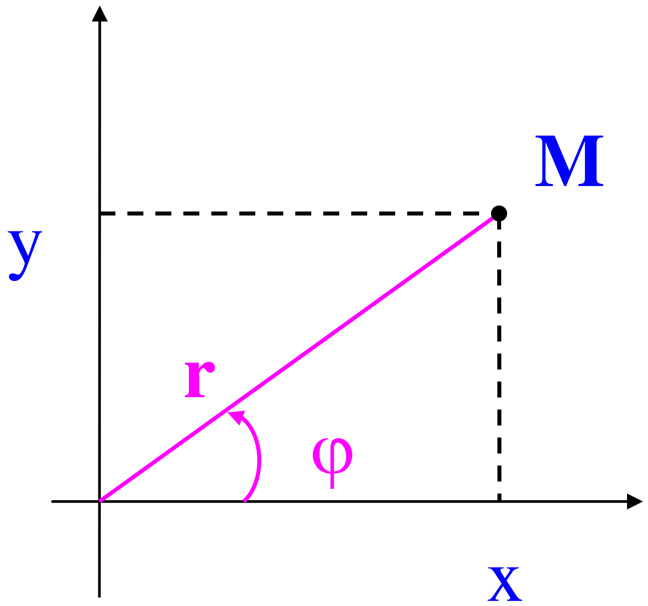
ĐỔI BIẾN TRONG TÍCH PHÂN KÉP

TỌA ĐỘ CỰC



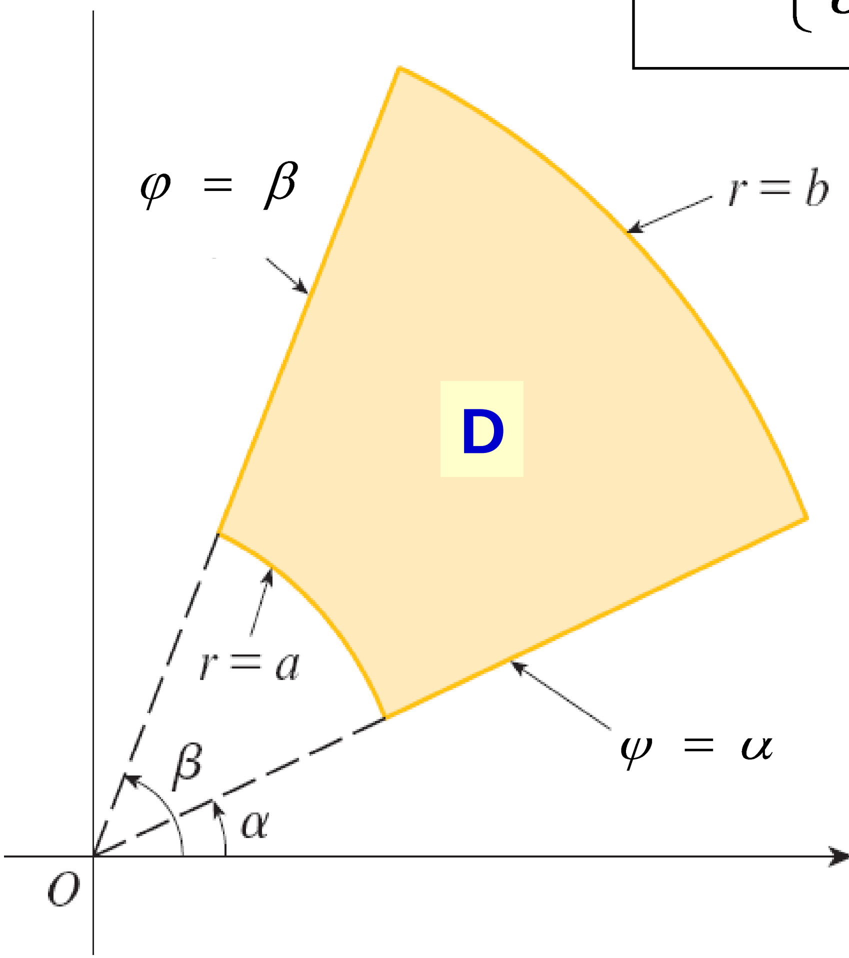
$$r = \sqrt{x^2 + y^2} \geq 0$$

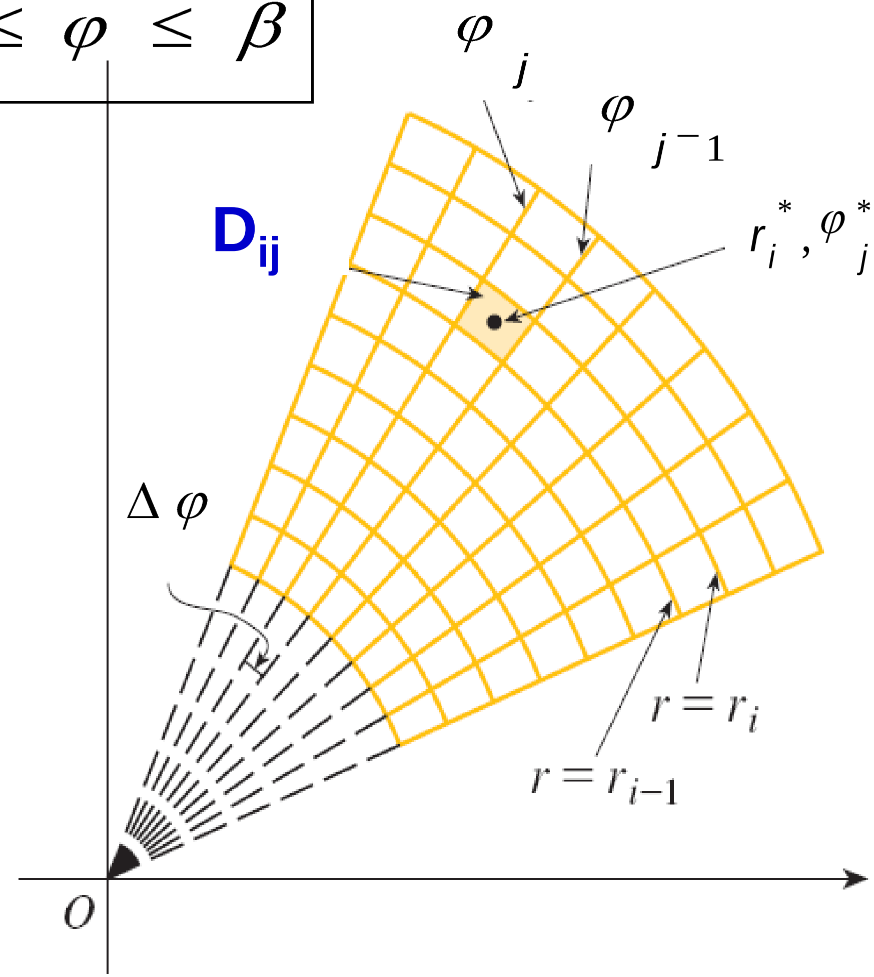
$$x = r\cos\varphi, \quad y = r\sin\varphi$$

$\varphi \in [0, 2\pi]$ hay $\varphi \in [-\pi, \pi]$

TÍCH PHÂN KÉP TRONG TỌA ĐỘ CỰC

$$D : \begin{cases} a \le r \le b \\ \alpha \le \varphi \le \beta \end{cases}$$





Tổng tích phân

$$S_n = \sum_{i,j} f(r_i^* \cos\varphi_j^*, r_i^* \sin\varphi_j^*) r_i^* \Delta r \Delta\varphi$$

$$\iint_D f(x,y)\,dx\,dy = \lim_{d \to 0} S_n$$

$$\lim_{d \to 0} S_n = \iint_D f(r\cos\varphi, r\sin\varphi)\, r\,dr\,d\varphi$$

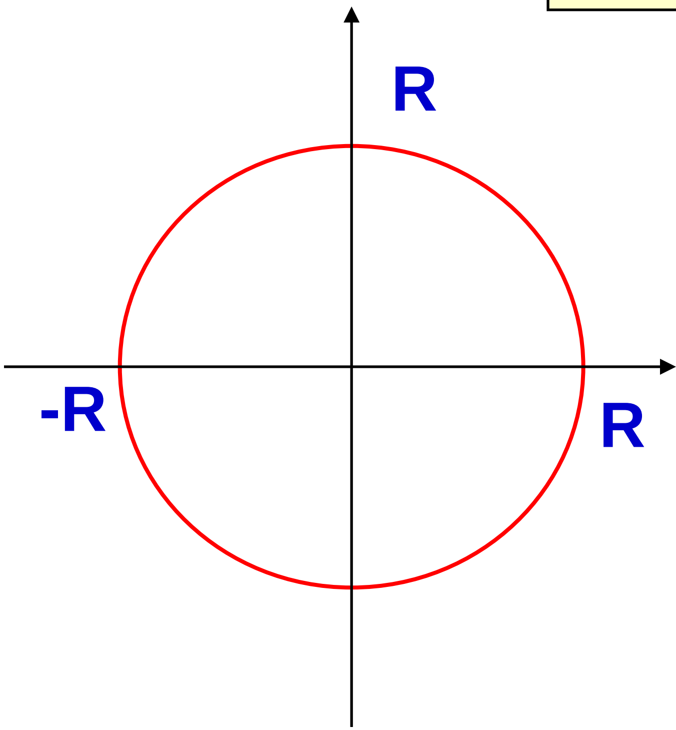
Công thức đổi biến sang tọa độ cực

$$x = r\cos\varphi,\ y = \sin\varphi$$

$$\iiint_D f(x,y)\,dx\,dy = \iint_D f(r\cos\varphi, r\sin\varphi)\,r\,dr\,d\varphi$$

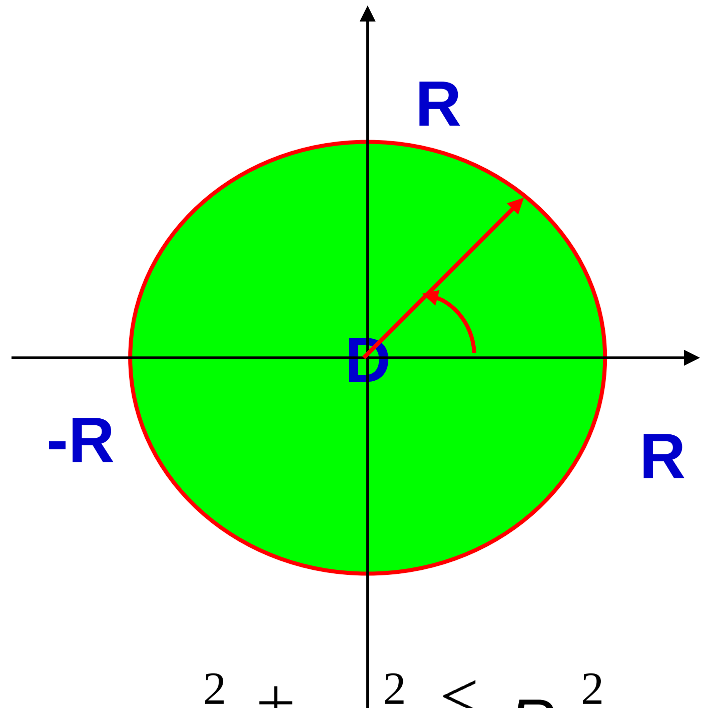
Một số đường cong và miền D trong tọa độ cực

$$x = r\cos\varphi,\quad y = r\sin\varphi$$



$$x^2 + y^2 = R^2$$

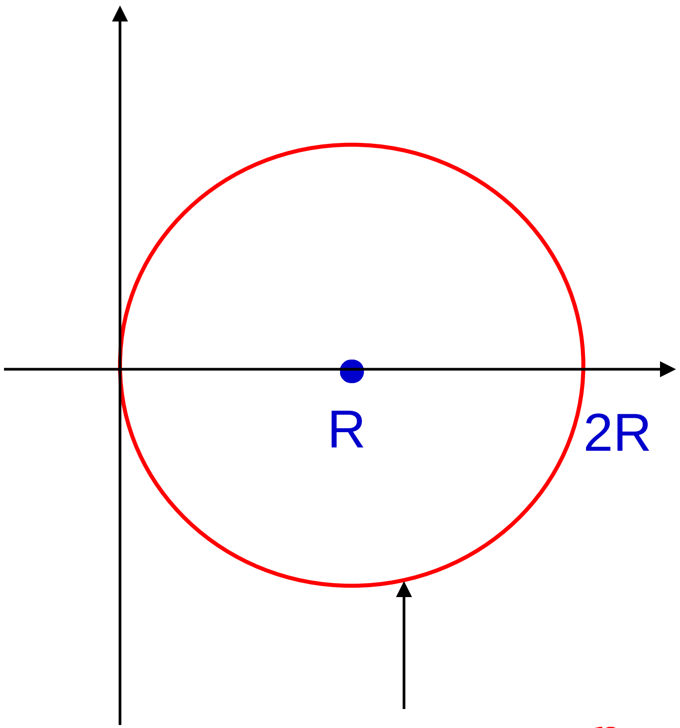
$$\Leftrightarrow r = R$$



$$x^2 + y^2 \le R^2$$

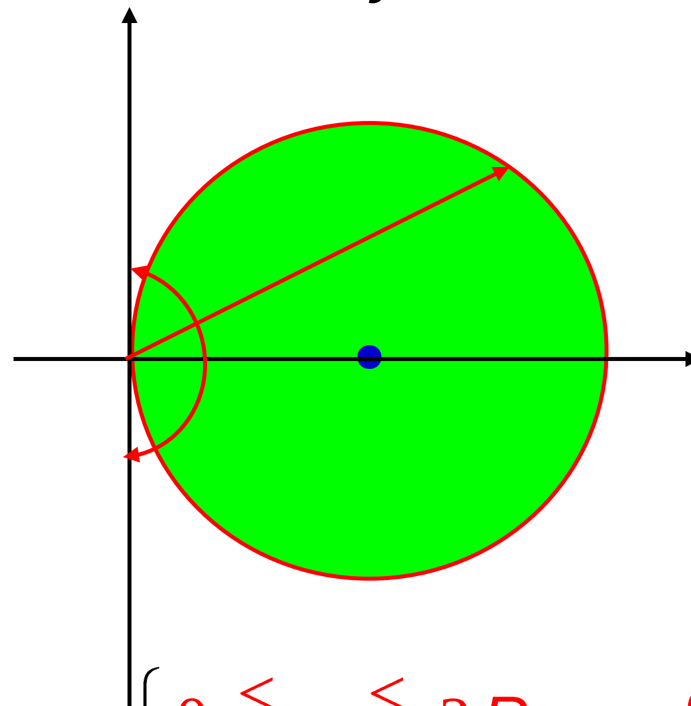
$$\Leftrightarrow \begin{cases} 0 \le r \le R \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$x^2 + y^2 = 2Rx$$



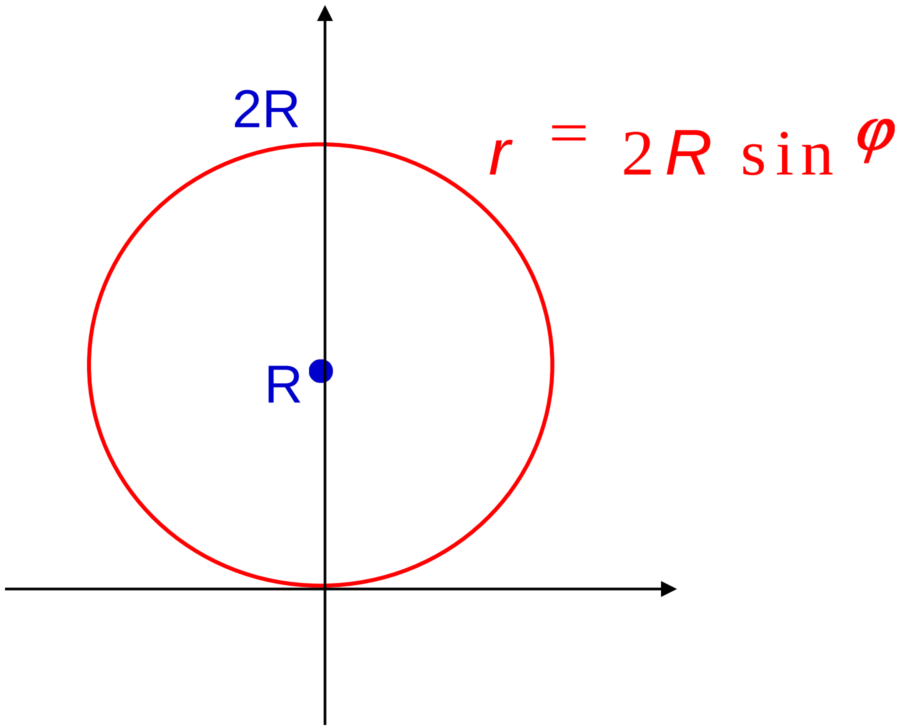
$$r = 2R\cos\varphi$$

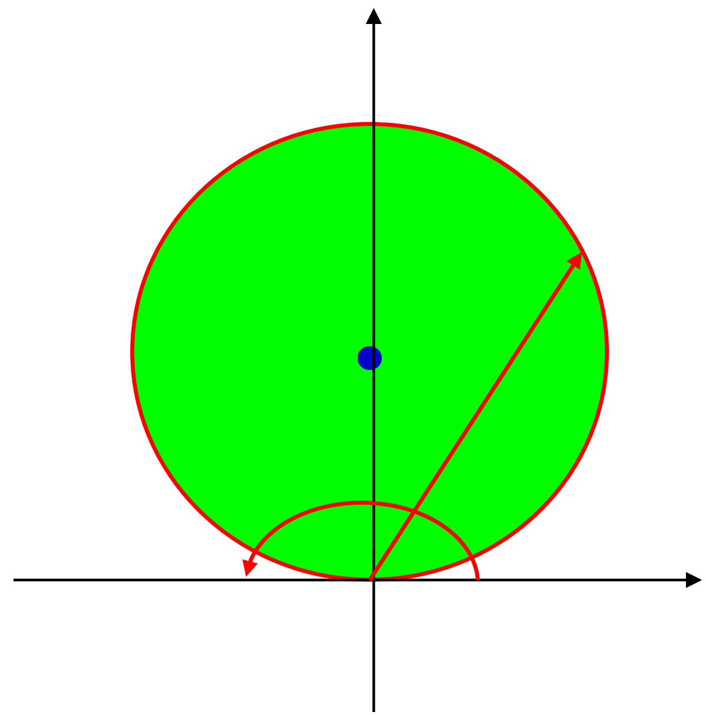
$$x^2 + y^2 \le 2Rx$$



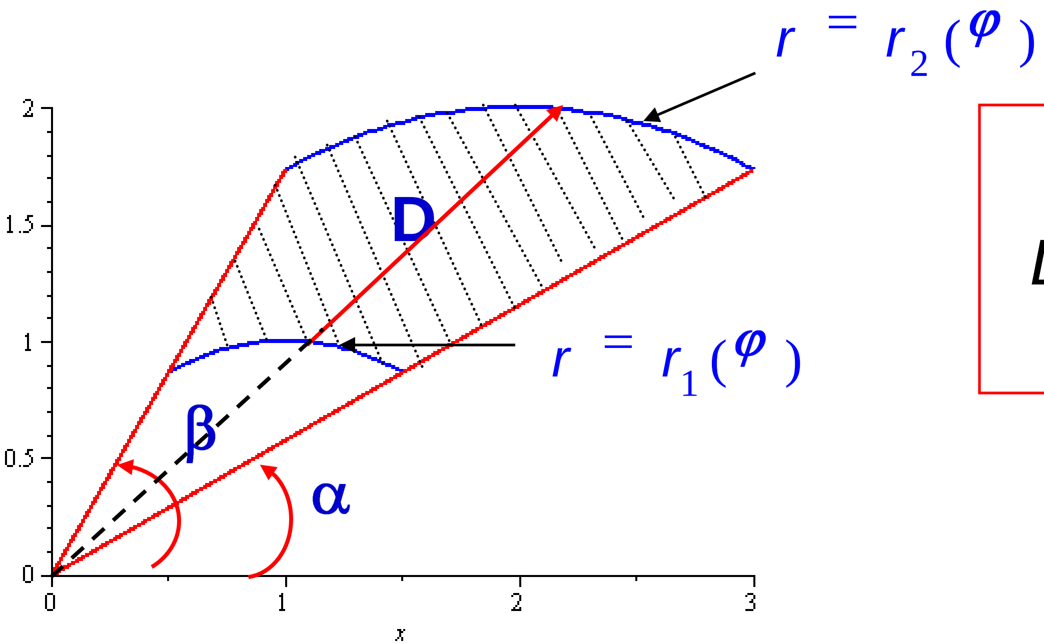
$$\begin{cases} 0 \le r \le 2R\cos\varphi \\ -\dfrac{\pi}{2} \le \varphi \le \dfrac{\pi}{2} \end{cases}$$

$$x^2 + y^2 = 2Ry \qquad\qquad x^2 + y^2 \le 2Ry$$





$$\begin{cases} 0 \le r \le 2R\sin\varphi \\ 0 \le \varphi \le \pi \end{cases}$$



$$D: \begin{cases} r_1(\varphi) \le r \le r_2(\varphi) \\ \alpha \le \varphi \le \beta \end{cases}$$

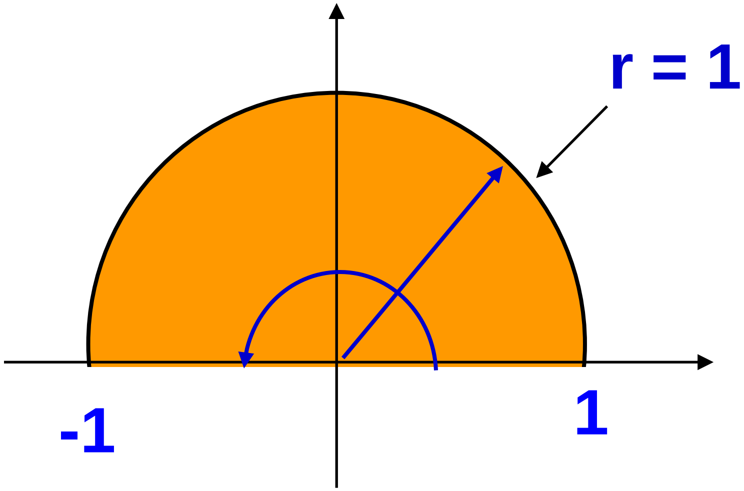
$$(0 < \beta - \alpha \le 2\pi)$$

$$\iint_D f(r\cos\varphi, r\sin\varphi)\, r\,dr\,d\varphi$$

$$= \int_\alpha^\beta d\varphi \int_{r_1(\varphi)}^{r_2(\varphi)} f(r\cos\varphi, r\sin\varphi)\, r\,dr$$

VÍ DỤ

1/ Tính: $I = \iint\limits_D \sqrt{x^2 + y^2}\, dxdy$ với $D: \begin{cases} x^2 + y^2 \le 1 \\ y \ge 0 \end{cases}$



$$x = r\cos\varphi, \quad y = r\sin\varphi$$

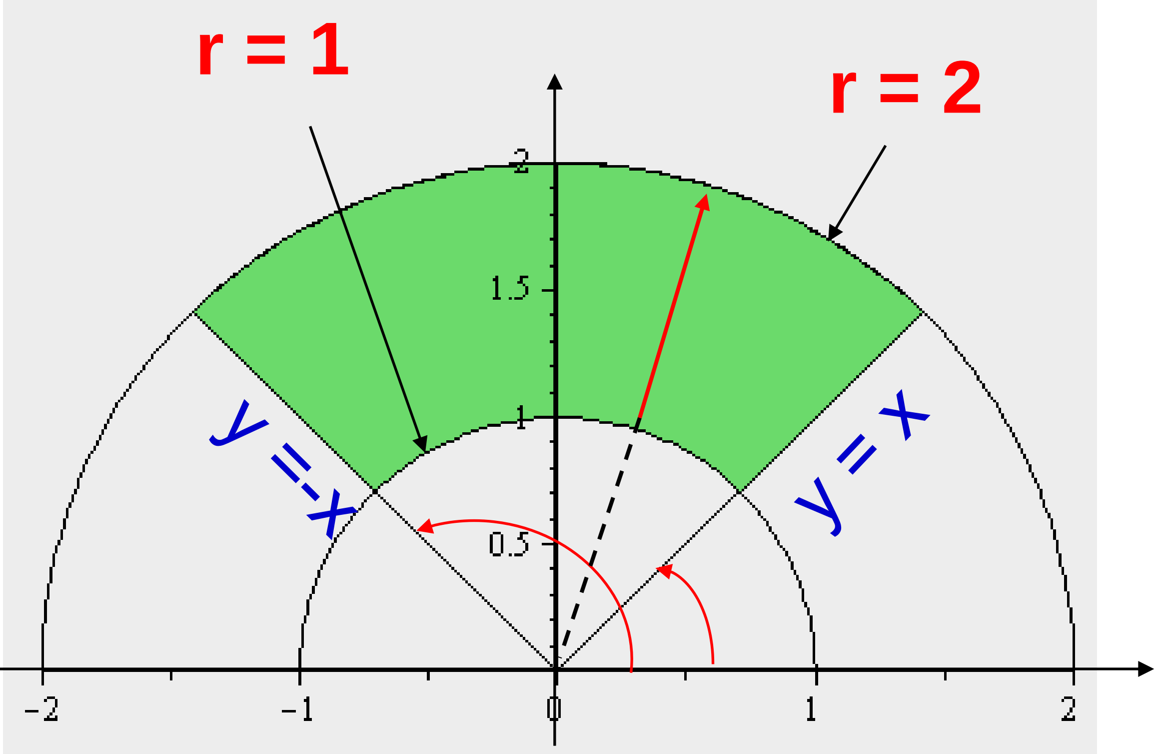
$$D: \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le \pi \end{cases}$$

$$I = \iint\limits_D r.rdrd\varphi = \int_0^{\pi} d\varphi \int_0^1 r^2 dr = \frac{1}{3}\int_0^{\pi} d\varphi = \frac{\pi}{3}$$

2/ Tính: $I = \iint_D (x-y)\,dx\,dy$

$$D: \begin{cases} 1 \le x^2 + y^2 \le 4 \\ y \ge x,\ y \ge -x \end{cases}$$

$$x = r\cos\varphi,\quad y = r\sin\varphi$$



$$D: \begin{cases} 1 \le r \le 2 \\ \dfrac{\pi}{4} \le \varphi \le \dfrac{3\pi}{4} \end{cases}$$

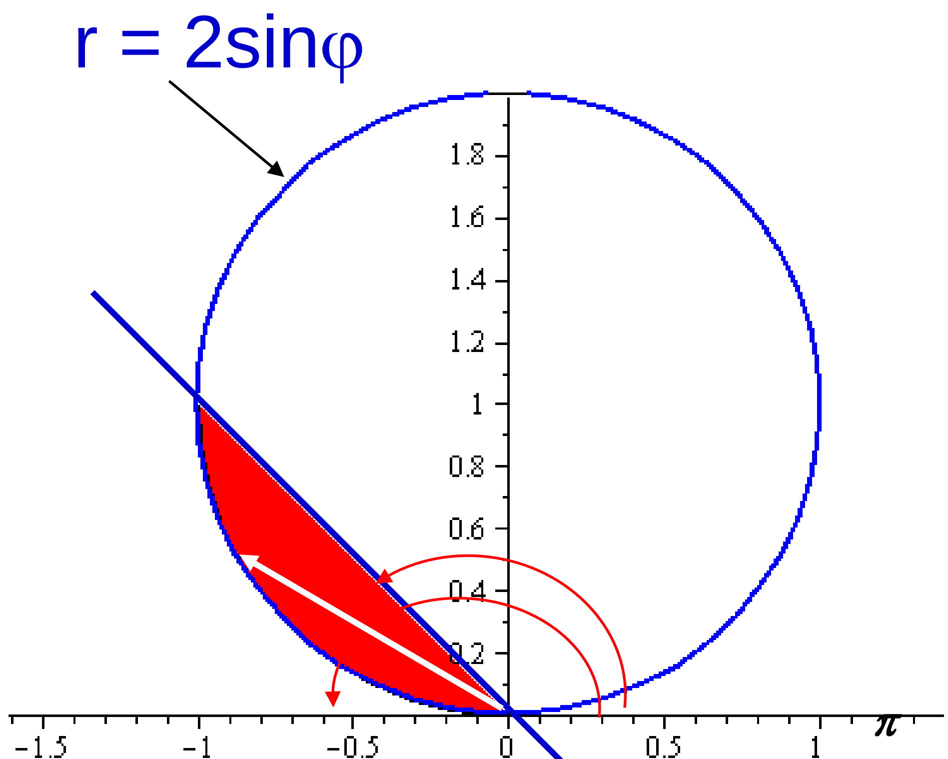
$$I = \iint_D (x - y)\,dx\,dy \qquad D: \begin{cases} 1 \le r \le 2 \\ \dfrac{\pi}{4} \le \varphi \le \dfrac{3\pi}{4} \end{cases}$$

$$= \iint_D (r\cos\varphi - r\sin\varphi).r\,dr\,d\varphi$$

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi \int_1^2 r^2(\cos\varphi - \sin\varphi)\,dr$$

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (\cos\varphi - \sin\varphi)\left(\frac{8}{3} - \frac{1}{3}\right)d\varphi = -\frac{7}{3}\sqrt{2}$$

3/ Tính: $I = \iint\limits_D x\,dx\,dy$ với $D:\begin{cases} x^2+y^2 \le 2y \\ y \le -x \end{cases}$



$$x = r\cos\varphi,\ y = r\sin\varphi$$

$$D:\begin{cases} 0 \le r \le 2\sin\varphi \\ \dfrac{3\pi}{4} \le \varphi \le \pi \end{cases}$$

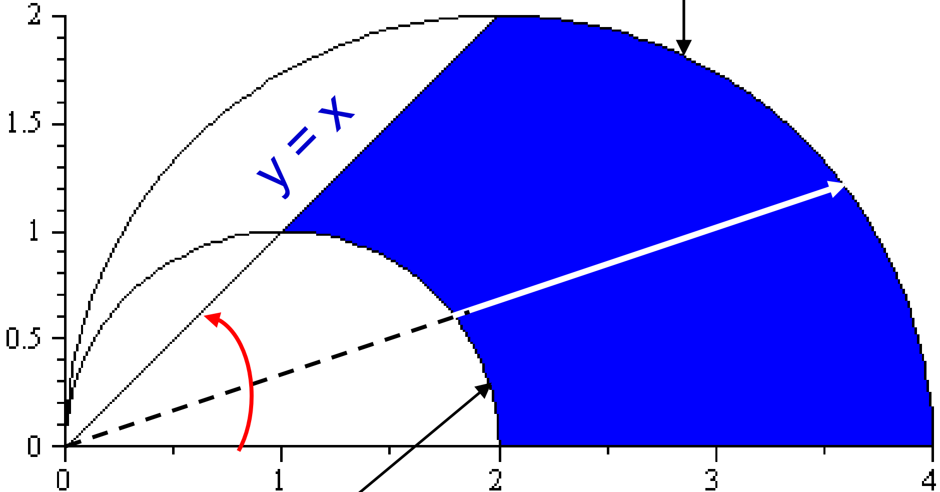
$$I = \iint\limits_D r\cos\varphi\, r\,dr\,d\varphi = \int_{\frac{3\pi}{4}}^{\pi} d\varphi \int_0^{2\sin\varphi} r^2\cos\varphi\, r\,dr = -\frac{1}{6}$$

4/ Tính diện tích miền D giới hạn bởi:

$$x^2 + y^2 = 4x, x^2 + y^2 = 2x, y = x, y = 0$$

r = 4cosφ

$$x = r\cos\varphi,\ y = r\sin\varphi$$



$$D\ \begin{cases} 0 \le \varphi \le \dfrac{\pi}{4} \\ 2\cos\varphi \le r \le 4\cos\varphi \end{cases}$$

r = 2cosφ

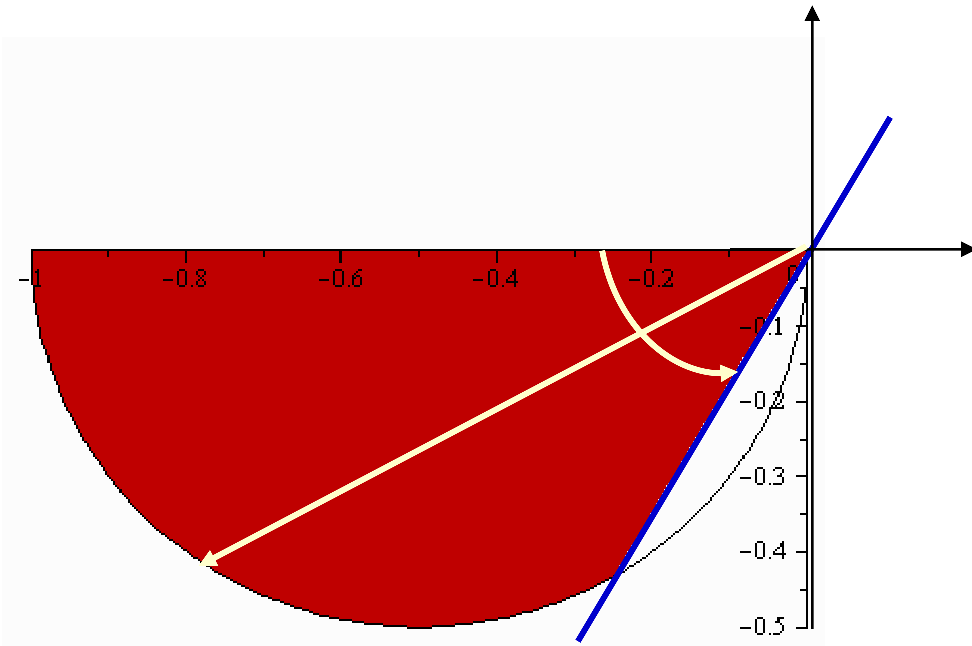
$$S(D) = \iint_D 1\,dx\,dy \qquad D \begin{cases} 0 \le \varphi \le \dfrac{\pi}{4} \\ 2\cos\varphi \le r \le 4\cos\varphi \end{cases}$$

$$= \iint_D r\,dr\,d\varphi$$

$$= \int_0^{\frac{\pi}{4}} d\varphi \int_{2\cos\varphi}^{4\cos\varphi} r\,dr$$

$$= \frac{3\pi}{4} + \frac{3}{2}$$

5/ Tính: $I = \iint\limits_D xy\,dx\,dy$ với $D: \begin{cases} x^2 + y^2 \le -x \\ \sqrt{3}x \le y \le 0 \end{cases}$

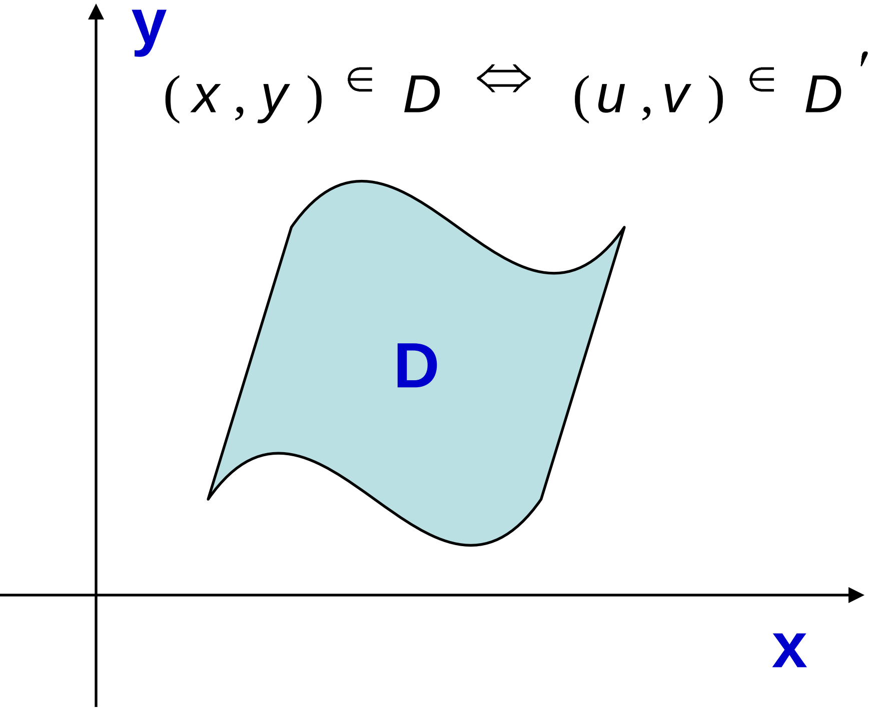


$y = \sqrt{3}x$

r = - cosφ

$$\begin{cases} 0 \le \varphi \le \dfrac{4\pi}{3} \\ 0 \le r \le -\cos\varphi \end{cases}$$

ĐỔI BIẾN TỔNG QUÁT



$$(x,y)\in D \Leftrightarrow (u,v)\in D'$$

x = x(u,v), y= y(u,v)

$$J=\frac{D(x,y)}{D(u,v)}=\begin{vmatrix} x'_u & x'_v \\ y'_u & y'_v \end{vmatrix}$$

$$J=\frac{1}{\dfrac{D(u,v)}{D(x,y)}}$$

Công thức đổi biến

$$\iint_D f(x,y)\,dx\,dy=\iint_{D'} f(x(u,v),y(u,v))\,|J|\,du\,dv$$

Áp dụng đổi biến tổng quát

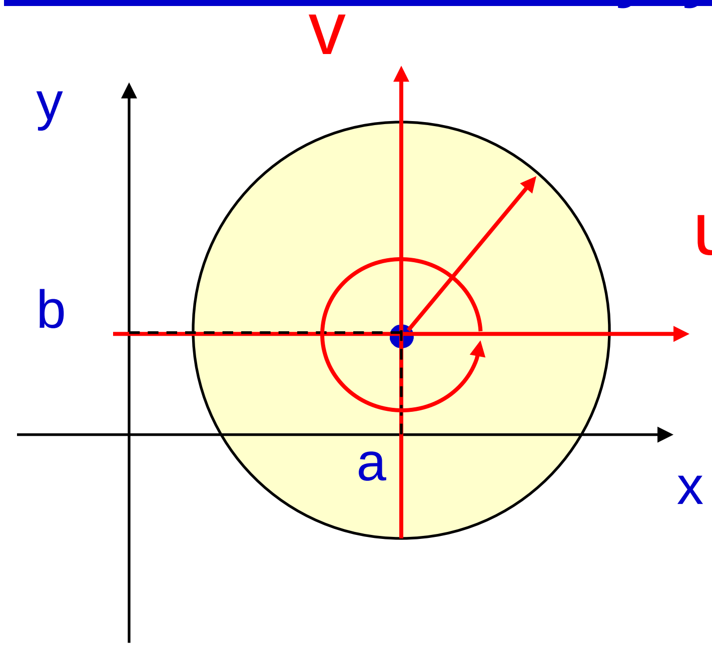
Tọa độ cực: $x = r\cos\varphi, \; y = r\sin\varphi$

$$J = \begin{vmatrix} x'_r & x'_\varphi \\ y'_r & y'_\varphi \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r$$

$$\boxed{\iint_D f(x,y)\,dxdy = \iint_{D'} f(r\cos\varphi, r\sin\varphi)\,rdrd\varphi}$$

Hình tròn tâm tùy ý:

$$D: (x-a)^2 + (y-b)^2 \le R^2$$



Dời gốc tọa độ đến tâm

$$x = u + a,\ y = v + b$$

$$J = \begin{vmatrix} x'_u & x'_v \\ y'_u & y'_v \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

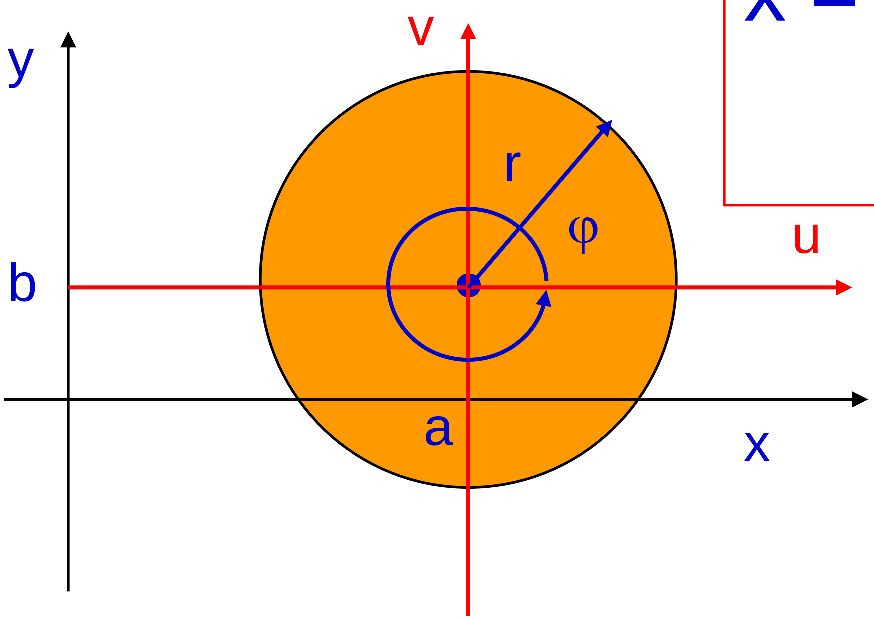
$$\iint_D f(x,y)\,dx\,dy = \iint_{u^2+v^2 \le R^2} g(u,v).1\,du\,dv$$

Đổi tiếp sang tọa độ cực:

$$u = r\cos\varphi,\ v = r\sin\varphi$$

Tóm tắt:

$$D: (x-a)^2 + (y-b)^2 \le R^2$$



$$x = a + r\cos\varphi,\ y = b + r\sin\varphi$$

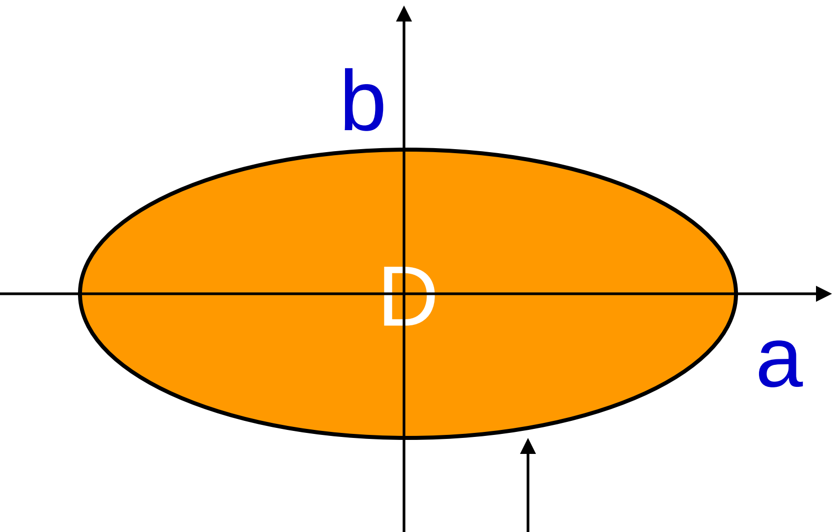
$$J = r$$

$$D': \begin{cases} 0 \le r \le R \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$\iint_D f(x,y)\,dx\,dy = \iint_{D'} f(a + r\cos\varphi, b + r\sin\varphi)\,r\,dr\,d\varphi$$

Đổi biến trong ellippse

$$D: \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2$$

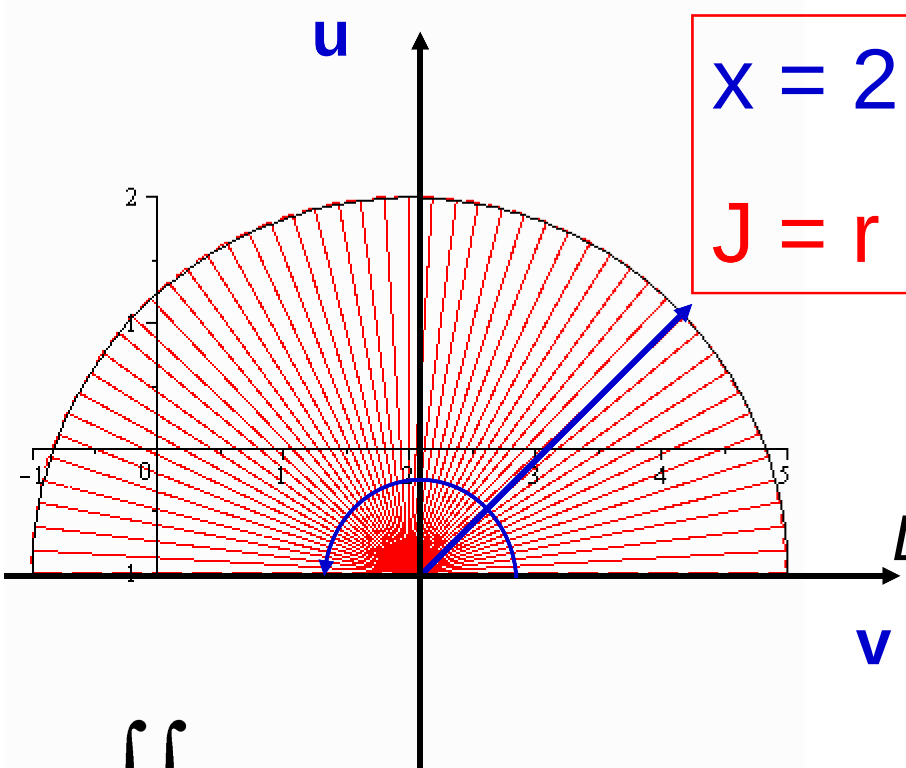
x = arcosφ, y = brsinφ

J = abr

$$D': \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$\iint_D f(x,y)\,dxdy = \iint_{D'} f(ar\cos\varphi, br\sin\varphi)\,abr\,drd\varphi$$

1/ Tính: $I = \iint_D xy\,dxdy$ với D là nửa trên của hình tròn: $(x-2)^2 + (y+1)^2 \le 9$



$x = 2 + r\cos\varphi,\ y = -1 + r\sin\varphi$

$J = r$

$$D': \begin{cases} 0 \le r \le 3 \\ 0 \le \varphi \le \pi \end{cases}$$

$$I = \iint_{D'} (2 + r\cos\varphi)(-1 + r\sin\varphi)\,r\,dr\,d\varphi$$

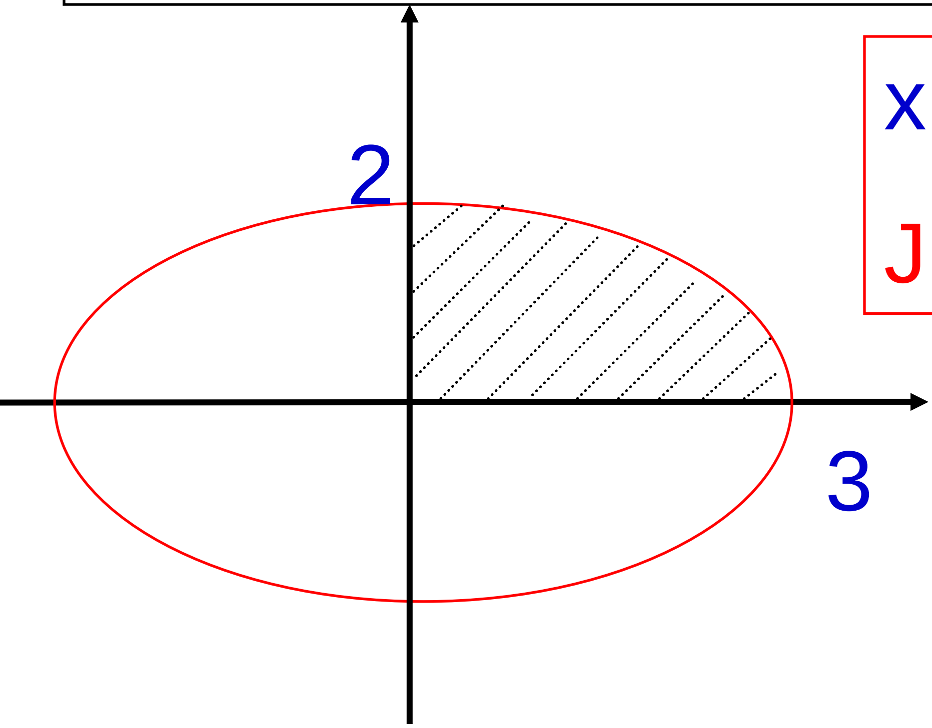
$$I = \iint_{D'} (2 + r\cos\varphi)(-1 + r\sin\varphi) r\,dr\,d\varphi$$

$$= \int_0^{\pi} d\varphi \int_0^{3} (-2 - r\cos\varphi + 2r\sin\varphi + r^2 \sin\varphi\cos\varphi) r\,dr$$

$$= -9\pi + 18$$

Ví dụ

2/ Tính: $I = \iint\limits_D xy\,dxdy, D: \frac{x^2}{9} + \frac{y^2}{4} \le 1; y \ge 0; x \ge 0$



x = 3rcosφ, y = 2rsinφ

J = 3.2.r = 6r

Miền D được viết lại:

$$\begin{cases} r^2 \le 1 \\ 3r\cos\varphi \ge 0, 2r\sin\varphi \ge 0 \end{cases}$$

$$\begin{cases} r^2 \le 1 \\ 3r\cos\varphi \ge 0, 2r\sin\varphi \ge 0 \end{cases} \Leftrightarrow \begin{cases} 0 \le r \le 1 \\ \cos\varphi \ge 0, \sin\varphi \ge 0 \end{cases}$$

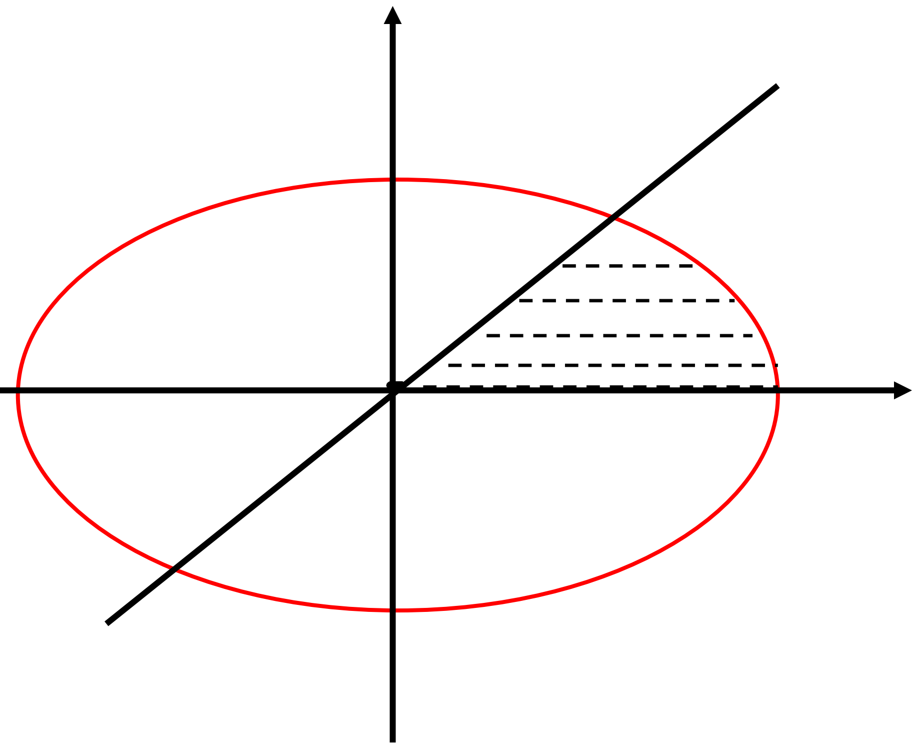
$$D': \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le \pi/2 \end{cases}$$

$$\iint\limits_D xy\,dx\,dy = \int_0^{\pi/2} d\varphi \int_0^1 3r\cos\varphi . 2r\sin\varphi . 6r\,dr$$

$$= \frac{9}{2}$$

3/ Tính diện tích miền giới hạn bởi

$$ellipse \quad \frac{x^2}{3} + y^2 = 1,\ y = 0,\ y = x,\ x \ge 0$$



$$x = \sqrt{3}r\cos\varphi,\ y = r\sin\varphi$$

$$J = \sqrt{3}r$$

Miền D được viết lại:

$$\frac{x^2}{3} + y^2 \le 1,\ 0 \le y \le x$$

$$\Leftrightarrow \begin{cases} 0 \le r \le 1, \\ 0 \le r\sin\varphi \le \sqrt{3}r\cos\varphi \end{cases}$$

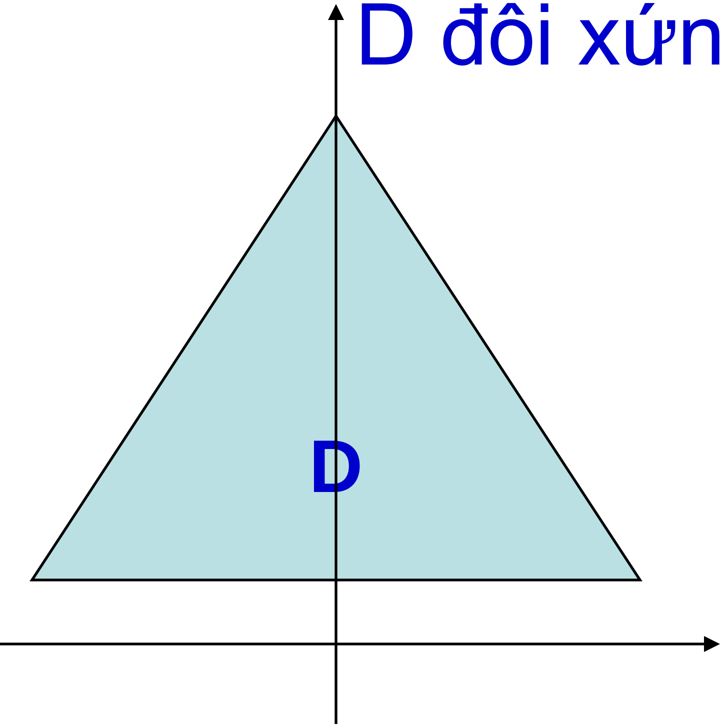
$$\begin{cases} 0 \le r \le 1, \\ 0 \le r\sin\varphi \le \sqrt{3}r\cos\varphi \end{cases}$$

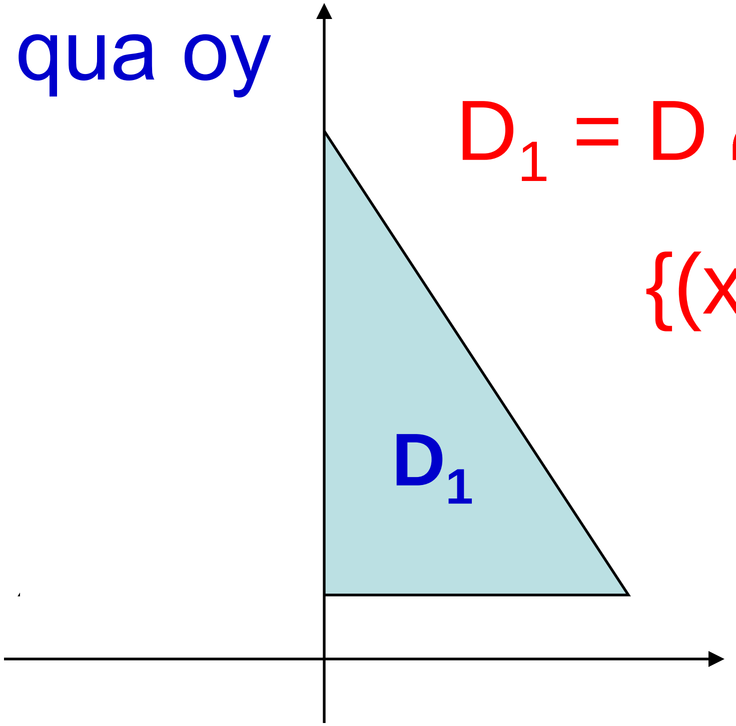
$$\Leftrightarrow \begin{cases} 0 \le r \le 1, \\ 0 \le \tan\varphi = \dfrac{\sin\varphi}{\cos\varphi} \le \sqrt{3} \end{cases} \Leftrightarrow \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le \dfrac{\pi}{3} \end{cases}$$

$$S(D) = \iint\limits_{D} dx\,dy = \int_{0}^{\frac{\pi}{3}} d\varphi \int_{0}^{1} \sqrt{3}r\,dr$$

Tính đối xứng của miền D trong tính tp kép

D đối xứng qua oy





$D_1 = D \cap$

$\{(x,y)/\ x \geq 0\}$

f(x,y) chẵn theo x: $$\iint_D f(x,y)\,dxdy = 2\iint_{D_1} f(x,y)\,dxdy$$

f(x,y) lẻ theo x: $$\iint_D f(x,y)\,dxdy = 0$$