ECE 307- Techniques for Engineering Decisions

Duality Concepts in Linear Programming

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DUALITY

- **Definition: A *LP* is in *symmetric form* if all the variables are restricted to be *nonnegative* and all the constraints are inequalities of the type:**

objective type	*corresponding inequality type*
max	$\leq$
min	$\geq$

DUALITY DEFINITIONS

- We define the *primal* problem as

$$\max \quad Z = \underline{c}^T \underline{x}$$

$$s.t.$$

$$\underline{A}\,\underline{x} \leq \underline{b} \qquad (P)$$

$$\underline{x} \geq \underline{0}$$

- The *dual* problem is therefore

$$\min \quad W = \underline{b}^T \underline{y}$$

$$s.t.$$

$$\underline{A}^T \underline{y} \geq \underline{c} \qquad (D)$$

$$\underline{y} \geq \underline{0}$$

DUALITY DEFINITIONS

- The problems (P) and (D) are called the *symmetric dual LP* problems

$$\max Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

s.t.

$$\left.\begin{array}{l} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \leq b_1 \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \leq b_2 \\ \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \leq b_m \\ x_1 \geq 0, \quad x_2 \geq 0, \ \ldots, \quad x_n \geq 0 \end{array}\right\} \quad (P)$$

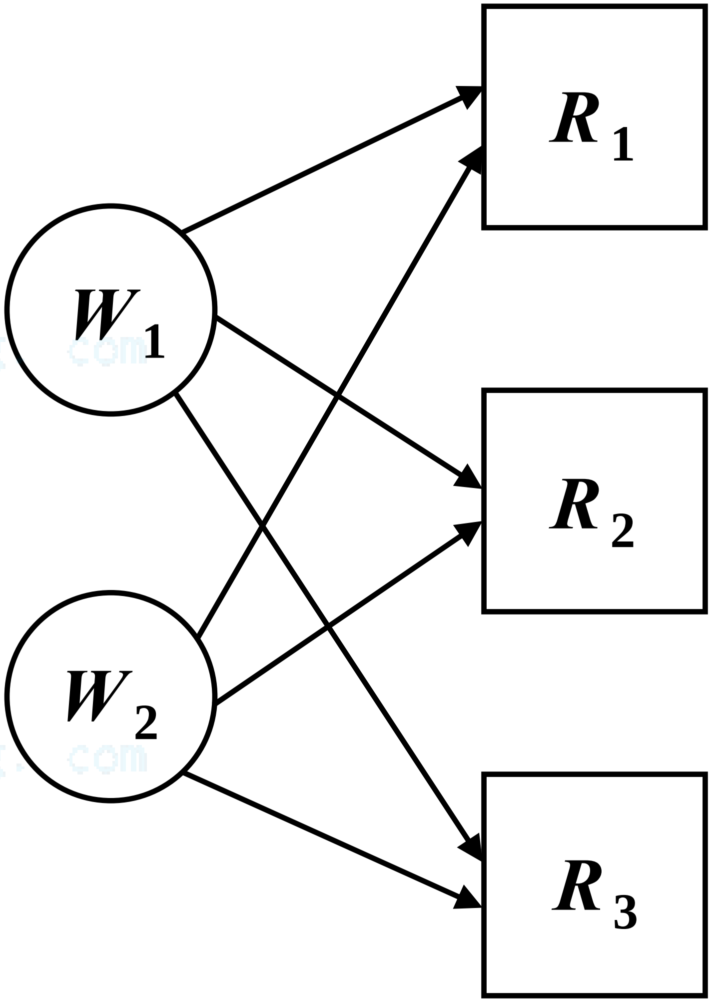
DUALITY DEFINITIONS

$$\left.\begin{aligned}
&\min W = b_1 y_1 + b_2 y_2 + \ldots + b_m y_m \\
&s.t. \\
&a_{11} y_1 + a_{21} y_2 + \ldots + a_{m1} y_m \geq c_1 \\
&a_{12} y_1 + a_{22} y_2 + \ldots + a_{m2} y_m \geq c_2 \\
&\vdots \\
&a_{1n} y_1 + a_{2n} y_2 + \ldots + a_{mn} y_m \geq c_n \\
&y_1 \geq 0, \quad y_2 \geq 0, \ldots, \quad y_m \geq 0
\end{aligned}\right\} (D)$$

EXAMPLE 1: MANUFACTURING TRANSPORTATION PROBLEM

shipment cost coefficients

warehouses	*retail stores*		
	R_1	R_2	R_3
W_1	2	4	3
W_2	5	3	4



EXAMPLE 1: MANUFACTURING TRANSPORTATION PROBLEM

- We are given that the *supplies needed* at Warehouse W_1 and W_2 are

$$W_1 \leq 300$$

$$W_2 \leq 600$$

- We are also specified the *demands needed* at retail stores R_1, R_2, and R_3 as

$$R_1 \leq 200$$

$$R_2 \leq 300$$

$$R_3 \leq 400$$

EXAMPLE 1: MANUFACTURING TRANSPORTATION PROBLEM

- The problem is to determine the *least-cost* shipping schedule

- We define the decision variable

$$x_{ij} \triangleq quantity\ shipped\ from\ W_1\ to\ R_j \quad i = 1, 2 \quad j = 1, 2, 3$$

- The shipping costs $c_{ij} \triangleq$ may be viewed as *element i,j of the transportation cost matrix*

FORMULATION STATEMENT

$$\min Z = \sum_{i=1}^{2}\sum_{j=1}^{3} c_{ij}\, x_{ij} = 2x_{11} + 4x_{12} + 3x_{13} + 5x_{21} + 3x_{22} + 4x_{23}$$

s.t.

$$x_{11} + x_{12} + x_{13} \leq 300$$

$$x_{21} + x_{22} + x_{23} \leq 600$$

$$x_{11} + x_{21} \geq 200$$

$$x_{12} + x_{22} \geq 300$$

$$x_{13} + x_{23} \geq 400$$

$$x_{ij} \geq 0 \quad i = 1, 2 \quad j = 1, 2, 3$$

DUAL PROBLEM SETUP

$$\min Z = \sum_{i=1}^{2} \sum_{j=1}^{3} c_{ij} \, x_{ij}$$

s.t.

$$
\begin{array}{llcccccccl}
y_1 & \leftrightarrow & -x_{11} & -x_{12} & -x_{13} & & & & \geq & -300 \\
y_2 & \leftrightarrow & & & & -x_{21} & -x_{22} & -x_{23} & \geq & -600 \\
y_3 & \leftrightarrow & x_{11} & & & +x_{21} & & & \geq & 200 \\
y_4 & \leftrightarrow & & x_{12} & & & +x_{22} & & \geq & 300 \\
y_5 & \leftrightarrow & & & x_{13} & & & +x_{23} & \geq & 400
\end{array}
$$

$$x_{ij} \geq 0 \quad i = 1, 2 \quad j = 1, 2, 3$$

DUAL PROBLEM SETUP

$$\max W = -300y_1 - 600y_2 + 200y_3 + 300y_4 + 400y_5$$

s.t.

$$\begin{aligned}
-y_1 \quad + y_3 \quad &\leq c_{11} = 2 \\
-y_1 \quad + y_4 \quad &\leq c_{12} = 4 \\
-y_1 \quad + y_5 &\leq c_{13} = 3 \\
-y_2 + y_3 \quad &\leq c_{21} = 5 \\
-y_2 \quad + y_4 \quad &\leq c_{22} = 3 \\
-y_2 \quad + y_5 &\leq c_{23} = 4
\end{aligned}$$

$$y_i \geq 0 \quad i = 1, 2, \ldots, 5$$

INTERPRETATION OF THE DUAL PROBLEM

- The moving company proposes to the manufac-turer to:

buy	all the	300	units at	W_1	at	$y_1 / unit$
buy	all the	600	units at	W_2	at	$y_2 / unit$
sell	all the	200	units at	R_1	at	$y_3 / unit$
sell	all the	300	units at	R_2	at	$y_4 / unit$
sell	all the	400	units at	R_3	at	$y_5 / unit$

- To convince the manufacturer to get the business, the mover ensures that the delivery is for less than the transportation costs the manufacturer would incur (the dual constraints)

INTERPRETATION OF THE DUAL PROBLEM

$$
\begin{aligned}
-y_1 \qquad + y_3 \qquad\qquad\qquad &\le c_{11} = 2 \\
-y_1 \qquad\qquad + y_4 \qquad\quad &\le c_{12} = 4 \\
-y_1 \qquad\qquad\qquad + y_5 &\le c_{13} = 3 \\
-y_2 + y_3 \qquad\qquad\qquad &\le c_{21} = 5 \\
-y_2 \qquad + y_4 \qquad\quad &\le c_{22} = 3 \\
-y_2 \qquad\qquad + y_5 &\le c_{23} = 4
\end{aligned}
$$

- The mover wishes to maximize profits, i.e.,

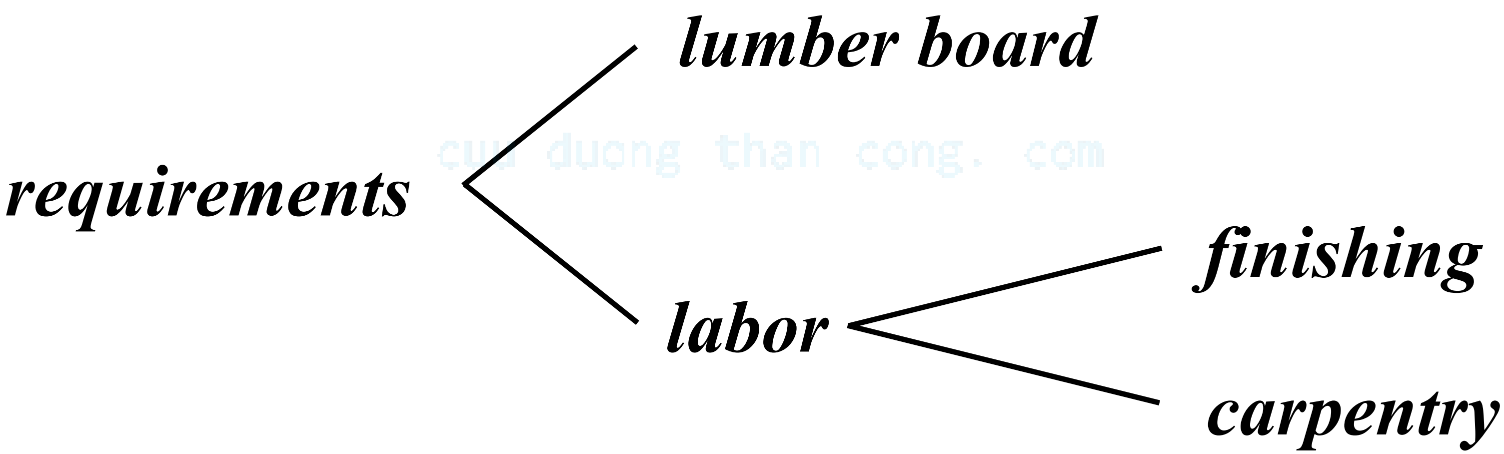
$$revenues - costs \Rightarrow dual\ cost\ objective\ function$$

$$\max W = -300\, y_1 - 600\, y_2 + 200\, y_3 + 300\, y_4 + 400\, y_5$$

EXAMPLE 2: FURNITURE PRODUCTS

- Resource requirements

item	*sales price ($)*
desks	60
tables	30
chairs	20



EXAMPLE 2: FURNITURE PRODUCTS

- The Dakota Furniture Company manufacturing:

resource	*desk*	*table*	*chair*	*available*
lumber board (ft)	8	6	1	48
finishing (h)	4	2	1.5	20
carpentry (h)	2	1.5	0.5	8

- We assume that the demand for desks, tables and chairs is unlimited and the available resources are already purchased
- The decision problem is to maximize *total revenues*

PRIMAL AND DUAL PROBLEM FORMULATION

- We define decision variables

$$x_1 = \textit{number of desks produced}$$

$$x_2 = \textit{number of tables produced}$$

$$x_3 = \textit{number of chairs produced}$$

- The Dakota problem is

$$\max \quad Z = 60x_1 + 30x_2 + 20x_3$$

$$s.t.$$

$$y_1 \leftrightarrow 8x_1 + 6x_2 + x_3 \le 48 \qquad \textit{lumber}$$

$$y_2 \leftrightarrow 4x_1 + 2x_2 + 1.5x_3 \le 20 \qquad \textit{finishing}$$

$$y_3 \leftrightarrow 2x_1 + 1.5x_2 + 0.5x_3 \le 8 \qquad \textit{carpentry}$$

$$x_1, x_2, x_3 \ge 0$$

PRIMAL AND DUAL PROBLEM FORMULATION

- The dual problem is

$$min \quad W = 48y_1 + 20y_2 + 8y_3$$

$$s.t.$$

$$8y_1 + 4y_2 + 2y_3 \geq 60 \quad desk$$

$$6y_1 + 2y_2 + 1.5y_3 \geq 30 \quad table$$

$$y_1 + 1.5y_2 + 0.5y_3 \geq 20 \quad chair$$

$$y_1, y_2, y_3 \geq 0$$

PRIMAL AND DUAL PROBLEM FORMULATION

$$\max \quad Z = 60x_1 + 30x_2 + 20x_3$$

$$
\begin{aligned}
y_1 &\leftrightarrow & 8x_1 &+ & 6x_2 &+ & x_3 &\leq 48 & \textit{lumber}\\
y_2 &\leftrightarrow & 4x_1 &+ & 2x_2 &+ & 1.5x_3 &\leq 20 & \textit{finishing}\\
y_3 &\leftrightarrow & 2x_1 &+ & 1.5x_2 &+ & 0.5x_3 &\leq 8 & \textit{carpentry}
\end{aligned}
$$

$$x_1, x_2, x_3 \geq 0$$

$$\min \quad W = 48y_1 + 20y_2 + 8y_3$$

$$
\begin{aligned}
8y_1 &+ & 4y_2 &+ & 2y_3 &\geq 60 & \textit{desk}\\
6y_1 &+ & 2y_2 &+ & 1.5y_3 &\geq 30 & \textit{table}\\
y_1 &+ & 1.5y_2 &+ & 0.5y_3 &\geq 20 & \textit{chair}
\end{aligned}
$$

$$y_1, y_2, y_3 \geq 0$$

INTERPRETATION OF THE DUAL PROBLEM

- An entrepreneur wishes to purchase all of Dakota's resources
- He thus needs to determine the price to pay for each unit of each resource

$$y_1 = \textit{price paid for } 1 \textit{ lumber board ft}$$

$$y_2 = \textit{price paid for } 1 \textit{ h of finishing}$$

$$y_3 = \textit{price paid for } 1 \textit{ h of carpentry}$$

- We solve the Dakota dual problem to determine y_1, y_2, and y_3

INTERPRETATION OF THE DUAL PROBLEM

- To induce Dakota to sell the raw resources, the resource prices must be set sufficiently high
- For example, the entrepreneur must offer Dakota at least $\$\,60$ for a combination of resources that includes $8\ ft$ of lumber board, $4\ h$ of finishing and $2\ h$ of carpentry since Dakota could use this combination to sell a desk for $\$\,60$: this consideration implies the construction of the dual constraint

$$8y_1 + 4y_2 + 2y_3 \geq 60$$

INTERPRETATION OF DUAL PROBLEM

- In the same way we obtain the two additional constraints for a table and for a chair

- The i^{th} primal variable corresponds to the i^{th} constraint in the dual problem statement

- The j^{th} dual variable corresponds to the j^{th} constraint in the primal problem statement

EXAMPLE 3: DIET PROBLEM

- A new diet requires that all food eaten come from one of the four "basic food groups": chocolate cake, ice cream, soda and cheesecake
- The four foods available for consumption are as given in the table
- Minimum requirements for each day are:
  - 500 *cal*
  - 6 *oz* chocolate
  - 10 *oz* sugar
  - 8 *oz* fat

EXAMPLE 3: DIET PROBLEM

food	*calories*	*chocolate (oz)*	*sugar (oz)*	*fat (oz)*	*costs (cents)*
brownie	400	3	2	2	50
chocolate ice cream (scoop)	200	2	2	4	20
cola (bottle)	150	*0*	4	1	30
pineapple cheesecake (piece)	500	*0*	4	5	80

PROBLEM FORMULATION

- Objective of the problem is to minimize the costs of the diet
- Decision variables are defined for each day's purchases

$$x_1 = \textit{number of brownies}$$

$$x_2 = \textit{number of chocolate ice cream scoops}$$

$$x_3 = \textit{number of bottles of soda}$$

$$x_4 = \textit{number of pineapple cheesecake pieces}$$

PROBLEM FORMULATION

- The problem statement is

$$\min \quad Z = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

s.t.

$$400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500 \; cal$$

$$3x_1 + 2x_2 \geq 6 \; oz$$

$$2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10 \; oz$$

$$2x_1 + 4x_2 + x_3 + 5x_4 \geq 8 \; oz$$

$$x_i \geq 0 \qquad i = 1,4$$

EXAMPLE 3: DIET PROBLEM

- The dual problem is

$$\max \quad W = 500\,y_1 + 6\,y_2 + 10\,y_3 + 8\,y_4$$

s.t.

$$400\,y_1 + 3\,y_2 + 2\,y_3 + 2\,y_4 \leq 50 \quad \text{brownie}$$

$$200\,y_1 + 2\,y_2 + 2\,y_3 + 4\,y_4 \leq 20 \quad \text{ice-cream}$$

$$150\,y_1 + 4\,y_3 + y_4 \leq 30 \quad \text{soda}$$

$$500\,y_1 + 4\,y_3 + 5\,y_4 \leq 80 \quad \text{cheesecake}$$

$$y_1, y_2, y_3, y_4 \geq 0$$

INTERPRETATION OF THE DUAL

- We consider a sales person of “nutrients” who is interested in assuming that each dieter meets daily requirements by purchasing calories, sugar, fat and chocolate

- The key decision is to determine the prices

$$y_i \;=\; \textit{price per unit to sell to dieters}$$

- Objective of sales person is to set the prices y_i so as to maximize revenues from selling to the dieter the daily ration of required nutrients

INTERPRETATION OF DUAL

- **Now, the dieter can purchase a brownie for 50 ¢ and have 400 *cal*, 30 *oz* of chocolate, 2 *oz* of sugar and 2 *oz* of fat**
- **Salesperson must set y_i sufficiently low to entice the buyer to get the required nutrients from the brownie:**

$$400 y_1 + 3 y_2 + 2 y_3 + 2 y_4 \leq 50 \longleftarrow \textit{brownie constraint}$$

- **We derive similar constraints for the ice cream, the soda and the cheesecake**

DUAL PROBLEMS

$$\left.\begin{aligned} \max \quad & Z = \underline{c}^T \underline{x} \\ \text{s.t.} \quad & \\ & \underline{A}\,\underline{x} \le \underline{b} \\ & \underline{x} \ge \underline{0} \end{aligned}\right\} (P)$$

$$\left.\begin{aligned} \min \quad & W = \underline{b}^T \underline{y} \\ \text{s.t.} \quad & \\ & \underline{A}^T \underline{y} \ge \underline{c} \\ & \underline{y} \ge \underline{0} \end{aligned}\right\} (D)$$

WEAK DUALITY THEOREM

- For any $\underline{x}$ feasible for (P) and any $\underline{y}$ feasible for (D)

$$\underline{c}^T \underline{x} \leq \underline{b}^T \underline{y}$$

□

- Proof:

$$\underline{A}^T \underline{y} \geq \underline{c} \Rightarrow \underline{c}^T \leq \underline{y}^T \underline{A} \Rightarrow \underline{c}^T \underline{x} \leq \underline{y}^T \underline{A}\underline{x}$$

$$\underline{c}^T \underline{x} \leq \underline{y}^T \underline{A}\underline{x} \leq \underline{y}^T \underline{b} = \underline{b}^T \underline{y}$$

□

COROLLARY 1 OF THE WEAK DUALITY THEOREM

$\underline{x}$ feasible for $(P) \Rightarrow \underline{c}^T \underline{x} \leq \underline{y}^T \underline{b}$

for any feasible $\underline{y}$ for (D)

$$\underline{c}^T \underline{x} \leq \underline{y}^{*T} \underline{b} = \min W$$

for any feasible $\underline{x}$ for (P),

$$\underline{c}^T \underline{x} \leq \min W$$

COROLLARY 2 OF THE WEAK DUALITY THEOREM

$\underline{y}$ feasible for (D) $\Rightarrow$ $\underline{c}^T \underline{x} \leq \underline{y}^T \underline{b}$

for every feasible $\underline{x}$ for (P)

$$max\, Z = max\, \underline{c}^T \underline{x} = \underline{c}^T \underline{x}^* < \underline{y}^{*T} \underline{b}$$

for any feasible $\underline{y}$ of (D),

$$\underline{y}^T \underline{b} \geq max\, Z$$

COROLLARIES 3 AND 4 OF THE WEAK DUALITY THEOREM

If (P) is feasible and $max\ Z$ is unbounded, i.e.,

$$Z \rightarrow +\infty \quad ;$$

then, (D) has no feasible solution

If (D) is feasible and $min\ Z$ is unbounded, i.e.,

$$Z \rightarrow -\infty \quad ;$$

then, (P) is infeasible

DUALITY THEOREM APPLICATION

- Consider the maximization problem

$$\max Z = x_1 + 2x_2 + 3x_3 + 4x_4 = \underbrace{[1,2,3,4]}_{\underline{c}^T} \underline{x}$$

s.t.

$$\underbrace{\begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 1 & 3 & 2 \end{bmatrix}}_{\underline{A}} \underline{x} \leq \underbrace{\begin{bmatrix} 20 \\ 20 \end{bmatrix}}_{\underline{b}}$$

$$\underline{x} \geq \underline{0}$$

(P)

DUALITY THEOREM APPLICATION

- The corresponding dual is given by

$$\left.\begin{array}{ll} \min & W = \underline{b}^T \underline{y} \\ s.t. & \\ & \underline{A}^T \underline{y} \geq \underline{c} \\ & \underline{y} \geq \underline{0} \end{array}\right\} \quad (D)$$

- With the appropriate substitutions, we have

DUALITY THEOREM APPLICATION

$$\min \quad W = 20y_1 + 20y_2$$

s.t.

$$y_1 + 2y_2 \geq 1$$

$$2y_1 + y_2 \geq 2$$

$$2y_1 + 3y_2 \geq 3$$

$$3y_1 + 2y_2 \geq 4$$

$$y_1 \geq 0, \quad y_2 \geq 0$$

GENERALIZED FORM OF THE DUAL

- Consider the primal decision

$$x_i = 1,\ i = 1, 2, 3, 4\ ;$$

decision is feasible for (P) with

$$Z = \underline{c}^T \underline{x} = 10$$

- The dual decision

$$y_i = 1, \quad i = 1, 2$$

is feasible for (D) with

$$W = \underline{b}^T \underline{y} = 40$$

DUALITY THEOREM APPLICATION

- Clearly,

$$Z\left(x_1, x_2, x_3, x_4\right) = 10 < 40 = W\left(y_1, y_2\right)$$

and so clearly, the feasible decision for (P) and (D) satisfy the *Weak Duality Theorem*

- Moreover, we have

corollary 1 $\Rightarrow 10 \leq \min W = W\left(y_1^*, y_2^*\right)$

corollary 2 $\Rightarrow \max Z = Z\left(x_1^*, x_2^*, x_3^*, x_4^*\right) \leq \underline{b}^T \underline{y} = 40$

CORROLARIES 5 AND 6

(P) is feasible and (D) is infeasible, then,

(P) is unbounded

(D) is feasible and (P) is infeasible, then,

(D) is unbounded

EXAMPLE

- Consider the primal dual problems:

$$\left.\begin{aligned} &\max Z = x_1 + x_2 \\ &s.t. \\ &-x_1 + x_2 + x_3 \le 2 \\ &-2x_1 + x_2 - x_3 \le 1 \\ &x_1, x_2, x_3 \ge 0 \end{aligned}\right\} (P) \qquad \left.\begin{aligned} &\min W = 2y_1 + y_2 \\ &s.t. \\ &-y_1 - 2y_2 \ge 1 \\ &y_1 + y_2 \ge 1 \\ &y_1 - y_2 \ge 0 \\ &y_1, y_2 \ge 0 \end{aligned}\right\} (D)$$

- Now

$$\underline{x} = \underline{0} \text{ is feasible for } (P)$$

EXAMPLE

but

$$-y_1 - 2y_2 \geq 1$$

is impossible for (D) since it is inconsistent with

$$y_1, y_2 \geq 0$$

- Since (D) infeasible, it follows from Corollary 5 that

$$Z \rightarrow \infty$$

- You should be able to show this result by solving (P) using the simplex scheme

OPTIMALITY CRITERION THEOREM

- We consider the primal-dual problems (P) and (D) with

$$\left.\begin{array}{l} \underline{x}^{0} \text{ is feasible for } (P) \\ \underline{y}^{0} \text{ is feasible for } (D) \\ \underline{c}^{T}\underline{x}^{0} = \underline{b}^{T}\underline{y}^{0} \end{array}\right\} \Rightarrow \begin{array}{l} \underline{x}^{0} \text{ is optimal for } (P) \\ \text{and} \\ \underline{y}^{0} \text{ is optimal for } (D) \end{array}$$

- We next provide the proof:

$$\begin{array}{l} \underline{x}^{0} \text{ is feasible for } (P) \\ \underline{y}^{0} \text{ is feasible for } (D) \end{array} \overset{Weak\ Duality}{\underset{Theorem}{\Rightarrow}} \underline{c}^{T}\underline{x}^{0} \leq \underline{b}^{T}\underline{y}^{0}$$

OPTIMALITY CRITERION THEOREM

but we are given that

$$\underline{c}^T \underline{x}^0 = \underline{b}^T \underline{y}^0$$

and so it follows that

$$\forall \text{ feasible } \underline{x}, \ \underline{y}^0 \text{ feasible} \qquad \underline{c}^T \underline{x} \leq \underline{b}^T \underline{y}^0 = \underline{c}^T \underline{x}^0$$

and therefore $\underline{x}^0$ is optimal ; similarly

$$\forall \text{ feasible } \underline{y}, \ \underline{x}^0 \text{ feasible} \qquad \underline{b}^T \underline{y} \geq \underline{c}^T \underline{x}^0 = \underline{b}^T \underline{y}^0$$

so it follows that $\underline{y}^0$ is optimal

MAIN DUALITY THEOREM

(P) is feasible and (D) is feasible; then,

$$\exists\ \underline{x}^{*}\ \textit{feasible for}\ (P)\ \textit{which is optimal and}$$

$$\exists\ \underline{y}^{*}\ \textit{feasible for}\ (D)\ \textit{which is optimal such that}$$

$$\underline{c}^{T}\underline{x}^{*} = \underline{b}^{T}\underline{y}^{*}$$

COMPLEMENTARY SLACKNESS CONDITIONS

- $\underline{x}^*$ and $\underline{y}^*$ are optimal for (P) and (D) respectively, if and only if

$$0 = \left(\underline{y}^{*T} \underline{A} - \underline{c}^T \right) \underline{x}^* + \underline{y}^{*T} \left(\underline{b} - \underline{A}\, \underline{x}^* \right)$$

$$= \underline{y}^{*T} \underline{b} - \underline{c}^T \underline{x}^*$$

- We prove this equivalence result by defining the slack variables $\underline{u} \in \mathbb{R}^m$ and $\underline{v} \in \mathbb{R}^n$ such that $\underline{x}$ and $\underline{y}$ are feasible; at the optimum,

$$\underline{A}\, \underline{x}^* + \underline{u}^* = \underline{b} \qquad \underline{x}^*, \underline{u}^* \geq \underline{0}$$

$$\underline{A}^T \underline{y}^* - \underline{v}^* = \underline{c} \qquad \underline{y}^*, \underline{v}^* \geq \underline{0}$$

COMPLEMENTARY SLACKNESS CONDITIONS

where the optimal values of the slack variables

$\underline{u}^{*}$ and $\underline{v}^{*}$ correspond to the optimal values

$\underline{x}^{*}$ and $\underline{y}^{*}$

- Now,

$$\underline{y}^{*T}\underline{A}\underline{x}^{*} + \underline{y}^{*T}\underline{u}^{*} = \underline{y}^{*T}\underline{b} = \underline{b}^{T}\underline{y}^{*}$$

$$\underbrace{\underline{x}^{*T}\underline{A}^{T}\underline{y}^{*}}_{\underline{y}^{*T}\underline{A}\underline{x}^{*}} - \underline{x}^{*T}\underline{v}^{*} = \underline{x}^{*T}\underline{c} = \underline{c}^{T}\underline{x}^{*}$$

COMPLEMENTARY SLACKNESS CONDITIONS

- This implies that

$$\underline{y}^{*T}\underline{u}^{*} + \underline{v}^{*T}\underline{x}^{*} = \underline{b}^{T}\underline{y}^{*} - \underline{c}^{T}\underline{x}^{*}$$

- We need to prove optimality which is true if and only if

$$\underline{y}^{*T}\underline{u}^{*} + \underline{v}^{*T}\underline{x}^{*} = 0$$

COMPLEMENTARY SLACKNESS CONDITIONS

- However,

$$\underline{x}^*, \underline{y}^* \text{ are optimal}$$

$$\xRightarrow[\text{Duality Theorem}]{\text{Main}}$$

$$\underline{c}^T \underline{x}^* = \underline{b}^T \underline{y}^* \Rightarrow \underline{y}^{*T} \underline{u}^* + \underline{v}^{*T} \underline{x}^* = 0$$

- Also,

$$\underline{y}^{*T} \underline{u}^* + \underline{v}^{*T} \underline{x}^* = 0 \Rightarrow \underline{b}^T \underline{y}^* = \underline{c}^T \underline{x}^*$$

$$\xRightarrow[\text{Criterion Theorem}]{\text{Optimality}}$$

$$\underline{x}^* \text{ is optimal for } (P) \text{ and } \underline{y}^* \text{ is optimal for } (D)$$

COMPLEMENTARY SLACKNESS CONDITIONS

- Note that

$$\underline{x}^*, \underline{y}^*, \underline{u}^*, \underline{v}^* > 0 \Rightarrow \text{component-wise each element} \geq 0$$

$$\underline{y}^{*T}\underline{u}^* + \underline{v}^*\underline{x}^* = 0 \Rightarrow y_i^* u_i^* = 0 \ \forall i = 1, \dots, m \ \text{ and}$$

$$v_j^* x_j^* = 0 \ \forall j = 1, \dots, n$$

- At the optimum,

$$y_i^* \left(b_i - \sum_{j=1}^{n} a_{ij} x_j^* \right) = 0 \quad i = 1, \dots, m$$

and

$$x_j^* \left(\sum_{i=1}^{m} a_{ji} y_i^* - c_j \right) = 0 \quad j = 1, \dots, n$$

COMPLEMENTARY SLACKNESS CONDITIONS

- Hence, for $i = 1, 2, \ldots, m$

$$y_i^* > 0 \Rightarrow b_i = \sum_{j=1}^{n} a_{ij} x_j^*$$

and

$$b_i - \sum_{j=1}^{n} a_{ij} x_j^* > 0 \Rightarrow y_i^* = 0$$

- Similarly for $j = 1, 2, \ldots, n$

$$x_j^* > 0 \Rightarrow \sum_{i=1}^{m} a_{ji} y_i^* = c_j$$

and

$$\sum_{i=1}^{m} a_{ji} y_i^* - c_j > 0 \Rightarrow x_j^* = 0$$

EXAMPLE

$$\max \quad Z = x_1 + 2x_2 + 3x_3 + 4x_4$$

$s.t.$

$$x_1 + 2x_2 + 2x_3 + 3x_4 \leq 20 \qquad (P)$$

$$2x_1 + x_2 + 3x_3 + 2x_4 \leq 20$$

$$x_i \geq 0 \quad i = 1, \ldots, 4$$

EXAMPLE

$$\min \quad W = 20y_1 + 20y_2$$

$$s.t.$$

$$y_1 + 2y_2 \geq 1$$

$$2y_1 + y_2 \geq 2$$

$$(D)$$

$$2y_1 + 3y_2 \geq 3$$

$$3y_1 + 2y_2 \geq 4$$

$$y_1 \geq 0$$

EXAMPLE

$\underline{x}^*, \underline{y}^*$ *optimal* $\Rightarrow$

$$y_1^*\left(20 - x_1^* - 2x_2^* - 2x_3^* - 3x_4^*\right) = 0$$

$$y_2^*\left(20 - 2x_1^* - x_2^* - 3x_3^* - 2x_4^*\right) = 0$$

$\underline{y}^* = \begin{bmatrix} 1.2 \\ 0.2 \end{bmatrix}$ **is given as an optimal solution with**

$$min\ W = 28$$

$\Rightarrow$

EXAMPLE

$$x_1^* + 2x_2^* + 2x_3^* + 3x_4^* = 20$$

$$2x_1^* + x_2^* + 3x_3^* + 2x_4^* = 20$$

$$y_1^* + 2y_2^* = 1.2 + 0.4 > 1 \Rightarrow x_1^* = 0$$

$$2y_1^* + y_2^* = 2.4 + 0.2 > 2 \Rightarrow x_2^* = 0$$

$$2y_1^* + 3y_2^* = 2.4 + 0.6 = 3$$

$$3y_1^* + 2y_2^* = 3.6 + 0.4 = 4$$

so that

$$\left.\begin{aligned} 2x_3^* + 3x_4^* &= 20 \quad \Rightarrow \quad x_3^* = 4 \\ 3x_3^* + 2x_4^* &= 20 \quad \Rightarrow \quad x_4^* = 4 \end{aligned}\right\} \max Z = 28$$

USES OF THE COMPLEMENTARY SLACKNESS CONDITION

- **Key applications use**
  - **finding optimal (P) solution given optimal (D) solution and vice versa**
  - **verification of optimality of solution (whether a feasible solution is optimal)**
- **We can start with a feasible solution and attempt to construct an optimal dual solution; if we succ-eed, then the feasible primal solution is *optimal***

DUALITY

$$
(P)\quad \begin{cases} \max & Z = \underline{c}^T \underline{x} \\ s.t. & \\ & \underline{A}\,\underline{x} \le \underline{b} \\ & \underline{x} \ge \underline{0} \end{cases}
$$

$$
(D)\quad \begin{cases} \min & W = \underline{b}^T \underline{y} \\ s.t. & \\ & \underline{A}^T \underline{y} \ge \underline{c} \\ & \underline{y} \ge \underline{0} \end{cases}
$$

DUALITY

- **Suppose the primal problem is minimization, then,**

$$\min \quad Z = \underline{c}^T \underline{x} \qquad (P)$$

$$\text{s.t.} \quad \underline{A}\,\underline{x} \geq \underline{b}$$

$$\underline{x} \geq \underline{0}$$

$$\max \quad W = \underline{b}^T \underline{y}$$

$$\text{s.t.} \qquad (D)$$

$$\underline{A}^T \underline{y} \leq \underline{c}$$

$$\underline{y} \geq \underline{0}$$

INTERPRETATION

- The economic interpretation is

$$Z^* = \max Z = \underline{c}^T \underline{x}^* = \underline{b}^T \underline{y}^* = W^* = \min W$$

$$\left.\begin{array}{l} b_i - \text{constrained resource quantities}, \\ y_i^* - \text{optimal dual variables} \end{array}\right\} i = 1, 2, \dots, m$$

- Suppose,

$$b_i \rightarrow b_i + \Delta b_i \Rightarrow \Delta Z = y_i^* \Delta b_i$$

- In words, the optimal dual variable for each primal constraint gives the net change in the optimal value of the objective function Z for a one unit change in the constraint on resources

INTERPRETATION

- Economists refer to this as a *shadow price* on the constraint resource
- The *shadow price* determines the value/worth of having an additional quantity of a resource
- In the previous example, the optimal dual variables indicate that the worth of another unit of resource 1 is 1.2 and that of another unit of resource 2 is 0.2

GENERALIZED FORM OF THE DUAL

- We start out with

$$\left.\begin{array}{lcccc} \max & Z & = & \underline{c}^T & \underline{x} \\ s.t. & & & & \\ & \underline{A}\,\underline{x} & = & \underline{b} & \\ & \underline{x} & \geq & \underline{0} & \end{array}\right\} (P)$$

- To find (D), we first put (P) in symmetric form

$$\begin{array}{lcrcr} \underline{y}_+ & \leftrightarrow & \underline{A}\underline{x} & \leq & \underline{b} \\ \underline{y}_- & \leftrightarrow & -\underline{A}\underline{x} & \leq & -\underline{b} \\ & & \underline{x} & \geq & \underline{0} \end{array} \qquad \begin{bmatrix} \underline{A} \\ -\underline{A} \end{bmatrix} \underline{x} \leq \begin{bmatrix} \underline{b} \\ -\underline{b} \end{bmatrix} \quad \textit{symmetric form}$$

GENERALIZED FORM OF THE DUAL

- Let

$$\underline{y} = \underline{y}_{+} - \underline{y}_{-}$$

- We rewrite the problem as

$$\min W = \underline{b}^{T} \underline{y}$$

$$s.t.$$

$$\underline{A}^{T} \underline{y} \geq \underline{c}$$

$$\underline{y} \text{ is unsigned}$$

- The *c.s.* conditions apply

$$\underline{x}^{*T} \left(\underline{A}^{T} \underline{y}^{*} - \underline{c} \right) = \underline{0}$$

EXAMPLE 5

$$\max Z = x_1 - x_2 + x_3 - x_4$$

s.t.

$$
\begin{array}{llr}
y_1 \leftrightarrow & x_1 + x_2 + x_3 + x_4 = & 8 \\
y_2 \leftrightarrow & x_1 \leq & 8 \\
y_3 \leftrightarrow & x_2 \leq & 4 \\
y_4 \leftrightarrow & -x_2 \leq & 4 \\
y_5 \leftrightarrow & x_3 \leq & 4 \\
y_6 \leftrightarrow & -x_3 \leq & 2 \\
y_7 \leftrightarrow & x_4 \leq & 10
\end{array}
\qquad (P)
$$

$$x_1, x_4 \geq 0$$

$$x_2, x_3 \ \textit{unsigned}$$

EXAMPLE 5

$$\min W = 8y_1 + 8y_2 + 4y_3 + 4y_4 + 4y_5 + 2y_6 + 10y_7$$

s.t.

$$\left.\begin{array}{llllllllll} x_1 \leftrightarrow & y_1 & + y_2 & & & & & & \geq & 1 \\ x_2 \leftrightarrow & y_1 & & + y_3 & - y_4 & & & & = & -1 \\ x_3 \leftrightarrow & y_1 & & & & + y_5 & - y_6 & & = & 1 \\ x_4 \leftrightarrow & y_1 & & & & & & + y_7 & \geq & -1 \end{array}\right\} (D)$$

$$y_2, \ldots, y_7 \geq 0$$

$$y_1 \;\; unsigned$$

EXAMPLE 5

- We are given that

$$\underline{x}^{*} = \begin{bmatrix} 8 \\ -4 \\ 4 \\ 0 \end{bmatrix}$$

is optimal for (P)

- Then the *c.s.* conditions obtain

$$x_1^{*}\left(y_1^{*} + y_2^{*} - 1\right) = 0$$

$$x_1^{*} = 8 > 0 \Rightarrow y_1^{*} + y_2^{*} = 1$$

EXAMPLE 5

- The other *c.s.* conditions obtain

$$y^*_i \left(\sum_{j=1}^{4} a_{ij} \, x^*_j - b_i \right) = 0$$

- Now, $x^*_4 = 0$ implies $x^*_4 - 10 < 0$ and so

$$y^*_7 = 0$$

- Also, $x^*_3 = 4$ implies

$$y^*_6 = 0$$

- Similarly, the *c.s.* conditions

$$x^*_j \left(\sum_{i=1}^{7} a_{ji} \, y^*_i - c_j \right) = 0$$

EXAMPLE 5

have implications on the y_i^* variable

- Since, $x_2^* = -4$ then, we have

$$y_3^* = 0$$

- Now, with $y_7^* = 0$ we have

$$y_1^* > -1$$

- Since, $x_1^* = 8$ we have

$$y_2^* = 1 - y_1^*$$

EXAMPLE 5

- Suspose

$$y_1^* = 1$$

and so,

$$y_2^* = 0$$

- Furthermore,

$$y_1^* + y_3^* - y_4^* = 1 - y_4^* = -1$$

implies

$$y_4^* = 2$$

EXAMPLE 5

- **Also**

$$y_1^* + y_5^* - y_6^* = 1$$

implies

$$1 + y_5^* = 1$$

and so

$$y_5^* = 0$$

EXAMPLE 5

- Therefore

$$W\left(\underline{y}^{*}\right) = (8)(1)+(8)(0)+(4)(0)+(4)(2)+$$

$$(4)(0)+(2)(0)+(10)(0)$$

$$= 16$$

and so

$$W^{*} = 16 = Z^{*} \Leftrightarrow$$ **optimality of** (P) **and** (D)

PRIMAL – DUAL TABLE

primal (maximize)	*dual* (minimize)
$\underline{A}$ (coefficient matrix)	$\underline{A}^T$ (transpose of the coefficient matrix)
$\underline{b}$ (right-hand side vector)	$\underline{b}$ (cost vector)
$\underline{c}$ (price vector)	$\underline{c}$ (right hand side vector)
i^{th} constraint is $=$ type	the dual variable y_i is unrestricted in sign
i^{th} constraint is $\leqq$ type	the dual variable $y_i \geqq 0$
i^{th} constraint is $\geqq$ type	the dual variable $y_i \leqq 0$
x_j is unrestricted	j^{th} dual constraint is $=$ type
$x_j \geqq 0$	j^{th} dual constraint is $\geqq$ type
$x_j \leqq 0$	j^{th} dual constraint is $\leqq$ type