ECE 307 – Techniques for Engineering Decisions

Introduction to the Simplex Algorithm

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SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

- We examine the solution of

$$\underline{A}\,\underline{x} = \underline{b}$$

  using Gauss—Jordan elimination

- We first use a simple example and then generalize to cases of general interest

- Consider the system of two equations in five unknowns:

SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

$$S_1 \begin{cases} x_1 - 2x_2 + x_3 - 4x_4 + 2x_5 = 2 & (i) \\ x_1 - x_2 - x_3 - 3x_4 - x_5 = 4 & (ii) \end{cases}$$

- **For this simple example, the number of unknowns exceeds the number of equations and so the system has multiple solutions; this is the principal reason that the *LP* solution is *nontrivial***

SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

- The Gauss — Jordan elimination uses *elementary row operations*:
  - multiplication of any equation by a nonzero constant
  - addition to any equation of a constant multiple of any other equation
- We transform S_1 into the set S_2 by multiplying equation (*i*) by –1 and adding it to equation (*ii*)

$$S_2 \begin{cases} x_1 - 2x_2 + x_3 - 4x_4 + 2x_5 = 2 \\ x_2 - 2x_3 + x_4 - 3x_5 = 2 \end{cases}$$

DEFINITIONS

- A *basic variable* is a variable x_i that appears with the coefficient 1 in an equation and with the coefficient *0* in all the other equations
- The variables x_j that are ***not*** basic are called ***nonbasic variables***
- In the system S_2 , x_1 appears as a *basic* variable; x_2, x_3, x_4 and x_5 are *nonbasic* variables
- Basic variables may be generated through the use of *elementary row operations*

DEFINITIONS

- A *pivot operation* is the sequence of elementary row operations that reduces a system of linear equations into the form in which a specified variable becomes a *basic variable*
- A *canonical system* is a set of linear equations obtained through *pivot operations* with the property that the system has the same number of *basic variables* as the number of equations in the set

CANONICAL SYSTEM FORM

- **We transform the system S_2 into the canonical form of system S_3:**

$$S_3 \begin{cases} x_1 - 3x_3 - 2x_4 - 4x_5 = 6 \\ x_2 - 2x_3 + x_4 - 3x_5 = 2 \end{cases}$$

- **The *basic solution* is obtained from a canonical system with all the nonbasic variables set to *0***

- **For the example, we set $x_3 = x_4 = x_5 = 0$ and so**

$$x_1 = 6 \text{ and } x_2 = 2$$

BASIC FEASIBLE SOLUTION

- A *basic feasible solution* is a basic solution in which the values of all the basic variables are nonnegative

- In the example of system S_2, we may choose any two variables to be basic

- In general for a system of m equations in n unknowns there are $\binom{n}{m}$ possible combinations of basic variables

BASIC FEASIBLE SOLUTION

- As n increases the number of combinations becomes large even though it is finite

- For the example, we have

$$\binom{5}{2} = \frac{5!}{3!\,2!} = 10$$

combinations of possible choices

THE SIMPLEX SOLUTION METHOD

- **We next use a simple example to construct the *simplex* solution method**
- **The *simplex method* is a *systematic* and *efficient* way of examining a subset of the basic feasible sol-utions of the *LP* to hone in on *an* optimal solution**
- **We apply the notions introduced in the definitions we introduced above**

SIMPLEX METHODOLOGY EXAMPLE

$$\max\ Z = 5x_1 + 2x_2 + 3x_3 - x_4 + x_5$$

$s.t.$

canonical form

$$x_1 + 2x_2 + 2x_3 + x_4 = 8 \qquad (*)$$

$$3x_1 + 4x_2 + x_3 + x_5 = 7 \qquad (**)$$

$$x_i \geq 0 \qquad i = 1, \ldots, 5$$

THE SIMPLEX SOLUTION METHOD

- The *canonical form* of the example allows the determination of a basic feasible solution

$$x_1 = x_2 = x_3 = 0 \qquad x_4 = 8, \; x_5 = 7$$

- The corresponding value of the objective is

$$Z = -8 + 7 = -1$$

- The next step is to improve the *basic feasible solution* by finding an *adjacent* basic feasible solution

ADJACENT FEASIBLE SOLUTION

- An *adjacent* feasible solution is one which differs from the current basic feasible solution in *exactly one* basic variable

- Note, we characterize a *basic feasible solution* by the following traits

$$\begin{aligned} \textit{basic} \quad \textit{variable} &\geq 0 \\ \textit{nonbasic variable} &= 0 \end{aligned}$$

ADJACENT FEASIBLE SOLUTION

- The search for an adjacent basic feasible solution uses the idea of making a *nonbasic* variable into a *basic* variable by increasing its value from *0* to the largest positive value without violating any constraints
- To make the search efficient, we choose the *nonbasic* variable that can improve the value of *Z* by the largest amount

ADJACENT FEASIBLE SOLUTION

- In the example, consider the *nonbasic* variable x_1, we leave $x_2 = x_3 = 0$ and examine the possibility of making x_1 into a basic variable
- The variable x_1 enters in both constraints

$$\begin{aligned} x_1 + x_4 \quad &= 8 \\ 3x_1 + \quad x_5 &= 7 \end{aligned}$$

ADJACENT FEASIBLE SOLUTION

- The largest value x_1 can assume without making either x_4 or x_5 negative is

$$min\left\{8,\ \frac{7}{3}\right\} = \frac{7}{3}$$

- We have the new *basic* variable with the value

$$x_1 = \frac{7}{3} \quad ,$$

and the other *basic* variable is

$$x_4 = \frac{17}{3}$$

ADJACENT FEASIBLE SOLUTION

and the three *nonbasic* variables are set to *0*:

$$x_2 = x_3 = 0 \text{ and } x_5 = 0$$

- Note that we have an improvement in Z since its value becomes

$$Z = 5 \cdot \frac{7}{3} - \frac{17}{3} = \frac{18}{3} = 6 > -1$$

- We next need to put the system of equations into *canonical form*:

SIMPLEX METHODOLOGY EXAMPLE

$$\max \; Z = 5x_1 + 2x_2 + 3x_3 - x_4 + x_5$$

s.t.

non - canonical form for x_1

$$x_1 + 2x_2 + 2x_3 + x_4 = 8 \qquad (*)$$

$$3x_1 + 4x_2 + x_3 + x_5 = 7 \qquad (**)$$

$$x_i \geq 0 \qquad i = 1, \ldots, 5$$

ADJACENT FEASIBLE SOLUTION

- multiply equation (**) by $-\frac{1}{3}$ and add to equation (*)

$$\frac{2}{3}x_2 + \frac{5}{3}x_3 + x_4 - \frac{1}{3}x_5 = \frac{17}{3}$$

- multiply equation (**) by $\frac{1}{3}$

$$x_1 + \frac{4}{3}x_2 + \frac{1}{3}x_3 + \frac{1}{3}x_5 = \frac{7}{3}$$

THE SIMPLEX SOLUTION METHOD

- **We continue this process until the *condition of optimality* is satisfied:**
  - **in a maximization problem, a *basic feasible solution* is *optimal* if and only if the relative profits of each *nonbasic variable* is ≤ 0**
  - **in a minimization problem, a basic feasible solution is optimal if and only if the relative costs of each *nonbasic variable* is ≥ 0**

THE SIMPLEX SOLUTION METHOD

- **The relative profits (costs) are given by the change in** Z **corresponding to a unit change in a** ***nonbasic variable***
- **We use this fact to select the next** ***nonbasic variable*** **to enter the basis**

SIMPLEX ALGORITHM FOR MAXIMIZATION

Step 1: start with an initial basic feasible solution with all constraint equations in *canonical form*

Step 2: check for optimality condition: if the relative profits are ≤ 0 for each *nonbasic variable*, then the *basic feasible solution* is optimal and *stop*; else, go to Step 3

SIMPLEX ALGORITHM FOR MAXIMIZATION

Step 3: select a *nonbasic variable* to become the new *basic variable*; check the limits on the *nonbasic variable* – the limiting constraint determines the *basic variable* that is being replaced by the selected *nonbasic variable*

Step 4: determine the *canonical form* for the new set of basic variables through *elementary row operations*; compute the *basic feasible solution*, Z and return to Step 2

THE SIMPLEX TABLEAU

- We use an efficient way to represent visually the steps in the simplex method through a sequence of so-called tableaus

- We illustrate the tableau for the simple example for the initial basic feasible solution

THE SIMPLEX TABLEAU

coefficients of the basic variables in Z

coefficient of x_j in Z

$\underline{c}_B$	basic variables \ c_j	5	2	3	– 1	1	constraint constants
		x_1	x_2	x_3	x_4	x_5	
– 1	x_4	1	2	2	1		8
1	x_5	3	4	1		1	7

THE SIMPLEX TABLEAU

- The optimality check requires the evaluation of

$$\tilde{c}_j = c_j - \left(\underline{c}_B^T \bullet \begin{array}{c} \text{column corresponding} \\ \text{to } x_j \text{ in } \textit{canonical form} \end{array} \right)$$

- For each *nonbasic variable* x_j, for our example, we have

$$\tilde{c}_1 = 5 - [-1, 1] \bullet \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3$$

$$\tilde{c}_2 = 2 - [-1, 1] \bullet \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 0$$

$$\tilde{c}_3 = 3 - [-1, 1] \bullet \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4$$

THE SIMPLEX TABLEAU

- We interpret as the change in Z corresponding to a unit increase in x_j

$\underline{c}_B$	basic variables \ c_j	5	2	3	- 1	1	constraint constants
		x_1	x_2	x_3	x_4	x_5	
- 1	x_4	1	2	2	1		8
1	x_5	3	4	1		1	7
$\underline{\tilde{c}}^T$		3	*0*	4	*0*	*0*	$Z = -1$

SIMPLEX TABLEAU

- Note that the optimality test indicates that

$$\tilde{c}_1 = 3 > 0 \qquad \text{and} \qquad \tilde{c}_3 = 4 > 0$$

  and so the *initial basic feasible solution* is not *optimal*

- Since $\tilde{c}_3 > \tilde{c}_1$, we pick x_3 as the *nonbasic variable* to become a *basic variable*

- We examine the limiting solution for x_3 in the two constraint equations:

THE SIMPLEX TABLEAU

equation	limiting basic variable	upper limit on x_3
1	x_4	$(8/2) = 4$
2	x_5	$(7/1) = 7$

and so the limiting value is

$$min \{ 4, 7 \} = 4$$

- We replace the basic variable x_4 by x_3

SIMPLEX METHODOLOGY EXAMPLE

$$max\ Z = 5x_1 + 2x_2 + 3x_3 - x_4 + x_5$$

$s.t.$

canonical form for x_4 and x_5

$$x_1 + 2x_2 + 2x_3 + x_4 = 8 \qquad (*)$$

$$3x_1 + 4x_2 + x_3 + x_5 = 7 \qquad (**)$$

$$x_i \geq 0 \qquad i = 1, \ldots, 5$$

THE SIMPLEX TABLEAU

- For the new basic feasible solution, we put the equations into canonical form
  - multiplying (*) by $\frac{1}{2}$ to produce (*†)
  - subtract (*†) from (**) to produce (**†)

$$\frac{1}{2}x_1 + x_2 + x_3 + \frac{1}{2}x_4 = 4 \quad (*\dagger)$$

$$\frac{5}{2}x_1 + 3x_2 - \frac{1}{2}x_4 + x_5 = 3 \quad (**\dagger)$$

- The adjacent basic feasible solution is

$$x_1 = x_2 = x_4 = 0 \qquad x_3 = 4,\ x_5 = 3$$

THE SIMPLEX TABLEAU

$\underline{c}_B$	c_j / basic variables	5	2	3	– 1	1	constraint constants
		x_1	x_2	x_3	x_4	x_5	
3	x_3	1/2	1	1	1/2		4
1	x_5	5/2	3		–1/2	1	3
$\tilde{\underline{c}}^T$		1	– 4	*0*	– 2	*0*	$Z = 15$

$x_3 = 4,\ x_5 = 3$

THE SIMPLEX TABLEAU

- Since $\tilde{c}_1 > 0$, the basic feasible solution is non-optimal
- We examine how to bring x_1 into the basis

equation	*limiting basic variable*	*upper limit on* x_1
(*†)	x_3	4/(1/2) = 8
(**†)	x_5	3/(5/2) = 6/5

THE SIMPLEX TABLEAU

- The variable x_1 enters the basis with the value

$$min \left\{ 8, \frac{6}{5} \right\} = \frac{6}{5}$$

  and x_5 is replaced as a basic variable by x_1

- We need to put the equations

$$\frac{1}{2} x_1 + x_2 + x_3 + \frac{1}{2} x_4 = 4 \quad (*\dagger)$$

$$\frac{5}{2} x_1 + 3 x_2 - \frac{1}{2} x_4 + x_5 = 3 \quad (**\dagger)$$

  into canonical form for the *basic variables* x_3 and x_1

THE SIMPLEX TABLEAU

- The following elementary row operations are used

  - multiply (**†) by – 1/5 and add to (*)

$$\frac{2}{5}x_2 + x_3 + \frac{3}{5}x_4 - \frac{1}{5}x_5 = \frac{17}{5}$$

  - multiply (**†) by 2/5

$$x_1 + \frac{6}{5}x_2 - \frac{1}{5}x_4 + \frac{2}{5}x_5 = \frac{6}{5}$$

and construct the corresponding tableau

THE SIMPLEX TABLEAU

$\underline{c}_B$	c_j / basic variables	5	2	3	– 1	1	constraint constants
		x_1	x_2	x_3	x_4	x_5	
3	x_3		2/5	1	3/5	– 1/5	17/5
5	x_1	1	6/5		– 1/5	2/5	6/5
$\underline{\tilde{c}}^T$		0	– 26/5	0	– 9/5	– 2/5	$Z = 81/5$

$\underbrace{}_{16.2\ >\ 15}$

$\tilde{c}_j \leq 0$ **implies optimality**

SIMPLEX TABLEAU EXAMPLE

$$\max \; Z = 3x_1 + 2x_2$$

$s.t.$

$$-x_1 + 2x_2 \leq 4$$

$$3x_1 + 2x_2 \leq 14$$

$$x_1 - x_2 \leq 3$$

$$x_1 \geq 0 \quad x_2 \geq 0$$

SIMPLEX TABLEAU EXAMPLE

- We put this problem into standard form:

$$\max Z = 3x_1 + 2x_2$$

canonical form

$$s.t.$$

$$\begin{aligned} -x_1 + 2x_2 + x_3 \qquad\qquad &= 4 \\ 3x_1 + 2x_2 \qquad + x_4 \qquad &= 14 \\ x_1 - x_2 \qquad\qquad + x_5 &= 3 \end{aligned}$$

$$x_1, \ldots, x_5 \geq 0$$

- x_3, x_4, x_5 are *fictitious* variables

SIMPLEX TABLEAU EXAMPLE

$\underline{c}_B$	c_j / basic variables	3	2	0	0	0	constraint constants
		x_1	x_2	x_3	x_4	x_5	
0	x_3	– 1	2	1			4
0	x_4	3	2		1		14
0	x_5	1	– 1			1	3
	$\underline{\tilde{c}}^T$	3	2	0	0	0	$Z = 0$

$\tilde{c}_j = c_j - \left(\underline{c}_B^T \bullet \text{ column corresponding to } x_j \right)$

SIMPLEX TABLEAU EXAMPLE

- The data in $\underline{\tilde{c}}^T$ indicates that the highest relative profits correspond to x_1 so want to make x_1 a basic variable
- To bring x_1 into the basis requires to evaluate

$$min \left\{ \infty, \frac{14}{3}, 3 \right\} = 3$$

and so x_1 replaces x_5 with the value 3

- We evaluate the basic variable at the adjacent basic feasible solution and convert into canonical form; the new tableau becomes

SIMPLEX TABLEAU EXAMPLE

$\underline{c}_B$	basic variables \ c_j	3 x_1	2 x_2	0 x_3	0 x_4	0 x_5	constraint constants
0	x_3		1	1		1	7
0	x_4		5		1	– 3	5
3	x_1	1	– 1			1	3
$\tilde{\underline{c}}^T$		0	5	0	0	– 3	$Z = 9$

SIMPLEX TABLEAU EXAMPLE

- We reproduce here the calculation of the $\underline{\tilde{c}}^{T}$ components

$$\tilde{c}_{j} = c_{j} - \left(\underline{c}_{B}^{T} \bullet \text{column corresponding to } x_{j} \right)$$

for each nonbasic variable x_j

- Note that $\tilde{c}_{i} = 0$ for each basic variable x_i by definition

SIMPLEX TABLEAU EXAMPLE

- The calculations give

$$\tilde{c}_1 = 0 \text{ by definition since } x_1 \text{ is in the basis}$$

$$\tilde{c}_2 = 2 - \begin{bmatrix} 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix} = 5$$

indicates possible improvement

$$\tilde{c}_3 = 0 \text{ by definition since } x_3 \text{ is in the basis}$$

$$\tilde{c}_4 = 0 \text{ by definition since } x_4 \text{ is in the basis}$$

$$\tilde{c}_5 = 0 - \begin{bmatrix} 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} = -3$$

SIMPLEX TABLEAU EXAMPLE

- Clearly, the only choice is to get x_2 into the basis and so we need to establish the limiting condition from the three equations by evaluating

$$min\{7, 1, \infty\} = 1$$

and so x_2 replaces x_4, which becomes a nonbasic variable

- We need to rewrite the equations into canonical form for x_3 and x_2 and construct the new tableau

SIMPLEX TABLEAU EXAMPLE

$\underline{c}_B$	basic variables \ c_j	3	2	*0*	*0*	*0*	constraint constants
		x_1	x_2	x_3	x_4	x_5	
0	x_3			1	–1/5	8/5	6
2	x_2		1		1/5	–3/5	1
3	x_1	1			1/5	2/5	4
$\underline{\tilde{c}}^T$		*0*	*0*	*0*	– 1		$Z = 14$

$\tilde{c}_j \leq 0 \;\forall\; j \quad \Rightarrow \quad$ ***optimum***

SIMPLEX TABLEAU EXAMPLE

- An optimum is at the solution of

$$
\begin{aligned}
x_3 \quad\quad\quad\quad\quad - \frac{1}{5}x_4 + \frac{8}{5}x_5 &= 6 \\
x_2 \quad\quad + \frac{1}{5}x_4 - \frac{2}{5}x_5 &= 1 \\
x_1 \quad\quad\quad\quad\quad\quad + \frac{1}{5}x_4 + \frac{2}{5}x_5 &= 4
\end{aligned}
$$

canonical form for x_1, x_2 and x_3

given by

$$x_4 = x_5 = 0$$

$$x_3 = 6$$

$$x_2 = 1$$

$$x_1 = 4$$

LINEAR PROGRAMMING EXAMPLE

- Consider the following *LP*

$$\max Z = 3x_1 + 2x_2$$

$$s.t.$$

$$-x_1 + 2x_2 \leq 4$$

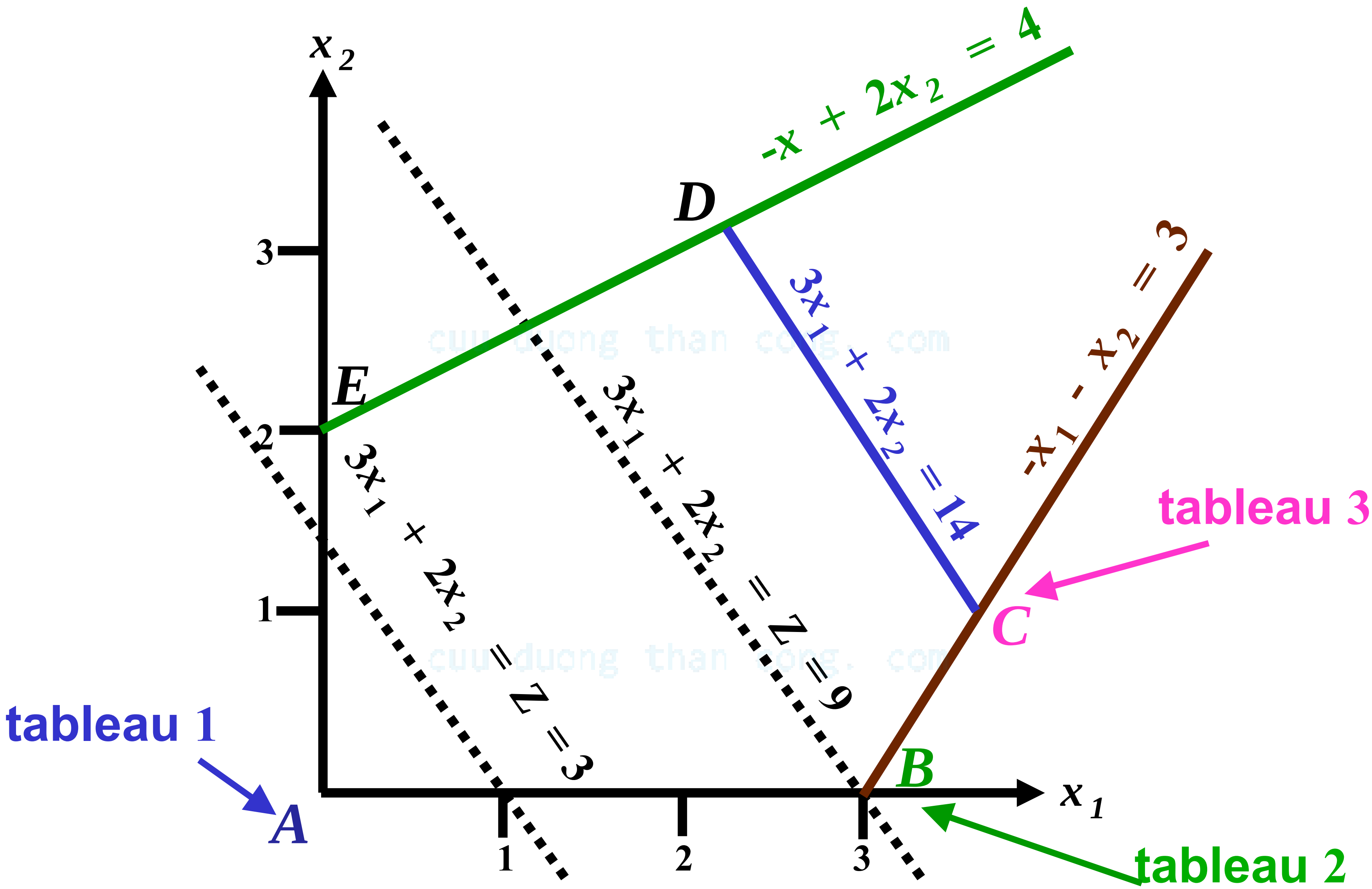
$$3x_1 + 2x_2 \leq 14$$

$$x_1 - x_2 \leq 3$$

$$x_1 \geq 0 \quad x_2 \geq 0$$

- The graphical representation corresponds to

LINEAR PROGRAMMING EXAMPLE



LINEAR PROGRAMMING EXAMPLE

- The tableau approach leads to C which is an optimal solution with

$$x_1 = 4,\ x_2 = 1,\ x_3 = 6,\ x_4 = 0,\ x_5 = 0$$

- Note that any point along CD has $Z = 14$ and as such D is another optimal solution corresponding to an adjacent basic feasible solution
- We may obtain D from C by bringing into the basis the nonbasic variable x_5 in Tableau 3; note that $\tilde{c}_5 = 0$

LINEAR PROGRAMMING EXAMPLE

- We may choose x_5 as a basic variable without effecting Z since the relative profits are 0 ; we compute the limiting value of x_5
- The limit is imposed by x_3 which, consequently, leaves the basis
- The corresponding tableau is:

LINEAR PROGRAMMING EXAMPLE

$\underline{c}_B$	basic variables \ c_j	3	2	0	0	0	constraint
		x_1	x_2	x_3	x_4	x_5	constants
0	x_3			5/8	–1/8	1	15/4
2	x_2		1	3/8	1/8		13/4
3	x_1	1		–1/4	1/4		5/2
$\underline{\tilde{c}}^T$		0	0	0	– 1	0	$Z = 14$

$$\tilde{c}_j \leq 0 \qquad \forall\ j$$

LINEAR PROGRAMMING EXAMPLE

- The adjacent feasible solution is given by

$$x_1 = \frac{5}{2}, \quad x_2 = \frac{13}{4}, \quad x_3 = x_4 = 0, \quad x_5 = \frac{15}{4}$$

- Note that at this basic feasible solution,

$$\tilde{c}_j \leq 0 \quad \forall \; j$$

and so this is also an optimal solution

ALTERNATE OPTIMAL SOLUTION

In general, an alternate optimal solutions is indicated whenever there exists a *nonbasic variable* x_j with $\tilde{c}_j = 0$ in an optimal tableau; such a situation corresponds to a *non unique optimum* for the *LP*

MINIMIZATION *LP*

- Consider a minimization problem

$$\min \qquad Z = \sum_{i=1}^{n} c_i \, x_i$$

$$s.t.$$

$$\underline{A}\underline{x} = \underline{b}$$

$$\underline{x} \geq \underline{0}$$

- In the simplex scheme, replace the optimality check by the following : if each coefficient $\tilde{c}_j$ is ≥ 0 stop; else, select the nonbasic variable with the *most negative* value in $\underline{\tilde{c}}$ to become the new basic variable

MINIMIZATION LP

- Every minimization LP may be solved as a maximization LP because of equivalence

min	$Z = \underline{c}^T \underline{x}$	max	$Z' = (-\underline{c}^T)\,\underline{x}$
$s.t.$		$s.t.$	
	$\underline{A}\,\underline{x} = \underline{b}$		$\underline{A}\,\underline{x} = \underline{b}$
	$\underline{x} \geq \underline{0}$		$\underline{x} \geq \underline{0}$

with the solutions of Z and Z' related by

$$min\{Z\} = -max\{Z'\}$$

COMPLICATIONS IN THE SIMPLEX METHODOLOGY

- Two variables x_j and x_k are tied in the selection of the *nonbasic variables* to replace a current basic variable when $\tilde{c}_j = \tilde{c}_k$; the choice of the new *nonbasic variable* to enter the basis is *arbitrary*
- Two or more constraints may give rise to the same *minimum ratio value* in selecting the basic variable to be replaced
- We consider the example of the following tableau

COMPLICATIONS IN THE SIMPLEX METHODOLOGY

$\underline{c}_B$	basic variables \ c_j	0	0	0	2	0	3/2	constraint constants
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_1	1			1	-1	0	2
0	x_2		1		2	0	1	4
0	x_3			1	1	1	1	3
$\underline{\tilde{c}}^T$		0	0	0	2	0	3/2	$Z = 0$

candidate for basic variable (→ 2 in the x_4 column)

COMPLICATIONS IN THE SIMPLEX METHODOLOGY

- in selecting the *nonbasic variable* x_4 to enter the basis, we observe that the first two constraints give the same minimum ratio: this means that when x_4 is first increased to 2 , both the basic variables x_1 and x_2 will reduce to zero even though only one of them can be made a *nonbasic variable*
- we *arbitrarily* decide to remove x_1 from the basis to get the new *basic feasible solution*:

COMPLICATIONS IN THE SIMPLEX METHODOLOGY

$\underline{c}_B$	basic variables \ c_j	0	0	0	2	0	3/2	constraint constants
		x_1	x_2	x_3	x_4	x_5	x_6	
2	x_4	1			1	– 1		2
0	x_2	– 2	1			2	1	0
0	x_3	– 1		1		1	1	1
$\tilde{\underline{c}}^T$		– 2	0	0	0	0	3/2	$Z = 4$

COMPLICATIONS IN THE SIMPLEX METHODOLOGY

- in the new basic feasible solution

$x_1 = 0, \quad x_2 = 0, \quad x_3 = 1, \quad x_4 = 2, \quad x_5 = 0,$ **and** $x_6 = 0$,

we treat x_2 **as a** ***basic variable*** **whose value is** 0**,**

the same as if it were a ***nonbasic variable***

DEGENERACY

- A *degenerate basic feasible* solution is one where one or more *basic variables* is *0*
- Degeneracy may lead to a number of complications in the simplex approach: an important implication is a minimum ratio of *0* , so that no new *nonbasic variable* maybe included in the basis and therefore the basis remains unchanged
- We consider the following example tableau

COMPLICATIONS IN THE SIMPLEX METHODOLOGY

$\underline{c}_B$	basic variables \ c_j	0	0	0	2	0	3/2	constraint constants
		x_1	x_2	x_3	x_4	x_5	x_6	
2	x_4		1/2		1		1/2	2
0	x_5	– 1	1/2			1	1/2	0
0	x_3	1	– 1	1			0	1
$\tilde{\underline{c}}^T$		0	– 1	0	0	0	1/2	$Z = 4$

DEGENERACY

the logical choice being the *nonbasic variable* x_6 to enter the basis; this leads to finding the limiting constraint from two equations

$$\frac{1}{2}x_6 = 2 - x_4$$

$$\frac{1}{2}x_6 = 0 - x_5$$

and no constraint in the third equation; thus

$$x_6 = min\{4, 0, \infty\}$$

DEGENERACY

- Degeneracy may result in the construction of new tableaus without improvement in the objective function value, thereby reducing the efficiency of the computations: theoretically, an infinite loop, the so-called *cycling*, is possible
- Whenever ties occur in the minimum ratio rule, an *arbitrary* decision is made regarding which *basic variable* is replaced, ignoring the theoretical consequences of degeneracy and cycling

MINIMUM RATIO RULE COMPLICATIONS

- The minimum ratio rule may not be able to determine the basic variable to be replaced: this is the case when all equations lead to ∞ as the limit

- Consider the example and corresponding tableau

$$\max \quad Z = 2x_1 + 3x_2$$

$$s.t.$$

$$\begin{aligned} x_1 - x_2 + x_3 \quad &= 2 \\ -3x_1 + x_2 \quad + x_4 &= 4 \end{aligned}$$

$$x_i \geq 0, \quad i = 1, \ldots, 4$$

MINIMUM RATIO RULE COMPLICATIONS

$\underline{c}_B$	basic variables \ c_j	2	3	*0*	*0*	constraint
		x_1	x_2	x_3	x_4	constants
0	x_3	1	– 1	1		2
0	x_4	– 3	1		1	4
$\underline{\tilde{c}}^T$		2	3	*0*	*0*	$Z = 0$

- The *nonbasic variable* x_2 enters the basis to replacing x_4 and the new tableau is

MINIMUM RATIO RULE COMPLICATIONS

$\underline{c}_B$	basic variables \ c_j	2	3	*0*	*0*	constraint
		x_1	x_2	x_3	x_4	constants
0	x_3	– 2		1	1	6
3	x_2	– 3	1		1	4
$\underline{\tilde{c}}^T$		11	*0*	*0*	– 3	$Z = 12$

- We select x_1 to enter the basis but we are unable to get limiting constraints from the two equations

MINIMUM RATIO RULE COMPLICATIONS

$$-2x_1 + x_3 = 6 \qquad x_1 = \frac{1}{2}x_3 - 3$$

$$-3x_1 + x_2 = 4 \qquad x_1 = \frac{1}{3}x_2 - \frac{4}{3}$$

- In fact, as x_1 increases so do x_2 and x_3 and Z and therefore, the solution is *unbounded*
- The failure of the minimum ratio rule to result in a bound at any simplex tableau implies that the problem has an *unbounded solution*