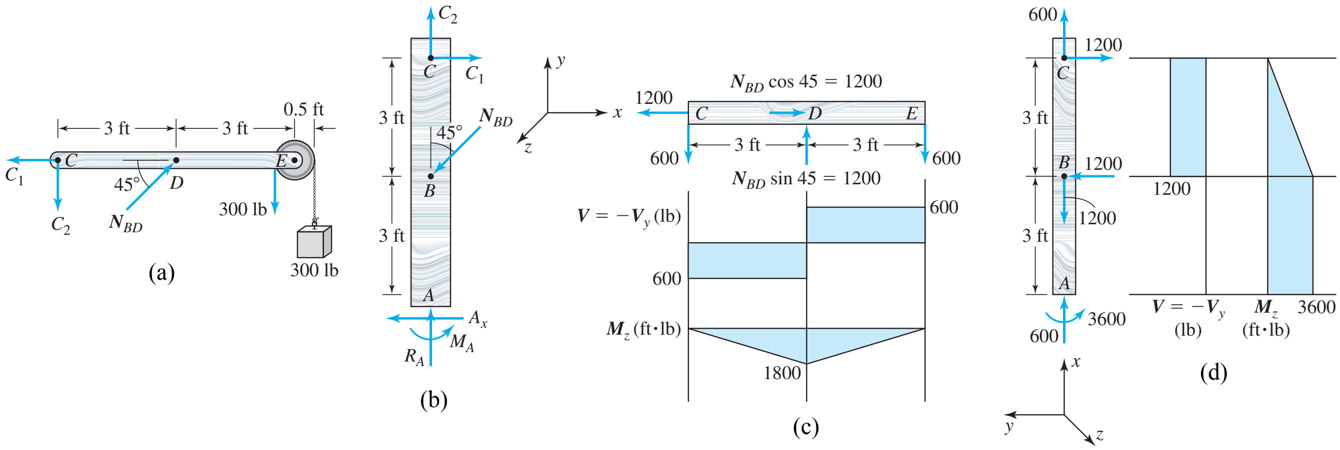


Figure 10.35 (a, b) Free-body diagram of member CDE and ABC. (c, d) Shear and moment diagrams of members CDE and ABC.

By equilibrium of forces in Figure 10.35*a,* we obtain

$$C_1 - N_{BD}\cos 45° = 0 \qquad \text{or} \qquad C_1 = 1200\text{ lbs} \tag{E2}$$

$$N_{BD}\sin 45° - C_2 - 600\text{ lbs} = 0 \qquad \text{or} \qquad C_2 = 600\text{ lbs} \tag{E3}$$

By equilibrium of forces and moment about point *A* in Figure 10.35*b*, we obtain

$$C_1 - N_{BD}\sin 45° - A_x = 0 \qquad \text{or} \qquad A_x = 0 \tag{E4}$$

$$R_A + C_2 - N_{BD}\cos 45° = 0 \qquad \text{or} \qquad R_A = 600\text{ lbs} \tag{E5}$$

$$M_A - C_1(6\text{ ft}) + (N_{BD}\cos 45°)(3\text{ ft}) = 0 \qquad \text{or} \qquad M_A = 3600\text{ ft}\cdot\text{lb} \tag{E6}$$

Bolt size calculations: The shear force acting on each bolt can be found from the forces calculated,

$$V_B = V_D = N_{BD} = 1697\text{ lbs} \qquad V_C = \sqrt{C_1^2 + C_2^2} = 1342\text{ lbs} \qquad V_E = 600\text{ lbs} \tag{E7}$$

The maximum shear stress will be in bolts *B* and *D*. This maximum shear stress should be less than 6 ksi. The cross-sectional area can be found and the diameter of the bolt calculated,

$$\tau_{max} = \frac{1697\text{ lbs}}{\pi d^2/4} \le 6000\text{ lbs/in.}^2 \qquad \text{or} \qquad d \ge 0.60\text{ in.} \tag{E8}$$

The nearest $\frac{1}{8}$-in. size that is greater than the numerical value in Equation (E8) is $d = 0.625$ in.

ANS. $\frac{5}{8}$-in.-size bolts should be used.

Lumber selection: The normal axial stress in member *BD* has to be less than 1200 psi. The cross-sectional area for member *BD* can be found as

$$\sigma_{BD} = \frac{N_{BD}}{A_{BD}} = \frac{1697\text{ lbs}}{A_{BD}} \le 1200\text{ lbs/in.}^2 \qquad \text{or} \qquad A_{BD} \ge 1.414\text{ in}^2 \tag{E9}$$

The 2-in. × 4-in. lumber has a cross-sectional area of 8 in.^{2}, which is the smallest cross section that meets the restriction of Equation (E9).

ANS. For member BD use lumber with the cross-section dimensions of 2 in. × 4 in**.**

Shear force and bending moment diagrams for members *ABC* and *CDE* can be drawn after resolving the force ***NBD*** into components parallel and perpendicular to the axis, as shown in Figure 10.35*c, d.* A local *x*, *y*, *z* coordinate system for each member is established to facilitate drawing the shear and moment diagrams.

From Figure 10.35*c* it can be seen that the maximum axial force is 1200 lbs tensile in segment *CD*. The bending moment is maximum on the cross section at *D* in member *CDE*. Due to bending, the top surface will be in tension and the bottom will be in compression. Thus the maximum normal stress in *CDE* will be at the top surface just before *D* and will be the sum of tensile stresses due to axial and bending loads. Using an axial force of 1200 lbs and a bending moment of 1800 ft·lbs = 21,600 in.·lbs, the maximum normal stress in *CDE* can be written as

$$\sigma_{CD} = \frac{1200\text{ lbs}}{A_{CDE}} + \frac{21{,}600\text{ in.}\cdot\text{lbs}}{S_{CDE}} \tag{E10}$$

where A_{CDE} and S_{CDE} are the cross-sectional area and the section modulus (with respect to the z axis) of member *CDE*.

From Figure 10.35*d* it can be seen that the axial force in *AB* is 600 lbs compressive and the axial force in *BC* is 600 lbs tensile. The bending moment is a maximum of 1800 ft · lbs = 43,200 in. · lbs throughout segment *AB*. Owing to bending, the right side of member *ABC* will be in compression and the left side will be in tension. Thus the maximum normal stress in member *ABC* will be on the right surface, just before *B* in segment *AB*, and will be the sum of compressive stresses due to axial and bending loads. With an axial force of 600 lbs and a bending moment of 3600 ft · lbs = 43,200 in. · lbs, the maximum compressive normal stress in *ABC* can be written as

$$\sigma_{AB} = \frac{600\ \text{lbs}}{A_{ABC}} + \frac{43{,}200\ \text{in.}\cdot\text{lbs}}{S_{ABC}} \tag{E11}$$

where A_{ABC} and S_{ABC} are the cross-sectional area and the section modulus (with respect to the z axis) of member *ABC*.

For the available lumber given in Table 10.2, the cross-sectional area A and the section modulus S can be determined assuming that the smaller dimension a is parallel to the z axis (bending axis or dimension out of the plane of the paper) and the larger dimension b is in the plane of the paper. With this stipulation $I_{zz} = ab^3/12$ and $y_{max} = b/2$. Hence $S = I_{zz}/y_{max} = ab^2/6$ (see the local coordinates in Figure 10.35*c, d*). Substituting the values of A and S into Equations (E10) and (E11), we find the stress values σ_{CD} and σ_{AB}. Table 10.2 can be created using a spreadsheet. Cross-section dimensions for which the normal stress is less than 1200 psi meet the strength limitation and are identified in bold in Table 10.2. But for the design of the lightest hoist we choose the cross section with the smallest area among the bold values.

ANS. For member *ABC*, use lumber with the cross-section dimensions of 4 in. × 8 in.
ANS. For member *CDE*, use lumber with the cross-section dimensions of 2 in. × 8 in.

TABLE 10.2 **Cross-section properties and stresses in Example 10.6**

a **(in.)**	***b*** **(in.)**	***A = ab*** **(in.^{2})**	***S = ab^2/6*** **(in.^{3})**	σ_{CD} **(psi)**	σ_{AB} **(psi)**
2	4	8	5.3	4200.0	8175.0
2	6	12	12.0	1900.0	3650.0
2	8	16	21.3	**1087.5**	2062.5
4	4	16	10.7	2100.0	4087.5
4	6	24	24.0	**950.0**	1825.0
4	8	32	42.7	**543.8**	**1031.3**
6	6	36	36.0	**633.3**	1216.7
6	8	48	64.0	**362.5**	**687.5**
8	8	64	85.3	**271.9**	**515.6**

COMMENTS

1. Members in axial compression such as *BD* must be designed for strength as well as checked for buckling failure, as will be elaborated in the next chapter.
2. In actual design it may be preferable to use two pieces of 1-in. × 8-in. lumber for member *CDE* so that the pulley is in the middle of the two members. This will change pins at *C*, *D*, and *E* from single shear into double shear, thus also reducing the shear stresses in the pins.
3. Equation (E10) defines the curve of the failure envelope for member *CDE*. Substituting for the area and the section modulus in terms of a and b, and solving for a in terms of b, we obtain the equation of the curve defining the failure envelope as $a = 0.5/b + 216/b^2$. Figure 10.36 shows the failure envelope. As can be seen, the three possible solutions in Table 10.2 for member *CDE* fall in the design space and the remaining cross sections fall in the failure space. If we were to choose any value for a and b, then the failure envelope would identify all possible solutions.

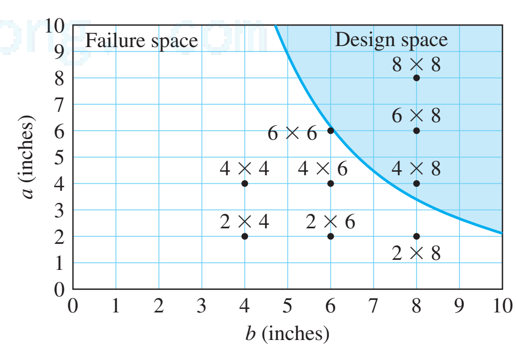


Figure 10.36 Failure envelope for member *CDE*.

4. The bending shear stress was not considered in selecting the lumber cross sections for members *ABC* and *CDE*. This is based on the consideration that the maximum bending normal stress is significantly (~10 times) greater than the maximum bending shear stress. Thus the principal stress at the top or bottom of the member, where the bending normal stress is maximum, will be greater than the

principal shear stress at the neutral axis, where bending shear stress is maximum. We check these statements for the selected sizes of members *ABC* and *CDE* as follows.

For rectangular cross sections it can be shown (see comment 1 in Example 6.14) that the maximum shear stress in bending at a cross section is $\tau_{max} = 1.5V/A$, where *A* is the cross-sectional area. From Figure 10.35 the maximum shear force is 600 lb and 1200 lb in members *CDE* and *ABC*, respectively. Substituting these shear force values and the values of 16 in.^{2} and 32 in.^{2} for the areas, we obtain the maximum shear stresses of $\tau_{CD} = 56.25$ psi and $\tau_{AB} = 56.25$ psi in members *CDE* and *ABC*, respectively. Comparing these maximum shear stress values to the maximum normal stress values of 1087.5 psi and 1031.3 psi for members *CDE* and *ABC*, which are given in Table 10.2, we conclude that the shear stresses can be ignored in the selection of lumber.

EXAMPLE 10.7

A rectangular wooden beam of 60 mm × 180 mm cross section is supported at the right end by an aluminum circular rod of 8-mm diameter, as shown in Figure 10.37. The allowable normal stress in the wood is 14 MPa and the allowable shear stress in aluminum is 60 MPa. The moduli of elasticity for wood and aluminum are $E_w = 12.6$ GPa and $E_{al} = 70$ GPa. Determine the maximum intensity *w* of the distributed load that the structure can support.

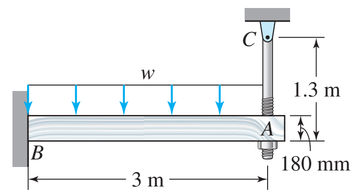


Figure 10.37 Beam in Example 10.7.

PLAN

We analyze the problem in two steps. First we find the forces and moments on individual members, and then we find the stresses.

1. To solve this statically indeterminate problem, the deflection of the beam at *A* can be equated to the axial deformation of the aluminum rod. This permits the calculation of the internal axial force in the aluminum rod in terms of *w*.
2. The axial stress in the aluminum rod in terms of *w* can be found from the internal axial force calculated in step 1. Using Mohr's circle, we can find the maximum shear stress in the axial rod in terms of *w*, and one limit on *w* can be obtained. The maximum bending moment at *B* can be found in terms of *w* and the maximum bending normal stress calculated in terms of *w*. Using the allowable value of 14 MPa, another limit on *w* can be found and a decision made on the maximum value of *w*.

SOLUTION

Calculation of forces on structural members: The area moment of inertia of the wood and the cross-sectional area of the aluminum rod can be calculated,

$$I_w = \frac{1}{12}(60\text{ mm})(180\text{ mm})^3 = 29.16(10^6)\text{ mm}^4 = 29.16(10^{-6})\text{ m}^4 \qquad A_{al} = \frac{\pi}{4}(8\text{ mm})^2 = 50.265\text{ mm}^2 = 50.265(10^{-6})\text{ m}^2 \quad \text{(E1)}$$

Making an imaginary cut through the aluminum rod, we obtain the beam and loading shown in Figure 10.38*a*. The total loading on the beam can be considered as the sum of the two loadings shown in Figure 10.38*b* and *c*.

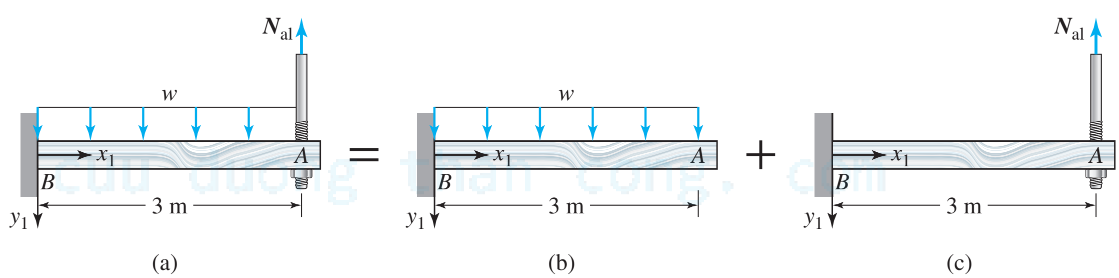


Figure 10.38 Superposition of deflection in Example 10.7.

Comparing the two beam loadings in Figure 10.38*b* and *c* to that shown for cases 1 and 3 in Table C.3, we obtain $P = -N_{al}$ N and $p = w$ N/m, $a = 3$ m, $b = 0$, $E = 12.6\,(10^9)$ N/m^2, and $I = I_w = 29.16\,(10^{-6})$ m^4. Noting that v_{max} shown in Table C.3 for the cantilever beam occurs at point *A*, we can substitute the load values and superpose to obtain deflection v_A,

$$v_A = \frac{(w\text{ N/m})(3\text{ m})^4}{8[12.6(10^9)\text{ N/m}^2][29.16(10^{-6})\text{ m}^4]} + \frac{(-N_{al}\text{ N})(3\text{ m})^3}{3[12.6(10^9)\text{ N/m}^2][29.16(10^{-6})\text{ m}^4]} \quad \text{or}$$

$$v_A = [27.56(10^{-6})\,w - 24.50(10^{-6})\,N_{al}]\text{ m} \qquad \text{(E2)}$$

The extension of the aluminum rod can be found using Equation (4.21),

$$\delta_{al} = \frac{N_{al}L_{al}}{E_{al}A_{al}} = \frac{(N_{al}\text{ N})(1.3\text{ m})}{[70(10^9)\text{ N/m}^2][50.265(10^{-6})\text{ m}^4]} = 0.369(10^{-6})N_{al}\text{ m} \tag{E3}$$

The extension of the aluminum rod should equal the deflection of the beam at A. Equating Equations (E2) and (E3) gives the internal force N_{al} in terms of w,

$$27.56(10^{-6})w - 24.50(10^{-6})N_{al} = 0.369(10^{-6})N_{al} \qquad \text{or} \qquad N_{al} = 1.11w \tag{E4}$$

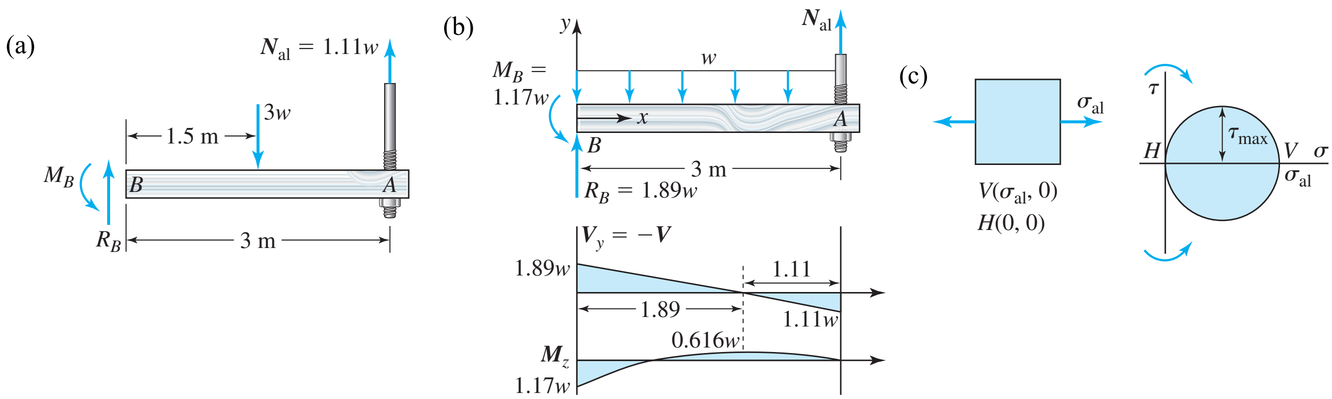


Figure 10.39 (a) Free-body diagram (b) Shear force and bending moment diagrams (c) Mohr's circle in Example 10.7.

Figure 10.39*a* shows the free-body diagram of the beam with the distributed force replaced by an equivalent force. By force and moment equilibrium we obtain

$$R_B - (3w) + N_{al} = 0 \qquad \text{or} \qquad R_B = 1.89w \tag{E5}$$

$$M_B - (3w)(1.5) + N_{al}(3) = 0 \qquad \text{or} \qquad M_B = 1.17w \tag{E6}$$

Figure 10.39*b* shows the shear and moment diagrams for the beam. From the moment diagram we see that the maximum moment is at the wall, and its value is

$$M_{max} = -1.17w \tag{E7}$$

Stress in aluminum: The axial stress in aluminum in terms of w is

$$\sigma_{al} = \frac{N_{al}}{A_{al}} = \frac{1.11w}{50.265(10^{-6})\text{ m}^2} = 22.04(10^3)w\text{ N/m}^2 \tag{E8}$$

Figure 10.39*c* shows the Mohr's circle for point on aluminum rod. The maximum shear stress maximum shear stress in should be less than 60 MPa,

$$\tau_{max} = \frac{\sigma_{al}}{2} = 11.02(10^3)w\text{ N/m}^2 \le 60(10^6)\text{ N/m}^2 \qquad \text{or} \qquad w \le 5.44(10^3)\text{ N/m} \tag{E9}$$

Stress in wood: The bending normal stress will be maximum at the top and bottom surfaces at the wall; that is, $y_{max} = \pm 0.09$ m. The magnitude of the maximum bending normal stress from Equation (10.3*a*) should be less than 14 MPa, yielding another limit on w,

$$\sigma_w = \left|\frac{M_{max}y_{max}}{I_w}\right| = \frac{(1.17w\text{ N}\cdot\text{m})(0.09\text{ m})}{29.16(10^{-6})\text{ m}^4} = 3.61(10^3)w\text{ N/m}^2 < 14(10^6)\text{ N/m}^2 \qquad \text{or} \qquad w \le 3.88(10^3)\text{ N/m} \tag{E10}$$

The value in Equation (E10) also satisfies the inequality in Equation (E9) giving the maximum intensity of the distributed load.

ANS. $w_{max} = 3.88\text{ kN/m}$

COMMENT

1. At joint A we ensured continuity of displacement by enforcing the condition that the deformation of the axial member be the same as the deflection of the beam. We also enforced equilibrium of forces by using the same force N_{al} in the axial member and acting on the beam. These two conditions, continuity of displacement and equilibrium of forces, must be satisfied by all joints in more complex structures.

EXAMPLE 10.8

A circular member was repaired by welding a crack at point A that was 30° to the axis of the shaft, as shown in Figure 10.40. The allowable shear stress at point A is 24 ksi and the maximum normal stress the weld material can support is 9 ksi (T). Calculations show that the stresses at point A are $\sigma_{xx} = 9.55P_2$ ksi (T) and $\tau_{xy} = -6.79P_1$ ksi. (a) Draw the failure envelope for the applied loads P_1 and P_2. (b) Determine the values of loads P_1 and P_2. (c) If $P_1 = 2$ kips and $P_2 = 1.5$ kips, determine the factor of safety.

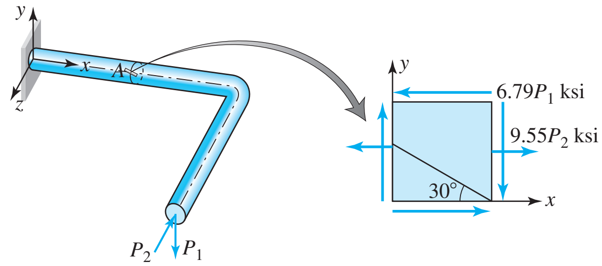


Figure 10.40 Problem geometry in Example 10.8.

PLAN

(a) By substituting the given stresses into Equation (8.12) we obtain the maximum shear stress in the material in terms of P_1 and P_2. Noting that the maximum shear stress is limited to 24 ksi, we obtain one equation relating P_1 and P_2. By substituting the given stresses into Equation (8.1) as well as the angle of the normal to the weld, we can obtain the normal stress on the weld, which gives us another equation relating P_1 and P_2. We can sketch both equations and obtain the failure envelope. (b) We can find the values of P_1 and P_2 that satisfy the two equations in part (a). (c) We can calculate the maximum in-plane shear stress in the material and the normal stress in the weld from the equations obtained in part (a) and compute two factors of safety. The lower value is the factor of safety for the system.

SOLUTION

(a) Substituting the values of the given stresses into Equation (8.12), we can obtain the maximum shear stress in the material. Noting that it should be less than 24 ksi, we obtain one equation on P_1 and P_2,

$$\tau_{max} = \sqrt{\left(\frac{9.55P_2\text{ ksi}}{2}\right)^2 + (-6.79P_1\text{ ksi})^2} = \sqrt{46.1P_1^2 + 22.8P_2^2}\text{ ksi} \le 24\text{ ksi} \quad \text{or} \quad 46.1P_1^2 + 22.8P_2^2 \le 576 \qquad \text{(E1)}$$

The normal to the weld makes an angle of 60° to the x axis. Substituting the given stresses and $\theta = 60°$ into Equation (8.1), the normal stress on the weld must be less than 9 ksi (T), we obtain another equation on P_1 and P_2,

$$\sigma_{weld} = (9.55P_2\text{ ksi})\cos^2 60° + 2(-6.79P_1\text{ ksi})\cos 60° \sin 60° \quad \text{or} \quad \sigma_{weld} = (2.387P_2 - 5.881P_1)\text{ ksi} \le 9\text{ ksi} \qquad \text{(E2)}$$

The maximum value of P_1 that will satisfy Equation (E1) corresponds to $P_2 = 0$. This maximum value of P_1 is 3.534 kips. We consider values of P_1 between zero and 3.534 in steps of 0.3 and solve for P_2 from Equation (E1). For the same values of P_1 we can also find values of P_2 from Equation (E2), as shown in Table 10.3, which was produced on a spreadsheet. We can plot the values in Table 10.3, as shown in Figure 10.41. The design space is the shaded region and the failure envelope is the boundary $ABCD$

TABLE 10.3 **Values of loads in Example 10.8**

P_1 (kips)	P_2 from Eq. (E1) (kips)	P_2 from Eq. (E2) (kips)
0.000	5.027	3.770
0.300	5.008	4.509
0.600	4.954	5.248
0.900	4.861	5.987
1.200	4.728	6.726
1.500	4.551	7.465
1.800	4.326	8.204
2.100	4.043	8.943
2.400	3.690	9.682
2.700	3.244	10.421
3.000	2.657	11.160
3.300	1.800	11.899

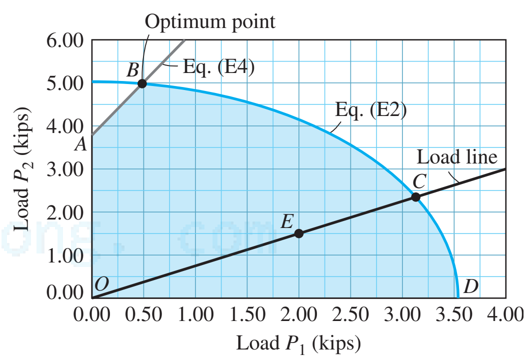


Figure 10.41 Failure envelope in Example 10.8.

(b) The values of P_1 and P_2 correspond to the maximum values of the loads that satisfy Equations (E1) and (E2). Using the equality sign in Equation (E2), we can solve for P_2,

$$P_2 = 3.770 + 2.4634P_1 \qquad \text{(E3)}$$

We substitute Equation (E3) into Equation (E1) with the equality sign to obtain a quadratic equation in P_1,

$$46.11P_1^2 + 22.80(4.19 + 2.46P_1)^2 = 576 \qquad \text{or} \qquad 184.45P_1^2 + 423.2P_1 - 252 = 0 \tag{E4}$$

Solving Equation (E3) we obtain two roots for P_1, 0.4905 and –2.784. Only the positive root is admissible. Substituting $P_1 = 0.4905$ into Equation (E3), we obtain $P_2 = 4.978$.

ANS. $P_1 = 0.49$ kips $\quad P_2 = 4.98$ kips

(c) Substituting $P_1 = 2$ kips and $P_2 = 1.5$ into Equations (E1) and (E2), we obtain

$$\tau_{max} = \sqrt{46.1 \times 2^2 + 22.8 \times 1.5^2} = 15.35 \text{ ksi} \qquad \sigma_{weld} = 2.387 \times 1.5 - 5.881 \times 2 = -8.18 \text{ ksi} \tag{E5}$$

For the given loads the normal stress in the weld is compressive. Hence it won't fail due to the specified failure in tension. The factor of safety is thus calculated from the maximum shear stress as

$$K = \frac{24 \text{ ksi}}{15.35 \text{ ksi}} = 1.56 \tag{E6}$$

ANS. $K = 1.56$

COMMENTS

1. In this example we generated the failure envelope using analytical equations. For more complex structures, the failure envelope can be created using numerical methods, such as the finite-element method described in Section 4.8.
2. In Figure 10.41 line *AB*, representing Equation (E1), would go downward if the direction of load P_1 were reversed [Substitute $-P_1$ in place of P_1 in Equation (E1)]. If line *AB* went downward, it would cut the design space considerably. Thus not only is the magnitude of the loads important in design, but the direction of the load can also be as critical. Failure envelopes can reveal such characteristics in a very visual manner.
3. The line joining the origin to point *E* is called *load line,* on which the loads vary proportionally. It's significance is that it can help give a graphical interpretation of the factor of safety. Along a load line, the distance of a point from the failure envelope is the margin of safety. In Figure 10.41 the factor of safety is the ratio of length *OC* to length *OE*. It can be verified that the coordinates of point *C* are $P_1 = 3.1263$ kips and $P_2 = 2.344$. Thus the length $OC = 3.9074$, whereas the length *OE* is 2.5. The factor of safety therefore is $K = 3.9074/2.5 = 1.56$, as before.

PROBLEM SET 10.2

Structural analysis and design

10.41 A brick chimney shown in Figure P10.41 has an outside diameter of 5 ft and a wall thickness of 6 in. The average specific weight of the brick and mortar is $\gamma = 120$ lb/ft^3. The height of the chimney is $H = 30$ ft. Determine the maximum wind pressure p that the chimney could withstand if there is to be no tensile stress.

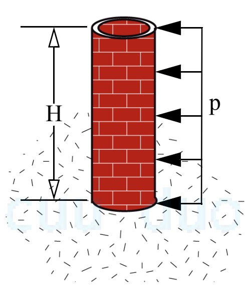


Figure P10.41

10.42 A brick chimney shown in Figure P10.41 has an outside diameter of 1.5 m and a wall thickness of 150 mm. The average specific density of the brick and mortar is $\gamma = 1800$ kg/m^3. The wind pressure acting on the chimney is $p = 800$ Pa. Determine the maximum height H of the chimney is if there is to be no tensile stress.

10.43 A hollow shaft that has an outside diameter of 100 mm and an inside diameter of 50 mm is loaded as shown in Figure P10.43. The normal stress and the shear stress in the shaft must be limited to 200 MPa and 115 MPa, respectively. (a) Determine the maximum value of the torque T that can be applied to the shaft. (b) Using the result of part (a), determine the strain that will be shown by the strain gage that is mounted on the surface at an angle of 35° to the axis of the shaft. Use $E = 200$ GPa, $G = 80$ GPa, and $\nu = 0.25$.

10.44 On the C clamp shown in Figure P10.44*a* determine the maximum clamping force *P* if the allowable normal stress is 160 MPa in tension and 120 MPa in compression.

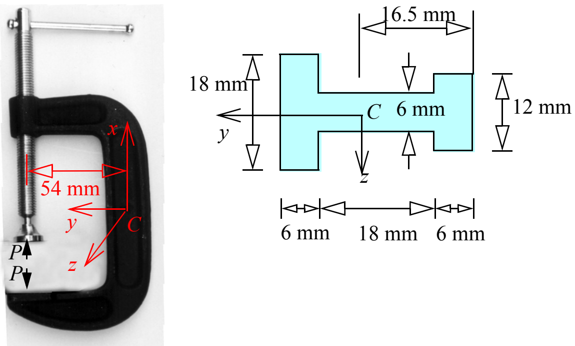


Figure P10.44

10.45 The T cross section of the beam was constructed by gluing two rectangular pieces together. A small crack was detected in the glue joint at section *AA*. Determine the maximum value of the applied load *P* if the normal stress in the glue at section *AA* is to be limited to 20 MPa in tension and 12 MPa in shear. The load *P* acts at the centroid of the cross section at *C*, as shown in Figure P10.45.

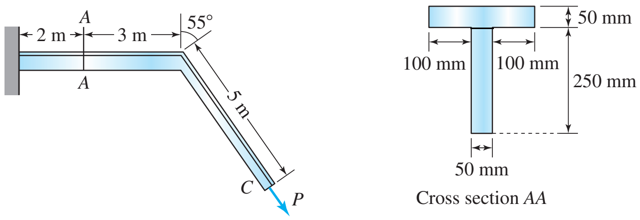


Figure P10.45

10.46 The bars in the pin connected structure shown in Figure P10.46 are circular bars of diameters that are available in increments of 5 mm. The allowable shear stress in the bars is 90 MPa. Determine the diameters of the bars for designing the lightest structure to support a force of $P = 40$ kN.

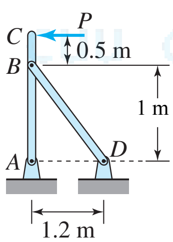


Figure P10.46

10.47 Member *AB* has a circular cross section with a diameter of 0.75 in. as shown in Figure P10.47 Member *BC* has a square cross section of 2 in. × 2 in. Determine the maximum normal stress in members *AB* and *BC*.

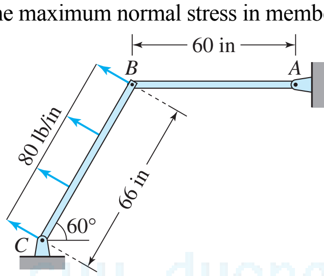


Figure P10.47

10.48 The members of the structure shown in Figure P10.48 have rectangular cross sections and are pin connected. Cross-section dimensions for members are 100 mm × 150 mm for *ABC*, 100 mm × 200 mm for *CDE*, and 100 mm × 50 mm for *BD*. The allowable normal stress in the members is 20 MPa. Determine the maximum intensity of the distributed load *w*.

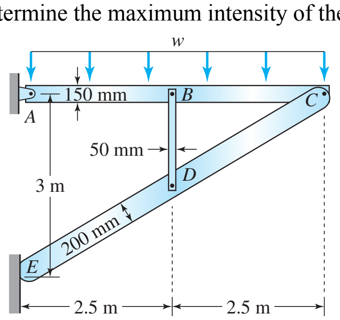


Figure P10.48

10.49 A hoist is to be designed for lifting a maximum weight $W = 300$ lb, as shown in Figure P10.49 The hoist will be installed at a certain height above ground and will be constructed using lumber and assembled using steel bolts. The lumber rectangular cross-section dimensions are listed in Table 10.2. The bolt joints will be modeled as pins in single shear. Same-size bolts will be used in all joints. The allowable normal stress in the wood is 1.2 ksi and the allowable shear stress in the bolts is 6 ksi. Design the lightest hoist by choosing the lumber from Table 10.2 and the bolt size to the nearest $\frac{1}{8}$-in. diameter.

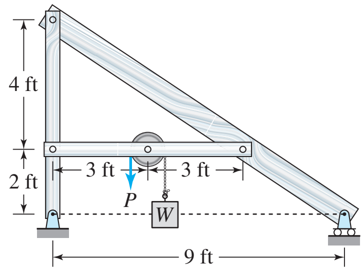


Figure P10.49

10.50 A rectangular wooden beam of 4-in. × 8-in. cross section is supported at the right end by an aluminum circular rod of $\frac{1}{2}$-in. diameter, as shown in Figure P10.50. The allowable normal stress in the wood is 1.5 ksi and the allowable shear stress in aluminum is 8 ksi. The moduli of elasticity for wood and aluminum are $E_w = 1800$ ksi and $E_{al} = 10{,}000$ ksi. Determine the maximum force P that the structure can support.

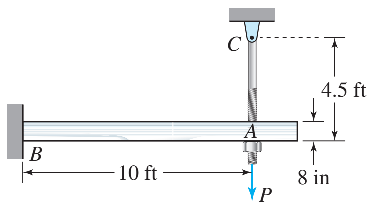
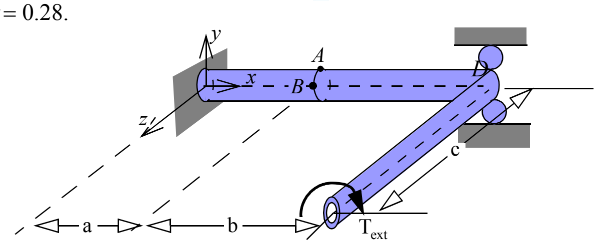


Figure P10.50

10.51 A steel pipe with an outside diameter of 1.5 in. and a wall thickness of $\frac{1}{4}$ in. is simply supported at D. A torque of $T_{ext} = 30$ in.·kips is applied as shown in Figure P10.51. If a = 12 in., b = 48 in., and c = 60 in., determine the normal and shear stresses at points A and B in the x, y, z coordinate system and show them on a stress cube. Points A and B are on the surface of the pipe. The modulus of elasticity is $E = 30{,}000$ ksi and Poisson's ratio is $\nu = 0.28$.



FigurP10.51

10.52 A composite beam is constructed by attaching steel strips at the top and bottom of a wooden beam, as shown in Figure P10.52 The beam is supported at the right end by an aluminum circular rod of 8-mm diameter. The allowable normal stresses in the wood and steel are 14 MPa and 140 MPa, respectively. The allowable shear stress in aluminum is 60 MPa. The moduli of elasticity for wood, steel, and aluminum are $E_w = 12.6$ GPa, $E_s = 200$ GPa, and $E_{al} = 70$ GPa, respectively. Determine the maximum intensity w of the distributed load that the structure can support.

10.53 A park structure is modeled with pin joints at the points shown in Figure P10.53. Members BD and CE have cross-sectional dimensions of 6 in. × 6 in., whereas members AB, AC, and BC have cross-sectional dimensions of 2 in. × 8 in. Determine the maximum normal and shear stresses in each of the members due to the estimated snow load shown on the structure.

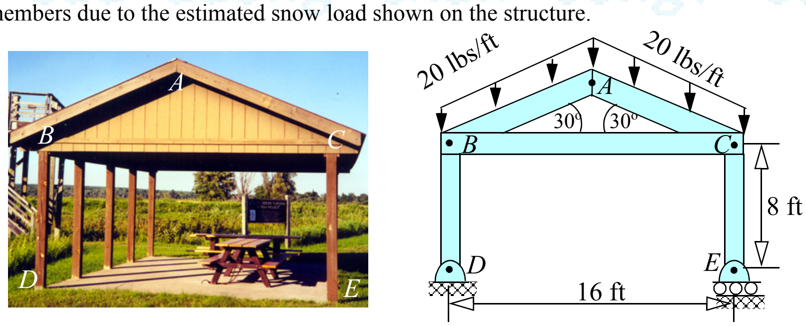


Figure P10.53

10.54 A highway sign uses a 16-in. hollow pipe as a vertical post and 12-in. hollow pipes for horizontal arms, as shown in Figure P10.54. The pipes are 1 in. thick. Assume that a uniform wind pressure of 20 lb/ft^2 acts on the sign boards and the pipes. Note that the pressure on the pipes acts on the projected area Ld, where L is the length of pipe and d is the pipe diameter. Neglecting the weight of the pipe, determine the normal and shear stresses at points A and B and show these stresses on stress cubes.

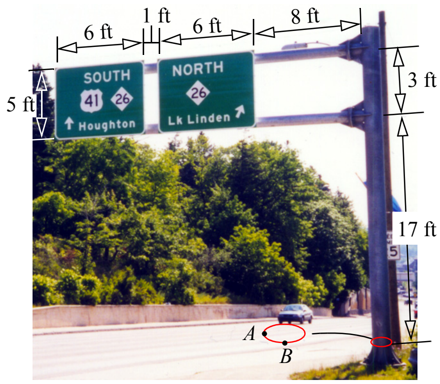


Figure P10.54

10.55 A bicycle rack is made from thin aluminum tubes of $\frac{1}{16}$-in. thickness and 1-in. outer diameter. The weight of the bicycles is supported by the belts from C to D and the members between C and B. Member AC carries negligible force and is neglected in the stress analysis, as shown on the model in Figure P10.55*b*. If the allowable normal stress in the steel tubes is 12 ksi and the allowable shear stress is 8 ksi, determine the maximum weight W to the nearest lb of each bicycle that can be put on the rack.

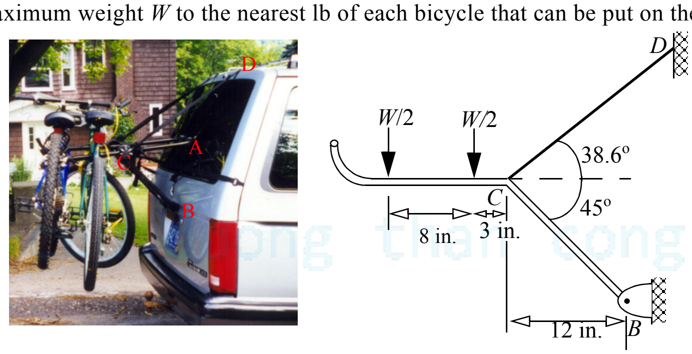


Figure P10.55

10.56 The hoist shown in Figure P10.56 was used to lift heavy loads in a mining operation. Member EF supported load only if the load being lifted was asymmetric with respect to the pulley; otherwise it carried no load and can be neglected in the stress analysis. If the allowable normal stress in steel is 18 ksi, determine the maximum load W that could be lifted using the hoist.[1]

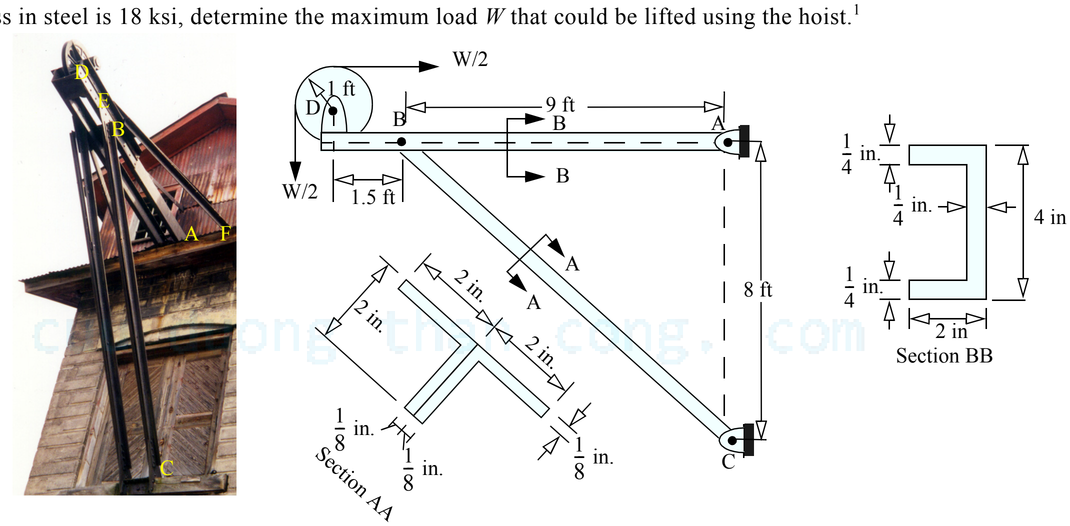


Figure P10.56

Failure envelopes

10.57 A solid shaft of 50-mm diameter is made from a brittle material that has an allowable tensile stress of 100 MPa, as shown in Figure P10.57. Draw a failure envelope representing the maximum permissible positive values of T and P.

[1]Though the load on section *BB* is not passing through the plane of symmetry, the theory of symmetric bending can still be used because of the structure symmetry.

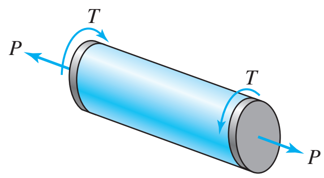


Figure P10.57

10.58 The shaft shown in Figure P10.57 is made from a ductile material and has an allowable shear stress of 75 MPa. Draw a failure envelope representing the maximum permissible positive values of T and P.

10.59 The shaft in Problem 10.57 is 1.5 m long and has a modulus of elasticity $E = 200$ GPa and a modulus of rigidity $G = 80$ GPa. Modify the failure envelope of Problem 10.57 to incorporate the limitation that the elongation cannot exceed 0.5 mm and the relative rotation of the right end with respect to the left end cannot exceed 3°.

10.60 A pipe with an outside diameter of 40 mm and a wall thickness of 10 mm is loaded as shown in Figure P10.60. At section AA the allowable shear stress is 60 MPa. Draw the failure envelope for the applied loads P_1 and P_2. Use a = 2.5 m, b = 0.4 m, c =0.1 m.

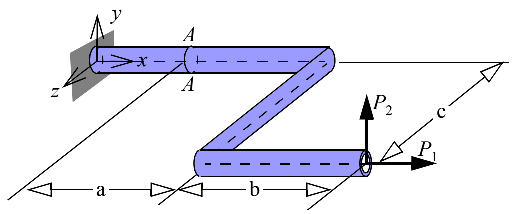


Figure P10.60

10.61 A bent pipe of 2-in. outside diameter and a wall thickness of $\frac{1}{4}$-in. is loaded as shown in Figure P10.61. The maximum shear stress the pipe material can support is 24 ksi. Draw the failure envelope for the applied loads P_1 and P_2. Use a = 16 in., b = 16 in., and c = 10 in.

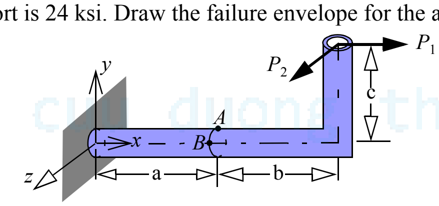


Figure P10.61

Computer problems

10.62 A hollow aluminum shaft of 5-ft length is to carry a torque of 200 in.·kips and an axial force of 100 kips. The inner radius of the shaft is 1 in. If the allowable shear stress in the shaft is 10 ksi, determine the outer radius of the lightest shaft.

10.63 The hollow cylinder shown in Figure P10.63 is fabricated from a sheet metal of 15-mm thickness. Determine the minimum outer radius to the nearest millimeter if the allowable normal stress is 150 MPa in tension or compression.

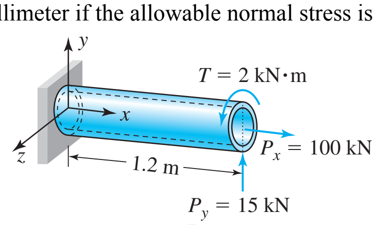


Figure P10.63

10.64 Table P10.64 shows the measured radii of the solid tapered member shown in Figure P10.64 at several points along the axis of the shaft. The member is subjected to a torque $T = 30$ kN·m and an axial force $P = 100$ kN. Plot the maximum normal and shear stresses as a function of x.

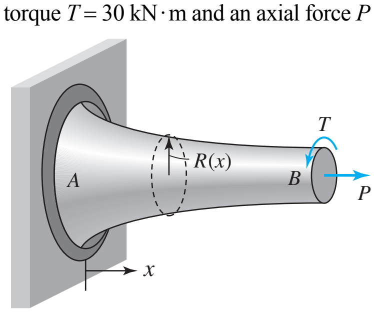


Figure P10.64

TABLE P10.64

x (m)	**R(x) (mm)**
0.0	100.6
0.1	92.7
0.2	82.6
0.3	79.6
0.4	75.9
0.5	68.8
0.6	68.0
0.7	65.9

TABLE P10.64

x (m)	**R(x) (mm)**
0.8	60.1
0.9	60.3
1.0	59.1
1.1	54.0
1.2	54.8
1.3	54.1
1.4	49.4
1.5	50.6

MoM in Action: Biomimetics

Nature produces the simplest and most efficient design by eliminating waste and inefficiency through the process of natural selection. This is the story about engineering design mimicking nature—that is, *Biomimetics*.

Tensegrity and adaptive, or smart, structures are just two of the latest in a long engineering history of mimicking nature. The flight of birds inspired the design of planes, while fish have inspired the sleek design of ship hulls. For a display of the technology of the Industrial Revolution, Joseph Paxton turned in 1851 to the structure of a lily pad. His wrought iron and glass building, the Crystal Palace, started an architectural trend. In Switzerland, George de Mestral invented Velcro in 1946 after observing the loops of seed-bearing burr clinging to his pants.

Tensegrity is the concatenation of tension and integrity. Tensegrity structures stabilize their shapes by continuous tension, like the camping tent in Figure 10.42*a*. Contrast these with stone arches, which achieve stability by continuous compression. Our own body is a tensegrity structure, with muscles supporting continual tension and bones in compression. Buckminster Fuller designed the first engineering tensegrity structure (Figure 10.42*b*) for the Expo 67 in Montreal, Canada. Such geodesic domes are structurally so efficient and stable that theoretically one could enclose New York City. Cells and the arrangement of carbon molecules called buckyballs in Fuller's honor are nature's tensegrity structures at the molecular level. They, in turn, are being emulated in carbon nano-tubes.

(a) (b)

Figure 10.42 Tensegrity structures: (a) a tent; (b) Montreal bio-sphere (Courtesy Mr. Philipp Hienstorfer).

Cracked bones heal, which means that the body senses a crack and then sends the material needed to seal the crack. To emulate this in metallic and masonry structures, three elements are needed: a *sensor* to detect a crack; a *controller* to decide if the crack is a threat; and a *smart material* that could be activated by the controller to seal the crack. Sensors could be electrical (like strain gages), acoustic (ultrasound), piezoelectric (producing current when pressured), or fiber optics. The controller on a computer chip is a central processing unit with decision-making algorithms. Finally, smart materials are those whose properties can be significantly altered in a controlled fashion by external stimuli—such as changes in temperature, moisture, pH, stress, or electric and magnetic fields.

With these three elements, *adaptive* or *smart structures* can adapt to the environment. Adaptive crack sealing would have applications to aircraft, bridges, buildings, and medical implants. Buildings that adapt to earthquake motion, aircraft wings that change shape during flight, pumps that dispense insulin to people with diabetes—all are possibilities on the research frontier of smart structures.

The healing of bones is only one example of the adaptive nature of our bodies. Bones and muscles get stronger in response to stress, a response that is still to be understood and mimicked. The orthotropic nature of bones also have lessons for the design of composite materials. Biomimetics is the formal acknowledgement that nature is smart and we would be smart to mimic it.

10.3 FAILURE THEORIES

In principle, the maximum strength of a material is its atomic strength. In bulk materials, however, the distribution of flaws such as impurities, microholes, or microcracks creates a local stress concentration. As a result the bulk strength of a material is orders of magnitude lower than its atomic strength. Failure theories assume a homogeneous material, so that effects of flaws have been averaged[2] in some manner. With this assumption we can speak of average material strength values, which are adequate for most engineering design and analysis.

For a homogeneous, isotropic material, the characteristic failure stress is either the yield stress or the ultimate stress, usually obtained from the uniaxial tensile test (Section 3.1.1). However, in the uniaxial tension test there is only one nonzero stress component. How do we relate this the stress components in two- and three-dimensions? Attempt to answer this question are called *failure theories* although no one answer is applicable to all materials. A **failure theory** relates the stress components to the characteristic value of material failure.

TABLE 10.3 **Synopsis of failure theories**

	Ductile Material	**Brittle Material**
Characteristic failure stress	Yield stress	Ultimate stress
Theories	1. Maximum shear stress	1. Maximum normal stress
	2. Maximum octahedral shear stress	2. Coulomb–Mohr

We shall consider the four theories listed in Table 10.3. The maximum shear stress theory and the maximum octahedral shear stress theory are generally used for ductile materials, in which failure is characterized by yield stress. The maximum normal stress theory and Mohr's theory are generally used for brittle material, in which failure is characterized by ultimate stress.

10.3.1 Maximum Shear Stress Theory

Maximum shear stress theory predicts that the maximum shear stress alone accounts for failure:

A material will fail when the maximum shear stress exceeds the shear stress at the yield point obtained from a uniaxial tensile test.

The theory gives reasonable results for ductile materials. Figure 10.43 shows that the maximum shear stress at yield in a tension test is half that of the normal yield stress. We obtain the following the failure criterion:

$$\tau_{max} \leq \frac{\sigma_{yield}}{2} \qquad (10.7)$$

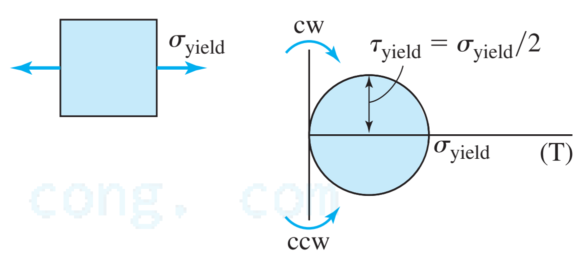


Figure 10.43 Shear stress at yield in tension test.

Equation (10.7) is also referred to as *Tresca's yield criterion*. The maximum shear stress at a point is given by Equation (8.13). If we substitute Equation (8.13) into Equation (10.7), we obtain

$$|\max(\sigma_1 - \sigma_2,\ \sigma_2 - \sigma_3,\ \sigma_3 - \sigma_1)| \leq \sigma_{yield} \qquad (10.8)$$

If we plot each principal stress on an axis, then Equation (10.8) gives us the failure envelope. For plane stress problems the failure envelope is shown in Figure 10.45 seen later.

[2]Micromechanics tries to account for the some of the flaws and nonhomogeneity in predicting the strength of a material, but extrapolating to macro levels requires some form of averaging.

10.3.2 Maximum Octahedral Shear Stress Theory

Figure P10.44 shows eight planes that make equal angles with the principal planes. These planes are called *octahedral planes* (from *octal*, meaning eight). The stress values on these planes are called *octahedral stresses*. The normal octahedral stress (σ_{oct}) and the octahedral shear stress (τ_{oct}) are given by Equations (8.16) and (8.17), written again here for convenience:

$$\sigma_{oct} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \tag{10.9}$$

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \tag{10.10}$$

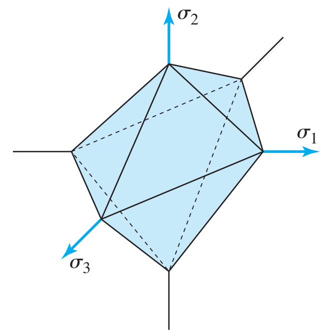


Figure 10.44 Octahedral Planes

The maximum octahedral shear stress theory for ductile materials states

A material will fail when the maximum octahedral shear stress exceeds the octahedral shear stress at the yield point obtained from a uniaxial tensile test.

Mathematically the failure criterion is

$$\tau_{oct} \le \overline{\tau_{yield}} \tag{10.11}$$

where $\overline{\tau_{yield}}$ is the octahedral shear stress at yield point in a uniaxial tensile test. Substituting $\sigma_1 = \sigma_{yield}$, $\sigma_2 = 0$, and $\sigma_3 = 0$ (the stresses at yield point in a uniaxial tension test) into the expression of octahedral shear stress, we obtain $\overline{\tau_{yield}} = \sqrt{2}\sigma_{yield}/3$. Substituting this and Equation (10.10) into Equation (10.11), we obtain

$$\frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \le \sigma_{yield} \tag{10.12}$$

The left-hand side of Equation (10.12) is referred to as *von Mises stress*. Because the von Mises stress σ_{von} is used extensively in the design of structures and machines, we formally define it as follows:

$$\sigma_{von} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \tag{10.13}$$

The failure criterion represented by Equation (10.12) is sometimes referred to as the *von Mises yield criterion* and is stated as follows:

$$\sigma_{von} \le \sigma_{yield} \tag{10.14}$$

At a point in a fluid the principal stresses are all compressive and equal to the hydrostatic pressure (p); that is, $\sigma_1 = \sigma_2 = \sigma_3 = -p$. Substituting this into Equation (10.9), we obtain $\sigma_{oct} = -p$; that is, octahedral normal stress corresponds to the hydrostatic state of stress. Thus this theory we are assumes that hydrostatic pressure has a negligible effect on the yielding of ductile material, a conclusion that is confirmed by experimental observation for very ductile materials like aluminum.

Equations (10.7) and (10.12) are failure envelopes[3] in a space in which the axes are principal stresses. For a plane stress ($\sigma_3 = 0$) problem we can represent these failure envelopes as in Figure 10.45. Notice that the maximum octahedral shear stress

[3]In drawing failure envelopes, the convention that $\sigma_1 > \sigma_2$ is ignored. If the convention were enforced, then there would be no envelope in the second quadrant, and only the envelope below a 45° line would be admissible in the third quadrant.A very strange looking envelope would result, rather than the symmetric envelope shown in Figure 10.45.

envelope encompasses the maximum shear stress envelope. Experiments show that, for most ductile materials, the maximum octahedral shear stress theory gives better results than the maximum shear stress theory. Still, the maximum shear stress theory is simpler to use.

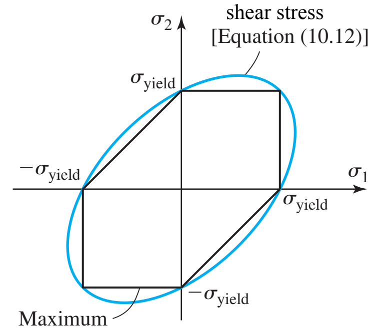


Figure 10.45 Failure envelopes for ductile materials in plane stress.

10.3.3 Maximum Normal Stress Theory

Maximum normal stress theory predicts that the maximum normal stress alone accounts for failure:

A material will fail when the maximum normal stress at a point exceeds the ultimate normal stress obtained from a uniaxial tension test.

The theory gives good results for brittle materials provided principal stress 1 is tensile, or if the tensile yield stress has the same magnitude as the yield stress in compression. Thus the failure criterion is given as

$$|\max(\sigma_1, \sigma_2, \sigma_3)| \leq \sigma_{ult} \tag{10.15}$$

For most materials the ultimate stress in tension is usually far less than the ultimate stress in compression because microcracks tend to grow in tension and tend to close in compression. But the simplicity of the failure criterion makes the theory attractive, and it can be used if principal stress 1 is tensile and is the dominant principal stress.

10.3.4 Mohr's Failure Theory

The Mohr's failure theory predicts failure using material strength from three separate tests in which the ultimate stress in tension, compression, and shear are determined.

A material will fail if a stress state is on the envelope that is tangent to three Mohr's circles—corresponding to ultimate stress in tension, compression, and pure shear.

By experiments, we can determine separately the ultimate stress in tension σ_T, the ultimate stress in compression σ_C, and the ultimate shear stress in pure shear τ_U. Figure 10.46*a* shows the stress cubes and the corresponding Mohr's circle for three stress states. We then draw an envelope tangent to the three circles to represent the failure envelope. If Mohr's circle corresponding to a stress state just touches the envelope at any point, then the material is at incipient failure. If any part of Mohr's circle for a stress state is outside the envelope, then the material has failed at that point.

We can also plot the failure envelopes of Figure 10.46*a* using principal stresses as the coordinate axes. In plane stress this envelope is represented by the solid line in Figure 10.46*b*. For most brittle materials the pure shear test is often ignored. In such a case the tangent line to the circles of uniaxial compression and tension would be a straight line in Figure 10.46*a*. The resulting simplification for plane stress is shown as dotted lines in Figure 10.46*b* and is called *modified Mohr's theory.*

Figure 10.46*b* emphasizes the following:

1. If both principal stresses are tensile, then the maximum normal stress has to be less than the ultimate tensile strength.
2. If both principal stresses are negative, then the maximum normal stress must be less than the ultimate compressive strength.

3. If the principal stresses are of different signs, then for the modified Mohr's theory the failure is governed by

$$\left|\frac{\sigma_2}{\sigma_C} - \frac{\sigma_1}{\sigma_T}\right| \le 1 \qquad (10.16)$$

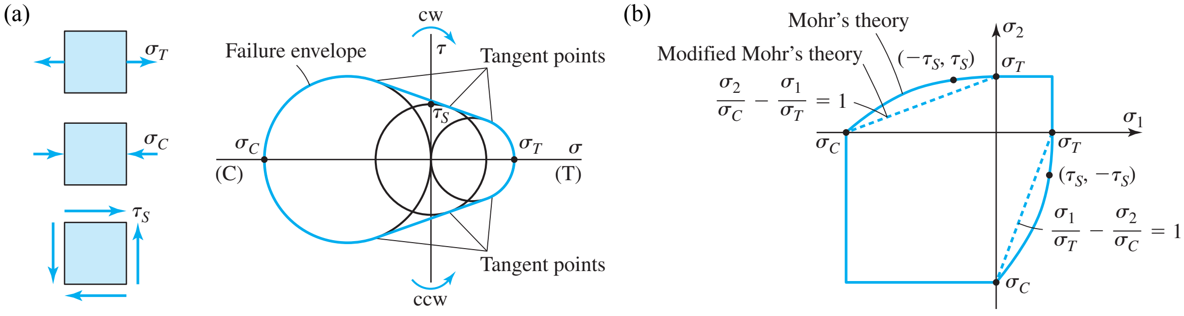


Figure 10.46 Failure envelopes for Mohr's failure theory in (a) normal and shear stress space; (b) principal stress space.

EXAMPLE 10.9

At a critical point on a machine part made of steel, the stress components were found to be $\sigma_{xx} = 100$ MPa (T), $\sigma_{yy} = 50$ MPa (C), and $\tau_{xy} = 30$ MPa. Assuming that the point is in plane stress and the yield stress in tension is 220 MPa, determine the factor of safety using (a) the maximum shear stress theory; (b) the maximum octahedral shear stress theory.

PLAN

We can find the principal stresses and maximum shear stress by Mohr's circle or by the method of equations. (a)From Equation (10.7) we know that failure stress for the maximum shear stress theory is half the yield stress in tension. Using Equation (3.10) we can find the factor of safety. (b)We can find the von Mises stress from Equation (10.13), which gives us the denominator in Equation (3.10), and noting that the numerator of Equation (3.10) is the yield stress in tension, we obtain the factor of safety.

SOLUTION

For plane stress: $\sigma_3 = 0$

Mohr's circle method: We draw the stress cube, record the coordinates of planes *V* and *H*, draw Mohr's circle as shown in Figure 10.47*a*. The principal stresses and maximum shear stress are

$$R = \sqrt{(75\ \text{MPa})^2 + (30\ \text{MPa})^2} = 80.8\ \text{MPa} \qquad (E1)$$

$$\sigma_1 = OC + OP_1 = 25\ \text{MPa} + 80.8\ \text{MPa} = 105.8\ \text{MPa} \qquad \sigma_2 = OC - OP_1 = 25\ \text{MPa} - 80.8\ \text{MPa} = -55.8\ \text{MPa} \qquad (E2)$$

$$\tau_{max} = R = 80.8\ \text{MPa} \qquad (E3)$$

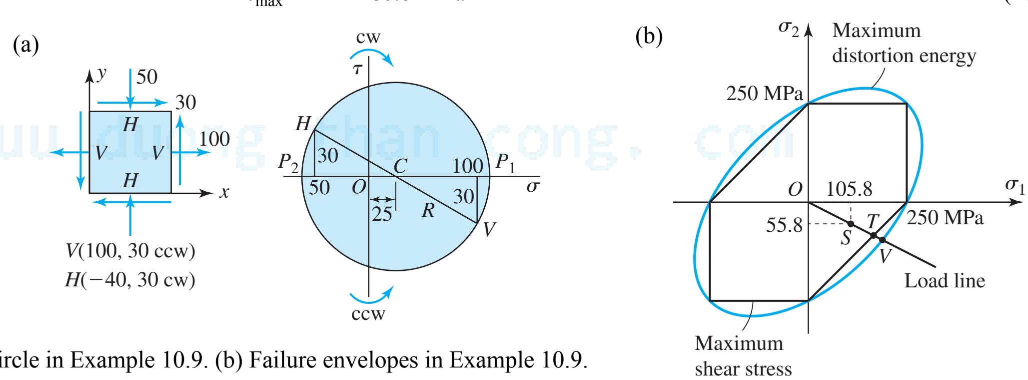


Figure 10.47 (a) Mohr's circle in Example 10.9. (b) Failure envelopes in Example 10.9.

Method of equations: From Equations (8.7) and (8.13) we can obtain the principal stresses and the maximum shear stress,

$$\sigma_{1,2} = \frac{100\ \text{MPa} - 50\ \text{MPa}}{2} \pm \sqrt{\left(\frac{100\ \text{MPa} + 50\ \text{MPa}}{2}\right)^2 + (30\ \text{MPa})^2} = 25\ \text{MPa} \pm 80.8\ \text{MPa} \qquad (E4)$$

$$\sigma_1 = 105.8\ \text{MPa} \qquad \sigma_2 = -55.8\ \text{MPa} \qquad \tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = 80.8\ \text{MPa} \qquad (E5)$$

(a) The failure shear stress is half the yield stress in tension, that is, 110 MPa. Following Equation (3.10), we divide this value by the maximum shear stress [Equation (E3) or (E5)] to obtain the factor of safety $K_\tau = (110\text{ MPa})/(80.8\text{ MPa})$.

ANS. $K_\tau = 1.36$

(b) The von-Mises stress can be found from Equation (10.13),

$$\sigma_{von} = \frac{1}{\sqrt{2}}\sqrt{[105.8\text{ MPa} - (-55.8\text{ MPa})]^2 + (-55.8\text{ MPa})^2 + (105.8\text{ MPa})^2} = 142.2\text{ MPa} \quad \text{(E6)}$$

The factor of safety is failure stress is 220 MPa divided by the von Mises stress, or $K_\sigma = (220\text{ MPa})/(142.2\text{ MPa})$

ANS. $K_\sigma = 1.55$

COMMENTS

1. The failure envelopes corresponding to the yield stress of 250 MPa are shown in Figure 10.47*b*. In comment 4 of Example 10.8 it was shown that graphically the factor of safety could be found by taking ratios of distances from the origin along the load line. If we plot the coordinates $\sigma_1 = 105.8$ MPa and $\sigma_2 = -55.8$ MPa, we obtain point *S*. If we join the origin *O* to point *S* and draw the line, we get the load line for the given stress values. It may be verified by measuring (or calculating coordinates of *T* and *V*) that the following is true: $K_\tau = OS/OT = 1.36$ and $K_\sigma = OS/OV = 1.55$.
2. Because the failure envelope for the maximum shear stress criterion is always inscribed inside the failure envelope of maximum octahedral shear stress criterion, the factor of safety based on the maximum octahedral shear stress will always be greater than the factor of safety based on maximum shear stress.

EXAMPLE 10.10

The stresses at a point on a free surface due to a load *P* were found to be $\sigma_{xx} = 3P$ ksi (C), $\sigma_{yy} = 5P$ ksi (T), and $\tau_{xy} = -2P$ ksi, where *P* is measured in kips. The brittle material has a tensile strength of 18 ksi and a compressive strength of 36 ksi. Determine the maximum value of load *P* that can be applied on the structure using the modified Mohr's theory.

PLAN

We can determine the principal stresses in terms of *P* by Mohr's circle or by the method of equations. As the given normal stresses are of opposite signs, we can expect that the principal stresses will have opposite signs. Using Equation (10.16) we can determine the maximum value of *P*.

SOLUTION

For plane stress: $\sigma_3 = 0$

Mohr's circle method: We draw the stress cube, record the coordinates of planes *V* and *H*, draw Mohr's circle as shown in Figure 10.48. The principal stresses and the maximum shear stress are

$$R = \sqrt{(4P\text{ ksi})^2 + (2P\text{ ksi})^2} = 4.47P\text{ ksi} \quad \text{(E1)}$$

$$\sigma_1 = OC + OP_1 = P\text{ ksi} + 4.47P\text{ ksi} = 5.57P\text{ ksi} \qquad \sigma_2 = OC - OP_1 = P\text{ ksi} - 4.47P\text{ ksi} = -3.37P\text{ ksi} \quad \text{(E2)}$$

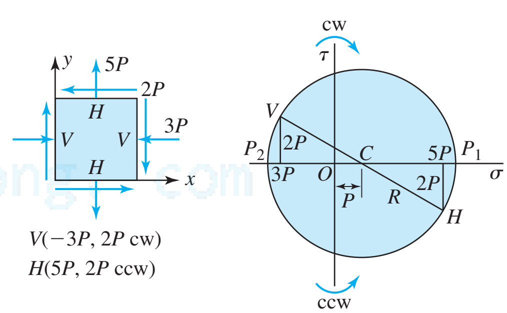


Figure 10.48 Calculation using Mohr's circle in Example 10.10.

Method of equations: From Equation (8.7) we can obtain the principal stresses as

$$\sigma_{1,2} = \frac{-3P\text{ ksi} + 5P\text{ ksi}}{2} \pm \sqrt{\left(\frac{-3P\text{ ksi} - 5P\text{ ksi}}{2}\right)^2 + (2P\text{ ksi})^2} = P\text{ ksi} \pm 4.47P\text{ ksi} \quad \text{(E3)}$$

$$\sigma_1 = 5.57P\text{ ksi} \qquad \sigma_2 = -3.37P\text{ ksi} \quad \text{(E4)}$$

Substituting the principal stresses into Equation (10.16) and noting that $\sigma_T = 18$ ksi and $\sigma_C = 36$ ksi, we can obtain the maximum value of *P*,

$$\left|\frac{(-3.37P\text{ ksi})}{(-36\text{ ksi})} - \frac{(5.57P\text{ ksi})}{(18\text{ ksi})}\right| \le 1 \quad \text{or} \quad 0.2158P \le 1 \quad \text{or} \quad P \le 4.633 \quad \text{(E5)}$$

ANS. $P_{max} = 4.63$ kips

COMMENT

1. We could not have used the maximum normal stress theory for this material, since the tensile and compressive strengths are significantly different.Here the compressive strength is the dominant strength, and not the tensile-strength.

PROBLEM SET 10.3

Failure theories

10.65 For a force P measured in kN, the stress components at a critical point that is in plane stress were found to be $\sigma_{xx} = 10P$ MPa (T), $\sigma_{yy} = 20P$ MPa (C), and $\tau_{xy} = 5P$ MPa. The material has a yield stress of 160 MPa as determined in a tension test. If yielding must be avoided, predict the maximum v force P using (a) maximum shear stress theory; (b) maximum octahedral shear stress theory.

10.66 For a force P, the stress components at a critical point that is in plane stress were found to be $\sigma_{xx} = 4P$ ksi (C), $\sigma_{yy} = 3P$ ksi (T), and $\tau_{xy} = -5P$ ksi. The material has a tensile rupture strength of 18 ksi and a compressive rupture strength of 32 ksi. Determine the maximum force P using the modified Mohr's theory.

10.67 A material has a tensile rupture strength of 18 ksi and a compressive rupture strength of 32 ksi. During usage a component made from this plastic showed the following stresses on a free surface at a critical point: $\sigma_{xx} = 9$ ksi (T), $\sigma_{yy} = 6$ ksi (T), and $\tau_{xy} = -4$ ksi. Determine the factor of safety using the modified Mohr's theory.

10.68 On a free surface of aluminum ($E = 10{,}000$ ksi, $\nu = 0.25$, $\sigma_{yield} = 24$ ksi) the strains recorded by the three strain gages shown in Figure P10.68 are $\varepsilon_a = -600$ μin./in., $\varepsilon_b = 500$ μin/in, and $\varepsilon_c = 400$ μin./in. By how much can the loads be scaled without exceeding the yield stress of aluminum at the point? Use the maximum shear stress theory.

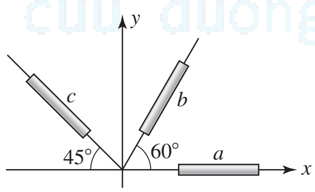


Figure P10.68

10.69 On a free surface of steel ($E = 200$ GPa, $\nu = 0.28$, $\sigma_{yield} = 210$ MPa) the strains recorded by the three strain gages shown in Figure P10.69 are $\varepsilon_a = -800$ μm/m, $\varepsilon_b = -300$ μm/m, and $\varepsilon_c = -700$ μm/m. By how much can the loads be scaled without exceeding the yield stress of steel at the point? Use the maximum octahedral shear stress theory.

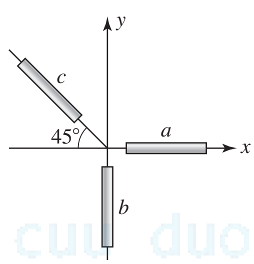


Figure P10.69

10.70 A thin-walled cylindrical gas vessel has a mean radius of 3 ft and a wall thickness of $\frac{1}{2}$ in. The yield stress of the material is 30 ksi. Using the von Mises failure criterion, determine the maximum pressure of the gas inside the cylinder if yielding is to be avoided.

10.71 A thin cylindrical boiler can have a minimum mean radius of 18 in. and a maximum mean radius of 36 in. The boiler will be subjected to a pressure of 750 psi. A sheet metal with a yield stress of 60 ksi is to be used with a factor of safety of 1.5. Construct a failure envelope with the mean radius R and the sheet metal thickness t as axes. Use the maximum octahedral shear stress theory.

10.72 For plane stress show that the von Mises stress of Equation (10.13) can be written as

$$\sigma_{von} = \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\tau_{xy}^2} \qquad (10.17)$$

Stretch Yourself

10.73 In Cartesian coordinates the von Mises stress in three dimensions is given by

$$\sigma_{von} = \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 - \sigma_{xx}\sigma_{yy} - \sigma_{yy}\sigma_{zz} - \sigma_{zz}\sigma_{xx} + 3\tau_{xy}^2 + 3\tau_{yz}^2 + 3\tau_{zx}^2} \quad (10.18)$$

Show that for plane strain Equation (10.18) reduces to

$$\sigma_{von} = \sqrt{(\sigma_{xx}^2 + \sigma_{yy}^2)(1 + \nu^2 - \nu) - \sigma_{xx}\sigma_{yy}(1 + 2\nu - 2\nu^2) + 3\tau_{xy}^2} \quad (10.19)$$

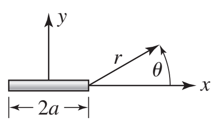
where ν is Poisson's ratio of the material.

10.74 Fracture mechanics shows that the stresses in model in the vicinity of the crack tip shown in Figure P10.74 are given by

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \qquad \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \qquad \tau_{xy} = \frac{K_I}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} \quad (10.20)$$

Notice that at $\theta = \pi$, that is, at the crack surface, all stresses are zero. In terms of K_I and r, obtain the von Mises stress at $\theta = 0$ and $\theta = \pi/2$, assuming plane stress.

Figure P10.74



10.4 CONCEPT CONNECTOR

In our problems thus far, we have relied on fixed values for the dimensions, material properties, and loads. Design that does not allow for variability in these parameters is called *deterministic*. Real structural members, however, are manufactured to dimensions only within a certain tolerance. Similarly, material properties may vary within a range, depending on material processing. Loads and support conditions, too, are at best an estimate, because wind pressure, snow weight, traffic loads, and other conditions are inherently variable.

Probabilistic design takes into account the variability in dimensions, material properties, and loads. Here we seek to achieve not just a given factor of safety but rather a specified *reliability*. This section offers a peek at how engineers approach probabilistic design. Section 10.4.1 discusses the concept of reliability, and Section 10.4.2 introduces a design methodology that incorporates it.

10.4.1 Reliability

Already in Section 3.3, we encountered the uncertainties regarding material properties, manufacturing processes, the control and estimate of loads, and so on. There we defined one measure of the margins of safety, the *factor of safety*. But choosing a factor of safety is always a compromise among several factors, including cost and human safety, based on experience. Such a compromise leaves an open question: *how reliable is our design?*

To understand the relationship between the factor of safety and reliability, suppose we wish to design an engine mount or other axial member with a factor of safety of 1.3. If the material strength of the axial member is 130 MPa, we have an allowable stress of 100 MPa.

The actual axial stress in the member, however, may be quite different, because of such factors as manufacturing tolerances, the variability of applied loads, temperature, and humidity. If we measured the axial stress in different members, we would get a range of values. We might ask instead, then, the frequency of occurrence of a given stress level. A plot of the number of members at that level would yield a distribution, perhaps like the left curve in Figure 10.49. Similarly, the material strength—that is, the failure stress for different batches of material—may vary due to impurities, material processing, and so on. The right curve in Figure 10.49 shows one possible distribution of material strengths.

In Figure 10.49, the mean axial stress in the members is 100 MPa, and the mean strength of all materials is 130 MPa. Naturally some axial members with stresses greater than 100 MPa will be made from materials that have failure stresses less than

130 MPa. Hence, those axial members are likely to fail. In the graph, these members occupy the region common to both distributions, labeled "possible failure." If we know the two distributions, we can always determine this possible failure region.

Statistical distributions are usually described by two parameters, their *mean value* and *standard deviation*. If the predicted reliability developed using these parameters is unacceptable, then the design can use a different factor of safety to obtain the desired reliability.

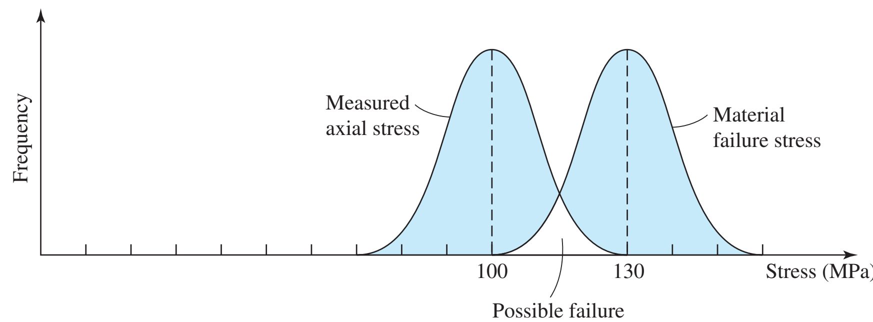


Figure 10.49 Load and resistance distribution curves.

10.4.2 Load and Resistance Factor Design (LRFD)

Load and resistance factor design (LRFD) allows civil engineers to design steel structures to a specified reliability. LRFD incorporates ideas from two other design methodologies, *allowable stress design* (ASD) and *plastic design* (PD).

ASD is based on elastic analysis, in which the factors of safety can vary with the primary function of the structural member. For example, a factor of safety of 1.5 is used for beams and 1.67 for tension members. These factors of safety are specified in building codes, usually based on statistical analysis.

Building codes also specify the kinds of load on a structure, and the methodologies that we are considering takes all these into account: the *dead load* (D) due to the weight of structural elements and other permanent features; *live load* (L) from people, equipment, and other movable objects during occupancy; *snow load* (S) and *rain or ice loads* (R) that appear on the roof of a structure; *roof load* L_r from cranes, air conditioners, and other movable objects during construction, maintenance, and occupancy; *wind load* W; and *earthquake load* E. In ASD, the sum total of the stresses from the various loads must be less than or equal to the allowable stress.

PD uses a single load factor for the design load on the structure. This factor varies with a combination of loads at hand. If only dead and live loads are considered, for example, then the load factor is 1.7 [written as 1.7 $(D+L)$]. If, however, the dead, live, and wind loads are considered, then the load factor is 1.3 [written as 1.3 $(D + L + W)$]. A *nonlinear analysis* can then determine the strength of the member at structure collapse. By nonlinear analysis, we mean that the stress values of many members fall in the plastic region—between the yield stress and ultimate stress. The member strengths must equal or exceed the required strengths calculated using factored loads.

The LRFD method overcomes shortcomings in both these methods. From the standpoint of consistent reliability in design, neither of the two methods is very accurate. In ASD the factor of safety is used to account for all variability in loads and material strength. In PD the variability in material strength is ignored. Furthermore, all loads do not have the same degree of variability.

TABLE 10.4 **Some resistance factors**

Tension members, failure due to yielding	0.90
Tension member, failure due to rupture	0.75
Axial compression	0.85
Beams	0.90
High-strength bolts, failure in tension	0.75

TABLE 10.5 **Load factors and load combinations**

$1.4D$
$1.2D + 1.6L + 0.5\,(L_r \text{ or } S \text{ or } R)$
$1.2D + 1.6\,(L_r \text{ or } S \text{ or } R) + (0.5L \text{ or } 0.8W)$
$1.2D + 1.3W + 0.5 + 0.5\,(L_r \text{ or } S \text{ or } R)$
$12D \pm 1.0E + 0.5L + 0.2S$
$0.9D \pm (1.3W \text{ or } 1.0E)$

In LRFD, the nominal failure strength of a member is multiplied by the appropriate resistance factor (from Table 10.4) to obtain the design strength. (The words *strength* and *resistance* for a material are often used interchangeably in LRFD.

Recall the historical evolution of the concept of strength from Section 1.6.) This accounts for variability in material strength and inaccuracies in dimensions and modeling.

The load factors in Table 10.7 account for the variability of the individual load components. It also takes into account the probability of combinations of loads acting together, such as live and snow loads. Using Table 10.5, factored loads are determined for a specific load combination. These factored loads are applied to the structure, and the member strength is calculated. This computed member strength must be less than or equal to the design strength computed using the resistance factor. Since variations of load and member strength are taken into account separately, LRFD gives a more consistent level of reliability.

10.5 CHAPTER CONNECTOR

This chapter synthesized and applied the concepts of all previous chapters. The use of subscripts and formulas to determine stress results in a systematic but slower approach to problem solving. Determining the stress directions by inspection can reduce the algebra significantly, but it requires care, depending on the problem being solved. For a given problem, it is important to find your own mix of these approaches. Whatever your preference, however, the importance of a systematic approach to the problem cannot be overstated. In the design and analysis of complex structures, without a systematic approach the chances of error rise dramatically.

So far we have based designs on material strength and structure stiffness. Instability, however, can cause a structure to fail at stresses far lower than the material strength. What is structure instability, and how can we incorporate it? The next chapter considers structure instability in the design of columns.

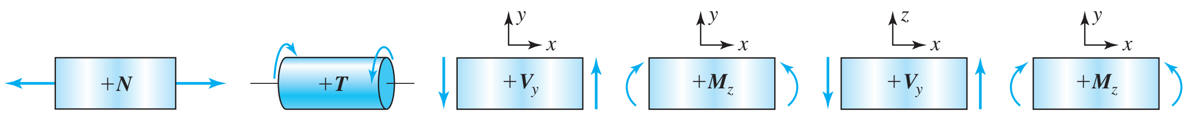
POINTS AND FORMULAS TO REMEMBER

- Superposition of stresses is addition or subtraction of stress components in the same direction acting on the same surface at a point.

$$\sigma_{xx} = \frac{N}{A} \quad (10.1) \qquad \tau_{x\theta} = \frac{T\rho}{J} \quad (10.2) \qquad \sigma_{xx} = -\frac{M_z y}{I_{zz}} \quad (10.3a) \qquad \tau_{xs} = -\frac{V_y Q_z}{I_{zz} t} \quad (10.3b)$$

$$\sigma_{xx} = -\frac{M_y z}{I_{yy}} \quad (10.4a) \qquad \tau_{xs} = -\frac{V_z Q_y}{I_{yy} t} \quad (10.4b)$$

- Sign convention for internal forces and moments:



- The internal forces and moments on a free-body diagram must be drawn according to the sign conventions if subscripts are to be used to determine the direction of stress components.
- The direction of the stress components must be determined by inspection if internal forces and moments are drawn on the free-body diagram to equilibrate the external forces and moments.
- A local x, y, z coordinate system can be established such that the x direction is along the axis of the long structural member.
- Stress components should be drawn on a stress cube and interpreted in the x, y, z coordinate system for use in the stress and strain transformation equations.
- Normal stresses perpendicular to the axis of the member are zero: $\sigma_{yy} = 0$, $\sigma_{zz} = 0$, and $\tau_{yz} = 0$.
- Normal strains perpendicular to the axis of the member can be obtained by multiplying the normal strains in the axis direction by Poisson's ratio.
- Superpose stresses, then use the generalized Hooke's law to obtain strains in combined loading:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} \qquad \varepsilon_{yy} = -\frac{\nu\sigma_{xx}}{E} \qquad \varepsilon_{zz} = -\frac{\nu\sigma_{xx}}{E} \qquad \gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{xz} = \frac{\tau_{xz}}{G} \qquad \gamma_{yz} = 0$$

- The allowable normal and shear stresses refer to the principal stresses and absolute maximum shear stress at a point, respectively.
- An individualized procedure that is a mix of subscripts, formulas, and inspection should be developed for analysis of stresses under combined loading.
- There are two major steps in the analysis and design of structures: (i) analysis of internal forces and moments that act on individual members; (ii) computation of stresses on members under combined loading.